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Characterization of compact subsets of curves with ω-continuous derivatives

by

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Abstract. We give a characterization of compact subsets of finite unions of disjoint finite-length curves in $\mathbb{R}^n$ with ω-continuous derivative and without self-intersections. Intuitively, our condition can be formulated as follows: *there exists a finite set of* regular *curves covering a compact set K iff every triple of points of K behaves like a triple of points of a* regular *curve.*

This work was inspired by theorems by Jones, Okikiolu, Schul and others that characterize compact subsets of rectifiable or Ahlfors-regular curves. However, their classes of curves are much wider than ours and therefore the condition we obtain and our methods are different.

1. Introduction

Notation. By $\mathbb{R}^n$ we denote the standard n-dimensional Euclidean space equipped with the l_2 norm $|\cdot|$ and the scalar product $\langle\cdot,\cdot\rangle$. Given a concave non-decreasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ and continuous at 0, we say that a function $f : \mathbb{R}^m \to \mathbb{R}^k$ is *ω-continuous* if for any different $x, y \in \mathbb{R}^m$ we have $|f(x) - f(y)| < \omega(|x - y|)$. We say that $f : \mathbb{R}^m \to \mathbb{R}^k$ is *r-locally ω-continuous* for some $r > 0$ if the aforementioned inequality holds whenever $|x - y| < r$.

Our results. In this paper we focus on characterizing compact subsets of regular curves in $\mathbb{R}^n$. We give a characterization of compact subsets of finite unions of disjoint embedded curves with ω-continuous derivative and without self-intersections. Namely, we prove the following theorems.

Theorem 1.1. *Let K be a compact subset of $\mathbb{R}^n$ satisfying the following condition: there exists $r_0 > 0$ and a concave non-decreasing function ω : $[0,\infty) \to [0,\infty)$ with $\omega(0) = 0$, continuous at 0, such that for all distinct*

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$x, y, z \in K$, if $|z-x| = \operatorname{diam}\{x, y, z\} < r_0$ then

$$\left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right| < \omega(\operatorname{diam}\{x, y, z\}). \tag{1.1}$$

Then there exists a finite family of finite-length curves without self-intersections with arc-length parametrizations $\gamma_i : A_i \to \mathbb{R}^n$, $i = 1, \ldots, N$, where A_i is either a circle or a closed segment, such that their images are disjoint, cover K and for every γ_i the derivative γ_i' is locally 342ω-continuous. Moreover, one can require that the total length of all curves γ_i is bounded by $5\mathcal{H}_1(K) + \varepsilon$, where $\mathcal{H}_1$ is the one-dimensional Hausdorff measure and $\varepsilon > 0$ is chosen arbitrarily.

THEOREM 1.2. *Let $\gamma : A \to \mathbb{R}^n$ be an arc-length parametrization of a finite-length curve without self-intersections, where A is a closed segment or a circle, such that γ' is locally ω-continuous for some concave non-decreasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ and continuous at 0. Then there exists $r_0 > 0$ such that for all distinct $x, y, z \in \gamma(A)$, if $|z - x| = \operatorname{diam}\{x, y, z\} < r_0$ then*

$$\left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right| < 6\omega(\operatorname{diam}\{x, y, z\}).$$

Let us make a few remarks here. First, note that both theorems include only conditions on the local continuity of the derivatives of the curves—that is, we are only interested in the regularity of the curves at small scales.

Second, the natural approach to prove Theorem 1.1 may be to do a construction at small scales and then use some compactness argument to make the family of curves finite and disjoint. We follow the idea of the first step in Section 3.2; however, note that the second part has to be more involved, as shown by the following easy example. Let K be the image of a *regular* curve, for example let K be a segment. As we require the images of the curves to be closed and disjoint, the only way to cover a segment is to use only one curve. Therefore we need some argument that allows us to merge neighboring curves constructed at small scales. This is done in Section 3.3.

Finally, if in Theorem 1.1 we assume that (1.1) holds not for the whole K, but for a subset $K \setminus Z$ with $\mathcal{H}_1(Z) = 0$, then we can basically apply Theorem 1.1 to the closure of $K \setminus Z$ and deduce that K is contained in the union of the images of a finite family of *regular* curves and a set of one-dimensional Hausdorff measure 0. We leave easy technical details to the reader.

Related work. A famous theorem by Jones [6] gives a characterization of compact subsets of rectifiable curves in $\mathbb{R}^2$. Jones's results were extended to $\mathbb{R}^n$ by Okikiolu [9] and to Hilbert spaces by Schul [10]. There exists an analogue of those theorems for rectifiable curves in general metric spaces [4, 5] and Heisenberg groups [3]. Ahlfors-regular subsets are treated by Schul

[10] and David and Semmes [1]. Lerman applied the idea of Jones's β numbers to rectifiable measures in $\mathbb{R}^n$ [8].

Inspired by these results, we have tried to find a characterization of compact subsets of much more regular curves, in particular, of curves with regular arc-length parametrization. Note that all aforementioned results focus on rectifiable or Ahlfors-regular sets and curves, which are much wider classes. The examples gathered in Section 4 show that Jones's β numbers carry insufficient information in our case and therefore we need a significantly different condition. Intuitively, Jones-like conditions concern the set K only at one scale and at a single location. Due to this, as our examples show, K might make infinitely many $90°$ turns while having various β numbers relatively small. Note that Jones's construction in [6] indeed yields Lipschitz curves which might turn and spiral without any control; this is the main difference from our work. Moreover, to cover the whole set K, we need a condition that involves *every* point of K, not just all points off a set of measure 0.

Therefore our characterization conditions are quite different than those given by Jones, Okikiolu and Schul [6, 9, 10]. In Section 4 we discuss Jones-like conditions applied to curves with arc-length parametrization with regular derivatives and we give two counter-examples showing that a Jones-like characterization does not seem to suit our needs.

Note that equation (1.1) is quite similar to a condition that bounds the inverse of the radius of the circumcircle of the triangle (x, y, z). This makes our result related to the analysis of Menger curvature [4, 5, 7]. In particular, the local construction presented in Section 3.2 seems to be similar to the constructions of Hahlomaa [4].

Finally, one may notice that the results in our paper seem to be similar to Whitney-type [11] theorems by Fefferman [2]. One can also view our results as stating that *there exists a* regular *curve through the whole set iff there exists a* regular *curve through every set of three neighboring points.*

Organization. In Section 2 we prove Theorem 1.2. This is a quite easy and straightforward corollary from the definition of a locally ω-continuous function. In Section 3 we prove Theorem 1.1, by providing a construction algorithm for a family of curves. In Section 4 we discuss Jones-like conditions for curves with arc-length parametrization with regular derivatives.

2. Properties of curves with ω-continuous derivative. In this section we prove Theorem 1.2. Let $\gamma : A \to \mathbb{R}^n$ be an arc-length parametrization of a finite-length curve without self-intersections, where A is a circle or a closed segment. Let $\omega : [0, \infty) \to [0, \infty)$ be a concave non-decreasing function with $\omega(0) = 0$, continuous at 0, and let $r_\omega > 0$ be such that γ' is r_ω-locally ω-continuous.

Let us start by choosing $r_1 > 0$ such that $r_1 < r_\omega$, $r_1 < \frac{1}{2} \operatorname{diam} A$ and $\omega(r_1) < 1/3$.

LEMMA 2.1. *There exists $r_2 > 0$ such that if $|\gamma(s) - \gamma(t)| < r_2$ then $|t - s| < r_1$.*

Proof. Let $r_2 = \inf_{s,t \in A : |s-t| \geq r_1} |\gamma(s) - \gamma(t)|$. By the compactness of the set $\{(s,t) \in A \times A : |s-t| \geq r_1\}$ there exist $s_0, t_0 \in A$ satisfying $|s_0 - t_0| \geq r_1$ and $r_2 = |\gamma(s_0) - \gamma(t_0)|$. Therefore if $|\gamma(s) - \gamma(t)| < r_2$ then $|s-t| < r_1$. We only need to prove that $r_2 > 0$. But $r_2 = |\gamma(s_0) - \gamma(t_0)|$ and γ does not have self-intersections. ∎

To prove Theorem 1.2, set $r_0 := \min(r_\omega, r_2)$. Take any distinct $x, y, z \in \gamma(A)$ satisfying $\operatorname{diam}\{x,y,z\} < r_0$. Let $x = \gamma(a)$, $y = \gamma(b)$, $z = \gamma(c)$. Since $\operatorname{diam}\{x,y,z\} < r_0 \leq r_2$, we have $\operatorname{diam}\{a,b,c\} < r_1 < \frac{1}{2} \operatorname{diam} A$. Therefore even if A is a circle, there is a natural order of a, b, c in the interior of one semicircle; we can assume $a < b < c \leq a + r_1$. Let $v = \gamma'(a)$. Then for all $s \in [a,c]$ we have

$$|v - \gamma'(s)| < \omega(s-a) \leq \omega(c-a) < 1/3. \tag{2.1}$$

Note that for all $a \leq s_1 \leq s_2 \leq c$ we have

$$\begin{aligned} |\gamma(s_2) - \gamma(s_1) - v(s_2 - s_1)| &= \Big| \int_{s_1}^{s_2} (\gamma'(s) - v)\, ds \Big| \\ &\leq \int_{s_1}^{s_2} |\gamma'(s) - v|\, ds < (s_2 - s_1)\omega(c-a). \end{aligned} \tag{2.2}$$

So for all $a \leq s_1 < s_2 \leq c$ (recall that γ is an arc-length parametrization),

$$\begin{aligned} s_2 - s_1 \geq |\gamma(s_2) - \gamma(s_1)| &\geq |v(s_2 - s_1)| - |\gamma(s_2) - \gamma(s_2) - v(s_2 - s_1)| \\ &> (1 - \omega(c-a))(s_2 - s_1) \end{aligned}$$

and

$$\begin{aligned} \left| \frac{\gamma(s_2) - \gamma(s_1)}{|\gamma(s_2) - \gamma(s_1)|} - \frac{\gamma(s_2) - \gamma(s_1)}{s_2 - s_1} \right| &= \left| 1 - \frac{|\gamma(s_2) - \gamma(s_1)|}{s_2 - s_1} \right| \\ &< \omega(c-a). \end{aligned} \tag{2.3}$$

Therefore by (2.2) and (2.3),

$$\begin{aligned} \left| \frac{\gamma(s_2) - \gamma(s_1)}{|\gamma(s_2) - \gamma(s_1)|} - v \right| &\leq \left| \frac{\gamma(s_2) - \gamma(s_1)}{|\gamma(s_2) - \gamma(s_1)|} - \frac{\gamma(s_2) - \gamma(s_1)}{s_2 - s_1} \right| \\ &\quad + \left| \frac{\gamma(s_2) - \gamma(s_1)}{s_2 - s_1} - v \right| \\ &< 2\omega(c-a). \end{aligned}$$

By taking every $(s_1, s_2) \in \{(a,b),(b,c),(a,c)\}$ we find that $\frac{y-x}{|y-x|}$, $\frac{z-y}{|z-y|}$ and $\frac{z-x}{|z-x|}$ differ from v by less than $2\omega(c-a)$ and we conclude that

$$\left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right| \le \left|\frac{y-x}{|y-x|} - v\right| + \left|v - \frac{z-y}{|z-y|}\right| < 2\omega(c-a) + 2\omega(c-a)$$
$$= 4\omega(c-a) < \frac{4}{3} < \sqrt{2}.$$

Note that this in particular means that the triangle with vertices x, y, z has an obtuse angle at vertex y and therefore $|z-x| = \operatorname{diam}\{x,y,z\}$. Since $|z-x| > (1-\omega(c-a))(c-a) > \frac{2}{3}(c-a)$ we have (recall that ω is a concave non-decreasing function)

$$\left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right| < 4\omega(c-a) \le 4\omega\left(\frac{3}{2}|z-x|\right) \le 6\omega(|z-x|).$$

That completes the proof of Theorem 1.2.

3. Characterization of subsets of curves with ω-continuous derivative. In this section we prove Theorem 1.1. This is done by an explicit construction of the desired curve family. In Section 3.1 we investigate the condition in (1.1) to prove that at small scales the set K lies approximately along a straight line. We use this observation in Section 3.2 to provide an explicit construction of *one* curve with properly regular derivative that covers $K \cap B$ for some small ball B. Finally, in Section 3.3 we show that all these small curves for different small balls B can be merged into the desired curve family.

3.1. Preliminaries. Let us fix some global coordinate system in the whole $\mathbb{R}^n$, so we can *compare points*. We use this comparison procedure to break ties in the constructing algorithm so that it is fully deterministic.

Definition 3.1. We say that $x < y$ for $x, y \in \mathbb{R}^n$ if for some $1 \le k \le n$ we have $x_i = y_i$ for $1 \le i < k$ and $x_k < y_k$ (i.e., we sort points lexicographically).

Given an isolated point $x \in K$, when we speak about the point of K (or some closed subset of K) *closest* to x, we mean that we break ties using the comparison procedure from Definition 3.1, i.e., we choose the smallest point among the set of closest points. Such a point exists since K is compact and the projection of a compact set onto any subspace is still a compact set.

Let $K \subset \mathbb{R}^n$ be a compact set, let $\omega : [0,\infty) \to [0,\infty)$ be a concave non-decreasing function with $\omega(0) = 0$ and continuous at 0 and let $r_0 > 0$ be such that for all distinct $x, y, z \in K$ with $|z-x| = \operatorname{diam}\{x,y,z\} < r_0$ we have

$$\left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right| < \omega(\operatorname{diam}\{x,y,z\}).$$

We can assume r_0 is sufficiently small to ensure that $\omega(r_0) < 0.001$. The constant 0.001 is far from optimal; we do not optimize constants in our proofs. As we shall see, we only need that all constants that appear in the proof are significantly smaller than 1.

LEMMA 3.2. *For any pairwise distinct $x, y, z \in K$, if*

$$|z-x| = \operatorname{diam}\{x,y,z\} < r_0$$

then the angle $\measuredangle(x,y,z)$ is obtuse and

$$\left|\frac{y-x}{|y-x|} - \frac{z-x}{|z-x|}\right| < \omega(\operatorname{diam}\{x,y,z\}),$$

$$\left|\frac{z-y}{|z-y|} - \frac{z-x}{|z-x|}\right| < \omega(\operatorname{diam}\{x,y,z\}).$$

Proof. Since

$$\left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right| < \omega(\operatorname{diam}\{x,y,z\}) \le \omega(r_0) < 0.001$$

the triangle with vertices x, y, z has an obtuse angle at y. Therefore the angle between the vectors $y-x$ and $z-y$ is acute and its measure is the sum of the measures of the angles between $y-x$ and $z-x$ and between $z-y$ and $z-x$. Therefore

$$\left|\frac{y-x}{|y-x|} - \frac{z-x}{|z-x|}\right| \le \left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right|,$$

$$\left|\frac{z-y}{|z-y|} - \frac{z-x}{|z-x|}\right| \le \left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right|,$$

which completes the proof. ∎

Take any $x_0 \in K$ and $r < \frac{1}{2}r_0$ and let $B = \{x : |x - x_0| \le r\}$ and $K_B = K \cap B$. Now we are going to prove that K_B lies approximately along one line. Later, we construct curves for all small balls B and then merge them into the desired family.

If K_B consists of a single point there is nothing to do, so let $x_1, x_2 \in K_B$ be such that $|x_2 - x_1| = \operatorname{diam} K_B > 0$ (they exist by compactness of K and B). Let $d = \operatorname{diam} K_B < r_0$ and set orthonormal coordinates with base vectors $(e_1, \ldots, e_n)$ so that $x_1 = 0$, $x_2 = (d, 0, \ldots, 0) = de_1$. Then $(x_2 - x_1)/|x_2 - x_1| = e_1$.

DEFINITION 3.3. We say that a point $y \in K_B$ is *to the right* [*to the left*] of $x \in K_B$ if $\langle y, e_1\rangle > \langle x, e_1\rangle$ [$\langle y, e_1\rangle < \langle x, e_1\rangle$]. We say that $y, y' \in K_B$ are *on the same side* [*on opposite sides*] of $x \in K_B$ if $0 \neq \operatorname{sgn}(\langle y - x, e_1\rangle) = \operatorname{sgn}(\langle y' - x, e_1\rangle)$ [$0 \neq \operatorname{sgn}(\langle y - x, e_1\rangle) \neq \operatorname{sgn}(\langle y' - x, e_1\rangle) \neq 0$].

LEMMA 3.4. *Let $x, y \in K_B$ satisfy $x \neq y$ and $\langle x, e_1\rangle \leq \langle y, e_1\rangle$. Then*

$$\left|\frac{y - x}{|y - x|} - e_1\right| < 2\omega(d) < 0.002.$$

In particular, different points in K_B have distinct first coordinate, and for any two distinct points $x, y \in K_B$ one always lies to the right of the other and $\operatorname{sgn}(\langle x - y, e_1\rangle) \neq 0$.

Proof. If $x = x_1$ and $y = x_2$ there is nothing to prove. By symmetry, we can assume $y \notin \{x_1, x_2\}$. By Lemma 3.2 for the triangle x_1, y, x_2 (note that $|x_2 - x_1| = d = \operatorname{diam}\{x_1, y, x_2\}$),

$$\left|\frac{y - x_1}{|y - x_1|} - e_1\right| < \omega(d). \tag{3.1}$$

If $x = x_1$ the proof is finished. Otherwise let us focus on the triangle x_1, x, y. By Lemma 3.2 one of the angles of this triangle is obtuse. We now prove that this is the angle $\measuredangle(x_1, x, y)$.

Note that $x \neq x_2$, since otherwise

$$\langle x_1 - x, y - x\rangle = -d(\langle e_1, y\rangle - \langle e_1, x\rangle) \leq 0$$

and $|y - x_1| > |x_2 - x_1|$, a contradiction. Thus we can use Lemma 3.2 for the triangle x_1, x, x_2, obtaining

$$\left|\frac{x - x_1}{|x - x_1|} - e_1\right| < \omega(d).$$

Therefore, by (3.1),

$$\begin{aligned}\left|\frac{x - x_1}{|x - x_1|} - \frac{y - x_1}{|y - x_1|}\right| &\leq \left|\frac{x - x_1}{|x - x_1|} - e_1\right| + \left|e_1 - \frac{y - x_1}{|y - x_1|}\right| \\ &< \omega(d) + \omega(d) < 0.002.\end{aligned}$$

Therefore the angle $\measuredangle(x, x_1, y)$ is acute.

Towards a contradiction, assume that $\measuredangle(x_1, y, x)$ is obtuse. In this case, by Lemma 3.2,

$$\left|\frac{x - y}{|x - y|} - e_1\right| \leq \left|\frac{x - y}{|x - y|} - \frac{x - x_1}{|x - x_1|}\right| + \left|\frac{x - x_1}{|x - x_1|} - e_1\right| < 2\omega(d) < 0.002.$$

This contradicts the assumption that $\langle x, e_1\rangle \leq \langle y, e_1\rangle$.

Therefore by applying Lemma 3.2 to the triangle x_1, x, y we get

$$\left|\frac{y - x}{|y - x|} - e_1\right| \leq \left|\frac{y - x}{|y - x|} - \frac{y - x_1}{|y - x_1|}\right| + \left|\frac{y - x_1}{|y - x_1|} - e_1\right| < 2\omega(d). \blacksquare$$

In the following two corollaries we replace any assumptions on how points x, y and z lie along the e_1 axis at the cost of using the sgn operator in the equations. Note that the term $\mathrm{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|}$ is equal to $\frac{y-x}{|y-x|}$ if y lies to the right of x, and to $\frac{x-y}{|x-y|}$ if y lies to the left of x. Thus, this term is always a unit vector parallel to $y-x$, but close to e_1, and not $-e_1$.

Merging Lemmas 3.2 and 3.4 we obtain

Corollary 3.5. *For $x, y, z \in K_B$, if $y \notin \{x, z\}$ then*

$$\left|\mathrm{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|} - \mathrm{sgn}(\langle z-y, e_1\rangle)\frac{z-y}{|z-y|}\right| < \omega(\mathrm{diam}\{x, y, z\}).$$

Applying Lemma 3.4 once again we obtain

Corollary 3.6. *For any distinct $x, y \in K_B$,*

$$\left|\mathrm{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|} - e_1\right| < 2\omega(d) < 0.002.$$

The following lemma is not used directly in the rest of the construction, but formalizes the intuition that all points of K_B lie along one line.

Lemma 3.7. *Let $x, y, z \in K_B$ be pairwise distinct points. Then $|z-x| = \mathrm{diam}\{x, y, z\}$ iff x and z lie on opposite sides of y.*

Proof. First assume x and z lie on opposite sides of y. Assume x lies to the left and z lies to the right. By Lemma 3.4,

$$\left|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\right| \le \left|\frac{y-x}{|y-x|} - e_1\right| + \left|e_1 - \frac{z-y}{|z-y|}\right| < 4\omega(d) < 0.004.$$

Therefore the angle $\measuredangle(x, y, z)$ is obtuse.

Now assume $|z-x| = \mathrm{diam}\{x, y, z\}$. If, say, z lies to the right of x then, by Lemmas 3.4 and 3.2,

$$\left|\frac{y-x}{|y-x|} - e_1\right| \le \left|\frac{y-x}{|y-x|} - \frac{z-x}{|z-x|}\right| + \left|\frac{z-x}{|z-x|} - e_1\right| < 3\omega(d) < 0.003.$$

Therefore y lies to the right of x. Making the same calculations for $z-y$ instead of $y-x$ we see that y lies to the left of z. ∎

Lemma 3.8. *Let $x \in K_B \setminus \{x_1, x_2\}$. Then $|x-x_1| < d$, $|x-x_2| < d$, i.e., the choice of x_1 and x_2 was unique up to numbering (i.e., up to a* 180° *rotation of the coordinate system).*

Proof. Since $\mathrm{diam}\{x_1, x, x_2\} = |x_2 - x_1|$, Lemma 3.2 implies that the angle $\measuredangle(x_1, x, x_2)$ is obtuse and the sides $x - x_1$ and $x - x_2$ of the triangle x_1, x, x_2 are shorter than the side $x_1 - x_2$. ∎

To sum up, we have proved that all the points in K_B lie approximately along the line x_1x_2. It is worth noting that a similar analysis appears in [4, Lemma 3.1], and there Hahlomaa uses the map $x \mapsto d(x, x_1)$, instead of the axis x_1x_2.

3.2. The local construction. In this section we provide an algorithm that allows us to construct *one* curve without self-intersections, with arc-length parametrization $\gamma : [a,b] \to \mathbb{R}^n$, such that γ' is 342ω-continuous and $K_B \subset \gamma([a,b])$.

LEMMA 3.9. *Let $y \in K_B$ be an accumulation point of K_B and let $(y_k)_{k=1}^{\infty}$ be a sequence of points of K_B different from y and convergent to y. Then the limit*

$$\lim_{k\to\infty} \operatorname{sgn}(\langle y_k - y, e_1\rangle)\frac{y_k - y}{|y_k - y|}$$

exists and is a vector of length 1.

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ satisfy $\omega(2\delta) < \varepsilon$. Assume $M \in \mathbb{N}$ satisfies: for all $k \geq M$ we have $|y_k - y| < \delta$. Let $k, l \geq M$. By Corollary 3.5 for the triangle y_k, y_l, y,

$$\begin{aligned}\left|\operatorname{sgn}(\langle y_k - y, e_1\rangle)\frac{y_k - y}{|y_k - y|} - \operatorname{sgn}(\langle y_l - y, e_1\rangle)\frac{y_l - y}{|y_l - y|}\right| &< \omega(\operatorname{diam}\{y, y_l, y_k\}) \\ &\leq \omega(2\delta) < \varepsilon.\end{aligned}$$

Therefore this sequence converges and its limit is a vector of length 1, since all terms have length 1. ∎

DEFINITION 3.10. For every $y \in K_B$ we define a unit vector $v_y \in \mathbb{R}^n$ as follows: if y is an accumulation point of K_B, then

$$v_y := \lim_{K_B \ni y' \to y} \operatorname{sgn}(\langle y' - y, e_1\rangle)\frac{y' - y}{|y' - y|},$$

i.e., v_y is a vector tangent to K_B at y. Otherwise, let y^* be the point of K_B closest to y (if there are many, choose the lexicographically smallest). Then

$$v_y := \operatorname{sgn}(\langle y^* - y, e_1\rangle)\frac{y^* - y}{|y^* - y|}.$$

Let us now prove some properties of the chosen vectors v_y.

LEMMA 3.11. *Let $y \in K_B$. Then $|v_y - e_1| \leq 2\omega(d) < 0.002$.*

Proof. By Corollary 3.6, for any $x \in K_B \setminus \{y\}$,

$$\left|\operatorname{sgn}(\langle x - y, e_1\rangle)\frac{x - y}{|x - y|} - e_1\right| \leq 2\omega(d).$$

But $|v_y - e_1|$ is a limit of such expressions (in the case of y being an accumulation point of K_B) or is equal to a single expression of that form. ∎

LEMMA 3.12. *Let $x, y \in K_B$, $x \neq y$. Then*

$$\left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y - x}{|y - x|} - v_x\right| < 2\omega(|y - x|).$$

Proof. If x is an accumulation point of K_B, then take $x^\circ \neq x$ sufficiently close to x such that

$$|x - x^\circ| < \frac{1}{2}|y-x|, \qquad \left|\operatorname{sgn}(\langle x^\circ - x, e_1\rangle)\frac{x^\circ - x}{|x^\circ - x|} - v_x\right| < \frac{1}{2}\omega(|y-x|).$$

Then $\operatorname{diam}\{x, y, x^\circ\} \le \frac{3}{2}|y-x|$ and using Lemma 3.2 and Corollary 3.5 we obtain

$$\begin{aligned}
&\left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y-x}{|y-x|} - v_x\right| \\
&\quad \le \left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y-x}{|y-x|} - \operatorname{sgn}(\langle x - x^\circ, e_1\rangle)\frac{x-x^\circ}{|x-x^\circ|}\right| \\
&\qquad + \left|\operatorname{sgn}(\langle x - x^\circ, e_1\rangle)\frac{x-x^\circ}{|x-x^\circ|} - v_x\right| \\
&\quad < \omega\left(\frac{3}{2}|y-x|\right) + \frac{1}{2}\omega(|y-x|) \le 2\omega(|y-x|).
\end{aligned}$$

Otherwise, note that by the definition of x^* we have $|x - x^*| \le |x - y|$ and $\operatorname{diam}\{x, x^*, y\} \le 2|x-y|$. Using Corollary 3.5 once again, we get

$$\begin{aligned}
&\left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y-x}{|y-x|} - v_x\right| \\
&\quad = \left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y-x}{|y-x|} - \operatorname{sgn}(\langle x - x^*, e_1\rangle)\frac{x-x^*}{|x-x^*|}\right| \\
&\quad < \omega(2|y-x|) \le 2\omega(|y-x|). \ \blacksquare
\end{aligned}$$

LEMMA 3.13. *Let $x, y \in K_B$, $x \neq y$. Then $|v_x - v_y| < 4\omega(|y-x|)$.*

Proof. We use Lemma 3.12 twice:

$$\begin{aligned}
|v_x - v_y| &\le \left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y-x}{|y-x|} - v_x\right| + \left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y-x}{|y-x|} - v_y\right| \\
&< 2\omega(|y-x|) + 2\omega(|y-x|) = 4\omega(|y-x|). \ \blacksquare
\end{aligned}$$

After these preparations, we now provide a construction of a sufficiently regular curve that connects two points of K_B.

LEMMA 3.14. *Let $x, y \in K_B$ and assume that y is to the right of x. Then there exists a smooth curve with arc-length parametrization $\gamma : [a, b] \to \mathbb{R}^n$ such that*

(i) $\gamma(a) = x$, $\gamma(b) = y$, $\gamma'(a) = v_x$, $\gamma'(b) = v_y$,
(ii) *γ' is $\frac{157\omega(|y-x|)}{|y-x|}$-Lipschitz continuous,*
(iii) *for any $t \in [a, b]$, $|\gamma'(t) - e_1| \le 138\omega(d) < 0.138$,*
(iv) *for any distinct $s, t \in [a, b]$, $|s - t| \ge |\gamma(s) - \gamma(t)| > 0.93|s - t|$.*

Proof. Let $\hat{d} = |y - x|$. We start by defining $\hat{\gamma} : [0, \hat{d}] \to \mathbb{R}^n$ satisfying: $\hat{\gamma} \in C^\infty$, $\hat{\gamma}(0) = x$, $\hat{\gamma}(\hat{d}) = y$, $\hat{\gamma}'(0) = v_x$, $\hat{\gamma}'(\hat{d}) = v_y$. However, $\hat{\gamma}$ will not be an arc-length parametrization. We will then bound $\hat{\gamma}'$ to reparametrize $\hat{\gamma}$ as desired.

Let us define

$$\hat{\gamma}(t) = \left(\frac{v_y - v_x}{\hat{d}^2} - 2\frac{y - x - \hat{d}v_x}{\hat{d}^3} \right) t^3 + \left(3\frac{y - x - \hat{d}v_x}{\hat{d}^2} - \frac{v_y - v_x}{\hat{d}} \right) t^2 + v_x t + x.$$

It is easy to check that $\hat{\gamma}(0) = x$ and $\hat{\gamma}(\hat{d}) = y$. Moreover

$$\hat{\gamma}'(t) = 3\left(\frac{v_y - v_x}{\hat{d}^2} - 2\frac{y - x - \hat{d}v_x}{\hat{d}^3} \right) t^2 + 2\left(3\frac{y - x - \hat{d}v_x}{\hat{d}^2} - \frac{v_y - v_x}{\hat{d}} \right) t + v_x.$$

It is easy to check that $\hat{\gamma}'(0) = v_x$ and $\hat{\gamma}'(\hat{d}) = v_y$. Note that by Lemmas 3.12 and 3.13,

$$|v_y - v_x| < 4\omega(\hat{d}), \qquad \left| \frac{y - x}{\hat{d}} - v_x \right| < 2\omega(\hat{d}).$$

Therefore for $0 \le s < t \le \hat{d}$ we have

$$\begin{aligned} (3.2) \qquad |\hat{\gamma}'(t) - \hat{\gamma}'(s)| &< \frac{t - s}{\hat{d}} \left(3\frac{4\omega(\hat{d})(t + s)}{\hat{d}} + 6\frac{2\omega(\hat{d})(t + s)}{\hat{d}} \right. \\ &\qquad \left. + 6 \cdot 2\omega(\hat{d}) + 2 \cdot 4\omega(\hat{d}) \right) \\ &\le (t - s)\frac{68\omega(\hat{d})}{\hat{d}}. \end{aligned}$$

In particular, for $s = 0$,

$$(3.3) \qquad |\hat{\gamma}'(t) - v_x| < 68\omega(\hat{d}) \le 68\omega(d) < 0.068.$$

And, by Lemma 3.11,

$$(3.4) \quad |\hat{\gamma}'(t) - e_1| \le |\hat{\gamma}'(t) - v_x| + |v_x - e_1| < 68\omega(\hat{d}) + 2\omega(d) \le 70\omega(d) < 0.07.$$

Therefore the projection of $\hat{\gamma}([0, \hat{d}])$ onto the first coordinate axis is injective, the curve $\hat{\gamma}$ goes roughly to the right (i.e., $\hat{\gamma}'$ points to the right) and it has no self-intersections.

Moreover, by (3.3) and the fact that $|v_x| = 1$ we get $0.932 < |\hat{\gamma}'(t)| < 1.068$, so by standard techniques one can modify the parametrization $\hat{\gamma}$ to obtain an arc-length parametrization γ of the curve $\hat{\gamma}([0, \hat{d}])$. Namely, if $L(t) = \int_0^t |\hat{\gamma}'(s)|\, ds$ is the length function of $\hat{\gamma}$, we define $\gamma : [0, L(\hat{d})] \to \mathbb{R}^n$ by $\gamma(u) = \hat{\gamma}(L^{-1}(u))$. Then

$$(3.5) \qquad \gamma'(u) = (L^{-1})'(u)\hat{\gamma}'(L^{-1}(u)) = \frac{\hat{\gamma}'(L^{-1}(u))}{|\hat{\gamma}'(L^{-1}(u))|}.$$

We now check all conditions for γ. Item (i) is obvious from the definition of $\hat\gamma$ and (3.5). By (3.3), $|\hat\gamma'(t) - \gamma'(\gamma^{-1}(\hat\gamma(t)))| < 68\omega(d)$. Therefore, by (3.4), $|\gamma'(u) - e_1| < 138\omega(d) < 0.138$ and (iii) is satisfied. To check (iv), note that by (3.4) for $0 \le t_1 < t_2 \le \hat d$,

$$|\hat\gamma(t_2) - \hat\gamma(t_1)| \ge \langle \hat\gamma(t_2) - \hat\gamma(t_1), e_1\rangle = \int_{t_1}^{t_2} \langle \hat\gamma'(s), e_1\rangle\, ds$$

$$> \int_{t_1}^{t_2} 0.93|\hat\gamma'(s)|\, ds = 0.93(L(t_2) - L(t_1)).$$

Conversely, setting $u_1 = L(t_1)$ and $u_2 = L(t_2)$ we obtain

$$|\gamma(u_2) - \gamma(u_1)| \ge 0.93(u_2 - u_1),$$

and (iv) is satisfied.

It remains to prove that γ' is $157\frac{\omega(\hat d)}{\hat d}$-Lipschitz continuous. Since all functions here are C^∞, we can compute

$$\gamma''(u) = ((L^{-1})')^2(u)\hat\gamma''(L^{-1}(u)) + (L^{-1})''(u)\hat\gamma'(L^{-1}(u)).$$

Since $0.932 < |\hat\gamma'(t)| < 1.068$ we obtain $0.932 < L'(t) < 1.068$ and $1/1.068 < (L^{-1})'(u) < 1/0.932$. Since $\hat\gamma'$ is $68\frac{\omega(\hat d)}{\hat d}$-Lipschitz continuous (by (3.2)), we have

$$|((L^{-1})')^2(u)\hat\gamma''(L^{-1}(u))| < \frac{68}{0.932^2}\,\frac{\omega(\hat d)}{\hat d} < 78.285\frac{\omega(\hat d)}{\hat d}.$$

Since $(L^{-1})'(u) = 1/|\hat\gamma'(L^{-1}(u))|$, we get

$$|(L^{-1})''(u)\hat\gamma'(L^{-1}(u))| = \left|\frac{\frac{d}{du}|\hat\gamma'(L^{-1}(u))|}{|\hat\gamma'(L^{-1}(u))|^2}\right| |\hat\gamma'(L^{-1}(u))|$$

$$< \left|\frac{1}{0.932}\cdot (L^{-1})'(u)|\hat\gamma''(L^{-1}(u))|\right|$$

$$< \frac{1}{0.932^2}\cdot 68\frac{\omega(\hat d)}{\hat d} < 78.285\frac{\omega(\hat d)}{\hat d}.$$

Therefore γ' is $157\frac{\omega(\hat d)}{\hat d}$-Lipschitz continuous. ■

Remark 3.15. Let us for a while swap the roles of x_1 and x_2, i.e., rotate the coordinate system by 180°. If we then connect x and y using Lemma 3.14, we obtain the same curve, but running backwards, since the formula for $\hat\gamma$ gives the unique polynomial of degree at most 3 that has fixed $\hat\gamma$ and $\hat\gamma'$ at the endpoints.

Lemma 3.16. *With the assumptions of Lemma* 3.14*, the constructed curve* γ *has* 169ω*-continuous derivative.*

Proof. Let $s, t \in [a, b]$. By (iv) above,

$$b - a \geq |y - x| > 0.93(b - a).$$

Since ω is non-decreasing,

$$\frac{\omega(|y-x|)}{|y-x|} < \frac{\omega(|b-a|)}{0.93(b-a)}.$$

Recall that γ' is $157\frac{\omega(|y-x|)}{|y-x|}$-Lipschitz continuous and ω is concave:

$$|\gamma'(t) - \gamma'(s)| < 157\frac{\omega(|y-x|)}{|y-x|}|t-s| < \frac{157}{0.93}\omega(b-a)\frac{|t-s|}{|b-a|} < 169\omega(|t-s|). \blacksquare$$

Let $\pi_{e_1} : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the e_1 axis. Since $x_1, x_2 \in K_B$, $[0, d] \setminus \pi_{e_1}(K_B)$ is a finite or countable family of disjoint open segments (p^i, q^i) with $p^i = \pi_{e_1}(x^i)$ and $q^i = \pi_{e_1}(y^i)$, $x^i, y^i \in K_B$. For every i, let us connect x^i with y^i by a curve γ_i constructed in Lemma 3.14. Let $\bar{K}_B$ be the set K_B together with all the images of γ_i.

By the properties of γ_i, for every $p \in [0, d]$ there exists exactly one $f(p) \in \bar{K}_B$ such that $p = \pi_{e_1}(f(p))$. We now prove some properties of the function f. Let us extend the definition of v_x to $x \in \bar{K}_B$. If x is in the image of γ_i, let $v_x = \gamma_i'(\gamma_i^{-1}(x))$, i.e., the tangent vector to γ_i at x. This definition works properly at the endpoints x^i, y^i of γ_i by the definition of γ_i.

Lemma 3.17. *For every $x \in \bar{K}_B$,*

$$|v_x - e_1| < 138\omega(d) < 0.138.$$

Proof. If $x \in K_B$, then Lemma 3.11 does the job. Otherwise, use Lemma 3.14(iii). $\blacksquare$

Lemma 3.18. *For any distinct $x, y \in \bar{K}_B$,*

$$|v_x - v_y| < 342\omega(|x - y|).$$

Proof. If $x, y \in K_B$, the statement is obvious by Lemma 3.13. Therefore let us assume $x \notin K_B$, so x is in the image of the curve γ_i. If y is also in the image of γ_i, Lemma 3.16 does the job. Otherwise, assume that y is to the right of x, say. Then it is to the right of y^i too.

If $y \in K_B$ then

$$|v_x - v_y| \leq |v_x - v_{y^i}| + |v_{y^i} - v_y| \leq 169\omega(|x - y^i|) + 4\omega(|y^i - y|) \leq 173\omega(|y - x|).$$

Otherwise, let y be in the image of γ_j. Then

$$\begin{aligned} |v_x - v_y| &\leq |v_x - v_{y^i}| + |v_{y^i} - v_{x^j}| + |v_{x^j} - v_y| \\ &\leq 169\omega(|x - y^i|) + 4\omega(|y^i - x^j|) + 169\omega(|x^j - y|) \\ &\leq 342\omega(|y - x|). \ \blacksquare \end{aligned}$$

Lemma 3.19. *For every $x \in \bar{K}_B$ the vector v_x is tangent at x to the set $\bar{K}_B$.*

Proof. For $x \in \bar{K}_B \setminus K_B$ the statement is obvious. Let now $x = f(p) \in K_B$. If x is an endpoint of some curve γ_i, then γ_i' at x equals v_x, whether x is a left or right endpoint of γ_i. On the other hand, if $x_n \to x$ and $x_n \neq x$, $x_n \in K_B$, then, by the definition of v_x,

$$\lim_{n\to\infty} \operatorname{sgn}(\langle x_n - x, e_1\rangle)\frac{x_n - x}{|x_n - x|} = v_x.$$

This finishes the proof. ∎

We conclude this section with the final theorem.

THEOREM 3.20. *Assume that K is a compact set and B is a closed ball of radius smaller than $\frac{1}{2}r_0$. Let $K_B = K \cap B$ and let d be the diameter of K_B. Pick $x_1, x_2 \in K_B$ such that $|x_1 - x_2| = d$. Introduce orthonormal coordinates $(e_1, \ldots, e_n)$ such that $x_1 = 0$, $x_2 = (d, 0, \ldots, 0)$, i.e., $x_2 - x_1 = de_1$. Then there exists a curve without self-intersections with arc-length parametrization $\gamma : [0, L] \to \mathbb{R}^n$ such that:*

(i) $\gamma(0) = x_1$, $\gamma(L) = x_2$, $d \leq L < 1.161d$;
(ii) γ' *is* 342ω*-continuous;*
(iii) *for all* $t \in [0, L]$ *we have* $|\gamma'(t) - e_1| < 138\omega(d) < 0.138$;
(iv) $K_B \subset \gamma([0, L])$;
(v) *for every* $t \in [0, L]$ *we have* $\gamma'(t) = v_{\gamma(t)}$;
(vi) *if moreover the center of* B *belongs to* K_B, *then* $\gamma([0, L]) \subset B$.

Proof. First note that by Lemma 3.19 the function $f : [0, d] \to \mathbb{R}^n$ is continuous and differentiable and $f'(p)/|f'(p)| = v_{f(p)}$ for every $p \in [0, d]$. Moreover $\langle f'(p), e_1\rangle = 1$, thus $f'(p) = v_{f(p)}/\langle v_{f(p)}, e_1\rangle$ and $|f'(p)| \geq 1$. Note that, by Lemma 3.17, $v_{f(p)}$ points roughly to the right and $|f'(p)| \leq 1/(1 - 0.138)$. Therefore one can parametrize $f([0, d]) = \bar{K}_B$ with an arc-length parametrization $\gamma : [0, L] \to \mathbb{R}^n$ such that $\gamma'(t) = v_{\gamma(t)}$.

(iv) follows directly from the construction. (iii) is a corollary of Lemma 3.17 and the fact that $d \leq L \leq d/(1 - 0.138) < 1.161d$. (ii) is a corollary of Lemma 3.18.

Now assume that the center x_c of B belongs to K_B. Integrating the inequality (iii) for $|\gamma'(t) - e_1|$, we obtain

$$\left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y - x}{|y - x|} - e_1\right| < 0.138$$

for all different $x, y \in \bar{K}_B$. Taking $x = x_1$, $x = x_2$ and $x = x_c$, we have $|x_c - x| \leq \max\{|x_c - x_1|, |x_c - x_2|\}$ and the proof is finished. ∎

REMARK 3.21. Note that we still keep the property from Remark 3.15. If we swap x_1 and x_2, i.e., rotate the local coordinate system by $180°$, we get the same curve $\bar{K}_B$, but running backwards. Indeed, v_x changes to $-v_x$ and by Remark 3.15 the images of the curves γ_i remain unchanged. Therefore, by Lemma 3.8, the constructed curve $\bar{K}_B$ does not depend on the chosen local coordinate system $(e_1, \ldots, e_n)$.

3.3. The global construction. In Section 3.2 we developed a way to pass one curve through all points of $K \cap B$ for a closed ball B of diameter smaller than r_0. Now we would like to extend this construction to the whole K. Naively, we would like to *take the union of all curves for all small balls B—it should look nice.* Indeed, this way we get a slightly weaker result than Theorem 1.1 quite immediately:

THEOREM 3.22. *With the assumptions of Theorem 1.1, there exists a finite family of finite-length curves whose images cover K and:*

(i) *the curves admit arc-length parametrizations $\gamma_i : [a_i, b_i] \to \mathbb{R}^n$, $i = 1, \ldots, N$,*
(ii) *each γ_i is injective, i.e., its image does not have self-intersections,*
(iii) *the derivative γ_i' is 342ω-continuous.*

Unlike in Theorem 1.1, the images of γ_i and γ_j for $i \neq j$ are not necessarily disjoint.

Proof. Take $r < \frac{1}{2} r_0$ and set

$$\mathcal{B}_0 = \{\{x : |x - x_0| \le r\} : x_0 \in K\}.$$

Take a finite subfamily $\mathcal{B}_0' \subset \mathcal{B}_0$ such that the interiors of the balls from $\mathcal{B}_0'$ cover K. For every $B \in \mathcal{B}_0'$ construct a curve γ_B using Theorem 3.20. The family $\{\gamma_B : B \in \mathcal{B}_0'\}$ is as desired. ∎

Note that it does not seem easy to make the images of the curves disjoint in the proof of Theorem 3.22. Consider an example when K is the image of a regular curve, e.g., a segment. The only way to cover a segment with a finite family of disjoint and *closed* images of curves is to use only one curve that covers the whole segment. Therefore, to obtain disjoint images, we need to merge curves γ_B for neighboring balls B.

We start by strengthening Remark 3.21, so that the local construction in the neighborhood of some $x \in K$ is totally independent even of the choice of the ball B covering that a neighborhood. The problem is that the choice of y^* in Definition 3.10 depends on the choice of B. However, this can be easily circumvented.

Take $r^* < \frac{1}{20} r_0$. Let $K_{\text{lonely}} \subset K$ be the set of isolated points from K that are at a distance of at least r^* from other points of K. K_{lonely} is finite and we can remove it from K: at the end of the construction, every non-covered point from K_{lonely} can be covered by a sufficiently small segment. Therefore we can assume that for every $x \in K$ there exists $y \in K \setminus \{x\}$ such that $|x - y| < r^*$.

Lemma 3.23. *Let B be a closed ball with radius r, $2r^* \le r < \frac{1}{2}r_0$, and suppose $x \in K \cap \frac{1}{2}B$ is not an accumulation point of K_B. Then x^*, taken in Definition* 3.10, *is in fact one of the points closest to x in the entire K, not only K_B, and among all the points closest to x, x^* is lexicographically smallest.*

Proof. For this x there exists $y \in K$ such that $|x - y| < r^*$. Moreover, since $r \ge 2r^*$, we have $\{y : |x-y| \le r^*\} \subset B$. Therefore all the points closest to x in K belong to B and the lemma is proved. ∎

Definition 3.24. Let B be a closed ball with radius $r < \frac{1}{2}r_0$. Assume there exist at least two points in $K \cap \frac{1}{2}B$. Construct the curve γ_B for the ball B, using Theorem 3.20. Let x be the first ($\frac{1}{2}B$ is closed) point of $K \cap \frac{1}{2}B$ on the image of γ_B, and let y be the last point (or, equivalently, x is the leftmost and y is the rightmost point of $K \cap \frac{1}{2}B$). We call the closed arc of γ_B from x to y the *inner curve* of γ_B and we denote its arc-length parametrization by $\bar{\gamma}_B$.

Lemma 3.25. *Let B_1, B_2 be closed balls with radii r_1, r_2 such that $2r^* \le r_2 < r_1 < \frac{1}{2}r_0$ and $B_2 \subset \frac{1}{2}B_1$. Assume that $K \cap \frac{1}{2}B_2$ consists of at least two points and the center of B_2 belongs to K. Then the curve $\bar{\gamma}_{B_2}$ is a subset of $\bar{\gamma}_{B_1}$.*

Proof. Construct γ_{B_1} using Theorem 3.20. Let x_l be the leftmost point of $K \cap B_2$ (in the coordinate system used to construct γ_{B_1}), and x_r the rightmost one. Since $K \cap \frac{1}{2}B_2$ consists of at least two points, x_l and x_r are well defined and are different. By Corollary 3.6 ball B_1, all points $x \in K_{B_1}$ between x_l and x_r (in particular, the center of B_2) are either equal to x_l or satisfy

$$\left|\frac{x - x_l}{|x - x_l|} - e_1\right| < 0.002, \tag{3.6}$$

and are either equal to x_r or satisfy

$$\left|\frac{x_r - x}{|x_r - x|} - e_1\right| < 0.002. \tag{3.7}$$

Therefore, as the center of B_2 is in K and between x_l and x_r, all points $x \in K \cap B_1$ between x_l and x_r are in B_2. Moreover, the angle $\measuredangle(x_l, x, x_r)$ is obtuse, and $|x_r - x_l| = \operatorname{diam} K \cap B_2$. Together with (3.6) and (3.7), this means that, to construct γ_{B_2} using Theorem 3.20, the e_1 axis connects x_l with x_r and the order (to the left or right) on this axis is the same as on the e_1 axis in B_1. Note that by Lemma 3.23, for every $x \in \frac{1}{2}B_2$ the point x^* closest to x in $K \cap B_2$ belongs to $B_2 \subset \frac{1}{2}B_1$ and it is in fact the point closest to x in the whole K. Therefore the choice of x^* is independent of whether we construct γ_{B_1} or γ_{B_2}.

Let now x_l' and x_r' be the leftmost and the rightmost points of $K \cap \frac{1}{2}B_2$. Recall that the notion of leftmost and rightmost points is the same whether we use the coordinate system of B_1 or B_2. Just as for B_2, from Corollary 3.6 we can deduce that all points of $K \cap B_1$ between x_l' and x_r' lie in $\frac{1}{2}B_2$. Together with Remark 3.21, this implies that the construction process in Theorem 3.20 for $\bar{\gamma}_{B_2}$ is part of the construction process for γ_{B_1}. Since $B_2 \subset \frac{1}{2}B_1$, $\bar{\gamma}_{B_2}$ is in fact a subset of $\bar{\gamma}_{B_1}$. ■

Lemma 3.26. *Let B_2, B_3 be closed balls with centers $y_2, y_3 \in K$ and radii $2r^* \le r_2, r_3 < \frac{1}{8}r_0$. Assume that $K \cap \frac{1}{2}B_2 \cap \frac{1}{2}B_3 \neq \emptyset$, and $\frac{1}{2}B_2$ and $\frac{1}{2}B_3$ have at least two points of K each. Then the union of the curves $\bar{\gamma}_{B_2}$ and $\bar{\gamma}_{B_3}$ is a curve with the same regularity as implied by Theorem* 3.20.

Proof. Just note that $B_2 \cup B_3$ can be covered by a closed ball $\frac{1}{2}B_1$ such that B_1 has radius smaller than $\frac{1}{2}r_0$. Then both $\bar{\gamma}_{B_2}$ and $\bar{\gamma}_{B_3}$ are subsets of $\bar{\gamma}_{B_1}$. Since $K \cap \frac{1}{2}B_2 \cap \frac{1}{2}B_3 \neq \emptyset$, the union of these curves is connected. ■

We can now finish the proof of Theorem 1.1. Take any finite family $\mathcal{B}$ of closed balls with centers in K and radii at least $2r^*$ and smaller than $\frac{1}{10}r_0$ such that $\{\mathrm{int}\frac{1}{2}B : B \in \mathcal{B}\}$ covers K. For each ball $B \in \mathcal{B}$ take $\bar{\gamma}_B$ and let $\bar{K}$ denote the union of all images of these curves. Note that $\bar{K}$ has finite length, since all the curves have finite length.

First, note that we can remove from $\mathcal{B}$, one by one, the balls B such that the image of $\bar{\gamma}_B$ is contained in the images of the curves $\bar{\gamma}$ for other balls in $\mathcal{B}$. Therefore we can assume that for every $B \in \mathcal{B}$ there exists a point in the image of $\bar{\gamma}_B$ that is not in any image of $\bar{\gamma}_{B'}$ for $B \neq B' \in \mathcal{B}$.

Take any $x \in \mathbb{R}^n$ and set

$$\mathcal{B}_x = \{B \in \mathcal{B} : \mathrm{dist}(x, B) \le r^*\}.$$

If $B_x = \{y : |y - x| \le 2r^* + \frac{2}{5}r_0\}$, then $\bigcup \mathcal{B}_x \subset \frac{1}{2}B_x$, and by Lemma 3.25 (note that the radius of B_x, i.e., $2r^* + \frac{2}{5}r_0$, is smaller than $\frac{1}{2}r_0$), all curves $\bar{\gamma}_B$ for $B \in \mathcal{B}_x$ are subsets of $\bar{\gamma}_{B_x}$. Intuitively, locally $\bar{K}$ looks like a single curve.

Let $x \in \bar{K}$ and assume that x is contained in the images of $\bar{\gamma}$ for (at least) three different $B_1, B_2, B_3 \in \mathcal{B}$. Then all images of $\bar{\gamma}_{B_i}$ for $i = 1, 2, 3$ are contained in the image of $\bar{\gamma}_{B_x}$ and one of the images must be contained in the union of the other two, which contradicts the previous assumption.

Now construct a graph with vertex set $\mathcal{B}$ and let B_1 and B_2 be connected in this graph if the images of $\bar{\gamma}_{B_1}$ and $\bar{\gamma}_{B_2}$ coincide. By the previous observations, every vertex $B \in \mathcal{B}$ has degree at most 2: no other curve image may be contained in the image of $\bar{\gamma}_B$ and at every endpoint the image of $\bar{\gamma}_B$ can coincide with at most one other curve. Therefore our graph consists only of paths and loops. By the previous observations, locally $\bar{K}$ looks like one curve, so in total it is a finite union of curves (and some of them may be closed, i.e., they may form images of circles).

What is left to prove is that for every curve in $\bar{K}$ its arc-length parametrization has locally 342ω-continuous derivative. However, note that if $x, y \in \bar{K}$ and $|x-y| < r^*$, then $\bar{K}$ in the neighborhood of x, y is a subset of $\bar{\gamma}_{B_x}$, and this curve has 342ω-continuous derivative.

We conclude this section with the following lemma concerning the total length of all constructed curves.

LEMMA 3.27. *For any $\varepsilon > 0$, by taking sufficiently small r^* and by taking the family $\mathcal{B}$ carefully in the above construction, the total length of all curves can be bounded by $5\mathcal{H}_1(K) + \varepsilon$, where $\mathcal{H}_1$ is the one-dimensional Hausdorff measure.*

Proof. Using the definition of $\mathcal{H}_1$, let $\mathcal{X}$ be a collection of arbitrary sets of diameters smaller than $\frac{1}{20}r_0$ that cover K and

$$\sum_{X \in \mathcal{X}} \operatorname{diam}(X) < \mathcal{H}_1(K) + \frac{1}{12}\varepsilon.$$

We may assume that every set in $\mathcal{X}$ contains a point of K. For every $X \in \mathcal{X}$ we take a ball B_X such that $X \subset B_X$, the center of B_X belongs to K, and B_X has diameter at most twice that of X. Let $\mathcal{B}_0 = \{B_X : X \in \mathcal{X}\}$; then every $B_X \in \mathcal{B}_0$ has diameter smaller than $\frac{1}{10}r_0$, has center in K and the family $\mathcal{B}_0$ covers K. Moreover

$$\sum_{B_X \in \mathcal{B}_0} \operatorname{diam}(B_X) < 2\mathcal{H}_1(K) + \frac{1}{6}\varepsilon.$$

Since K is a compact set, we may assume that $\mathcal{B}_0$ is finite. Let $r^* = \frac{1}{2}\min_{B_X \in \mathcal{B}_0} \operatorname{diam}(B_X)$ and let $\mathcal{B} = \{\operatorname{cl}(2B_X) : B_X \in \mathcal{B}_0\}$. Then $\mathcal{B}$ is a collection of closed balls of radii at least $2r^*$ and smaller than $\frac{1}{10}r_0$, and $\{\operatorname{int}\frac{1}{2}B : B \in \mathcal{B}\}$ covers K. Moreover

$$\sum_{B \in \mathcal{B}} \operatorname{diam} B < 4\mathcal{H}_1(K) + \frac{1}{3}\varepsilon.$$

Since we have chosen r^*, we may now remove the set K_{lonely} from K. Note that K_{lonely} can be covered by segments of total length at most $\frac{1}{2}\varepsilon$.

We now construct $\bar{K}$ using our new family $\mathcal{B}$. By Theorem 3.20(i), the curve γ_B for $B \in \mathcal{B}$ has length smaller than $1.161 \operatorname{diam}(B)$. Therefore $\bar{K}$ has total length at most

$$\frac{1}{2}\varepsilon + 1.161 \cdot \left(4\mathcal{H}_1(K) + \frac{1}{3}\varepsilon\right) < 5\mathcal{H}_1(K) + \varepsilon. \blacksquare$$

4. Counter-examples for Jones-style conditions. In this section we give two counter-examples that show that any Jones-like conditions seem to work worse in the field of curves with regular derivatives. Intuitively, these curves cannot *turn* fast, and therefore the condition should involve *every* point in K, and not *all points off a set of measure* 0.

In this section we work only in $\mathbb{R}^2$, i.e., we do not use any high-dimensional tricks. To simplify the notation, we treat $\mathbb{R}^2$ as the complex field $\mathbb{C}$, for example iv is the vector v rotated by $90°$ counterclockwise.

Let us recall the main techniques used by Jones [6]. Assume we have a compact set $K \subset \mathbb{R}^2$. Given a square Q with sidelength $l(Q)$ we denote by $S_K(Q)$ one of the narrowest strips covering $K \cap 3Q$, and by $\beta_K(Q)$ the ratio $\mathrm{width}(S_K(Q))/l(Q)$.

The main result of Jones [6] is that a compact set $K \subset \mathbb{R}^2$ is a subset of a rectifiable curve iff the following sum is finite:

$$\beta^2(K) = \sum_{Q\,\mathrm{dyadic}} \beta_K^2(Q) l(Q).$$

Our examples show that measuring $\beta_K(Q)$ does not allow one to decide whether K can be covered by a more regular curve.

Let $\omega : [0,\infty) \to [0,\infty)$ be a concave non-decreasing function continuous at 0 with $\omega(0) = 0$ and $\omega(x) > 0$ for $x > 0$. We assume that $\omega(1)$ is much smaller than 1; otherwise we may need some rescaling in the examples. We are about to give two compact sets in $\mathbb{R}^2$ that cannot be covered by a finite family of disjoint finite-length curves with locally ω-continuous derivative (we call such curves *regular curves* for short). However, in both examples the widths of $S_K(Q)$ are small and do not seem to carry much information. Note that both sets satisfy the Jones condition and therefore they are subsets of rectifiable curves.

Moreover, the set from the second example has two more properties. First, it can be covered by four regular curves, but with an intersection, showing that the condition in our result that the curves are disjoint is essential. This example shows that the question whether a compact set K can be covered by a finite union of regular curves, not necessarily disjoint, is significantly different than the question this paper answers.

Second, if one applies Jones's algorithm to construct a rectifiable curve covering the set K from the second example, the curve obtained will have an infinite number of turns by roughly $90°$ and therefore cannot be easily smoothed to a curve with a regular derivative and finite length.

We are not going to give all details of the examples, but we want to give an intuition why Jones's approach seems to break down here.

4.1. Vertical strokes example. Let

$$K := \{(0,0)\} \cup \{[(2^{-n},0),(2^{-n},2^{-n}\omega(2^{-n}))] : n \in \mathbb{Z}_+\}.$$

This set consists of countably many vertical strokes of length $2^{-n}\omega(2^{-n})$ that converge to $(0,0)$. It is obviously compact, but for every n the triangle $(0,0),(2^{-n},0),(2^{-n},2^{-n}\omega(2^{-n}))$ has diameter smaller than $2\cdot 2^{-n}$ and right angle at $(2^{-n},0)$. Therefore, by Theorem 1.2, it cannot be covered by a finite number of disjoint *regular* curves.

Let us explain, without using Theorem 1.2, why it cannot be covered. Assume, towards a contradiction, that it is covered by a finite family $\{\gamma_i\}_{1\le i\le N}$. By easy measure arguments, for every point $(2^{-n},0)$ there must be a curve γ_i tangent to the stroke $[(2^{-n},0),(2^{-n},2^{-n}\omega(2^{-n}))]$ at this point. Therefore there must be one curve γ_{i_0} that is tangent to an infinite number of strokes at their bottom endpoints. Note that between two such points the curve must turn by 180°, and since γ_{i_0} has locally ω-continuous derivative and finite length, this cannot happen infinitely many times.

Let us now look at K from the point of view of the numbers $\beta_K(Q)$. If there is only one stroke in $3Q$, then $\beta_K(Q)=0$. Apart from dyadic squares with corner at $(0,0)$, this is the case for sufficiently small dyadic squares. However, for dyadic squares Q such that $3Q$ intersects more than one stroke we have $\beta_K(Q)\le c\omega(l(Q))$ for some universal constant c, which means that these values are comparable to the values $\beta_K(Q)$ for a *regular* curve. Therefore the numbers $\beta_K(Q)$ do not seem to yield any information in this case.

4.2. Snail example. Let $a_0=1$ and $a_{n+1}=\omega(a_n)a_n$. By the assumption that $\omega(1)$ is much smaller than 1, the sequence a_n is exponentially quickly decreasing and it converges to 0. Let $z_0=0\in\mathbb{C}$ and $z_{n+1} = z_n + a_n i^n\cdot$. Then the points z_n form a snail-like structure and they converge to some point $z=\lim_{n\to\infty} z_n$. Moreover, since ω is small on $[0,1]$, the point z_{n+1} is much closer to z_n than z_n is to z_{n-1}. Let $K=\{z\}\cup\{z_n : n\ge 0\}$.

Since $\measuredangle(z_{n-1},z_n,z_{n+1})$ is a right angle, and $\operatorname{diam}\{z_{n-1},z_n,z_{n+1}\} < a_{n-1} + a_n$, by Theorem 1.2, K is not covered by a finite family of disjoint *regular* curves.

Let us now see what happens if we construct a *rectifiable* curve through K, using the algorithm by Jones [6]. The algorithm takes a sequence $L_0\subseteq L_1\subseteq L_2\subseteq\cdots$ of nets in K; the net L_m is an inclusion maximal subset of points in K such that if $x,y\in L_n$, $x\ne y$, then $|x-y|\ge 2^{-m}$. The algorithm constructs a converging sequence of curves Γ_m such that L_m is covered by Γ_m. The curve Γ_{m+1} is obtained from Γ_m by some local modifications. In our example, the points z_n are included in the consecutive nets one by one (since z_{n+1} is much closer to z_n than z_n is to z_{n-1}), and thus the construction ends up with some sort of spiral and an infinite number of 90° (or close to 90°) turns.

On the other hand, precise calculations show that we can pass one *regular* curve through all points of each of the sets $K_l = \{z\} \cup \{z_{4k+l} : k \in \mathbb{N}\}$ for $l = 0, 1, 2, 3$. However, these curves intersect at z, and since this is a limit of the whole sequence z_n, the intersection there is unavoidable.

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