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Statistical stability of geometric Lorenz attractors

by

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Abstract. We consider the robust family of geometric Lorenz attractors. These attractors are chaotic, in the sense that they are transitive and have sensitive dependence on initial conditions. Moreover, they support SRB measures whose ergodic basins cover a full Lebesgue measure subset of points in the topological basin of attraction. Here we prove that the SRB measures depend continuously on the dynamics in the weak* topology.

1. Introduction. The theory of dynamical systems was initiated by Poincaré's work on the three-body problem of celestial mechanics and it studies processes which are evolving in time. The description of the processes is given in terms of flows when the time is continuous or iterations of maps when the time is discrete. An orbit is a time-ordered collection of states of the system, obtained by starting from a specific state and applying the flow or the map. The main goals of this theory are to describe the typical behavior of orbits as time goes to infinity, and to understand how this behavior changes when we perturb the system or to which extent it is stable. In this work we are concerned with the stability of systems.

Ergodic theory deals with measure preserving processes in a measure space. One in particular tries to describe the average time spent by typical orbits in different regions of the phase space. According to Birkoff's Ergodic Theorem, such times are well defined for almost all points, with respect to any invariant probability measure. However, the notion of typical orbit is usually meant in the sense of volume (Lebesgue measure), which is not always an invariant measure. It is a fundamental open problem to understand under which conditions the behavior of typical (with respect to Lebesgue measure) orbits is well defined from the statistical point of view. This problem can be precisely formulated by means of *Sinai–Ruelle–Bowen* (*SRB*) *measures* which were introduced by Sinai for Anosov diffeomorphisms [Si72]

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and later extended by Ruelle and Bowen to Axiom A diffeomorphisms and flows [BR75, Ru76].

Definition 1.1 (SRB measures). Let μ be an invariant Borel probability measure for a flow $(X^t)_t$ on the Borel sets of a manifold M. The *basin* of μ is the set of points $x \in M$ such that for any continuous $\varphi : M \to \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(X^t(x))\,dt = \int \varphi\,d\mu. \tag{1.1}$$

The measure μ is called an *SRB measure* if its basin has positive Lebesgue measure.

The notions of basin and SRB measure can be easily extended to discrete time dynamical systems, simply by replacing the integral by a time series in (1.1).

A fairly good description of the statistical behavior of orbits can be given by an SRB measure in the sense that, for a "big" (meaning positive volume) set of points, the time averaging of a physical observable (a continuous function on the manifold) of the system is accomplished simply by integrating the observable with respect to SRB measure (space average).

In trying to capture the persistence of the statistical properties of a dynamical system, Alves and Viana [AV02] proposed a notion, called *statistical stability*, which expresses the continuous variation of SRB measures as a function of the dynamical system. This is a kind of stability in the sense that the outcome of evaluating continuous functions along orbits does not change much under small perturbations of the system. This is what may be observed in computer experiments, where typically the picture obtained by plotting an orbit seems to be independent of the starting point and truncation errors.

Next we introduce the notion of statistical stability for vector fields.

Definition 1.2 (Statistical stability). Assume we have a family $\mathcal{X}$ of vector fields endowed with a topology, admitting a common trapping region U on which each $X \in \mathcal{X}$ has a unique SRB measure μ_X. We say that $\mathcal{X}$ is *statistically stable* (in U) if the map $\mathcal{X} \ni X \mapsto \mu_X$ is continuous, where the space of probability measures is equipped with the weak* topology.

Our goal in this work is to prove the statistical stability of a family of vector fields associated to the Lorenz equations.

1.1. Lorenz equations. Lorenz [Lo63] studied numerically a vector field X defined by the system of equations

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = bx - y - xz, \\ \dot{z} = xy - cz, \end{cases}$$

for the parameters $a = 10$, $b = 28$ and $c = 8/3$. The following properties of this vector field are well known:

(1) X has a *singularity* at the origin where $DX(0)$ has real eigenvalues
$$0 < -\lambda_3 \approx 2.6 < \lambda_1 \approx 11.83 < -\lambda_2 \approx 22.83;$$
(2) there is an open set U, the *trapping region*, such that $X^t(\bar{U}) \subseteq U$ for all $t > 0$; the maximal invariant set in U, $\Lambda = \bigcap_{t>0} X^t(U)$, is an attractor and the origin is the only singularity contained in U;
(3) the divergence of X is negative:
$$\operatorname{div} X = \partial\dot{x}/dx + \partial\dot{y}/dy + \partial\dot{z}/dz = -(a + 1 + c) < 0.$$
By Liouville's formula, the flow of X contracts volume. Thus, Λ has zero volume.

Lorenz found with his experimental computations that the flow is sensitive to initial conditions near the attractor, i.e. even a small initial error leads to enormous differences in the outcome. It was a challenging problem to give a rigorous mathematical proof for this experimental evidence. Tucker [Tu99] gave a computer assisted proof that the original Lorenz system indeed corresponds to a robustly transitive non-hyperbolic attractor containing a singularity. Moreover, he proved that the Lorenz equations define a dynamical system with the behavior of the geometric model introduced by Guckenheimer and Williams [GW79] that we describe next.

1.2. Geometric model. Here we briefly describe the geometric model of the Lorenz attractor (see e.g. [AP10] for more details). The model is given by a vector field X_0 which is linear in a neighborhood of the origin. The real eigenvalues λ_1, λ_2 and λ_3 of $DX_0(0)$ with the eigenvectors along the coordinate axes satisfy $0 < -\lambda_3 < \lambda_1 < -\lambda_2$. We consider the square given by
$$\Sigma = \{(x, y, 1) : -1/2 \le x, y \le 1/2\},$$
and let Γ be the intersection of Σ with the local stable manifold of the singularity. The segment Γ divides Σ into two parts
$$\Sigma^+ = \{(x, y, 1) \in \Sigma : x > 0\} \quad \text{and} \quad \Sigma^- = \{(x, y, 1) \in \Sigma : x < 0\},$$
The images of $\Sigma^\pm$ under this map are curvilinear triangles $S^\pm$ without the vertices $(\pm 1, 0, 0)$, and every line segment in $\mathcal{F} = \{x = \text{const} \cap \Sigma\}$ except Γ is mapped to a segment in $\{z = \text{const} \cap S^\pm\}$. The time τ it takes for each $(x, y, 1) \in \Sigma \setminus \Gamma$ to reach $S^\pm$ is given by $\tau(x, y, 1) = -\frac{1}{\lambda_1} \log |x|$. Now we suppose that the flow takes the triangles back to Σ in a smooth way as

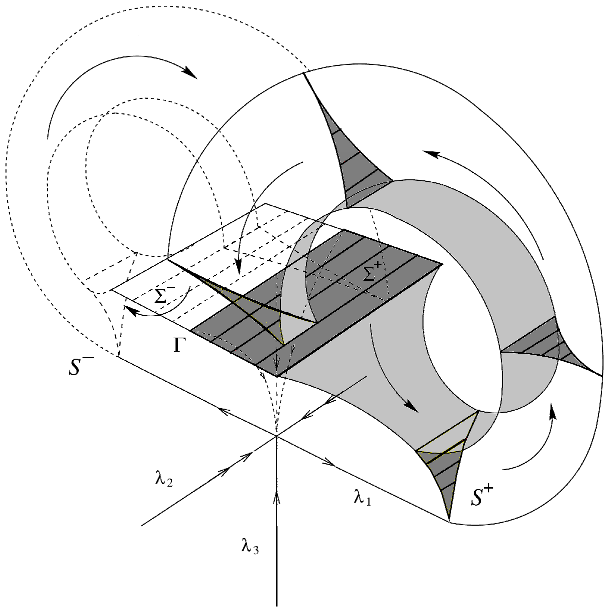

Fig. 1. Geometric Lorenz flow

shown in Figure 1. The resulting Poincaré map from $\Sigma \setminus \Gamma$ into Σ has the form

$$P(x,y) = (f(x), g(x,y)) \tag{1.2}$$

for some $f : I \setminus \{0\} \to I$ and $g : I \setminus \{0\} \times I \to I$, where $I = [-1/2, 1/2]$. The one-dimensional map f is as described in Figure 2 and satisfies:

(1) f has a discontinuity at $x = 0$ with one-sided limits $f(0^+) = -1/2$ and $f(0^-) = 1/2$;

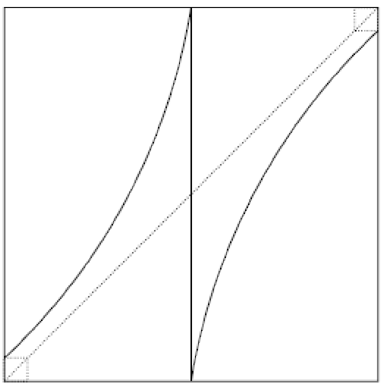

Fig. 2. Lorenz map

(2) f is differentiable on $I \setminus \{0\}$ and there is $c > 1$ such that $f'(x) \geq c$ for all $x \in I \setminus \{0\}$;
(3) the limit of $f'(x)$ is infinity as x approaches $0^{\pm}$;
(4) $f''(x) > 0$ for $x \in [-1/2, 0)$ and $f''(x) < 0$ for $x \in (0, 1/2]$;
(5) f is transitive.

The map g in (1.2) is defined in such a way that the stable foliation $\mathcal{F}$ is uniformly contracting: there exist constants $C' > 0$ and $0 < \rho < 1$ such that for any given leaf γ of the foliation and $\xi_1, \xi_2 \in \gamma$ and $n \geq 1$,

$$\operatorname{dist}(P^n(\xi_1), P^n(\xi_2)) \leq C'\rho^n \operatorname{dist}(\xi_1, \xi_2).$$

1.3. Statement of results. A crucial fact about the geometric Lorenz attractor is that it is *robust*, i.e. vector fields sufficiently close in the C^1 topology to the original one constructed as above also have strange attractors. Indeed, there exist an open neighborhood U in $\mathbb{R}^3$ containing the geometric Lorenz attractor Λ and an open neighborhood $\mathcal{U}$ of X_0 in the C^1 topology such that for all vector fields $X \in \mathcal{U}$, the maximal invariant set $\Lambda_X = \bigcap_{t \geq 0} X^t(U)$ is a transitive set which is invariant under the flow of X. This is a consequence of the persistence of an invariant contracting foliation $\mathcal{F}_X$ on the cross section Σ for $X \in \mathcal{U}$ (see [AP10, Theorem 3.10]).

Under some conditions on the eigenvalues of the singularity, for a vector field X C^2-close to X_0, the leaves of $\mathcal{F}_X$ are C^2 close to those of $\mathcal{F}$ and it follows that f_X is C^2 close to f (see [Ro81, Ro84]). Thus, there exists $o \in [-1/2, 1/2]$ which plays for f_X the same role as 0 for f, and the properties of f in Subsection 1.2 are still valid for f_X on a subinterval $[-b, b]$, for some $0 < b < 1/2$ close to $1/2$.

Definition 1.3. We define the family $\mathcal{X}$ of *geometric Lorenz vector fields* as a C^2 neighborhood of X_0 with the following properties:

(1) for each $X \in \mathcal{X}$, the maximal forward invariant set Λ_X inside U is an attractor containing a hyperbolic singularity;
(2) for each $X \in \mathcal{X}$, Σ is a cross-section for the flow with a return time τ_X and a Poincaré map P_X;
(3) for each $X \in \mathcal{X}$, the map P_X admits a C^2 uniformly contracting invariant foliation $\mathcal{F}_X$ on Σ with projection along the leaves of $\mathcal{F}_X$ onto I given by a map π_X;
(4) for each $X \in \mathcal{X}$, the map f_X on the quotient space I by the leaves in $\mathcal{F}_X$ is transitive C^2 piecewise expanding with two branches; moreover, there is $c > 1$ such that $f'_X(x) \geq c$ except at the discontinuity point O_X and $\lim_{x \to O_X^{\pm}} f'_X(x) = +\infty$;
(5) there is some constant $C > 0$ such that for each $X \in \mathcal{X}$,

$$\tau_X(\xi) \leq -C \log |\pi_X(\xi) - O_X|. \tag{1.3}$$

Observe that as the length of I is 1, we have $|\pi_X(\xi) - O_X| < 1$ for all $X \in \mathcal{X}$. For a detailed exposition of the properties of geometric Lorenz flows see e.g. [AP10, Section 2.3]; see also [AV12, equation (9)] for the last property. The main goal of this work is to prove the following result.

Theorem 1.4. *Geometric Lorenz vector fields are statistically stable.*

2. Preliminaries. Consider the family $\mathcal{X}$ of geometric Lorenz vector fields as in Definition 1.3. We assume that for each $X \in \mathcal{X}$ the derivative f'_X is monotonic on each branch. On the other hand, $1/f'_X$ is bounded because $f'_X > 1$. Therefore $1/f'_X$ is monotonic and bounded and hence is of bounded variation. It follows from [Vi97, Corollary 3.4] that each f_X admits a unique ergodic invariant probability $\bar{\mu}_X$ which is absolutely continuous with respect to Lebesgue measure λ, whose density $d\bar{\mu}_X/d\lambda$ is a bounded variation function and, in particular, it is bounded.

We point out that statistical stability results for piecewise expanding maps have been obtained in [Ke82]. According to [Ke82, Corollary 14.] or [BG97, Theorem 11.2.2], the family f_X with $X \in \mathcal{X}$ satisfies the conditions of Keller's results. Moreover, the density $d\bar{\mu}_X/d\lambda$ can be obtained by means of the Lasota–Yorke inequality whose constants can be taken the same for all Lorenz maps (see [Vi97, Proposition 3.1]). Therefore the density functions $d\bar{\mu}_X/d\lambda$ are uniformly bounded [Vi97, Corollary 3.4]. Hence we have:

Proposition 2.1. *Each f_X with $X \in \mathcal{X}$ is strongly statistically stable, i.e. $f_X \mapsto d\bar{\mu}_X/d\lambda$ is continuous with respect to the L^1-norm in the space of densities. Moreover, there exists $M > 0$ such that $d\bar{\mu}_X/d\lambda < M$ for all $X \in \mathcal{X}$.*

For any bounded function $\phi : \Sigma \to \mathbb{R}$, we define $\phi^{\pm} : I \to \mathbb{R}$ by

$$\phi^+(x) = \sup_{\xi \in \pi_X^{-1}(x)} \phi(\xi) \quad \text{and} \quad \phi^-(x) = \inf_{\xi \in \pi_X^{-1}(x)} \phi(\xi), \tag{2.1}$$

where $\pi_X : \Sigma \to I$ is the canonical projection along stable leaves. The next result is proved in [APPV09, Corollary 6.2].

Lemma 2.2. *There is a unique P_X-invariant ergodic probability measure $\tilde{\mu}_X$ on Σ such that for every continuous function $\phi : \Sigma \to \mathbb{R}$,*

$$\int \phi \, d\tilde{\mu}_X = \lim_{n\to\infty} \int (\phi \circ P_X^n)^- \, d\bar{\mu}_X = \lim_{n\to\infty} \int (\phi \circ P_X^n)^+ \, d\bar{\mu}_X.$$

The measure $\tilde{\mu}_X$ is an SRB measure for P_X that we shall call the *lift* of $\bar{\mu}_X$. Indeed, the uniform contraction of the stable leaves implies that the forward time averages of any pair ξ_1, ξ_2 of points on the same stable leaf for a continuous function $\phi : \Sigma \to \mathbb{R}$ are equal:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(P_X^j(\xi_1)) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(P_X^j(\xi_2)).$$

Hence the inverse image of the basin of $\bar{\mu}_X$ under π_X is contained in the basin of $\tilde{\mu}_X$. This shows that the basin of $\tilde{\mu}_X$ contains an entire strip of positive Lebesgue measure, because the basin of $\bar{\mu}_X$ is a subset of positive Lebesgue measure.

On the other hand, since the density $d\bar{\mu}_X/d\lambda$ is bounded, we conclude that the return time is integrable with respect to $\tilde{\mu}_X$. Thus we can saturate this measure along the flow to obtain a unique SRB measure μ_X for the flow, supported on the attractor Λ_X, whose ergodic basin covers a full Lebesgue measure subset of points on the topological basin of attraction (see [APPV09, Section 7]).

PROPOSITION 2.3. *The flow of each $X \in \mathcal{X}$ has a unique SRB measure μ_X given for any continuous map $\varphi : U \to \mathbb{R}$ by*

$$\int \varphi \, d\mu_X = \frac{1}{\tilde{\mu}_X(\tau_X)} \int \int_0^{\tau_X(\xi)} \varphi(X(\xi,t)) \, dt \, d\tilde{\mu}_X(\xi),$$

where $\tilde{\mu}_X(\tau_X) = \int \tau_X \, d\tilde{\mu}_X$.

3. Statistical stability for the Poincaré map. Here we prove the statistical stability of the Poincaré maps on the cross-section Σ, i.e. the SRB measures $\tilde{\mu}_X$ depend continuously on the vector fields. Let $(X_n)_{n\geq 1}$ be any sequence in $\mathcal{X}$ converging to $X \in \mathcal{X}$ in the C^2 topology. To shorten notation, in subscripts we shall use n instead of X_n, for $n \geq 1$, and no subscript instead of X.

Let $\phi : \Sigma \to \mathbb{R}$ be an arbitrary continuous function.

LEMMA 3.1. *Given $m \geq 1$ and $\epsilon > 0$, there is $n_0 = n_0(m,\epsilon)$ such that for all $n \geq n_0$,*

$$\int |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \, d\lambda < \epsilon.$$

Proof. Given $m \geq 1$, we can write $\int |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \, d\lambda$ as the sum

$$\int_{B_n} |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \, d\lambda + \int_{B_n^c} |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \, d\lambda, \tag{3.1}$$

where $B_n = \{\sum_{i=0}^{m-1} \tau_n \circ P_n^i > N\}$ and $N = N(m)$ is some large number. Now, by the last property in the definition of the geometric Lorenz flow and the fact that the leaves of $\mathcal{F}_n$ are nearly vertical lines, there is some constant

$C_1 > 0$ such that

$$\lambda(B_n) \le C_1 \sum_{i=0}^{m-1} |\{x \in I : -C \log |f_n^i(x) - O_n| > N\}|$$
$$\le C_1 \sum_{i=0}^{m-1} |f_n^{-i}(O_n - e^{-N/C}, O_n + e^{-N/C})| \le C_1 \sum_{i=0}^{m-1} (2/c)^i e^{-N/C},$$

where $c > 1$ is the uniform lower bound for the derivative. As ϕ is bounded, the first integral in (3.1) can be made arbitrarily small, provided N is large enough.

We now estimate the second integral in (3.1). Considering

$$A_n = \Big\{\xi : \Big| \sum_{i=0}^{m-1} (\tau_n \circ P_n^i)(\xi) - \sum_{i=0}^{m-1} (\tau \circ P^i)(\xi)\Big| \ge 1\Big\},$$

we easily see that the second integral in (3.1) is bounded by

$$\int_{\{\sum_{i=0}^{m-1} \tau \circ P^i \le N+1\}} |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \, d\lambda + \int_{A_n} |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \, d\lambda.$$

Observe that

$$\lambda(A_n) \to 0 \quad \text{as } n \to \infty, \tag{3.2}$$

because for large n, a point belongs to A_n only if it belongs to some small neighborhood of the (finite) set of discontinuity lines of P^m. As ϕ is bounded, the second term in the last sum is bounded by $2\lambda(A_n) \sup \phi$. Then, (3.2) implies that the second term in that sum is small for sufficiently large n.

It remains to control the first term

$$\int_{\{\sum_{i=0}^{m-1} \tau \circ P^i \le N+1\}} |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \, d\lambda.$$

Observe that the points in $\{\sum_{i=0}^{m-1} \tau \circ P^i \le N+1\}$ must necessarily be outside a neighborhood of the discontinuity lines of the map P^m. If n is sufficiently large, then the same holds for P_n^m. This means that the return time associated to these maps is uniformly bounded for large n. Then, just by the continuous variation of trajectories in finite periods of time, we can make $|(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+|$ small for large n. ∎

LEMMA 3.2. *For any $m \ge 1$ we have*

$$\lim_{n\to\infty} \int (\phi \circ P_n^m)^+ \, d\bar{\mu}_n = \int (\phi \circ P^m)^+ \, d\bar{\mu}.$$

Proof. Given $m \in \mathbb{N}$, then

$$\Big|\int (\phi \circ P_n^m)^+ \, d\bar{\mu}_n - \int (\phi \circ P^m)^+ d\bar{\mu}\Big| \le \Big|\int (\phi \circ P_n^m)^+ \, d\bar{\mu}_n - \int (\phi \circ P^m)^+ \, d\bar{\mu}_n\Big| + \Big|\int (\phi \circ P^m)^+ \, d\bar{\mu}_n - \int (\phi \circ P^m)^+ \, d\bar{\mu}\Big|.$$

Since the density of $\bar{\mu}_n$ converges to the density of $\bar{\mu}$ in the L^1-norm, by Proposition 2.1 and the fact that ϕ is bounded, we easily see that the second term in the sum above tends to zero as $n \to \infty$. So, it remains to prove that the first term converges to zero. In fact, using the uniform boundedness of the densities in Proposition 2.1, we obtain

$$\Big|\int (\phi \circ P_n^m)^+ d\bar{\mu}_n - \int (\phi \circ P^m)^+ d\bar{\mu}_n\Big| \le \int |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \left|\frac{d\bar{\mu}_n}{d\lambda}\right| d\lambda \le M \int |(\phi \circ P_n^m)^+ - (\phi \circ P^m)^+| \, d\lambda,$$

which, by Lemma 3.1, can be arbitrarily small for n sufficiently large. ∎

Proposition 3.3. $\lim_{n\to\infty} \int \phi \, d\tilde{\mu}_n = \int \phi \, d\tilde{\mu}$.

Proof. The compactness of Σ implies that ϕ is uniformly continuous, and therefore given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\phi(\xi_1) - \phi(\xi_2)| < \epsilon \quad \text{for all } \xi_1, \xi_2 \in \Sigma \text{ with } \operatorname{dist}(\xi_1, \xi_2) < \delta. \tag{3.3}$$

As we know, the rate of contraction of the stable foliation on Σ is uniform for all vector fields in $\mathcal{X}$. So, the first return maps are uniformly contractive. In particular, given $\delta > 0$ there exists $m_0 > 0$ such that for all n we have

$$\operatorname{diam} P_n^m(\gamma) \le \delta \quad \text{for all } \gamma \in \mathcal{F}_n \text{ and } m \ge m_0. \tag{3.4}$$

Take arbitrary numbers m_1, m_2 with $m_2 \ge m_1 \ge m_0$. Given $x \in I$, let γ be the leaf in $\mathcal{F}_n$ containing x and $\gamma_{m_2-m_1}$ be the leaf in $\mathcal{F}_n$ containing $P_n^{m_2-m_1}(\gamma)$. We have

$$(\phi \circ P_n^{m_2})^+(x) = \sup \phi|_{P_n^{m_2}(\gamma)} = \sup \phi|_{P_n^{m_1}(P_n^{m_2-m_1}(\gamma))}.$$

As $f^{m_2-m_1}(x) \in \gamma_{m_2-m_1}$, we also have

$$(\phi \circ P_n^{m_1})^+(f^{m_2-m_1}(x)) = \sup \phi|_{P_n^{m_1}(\gamma_{m_2-m_1})}$$

Then, since $\gamma_{m_2-m_1}$ contains $P_n^{m_2-m_1}(\gamma)$, it follows from (3.3) and (3.4) that

$$|(\phi \circ P_n^{m_2})^+(x) - (\phi \circ P_n^{m_1})^+(f^{m_2-m_1}(x))| < \epsilon.$$

Knowing that $\int (\phi \circ P_n^{m_1})^+ \, d\bar{\mu}_n = \int (\phi \circ P_n^{m_1})^+ \circ f_n^{m_2-m_1} \, d\bar{\mu}_n$, because $\bar{\mu}_n$ is an f_n-invariant probability measure, we obtain

$$\Big|\int (\phi \circ P_n^{m_2})^+ d\bar{\mu}_n - \int (\phi \circ P_n^{m_1})^+ d\bar{\mu}_n\Big| \le \epsilon.$$

Consequently, the sequence $(\int (\phi\circ P_n^m)^+\,d\bar\mu_n)_{m,n}$ is uniformly Cauchy, because m_0 does not depend on n. Hence,

$$\lim_{n\to\infty}\lim_{m\to\infty}\int(\phi\circ P_n^m)^+\,d\bar\mu_n=\lim_{m\to\infty}\lim_{n\to\infty}\int(\phi\circ P_n^m)^+\,d\bar\mu_n.$$

Therefore,

$$\lim_{n\to\infty}\int\phi\,d\tilde\mu_n=\lim_{n\to\infty}\lim_{m\to\infty}\int(\phi\circ P_n^m)^+\,d\bar\mu_n=\lim_{m\to\infty}\lim_{n\to\infty}\int(\phi\circ P_n^m)^+\,d\bar\mu_n,$$

and by Lemma 3.2,

$$\lim_{n\to\infty}\int(\phi\circ P_n^m)^+\,d\bar\mu_n=\int(\phi\circ P^m)^+\,d\bar\mu.$$

Letting $m\to\infty$ we complete the proof, by definition of $\tilde\mu$. ∎

4. Statistical stability for the flow. Now we prove Theorem 1.4. Let $(X_n)_{n\ge1}$ be a sequence in $\mathcal X$ converging to $X\in\mathcal X$ in the C^2 topology. Using again shortened subscript notation as in Section 3, we need to prove that $\mu_n\to\mu$ in the weak* topology. Let $\varphi:\bar U\to\mathbb R$ be any continuous function. We have

$$\int\varphi\,d\mu_n=\frac{1}{\tilde\mu_n(\tau_n)}\int\int_0^{\tau_n(\xi)}\varphi(X_n(\xi,t))\,dt\,d\tilde\mu_n(\xi).$$

Adding and subtracting the term

$$\frac{1}{\tilde\mu_n(\tau_n)}\int\int_0^{\tau(\xi)}\varphi(X(\xi,t))\,dt\,d\tilde\mu(\xi),$$

we have $|\int\varphi\,d\mu_n-\int\varphi\,d\mu|$ bounded by the sum of two terms,

$$\left|\frac{1}{\tilde\mu_n(\tau_n)}-\frac{1}{\tilde\mu(\tau)}\right|\int\int_0^{\tau(\xi)}|\varphi(X(\xi,t))|\,dt\,d\tilde\mu(\xi)\tag{4.1}$$

and

$$\frac{1}{\tilde\mu_n(\tau_n)}\left|\int\int_0^{\tau_n(\xi)}\varphi(X_n(\xi,t))\,dt\,d\tilde\mu_n(\xi)-\int\int_0^{\tau(\xi)}\varphi(X(\xi,t))\,dt\,d\tilde\mu(\xi)\right|.\tag{4.2}$$

Our goal now is to show that (4.1) and (4.2) converge to zero as $n\to\infty$.

Lemma 4.1. $\lim_{n\to\infty}\int\tau_n\,d\tilde\mu_n=\int\tau\,d\tilde\mu$.

Proof. Define

$$\tau_N(\xi)=\min\{\tau(\xi),N\}\quad\text{and}\quad\tau_{n,N}(\xi)=\min\{\tau_n(\xi),N\}.$$

Observe that for each fixed $N \geq 1$ the functions τ_N and $\tau_{n,N}$ are bounded and continuous, and $\tau_{n,N}$ converges uniformly to τ_N as $n \to \infty$. We have

$$(4.3)\quad \left|\int \tau_n \, d\tilde{\mu}_n - \int \tau \, d\tilde{\mu}\right| \leq \left|\int \tau_n \, d\tilde{\mu}_n - \int \tau_{n,N} \, d\tilde{\mu}_n\right| + \left|\int \tau_{n,N} \, d\tilde{\mu}_n - \int \tau_N \, d\tilde{\mu}\right| + \left|\int \tau_N \, d\tilde{\mu} - \int \tau \, d\tilde{\mu}\right|.$$

Now we prove that the first term on the right hand side is small for large N (uniformly in n); the calculation for the third term is similar. We have

$$(4.4)\quad \left|\int \tau_n \, d\tilde{\mu}_n - \int \tau_{n,N} \, d\tilde{\mu}_n\right| = \int (\tau_n - \tau_{n,N}) \, d\tilde{\mu}_n \leq \int (\tau_n - \tau_{n,N})^+ \, d\tilde{\mu}_n.$$

Now, for $n, N, m \geq 1$ define

$$(\tau_n - \tau_{n,N})^+_m(\xi) = \min\{(\tau_n - \tau_{n,N})^+(\xi), m\}.$$

Note that $(\tau_n - \tau_{n,N})^+_m$ converges monotonically to $(\tau_n - \tau_{n,N})^+$ as $m \to \infty$. Moreover the functions $(\tau_n - \tau_{n,N})^+_m$ and $(\tau_n - \tau_{n,N})^+$ can be interpreted as the functions defined in I introduced in (2.1). Using the Monotone Convergence Theorem, the definition of $\tilde{\mu}_n$ and the uniform boundedness on the density of $\bar{\mu}_n$ we can write

$$\begin{aligned}\int (\tau_n - \tau_{n,N})^+ \, d\tilde{\mu}_n &= \lim_{m\to\infty} \int (\tau_n - \tau_{n,N})^+_m \, d\tilde{\mu}_n = \lim_{m\to\infty} \int (\tau_n - \tau_{n,N})^+_m \, d\bar{\mu}_n \\ &\leq M \lim_{m\to\infty} \int (\tau_n - \tau_{n,N})^+_m \, d\lambda = M \int (\tau_n - \tau_{n,N})^+ \, d\lambda.\end{aligned}$$

Now, defining $A_{n,N} = \{x \in I : -C\log(x - O_n) > N\}$ by (1.3) we have

$$\int (\tau_n - \tau_{n,N})^+ \, d\lambda \leq \int_{A_{n,N}} -C\log(x - O_n) \, dx = \int_0^{e^{-C/N}} -C\log x \, dx,$$

and this last integral is clearly small for large N (uniformly in n).

Finally, for each $N \geq 1$, the second term on the right hand side in (4.3) converges to zero as $n \to \infty$, since $\tilde{\mu}_n$ converges weakly to $\tilde{\mu}$, by Proposition 3.3, and the functions $\tau_{n,N}$ are continuous and converge uniformly to τ_N as $n \to \infty$. ■

Lemma 4.1 implies that (4.1) converges to zero as $n \to \infty$, since

$$\left|\frac{1}{\tilde{\mu}_n(\tau_n)} - \frac{1}{\tilde{\mu}(\tau)}\right| \int \int_0^{\tau(\xi)} |\varphi(X(\xi,t))| \, dt \, d\tilde{\mu}(\xi) \leq \left|\frac{1}{\tilde{\mu}_n(\tau_n)} - \frac{1}{\tilde{\mu}(\tau)}\right| \|\varphi\|_\infty \tilde{\mu}(\tau).$$

The next result implies that (4.2) converges to zero as $n \to \infty$.

Lemma 4.2.

$$\lim_{n\to\infty} \int \int_0^{\tau_n(\xi)} \varphi(X_n(\xi,t)) \, dt \, d\tilde{\mu}_n(\xi) = \int \int_0^{\tau(\xi)} \varphi(X(\xi,t)) \, dt \, d\tilde{\mu}(\xi).$$

Proof. We define

$$h(\xi) = \int_0^{\tau(\xi)} \varphi(X(\xi,t))\,dt, \quad h_n(\xi) = \int_0^{\tau_n(\xi)} \varphi(X_n(\xi,t))\,dt,$$

and using the notations of the proof of Lemma 4.1, we also define, for $N \geq 1$,

$$h_N(\xi) = \int_0^{\tau_N(\xi)} \varphi(X(\xi,t))\,dt, \quad h_{n,N}(\xi) = \int_0^{\tau_{n,N}(\xi)} \varphi(X_n(\xi,t))\,dt.$$

The difference

$$\Big|\int \int_0^{\tau_n(\xi)} \varphi(X_n(\xi,t))\,dt\,d\tilde{\mu}_n(\xi) - \int \int_0^{\tau(\xi)} \varphi(X(\xi,t))\,dt\,d\tilde{\mu}(\xi)\Big|$$

is bounded by

$$\Big|\int h_n d\tilde{\mu}_n - \int h_{n,N} d\tilde{\mu}_n\Big| + \Big|\int h_{n,N}\,d\tilde{\mu}_n - \int h_N\,d\tilde{\mu}\Big| + \Big|\int h_N d\tilde{\mu} - \int h\,d\tilde{\mu}\Big|.$$

The first term is bounded by

$$\|\varphi\|_\infty \int (\tau_n - \tau_{n,N})\,d\tilde{\mu}_n \leq \|\varphi\|_\infty \int (\tau_n - \tau_{n,N})^+\,d\tilde{\mu}_n.$$

As we saw in the proof of Lemma 4.1, this last term is small for large enough N (uniformly in n), and a similar conclusion holds for the third term. The second term is also handled as in the proof of Lemma 4.1, because the functions $h_{n,N}$ are continuous and converge uniformly to h_N as $n \to \infty$. ∎

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