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Topology and measure of buried points in Julia sets

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Abstract. It is well-known that the set of *buried points* of a Julia set of a rational function (also called the *residual Julia set*) is topologically "fat" in the sense that it is a dense G_δ if it is non-empty. We show that it is, in many cases, a full-measure subset of the Julia set with respect to conformal measure and the measure of maximal entropy. We also address Hausdorff dimension of buried points in the same cases, and discuss connectivity and topological dimension of the set of buried points. Finally, we present a non-dynamical example of a plane continuum whose set of buried points is a dense and hereditarily disconnected (components are points) G_δ, but not totally disconnected (not all quasi-components are points).

1. Introduction. Let $\mathbb{C}_\infty$ denote the Riemann sphere, and let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational function. Then $\mathbb{C}_\infty$ decomposes into the disjoint union of two sets: the Fatou set $F(R)$, which is the maximal domain of normality of the family $\{R^i \mid i \geq 0\}$ of iterates of R, and the Julia set $J(R)$, which is a compact set upon which R exhibits chaotic behavior. The Fatou set often consists of infinitely many components, which are called *Fatou domains*. The Julia set $J(R)$ is either all of $\mathbb{C}_\infty$ or nowhere dense in $\mathbb{C}_\infty$. In this paper, we consider only the case where the Julia set is nowhere dense in $\mathbb{C}_\infty$. General references for studying Fatou and Julia sets include [Bea91, Mil06, CG93]; any unreferenced facts can be found in any of these sources.

We are particularly interested in the set of *buried points* of $J(R)$, defined to be the set of points of $J(R)$ which are not on the boundary of any Fatou domain of R. (Note that, while every point of $J(R)$ will be on the boundary of the *Fatou set* of R, it is not the case that every point of $J(R)$ will be on the

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boundary of a *component* of the Fatou set.) For learning about buried points of Julia sets (also called the *residual Julia set*), we recommend [CM10], and especially [DF08] for transcendental functions.

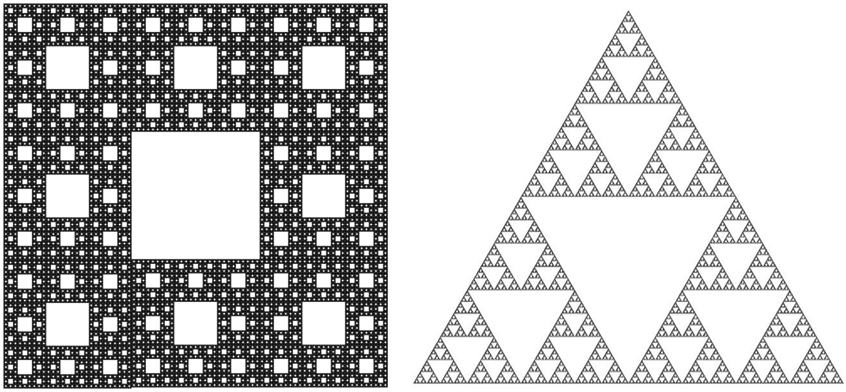
The Julia set tends to be an interesting set, and its set of buried points tends to be as interesting. In this paper, we will show that the Hausdorff dimension of the residual Julia set is often equal to that of the Julia set itself (Corollary 4.3), by showing that the conformal measure of the set of non-buried points is zero (Theorem 4.1). We also show that the residual Julia set supports the measure of maximal entropy. In terms of continuum theory, we show that the set of buried points of a plane continuum reflects the topological complexity of the continuum (Theorem 3.5). For example, a locally connected continuum is Suslinian if and only if its buried point set is Suslinian. Finally, we construct a non-dynamical example, inspired by one of Lelek [L], of a continuum Z such that Z is locally connected, and $\mathrm{Bur}(Z)$ is dense and hereditarily disconnected, but not totally disconnected.

2. Previous work. The notion of a *buried point* is purely topological; we apply it in this paper to the case of Julia sets not equal to the Riemann sphere.

DEFINITION 2.1 (Buried points). Let $X \subset \mathbb{C}_\infty$ be a nowhere dense continuum. A point of X is said to be *buried* if it does not belong to the boundary of any component of $\mathbb{C}_\infty \setminus X$. We denote the set of buried points of X by $\mathrm{Bur}(X)$.

In case X is the Julia set of a rational function R, the set of all buried points of $J(R)$ is also called the *residual Julia set* and is denoted $\mathrm{Bur}(J(R))$. Julia sets of polynomials have no buried points, because the Fatou domain containing ∞ has the Julia set as its boundary. However, non-empty residual Julia sets exist, with examples given by [McM88, Mor97, Mor00, MT93, DLU05]. Singularly perturbed polynomials (i.e., complex polynomials plus terms of the form $\lambda(z-a)^{-d}$ for $a \in \mathbb{C}$ and $d \geq 1$) provide a rich family of examples, as demonstrated in [BDGMR, BDGR, BlaDev06, D05, DRS07].

There are two classes of examples which are important for intuition. The first are Julia sets which are homeomorphic to the Sierpiński carpet (Figure 1(a)) as first exhibited by Milnor and Tan [MT93] and with many examples in [BlaDev06]. The residual Julia set of such a map is one-dimensional, and in fact arcwise connected (every pair of points in the residual Julia set is contained in a homeomorphic image of the unit interval $[0, 1]$, also contained in the residual Julia set). Another important class of examples is given in [BlaDev06] and particularly in [DRS07], where the Julia sets are generalizations of the Sierpiński gasket (Figure 1(b)), and turn out to have zero-dimensional residual Julia sets.



(a) The Sierpiński carpet (b) The Sierpiński gasket

Fig. 1. Two prototypical examples of continua with buried points

Moreover, understanding buried points has figured significantly in the partial solutions to a long-standing problem in complex dynamics called the Makienko Conjecture [Qia97], [Mor00], [SY03], [CMMR09]. The best partial solution thus far shows that the Julia set of a counter-example to the Makienko Conjecture must be an indecomposable continuum, a level of topological complexity we do not address in this paper.

Since $R|_{J(R)}$ is topologically exact (that is, each non-empty open subset of $J(R)$ is eventually mapped onto $J(R)$), we see that the residual Julia set is non-empty if and only if the boundary of each Fatou domain is nowhere dense in $J(R)$. As a consequence, the residual Julia set is a dense G_δ subset of $J(R)$ whenever it is non-empty, as there are only countably many Fatou domains. It is also nowhere locally compact (because the union of the boundaries of all Fatou domains is also dense in $J(R)$). The residual Julia set and the union of the boundaries of the Fatou domains are each fully invariant subsets of $J(R)$.

3. Topology of buried points. In [CM10] we asked, among others, the following two questions, to which we provide partial answers in this paper. Other recent papers have addressed Question 1 from a continuum-theoretic viewpoint [vMT, vMTTV].

Question 1. What can be said about the topological dimension of the set of buried points [of a Julia set]? For example, if the set of buried points is totally disconnected, is it zero-dimensional?

Question 2. What can be said about the Hausdorff dimension of the set of buried points? For example, can a Julia set and its (non-empty) residual

Julia set have different Hausdorff dimensions? In particular, what about a Sierpiński gasket Julia set?

At that time, we did not distinguish between several inequivalent definitions of "totally disconnected." We do so now.

DEFINITION 3.1 ([E, pp. 31 ff.]). Let X be a Hausdorff space. We say that X is

(1) *zero-dimensional* if the topology of X has a basis of sets which are both closed and open,
(2) *totally disconnected* if quasi-components of X are points,
(3) *hereditarily disconnected* if components of X are points, and
(4) *punctiform* if all subcontinua of X are points.

The list is presented in order of decreasing strength; for example, spaces which are zero-dimensional are automatically totally disconnected, hereditarily disconnected, and punctiform. However, these are truly different concepts; for any two properties in the list, there is a one-dimensional planar set which satisfies one but not the other. Nevertheless, these concepts are all the same for locally compact spaces.

There is an applicable sufficient condition for the buried point set of a plane continuum to be zero-dimensional. A continuum is said to be *rim-finite* (or *regular*) if it has a basis of open sets with finite boundaries. The following theorem applies to the topological example of the Sierpiński gasket above and to all the "gasket-like" rational Julia sets described in [DRS07]. We denote the boundary of a set A by ∂A and the closure of A by $\overline{A}$.

THEOREM 3.2. *Let X be a planar rim-finite continuum. If non-empty, the set $\mathrm{Bur}(X)$ of buried points of X is zero-dimensional.*

Proof. According to [W, Corollary 3.12], for every $x \in X$ and for every $\epsilon > 0$ there is a simple closed curve $S \subset \mathbb{C}_\infty \setminus \{x\}$ such that $S \cap X$ is finite and the diameter of the component U of $\mathbb{C}_\infty \setminus S$ containing x is less than ϵ. However, the finitely many open arcs comprising $S \setminus X$ serve to show that $\partial_X(U \cap X)$ consists of points accessible from $\mathbb{C}_\infty \setminus X$, so $S \cap \mathrm{Bur}(X) = \emptyset$. Hence, $U \cap \mathrm{Bur}(X)$ is a neighborhood of x in $\mathrm{Bur}(X)$ with empty boundary, and whose diameter is less than ϵ. ∎

How the boundaries of Fatou domains meet each other plays an important role in the character of the set of buried points.

THEOREM 3.3. *Let $X \subset \mathbb{C}_\infty$ be a continuum. If $X \setminus \mathrm{Bur}(X)$ is not connected, then a subcontinuum of $\mathrm{Bur}(X)$ separates X. In particular, $\mathrm{Bur}(X)$ is not punctiform.*

Proof. Because $\mathbb{C}_\infty$ is unicoherent, there is a continuum $Y \subset \mathbb{C}_\infty$, disjoint from $X \setminus \mathrm{Bur}(X)$, which separates $\mathbb{C}_\infty$ between some pair of points of

$X \setminus \mathrm{Bur}(X)$. If U is a complementary component of X and $Y \cap U \neq \emptyset$, then the connectedness of Y implies that $Y \subset U$ since $Y \cap \partial U \subset Y \cap (X \setminus \mathrm{Bur}(X)) = \emptyset$, and therefore $Y \cap \partial U$ is empty. Hence, either Y is a subset of $\mathbb{C}_\infty \setminus X$, or $Y \subset \mathrm{Bur}(X)$. In the first case, X is not connected, contrary to our assumption, so we conclude that $Y \subset \mathrm{Bur}(X)$ is a non-degenerate continuum separating $\mathbb{C}_\infty$ between points of X. ∎

DEFINITION 3.4. A space X is *Suslinian* if every pairwise disjoint collection of non-degenerate continua in X is countable.

Note that the Sierpiński gasket is Suslinian, while the Sierpiński carpet is not, and both are locally connected. The cone over the convergent sequence $\{1/n\}_{n=1}^{\infty} \cup \{0\}$ is Suslinian, while the cone over the middle-third Cantor set is not, and both are non-locally connected. We prove the following theorem which relates the topological complexity of the set of buried points to that of the whole Julia set. After Moore [Moo29], we say a continuum is a *triod* if it is the union of three continua such that the common part of all three is a proper subcontinuum of each and is also the common part of any two of them.

THEOREM 3.5. *Let X be a plane continuum and suppose the boundary of each component of $\mathbb{C}_\infty \setminus X$ is locally connected. If* $\mathrm{Bur}(X)$ *is Suslinian, then X is Suslinian.*

Proof. Suppose X is not Suslinian. Let $\mathcal{C}$ be an uncountable collection of pairwise disjoint continua in X. Let $\mathcal{C}'$ denote the subcollection $\mathcal{C}$ which meets the set of non-buried points of X in at least three points. Because the boundaries of complementary domains are locally connected, each point in the boundary of each complementary domain is accessible. Hence, for each $C \in \mathcal{C}'$ there are three disjoint arcs A_C^1, A_C^2, and A_C^3, each contained in the complement of X except for one endpoint, which is in C. Then $C \cup \bigcup_{i=1}^{3} A_C^i$ is a triod. If one chooses the arcs A_C^i to be short and "conformally radial" according to selected Riemann maps of the complementary domains, one can ensure that any two added arcs in a given complementary domain are disjoint. Hence, the triods corresponding to distinct elements of $\mathcal{C}'$ are disjoint. By a theorem of Moore [Moo29], the plane does not contain uncountably many disjoint triods. Hence, the collection $\mathcal{C}'$ is countable.

Therefore, the collection $\mathcal{C} \setminus \mathcal{C}'$ is uncountable and consists of continua which meet $X \setminus \mathrm{Bur}(X)$ in at most two points. There are only countably many complementary domains of X, so there exist complementary domains U_1 and U_2 (perhaps the same) such that uncountably many elements of $\mathcal{C} \setminus \mathcal{C}'$ meet $X \setminus \mathrm{Bur}(X)$ only in $\overline{U_1 \cup U_2}$. We can then find for each $C \in \mathcal{C} \setminus \mathcal{C}'$, by an application of the Boundary Bumping Theorem [Nad92, p. 73], a non-degenerate subcontinuum $C'' \subset C$ such that $C'' \cap (X \setminus \mathrm{Bur}(X)) = \emptyset$. The collection $\mathcal{C}''$ of such non-degenerate C'' is in one-to-one correspondence with

$\mathcal{C} \setminus \mathcal{C}'$, so it is uncountable, and each element of $\mathcal{C}''$ is contained in $\mathrm{Bur}(X)$. Therefore, $\mathrm{Bur}(X)$ is not Suslinian. ∎

COROLLARY 3.6. *Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map. If $J(R)$ has locally connected boundaries of Fatou domains, and* $\mathrm{Bur}(J(R))$ *is punctiform, then $J(R)$ is Suslinian.*

Proof. Suppose $J(R)$ has locally connected boundaries of Fatou domains, and the set $\mathrm{Bur}(J(R)) \neq \emptyset$ is punctiform. Then, trivially, $\mathrm{Bur}(J(R))$ is Suslinian. It follows from Theorem 3.5 that $J(R)$ is Suslinian. ∎

REMARK 3.7. It follows that if the set of buried points of a Julia set has any one of the properties (1)–(4) of Definition 3.1, and the boundaries of all Fatou domains are locally connected, then the Julia set itself can be only mildly complex: it must be Suslinian, even if it is not locally connected. The generalized Sierpiński gasket Julia sets described in [DRS07] have punctiform, in fact zero-dimensional, sets of buried points, and Fatou domain boundaries are simple closed curves, so the Julia sets are Suslinian by the above corollary. But these Julia sets are known, for dynamical reasons, to be locally connected. It is not known if there is any rational Julia set whose Fatou domain boundaries do not form a null sequence. If the boundaries do form a null sequence, and are locally connected, then the Julia set is locally connected. This leads to Question 4 below.

4. Measurable dynamics of the buried points. Our partial answer to Question 2 turns on the measure theory of the Julia set. A rational function R and its Julia set $J(R)$ are said to be *hyperbolic* if the closure of the forward orbit of all critical points is disjoint from the Julia set. Hyperbolic Julia sets, for example, support a Borel probability measure μ which is positive on non-empty open sets, invariant (that is, $\mu(R^{-1}(A)) = \mu(A)$ for all Borel A), and for which R sends sets of μ-measure 0 to sets of μ-measure 0. These measure-theoretic properties, combined with the dynamical properties of rational maps and the topological properties of Julia sets, are just what is needed to prove the following theorem.

THEOREM 4.1. *Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with Julia set $J(R)$ and non-empty buried point set* $\mathrm{Bur}(J(R))$. *Suppose $J(R)$ supports an invariant Borel probability measure μ, positive on non-empty open sets, and R sends sets of μ-measure* 0 *to sets of μ-measure* 0*. Then* $\mu(\mathrm{Bur}(J(R))) = 1$.

REMARK 4.2. There are two measures to which this theorem immediately applies. The first is the measure of maximal entropy on a Julia set constructed in [FMS]. The second is an invariant probability measure constructed in [R-LS], which is absolutely continuous with respect to an ergodic conformal measure.

Proof of Theorem 4.1. Since $J(R) \setminus \mathrm{Bur}(J(R))$, the union of the boundaries of all Fatou domains, is fully invariant under R, it suffices to show that $\mu(\partial U) = 0$ for each Fatou domain U of the countably many Fatou domains. In fact, we will show that the grand orbit of the boundary of each Fatou domain has μ-measure 0.

By Sullivan's Classification Theorem, every Fatou domain is eventually periodic. Recall that the Julia set of R is also the Julia set of each iterate of R. So without loss of generality, let U be a Fatou domain fixed by R and let $A = \bigcup_{k=0}^{\infty} R^{-k}(\partial U)$ be the grand orbit of ∂U. Since A is an increasing union and $\partial U \subset A$, it follows from the invariance property of μ with respect to R that $\mu(A \setminus \partial U) = 0$.

Let $z \in \partial U$. We claim that z has a neighborhood V in A such that $\mu(V) = 0$. If so, then by compactness ∂U is contained in the finite union of such neighborhoods; hence, $\mu(\partial U) = 0$, from which it follows that $\mu(A) = 0$, as desired.

The backward orbit of z is dense in $J(R)$ and $\partial U \neq J(R)$, else there are no buried points (since under our assumptions $J(R)$ is at most one-dimensional). Hence, there is a $k \geq 1$ and $w \in A \setminus \partial U$ such that $R^k(w) = z$. Given any $\epsilon > 0$, we can find a neighborhood W of w in A such that $R^k(W) \subset B(z, \epsilon)$. We may choose W so that $W \cap \partial U = \emptyset$ and $R^k(W) \cap W = \emptyset$. Let $V = R^k(W)$. Since A is fully invariant, and R is an open map on $J(R)$, R is open on A. Hence, V is a neighborhood of z in A. Since $\mu(W) = 0$, and μ sends sets of measure 0 to sets of measure 0, we have $\mu(V) = 0$, as claimed. ∎

Even when a Julia set is locally connected, it often has complexity indicated by a Hausdorff dimension (denoted HD $J(R)$) exceeding its topological dimension. The above theorem, combined with [R-LS, Main Theorem], gives the following corollary; see [R-LS] for the current state of the matter, as well as for the definitions of terms left undefined in the result below.

Corollary 4.3. *Let R be a rational map which is expanding away from critical points. If* $\mathrm{Bur}(J(R))$ *is non-empty, then its Hausdorff dimension is the same as that of $J(R)$. If, in addition, there are no critical points on the boundary of the (necessarily hyperbolic) periodic Fatou domains of R, then the set of non-buried points is of strictly smaller dimension.*

Remark 4.4. Many good sorts of maps satisfy the hypotheses of the above, including hyperbolic, geometrically finite, and topological Collett–Eckmann maps (see [Prz07]). However, rational maps with parabolic points are not addressed by this corollary.

Proof of Corollary 4.3. According to [R-LS, Main Theorem], such a rational function admits a unique conformal measure μ whose exponent is equal to HD $J(R)$. The dimension of the measure is also HD $J(R)$, meaning that every full μ-measure set $A \subset J(R)$ satisfies HD $A =$ HD $J(R)$. Also, we are guaranteed the existence of a measure η, absolutely continuous with respect to μ, which is an invariant measure satisfying the requirements of Theorem 4.1. Therefore, $\eta(\mathrm{Bur}(J(R))) = 1$. We deduce by absolute continuity that $\mu(\mathrm{Bur}(J(R)))$ is positive; that it is full follows from ergodicity and the fact that $\mathrm{Bur}(J(R))$ is fully invariant. Hence, HD $\mathrm{Bur}(J(R)) =$ HD $J(R)$.

Now, suppose that no periodic Fatou domain of R has a critical point on its boundary. Let X denote the union of the boundaries of all periodic Fatou domains. (Note that, due to Sullivan's No Wandering Domains Theorem, all non-buried points eventually map into X.) We see that X is a non-empty, compact, forward-invariant subset of $J(R)$. Since X does not contain any critical points and R is expanding away from critical points, X is a hyperbolic subset of $J(R)$.

We now use some concepts from the thermodynamical formalism. For a forward-invariant subset $K \subset J(R)$, it is fruitful to consider the *topological pressure* $P(t)$ of $-t \log |f'|$ on $J(R)$, where t is a real variable [U03]. Generally, the least zero of P_K detects the Hausdorff dimension of a conformal repeller. Specifically, the least zero t_J of $P_{J(R)}$ is the *hyperbolic dimension* of $J(R)$, i.e. the supremum of the Hausdorff dimensions of hyperbolic subsets of $J(R)$ (cf. [P99]). Also, the least zero t_X of P_X is the Hausdorff dimension of X (cf. [B79, R82]). By [PL11, Lemma 6.2], $P_X(t) < P_{J(R)}(t)$ for all t, so we find that HD $X <$ HD $J(R)$. Since Hausdorff dimension is not increased by countable unions and Lipschitz maps, we conclude that HD $\bigcup_{n=0}^{\infty} f^{-n}(X) = 0$. ■

5. Example. In this section we describe the topological example to which we referred in the Introduction. Our example is strongly inspired by one of Lelek [L]. It bears a passing resemblence to the Sierpiński triangular gasket (see Figure 1(b)) in that it is obtained from a triangle by removing countably many triangles (with some identification afterward). However, our end result will not have zero-dimensional buried point set; in fact, though its buried points will have no non-degenerate components, some of the quasi-components of the set of buried points will not be singletons.

Example. There is a locally connected plane continuum Z such that $\mathrm{Bur}(Z)$ is a dense, hereditarily disconnected, though not totally disconnected, G_δ.

We construct the continuum in stages.

(1) We will first construct a non-locally connected (though Suslinian) continuum W as a nested intersection of continua such that any

subset of W separating W between points of $\mathrm{Bur}(W)$ must either contain one of a countable collection $\mathcal{A}$ of arcs, or intersect $\mathrm{Bur}(W)$ in an uncountable set.

(2) We will add to W a null sequence of disjoint arcs to form a continuum X so as to bury a dense subset of each of those countably many arcs in $\mathcal{A}$. Hence $\mathrm{Bur}(X)$ will be hereditarily disconnected but not totally disconnected.

(3) We will add to X a null sequence of arcs to obtain a locally connected continuum Y with $\mathrm{Bur}(Y) = \mathrm{Bur}(X)$.

(4) We will perform a finite identification on infinitely many complementary domains of Y to obtain a locally connected continuum Z such that $\mathrm{Bur}(Z)$ is homeomorphic to $\mathrm{Bur}(Y)$ and is a dense G_δ in Z.

5.1. A continuum with strange separation properties. Here we construct the continuum W mentioned above. In what follows, we use vector arithmetic to specify various points used in the construction. If A, B, and C are (non-collinear) points in the plane, then $\overline{AB}$ denotes the closed straight line segment joining A and B, and $\Delta(A, B, C)$ denotes the closed triangle with vertices A, B, and C.

Let τ be an ordered triple (T^τ, L^τ, R^τ) of non-collinear points in the plane. Here we describe how to define the family of triples $\mathcal{T}(\tau)$ corresponding to subtriangles of $\Delta\tau$. The sequences p_i, L_i, R_i, and Q_i in the following depend, strictly speaking, upon τ; for the moment we will suppress this, and will indicate dependence with a superscript only when confusion may arise.

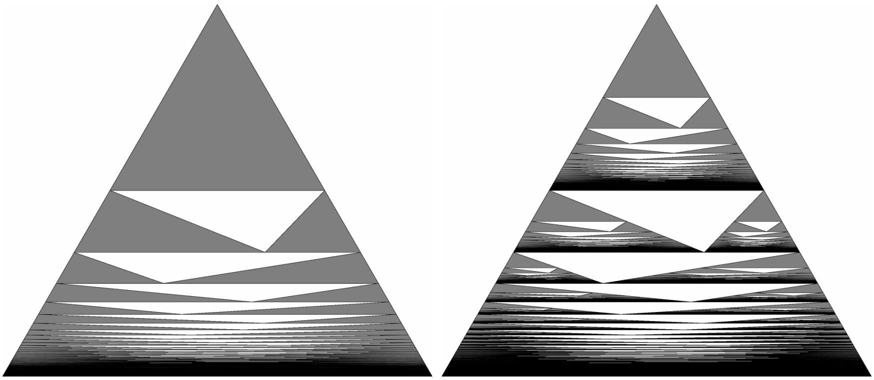
Choose a sequence $(p_i)_{i=1}^\infty$ of numbers in $[1/3, 2/3]$ so that $\limsup_{i\to\infty} p_i = [1/3, 2/3]$. Define the points

$$\begin{aligned} L_i &= \frac{T + iL}{i+1}, && i \geq 1, \\ R_i &= \frac{T + iR}{i+1}, && i \geq 1, \\ Q_i &= p_i R_i + (1 - p_i) L_i, && i \geq 2. \end{aligned}$$

Hence, L_i converges to L, R_i converges to R, and the $\limsup$ of Q_i is the closed middle-third interval of $\overline{LR}$. Now define triples corresponding to subtriangles of $\Delta\tau$ (see Figure 2 for an illustration):

$$\mathcal{T}(\tau) = \{(T, L_1, R_1)\} \cup \bigcup_{i\geq 1} \{(L_i, L_{i+1}, Q_{i+1}), (R_i, Q_{i+1}, R_{i+1})\}.$$

For the sake of definiteness, we set $\tau_0 = ((1/2, \sqrt{3}/2), (0, 0), (1, 0))$, i.e., the equilateral triangle in $\mathbb{R}^2$ whose base is $[0, 1] \times \{0\}$ and whose other vertex is above the x-axis. We define a sequence of collections of triples,



(a) Level 1 (b) Level 2

Fig. 2. Two levels of the construction of the continuum W

naturally viewed as simplicial complexes:

$$S_n = \begin{cases} \{\tau_0\} & \text{if } n = 0, \\ \bigcup\{\mathcal{T}(\tau') \mid \tau' \in S_{n-1}\} & \text{if } n \geq 1. \end{cases}$$

The solid triangles coming from S_n are therefore subsets of solid triangles coming from S_{n-1}; if $\tau' \in S_{n-1}$ and $\tau \in S_n$ are such that $\Delta\tau \subset \Delta\tau'$, we call τ a *child* of τ', and τ' the *parent* of τ. We will also speak of triangles being *ancestors* (respectively, *descendants*) of other triangles, if they are comparable with respect to the relation "is a child of" (respectively, "is the parent of"). It will be convenient later to refer to *left* children, *right* children, and *top* children, depending upon which edges of $\Delta\tau$ are subsets of $\partial\Delta\tau'$. By a *left descendant* (respectively, *right descendant*) of τ we mean any descendant of τ which is a left (respectively, right) child of its parent.

For each $n \geq 0$, we set

$$W_n = \overline{\bigcup\{\Delta\tau \mid \tau \in S_n\}},$$

and let $W = \bigcap_{n\geq 0} W_n$. In Figure 2, we have illustrated one possibility for W_1 and W_2.

It is not difficult to see that W is a one-dimensional continuum, since the intersection of a nest of continua is a continuum. To see that $\mathrm{Bur}(W)$ is zero-dimensional, consider the families of triangles defined by elements of S_n and used in the construction of W. The diameters of the triples in S_n tend uniformly to zero in n: the maximum height of $\Delta\tau$ for $\tau \in S_n$ is $1/2^n$ and the maximum width is $(2/3)^n$. Since $\partial\Delta\tau \subset W \setminus \mathrm{Bur}(W)$, the sets $\Delta\tau \cap \mathrm{Bur}(W)$, for $\tau \in \bigcup_{n=1}^{\infty} S_n$, define a null local basis for every buried point. So $\mathrm{Bur}(W)$ is zero-dimensional. Because every continuum in W must contain a vertex of some triangle $\Delta\tau$, $\mathrm{Bur}(W)$ contains no non-degenerate continua. Therefore,

W is Suslinian by Theorem 3.5 (it is easy to observe this directly as well). Since the property of being Suslinian is preserved by adding countably many arcs and by monotone mappings, our future modifications of W will also be Suslinian.

Though W is relatively tame, we can show that it is difficult to separate points in the horizontal segments in W.

LEMMA 5.1. *If $C \subset W$ is a closed subset of W which separates W between two points in $\overline{L^{\tau_0}R^{\tau_0}}$, then either $C \cap \mathrm{Bur}(W)$ is uncountable, or, for some $n \geq 0$ and $\tau \in \mathcal{T}^n(\tau)$, C contains a subinterval of $\overline{L^\tau R^\tau}$.*

REMARK 5.2. Both types of separators are possible; containing an interval would be a by-product of containing all but finitely many $Q_i^{\tau_0}$. On the other hand, the separator obtained by intersecting the line $x = 1/2$ with W would necessarily contain many buried points of W.

Proof of Lemma 5.1. Assume C contains no subintervals. It is evident that a point is buried if and only if it is contained in infinitely many left and in infinitely many right triangles. For brevity, let us say a triple $\tau \in S_n$ is *cut* if $C \cap \Delta\tau$ separates $\Delta\tau \cap W$ between points of its base $\overline{L^\tau R^\tau}$.

CLAIM. *Let $\tau \in S_n$ be such that only finitely many of its left descendants are cut. Then τ is not cut.* (A symmetric proof shows the corresponding fact for right descendants.)

Proof. Choose a sequence $x_i \in \overline{L^\tau R^\tau}$ so that $x_i \to R^\tau$ monotonically, $x_0 = L^\tau$, x_1 is in the middle third of the base $\overline{L^\tau R^\tau}$, $x_i \notin C$ for all i, and

$$\frac{1}{3} < \frac{|x_{i+2} - x_{i+1}|}{|R^\tau - x_{i+1}|} < \frac{2}{3} \quad \text{for all } i. \tag{5.1}$$

We now show inductively that $W \cap \Delta\tau$ is not separated between two points of $\overline{L^\tau R^\tau}$. Note that by condition (5.1), x_{i+1} is in the middle third of the interval $\overline{x_i R^\tau}$. Choose a sequence i_k^1 so that $Q_{i_k^1} \to x_1$. Since C is closed, $Q_{i_k^1} \in C$ for at most finitely k. Let $\tau_{i_k^1} \in S_{n+1}$ denote the left child of τ with right vertex $Q_{i_k^1}$. Since at most finitely many $\tau_{i_k^1}$ are cut, this serves to show that $W \cap \Delta\tau$ is not separated between points of $\overline{x_0 x_1}$.

Correspondingly, let $\tau'_{i_k^1}$ be the sequence of right children of τ with vertex $Q_{i_k^1}$. Let $\tau_{i_k^2}$ be a left child of $\tau'_{i_k^1}$ which is not cut, with the sequence chosen so that the right vertex converges to x_2. (This can be done because of (5.1).) Additionally, we can choose these triangles so that the index of the left vertex of $\tau_{i_k^2}$ in its parent is at least k. Consequently, the sequence of left vertices of the $\tau_{i_k^2}$ converge to x_1. Thus, the sequence $\Delta\tau_{i_k^2}$ of triangles converges to the interval $\overline{x_1 x_2}$, and W is not cut between two points of $\overline{x_1 x_2}$.

This serves to show that W is not separated between two points of $\overline{x_0x_2}$ since $x_1 \notin C$. We would continue by setting $\tau'_{i_k^2}$ to be the right triples sharing a vertex with $\tau_{i_k^2}$, and finding left subtriangles thereof converging to $\overline{x_2x_3}$. Proceeding inductively, we find that W is not separated between points of $\overline{x_0x_n} \setminus C$ for all n, so τ is not cut. This concludes the proof of the Claim. ∎

Suppose then that W is separated between points of $\overline{L^{\tau_0}R^{\tau_0}}$. Then infinitely many left and infinitely many right children of τ_0 are cut. Every separated child has infinitely many left and infinitely many right cut children; reasoning inductively, we see that there are therefore uncountably many sequences $(\tau_i)_{i=1}^{\infty}$, where τ_i is the parent of τ_{i+1}, that consist of infinitely many left and infinitely many right triples which are cut. The corresponding intersections $\bigcap \Delta\tau_i$ are buried points of W, and are contained in C since C is closed. ∎

5.2. Hereditarily disconnected but not totally disconnected sets of buried points. We now describe a method for burying a dense G_δ subset of points on bottom segments of triangles with a null sequence of arcs. This can be achieved by gluing infinitely many copies of an auxiliary continuum K to each of the bottom segments.

We define the auxilliary continuum K as follows. Let $K_{-1} = [0, 1] \times \{0\}$. Let K_0 be the semicircle in the closed lower half-plane with center $(2^{-1}, 0)$ and radius 2^{-1}. Suppose K_n is defined. Let K_{n+1} be the union of the 3^n semicircles in the closed lower half-plane with centers the midpoints of the components of $K_{-1} \setminus \bigcup_{i=0}^{n} K_i$ and diameters half the length of the corresponding component. Let $K = \bigcup_{i=-1}^{\infty} K_i$. See Figure 3 for an illustration. Let M denote the set of local cutpoints of K in K_{-1} (a dense F_σ in K_{-1} by construction). It is easy to see that no point in $K_{-1} \setminus M$ is contained in a bounded complementary domain of K.

It is evident that we can embed countably infinitely many copies $\{C_i\}_{i=1}^{\infty}$ of K into $\mathbb{R}^2$ by Euclidean similarities $\{f_i\}_{i=1}^{\infty}$ so that the following conditions are met:

(1) For any triple τ, each open subset of $\overline{L^\tau R^\tau}$ meets at least one C_i.
(2) $C_i \cap C_j = \emptyset$ whenever $i \neq j$.
(3) $C_i \cap W = f_i(K_{-1})$ for each i.

Note that by our construction, each horizontal edge of a triangle in W meets countably infinitely many copies of K, as shown in Figure 3 bottom. Let X be the continuum $W \cup \bigcup_{i=1}^{\infty} C_i$. Then the points in $\bigcup_{i=1}^{\infty} f_i(K_{-1} \setminus M)$ corresponding to points in $K_{-1} \setminus M$ (which comprise a dense G_δ of every horizontal edge in W) are buried in X. Thus, $\mathrm{Bur}(X)$ consists of the set of buried points of W together with the sets of newly buried points in each horizontal edge in W.

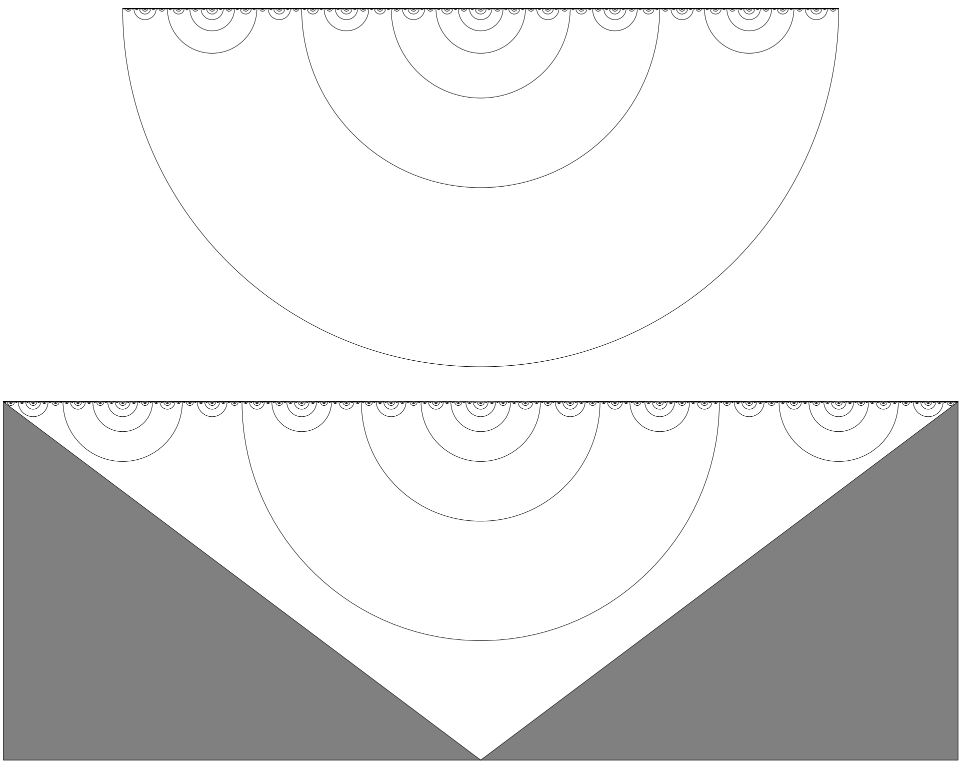


Fig. 3. The auxiliary continuum, and embeddings on an edge $\overline{L^\tau R^\tau}$ of a typical complementary domain of X

It is now easy to see that the buried points of X are hereditarily disconnected. For let

$$F = \Big[\bigcup_{n\geq 1}\Big(\bigcup S_n\Big)\Big] \cup \Big[\bigcup_{n\geq 1} f_n(M)\Big] \subset X$$

be the union of all vertices of all triangles used in the construction together with all embedded copies of M. Then F consists entirely of non-buried points, and it is easy to verify that every connected subset of X intersects F. It follows that $\mathrm{Bur}(X)$ is hereditarily disconnected. By Lemma 5.1, $\mathrm{Bur}(X)$ is not totally disconnected.

5.3. Local connectivity and density of buried points. For a locally connected example, we must perform some identifications. We will first embed X in a locally connected plane continuum Y so that $\mathrm{Bur}(X) = \mathrm{Bur}(Y)$. Enumerate the bounded complementary domains of X as $(U_i)_{i=1}^{\infty}$, and let the height of U_i be ϵ_i. It is then apparent that we can find finitely many pairwise disjoint vertical arcs $A_1^i, \ldots, A_{n_i}^i$ in $\overline{U_i}$ such that

(1) for each j, both endpoints of A^i_j are in ∂U_i;
(2) one endpoint of A^i_j is in the horizontal edge of ∂U_i, but not in $\bigcup_{i\geq 1} C_i$;
(3) for each j, the other endpoint of A^i_j is in a non-horizontal edge of ∂U_i;
(4) the diameter of every component of $U \setminus (A^i_1 \cup \cdots \cup A^i_{n_i})$ is at most $2\epsilon_i$.

Hence, the continuum $Y = X \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i} A^i_j$ is locally connected, since its complementary domain boundaries are locally connected and form a null sequence. (For any ϵ, only finitely many complementary domains of X have height exceeding ϵ.) Since only finitely many arcs are added in each $\overline{U_i}$, $\mathrm{Bur}(Y) = \mathrm{Bur}(X)$. Therefore, Y is a locally connected continuum whose buried point set is hereditarily disconnected and not totally disconnected.

In order to obtain an example Z where the buried point set is a dense G_δ, it is enough to notice that $Y \setminus W$ is the union of a null sequence of open arcs whose closures are pairwise disjoint. Hence, the equivalence relation $\sim$ defined by "$x \sim y$ if and only if $x = y$ or x and y are contained in the closure of a free arc in Y" is an upper semicontinuous equivalence relation with connected, non-separating classes. (An open arc in Y is *free* if it is an open set in Y.) The quotient $\mathbb{R}^2/{\sim}$ is therefore homeomorphic to $\mathbb{R}^2$ by Moore's Plane Decomposition Theorem, and the corresponding quotient map $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism on $\mathrm{Bur}(Y)$. Hence, $Z = \pi(Y)$ has buried point set homeomorphic to that of Y, so this set is hereditarily disconnected but not totally disconnected, and a dense G_δ in Z.

6. Questions. We conclude with some questions related to the above that remain open. We state the questions for Julia sets, but one is free to phrase them for plane continua with appropriate topological (and perhaps dynamical) properties.

Question 3. If the set $\mathrm{Bur}(J(R))$ is totally disconnected (quasi-components are points), is it zero-dimensional? What if $J(R)$ is locally connected?

A Suslinian continuum is connected im kleinem at a dense set of points (see [FL]). (A continuum X is *connected im kleinem* at a point $x \in X$ provided that x has arbitrarily small connected, but not necessarily open, neighborhoods.) There are planar topological examples of such continua which are not locally connected. This motivates the question below.

Question 4. If the Julia set $J(R)$ of a rational function is Suslinian, is it locally connected?

There are polynomial Julia sets which are connected im kleinem at a dense set of points, are not locally connected, and are not Suslinian [BO09, BBCO]. As a start on finding a Suslinian but not locally connected Julia set, one could consider the non-locally connected Julia sets of Sørensen [S] and Roesch [R].

As we were completing this paper, we received a preprint from Jan van Mill and co-authors ([vMTTV]) which shows that there is indeed a locally connected continuum in the plane with a totally disconnected, one-dimensional, dense G_δ set of buried points. The example is constructed by "burying" a one-dimensional set originally constructed by Kuratowski ([E, p. 19]). The example answers our Question 1 in the negative from a topological viewpoint. Kuratowski's set is uncountable, but one-dimensional only on a countable set. They go on to prove that a totally disconnected, buried one-dimensional set in a plane continuum, where the boundaries of complementary domains are locally connected, must be one-dimensional on only a countable set, showing their topological example is sharp. The question (Question 3) is still open for the set of buried points of a Julia set. It seems likely that the increased local structure available in the Julia set case, coming from the existence of the rational function under which the Julia set is fully invariant, will result in an affirmative answer to the question for Julia sets.

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