FUNDAMENTA
MATHEMATICAE
225 (2014)

Extending the Dehn quandle to shears and foliations on the torus

by

Reza Chamanara (Brooklyn, NY),
Jun Hu (Brooklyn, NY, and New York, NY) and
Joel Zablow (Brooklyn, NY)

Abstract. The Dehn quandle, Q, of a surface was defined via the action of Dehn twists about circles on the surface upon other circles. On the torus, $\mathbb{T}^2$, we generalize this to show the existence of a quandle $\hat{Q}$ extending Q and whose elements are measured geodesic foliations. The quandle action in $\hat{Q}$ is given by applying a shear along such a foliation to another foliation. We extend some results which related Dehn quandle homology to the monodromy of Lefschetz fibrations. We apply certain quandle 2-cycles to yield factorizations of elements of $\mathrm{SL}_2(\mathbb{R})$ fixing specified vectors (circles, foliations) and give examples. Using these, we show the quandle homology of $\hat{Q}$ is nontrivial in all dimensions.

1. Introduction and background information on quandles. The Dehn quandle of an oriented surface F is a quandle structure on the collection of isotopy classes of simple closed curves (circles) on the surface, with an action by circles on circles, given by Dehn twists about such circles. It was defined in [12]. In the present article, we will look at a way of generalizing the Dehn quandle on the torus $\mathbb{T}^2$. In particular, we will show that the action given by shearing along a measured geodesic foliation on a torus, as applied to another such foliation, gives a quandle structure to the collection of such foliations. This directly generalizes the original Dehn quandle on the torus.

We will develop this extended Dehn quandle on the flat torus $\mathbb{T}^2$, obtained by identifying opposite sides of the unit square. The mapping class group $\mathrm{MCG}(\mathbb{T}^2)$ of the torus is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$, and its action on circles on $\mathbb{T}^2$ is well understood and relatively easy to compute in that we may describe any such circle as a 2-dimensional vector $\mathbf{v} = \langle x, y\rangle$, where $x, y \in \mathbb{Z}$

2010 *Mathematics Subject Classification*: 57Mxx, 17Dxx.
Key words and phrases: Dehn quandle, quandle homology, mapping class group, foliation, elliptic fibration.

DOI: 10.4064/fm225-1-1

are relatively prime. The action by an element of $\mathrm{MCG}(\mathbb{T}^2)$ on a circle will be given by applying the matrix corresponding to the homeomorphism in $\mathrm{MCG}(\mathbb{T}^2)$, *on the right*, to the vector corresponding to the circle in question. These actions, written on the right, correspond to the notation that has been used for the action by Dehn twists in the original Dehn quandle, e.g. in [12] and [13].

If x, y are relatively prime integers, the vector $\mathbf{v}$ corresponds to an imbedded circle (a knot) in $\mathbb{T}^2$ corresponding to the slope y/x of the line segment on the torus viewed as a square of area 1 with sides identified in pairs. If x, y are not relatively prime we may think of the vector as corresponding to a link in $\mathbb{T}^2$. We shall make use of a simple correspondence between vectors representing circles on $\mathbb{T}^2$ and the matrices in $\mathrm{PSL}_2(\mathbb{Z})$ corresponding to Dehn twists about such circles, given in [9] or [11].

In the proof of the main result, we essentially extend the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{Z}$-valued vectors to an action by elements of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{R}$-valued vectors. See a related discussion of the $\mathbb{T}^2$ Dehn quandle in [9].

Here are some definitions, background facts, and notations that will be used throughout what follows, regarding quandles in general and the Dehn quandle in particular. For a closed orientable surface F of genus g, the mapping class group is denoted $\mathrm{MCG}(F)$.

Definition 1.1. A *quandle* is a quadruple $(S, G, \overline{|\ \ }, \overline{\ \ |})$, where G is a group acting on the right on a set S, together with a pair of maps $\overline{\ \ |}, \overline{|\ \ } : S \to G$ for which the following axioms hold:

$$\text{(Q1)} \qquad a\,\overline{a|} = a\,\overline{|a} = a \qquad \forall a \in S \qquad (\text{“idempotence”}),$$

$$\text{(Q2)} \qquad a\,\overline{|b}\,\overline{b|} = a = a\,\overline{b|}\,\overline{|b} \qquad \forall a, b \in S \qquad (\text{“inverses”}),$$

$$\text{(Q3)} \qquad x\,\overline{a\,\overline{b|}\,|} = x\,\overline{|b}\,\overline{a|}\,\overline{b|} \qquad \forall a, b, x \in S \qquad (\text{“conjugation”}).$$

There are three other variants of the expression in (Q3), corresponding to all the left/right possibilities for the inner and outer brackets. The quandle axioms are algebraic analogs of the Reidemeister moves I, II, III, governing ambient isotopy of knots. An algebraic structure satisfying only axioms (Q2) and (Q3) is called a *rack*. We note that (Q2) and (Q3) are equivalent to (Q2) and right distributivity of the operation over itself. All words are considered to be left-associated.

The bracket notation for rack and quandle actions used here is adopted from [7]. Applying right $\overline{\ \ |}$ or left $\overline{|\ \ }$ brackets to elements turns them into operators on other elements (operating from the right). Quandles are also known as distributive groupoids, and racks as automorphic sets, in the literature. Many other notations exist for the quandle action as well.

Examples 1.2.

1. Any group G forms a quandle under the operation of conjugation, where for $a, g \in G$, we take $a\overline{g}\rvert = gag^{-1}$ and $a\lvert\overline{g} = g^{-1}ag$.
2. The set of $(n-1)$-dimensional planes in $\mathbb{R}^n$ forms a quandle, where both the operations $\overline{x}\rvert$ and $\lvert\overline{x}$ correspond to reflection in the hyperplane x.
3. The quandle $Q(K)$ associated to an oriented knot diagram K is an invariant of ambient isotopy. The elements are labeled arcs of the diagram, and $a\overline{b}\rvert = c$ is interpreted as "arc b crosses over arc a from the right, to produce arc c". See e.g. [6] or [7].
4. Any finite cyclic group $\mathbb{Z}_n$ may be given a quandle structure. For any $a, b \in \mathbb{Z}_n$, define $a\,\overline{b}\rvert = 2b - a = a\lvert\overline{b}$. This is the *dihedral quandle* denoted R_n.
5. Let M be a free module over a commutative ring R. Suppose $\langle\ ,\ \rangle : M \times M \to R$ is a non-degenerate bilinear form on M with $\langle \mathbf{x}, \mathbf{x}\rangle = 0$. Then we get a quandle structure on M with the operations

$$\mathbf{x}\,\overline{\mathbf{y}}\rvert = \mathbf{x} + \langle \mathbf{x}, \mathbf{y}\rangle \mathbf{y} \quad \text{and} \quad \mathbf{x}\lvert\overline{\mathbf{y}} = \mathbf{x} - \langle \mathbf{x}, \mathbf{y}\rangle \mathbf{y}, \tag{1}$$

for vectors $\mathbf{x}, \mathbf{y} \in M$. This is called an *alternating quandle* by Yetter [11], or a *symplectic quandle*. For further details see [8]. It will be of particular interest to us, as the quandle extension we will describe may be expressed as a quotient of such a quandle.

For an oriented surface F as above, and the collection P of simple closed curves (henceforth, circles) on F considered up to isotopy, it was shown in [12] that $(P, \mathrm{MCG}(F), \lvert\overline{\ \ }, \overline{\ \ }\rvert)$ forms a quandle, where $\mathrm{MCG}(F)$ acts on elements of P (on the right). For all circles $a \in P$, the operators $\lvert\overline{a}, \overline{a}\rvert$ correspond, respectively, to left and right Dehn twists about the circle a. Mnemonically,

$$\text{bracket on left} = \text{left Dehn twist}, \quad \text{bracket on right} = \text{right Dehn twist}.$$

So, if a, b are circles in F,

$$b\,\overline{a}\rvert = \text{right Dehn twist about } a \text{, applied to } b,$$

$$b\lvert\overline{a} = \text{left Dehn twist about } a \text{, applied to } b.$$

The quandle formed is called the standard *Dehn quandle* of the surface F.

Quandle homology. There is a homology theory for quandles. Many of the main ideas, definitions, and conventions can be found in [2] or [3]. For completeness and context we now review some useful facts and features.

For a quandle Q, let $C_n^R(Q)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of Q. This is the *rack chain group*. Such a tuple will be referred to as an *n-simplex*. Note that this differs from the conventional definition where a k-simplex involves $k+1$ terms. We assume $C_0 = 0$.

The boundary homomorphism $\partial_n : C_n \to C_{n-1}$ is defined by

$$\partial_n(x_1,\dots,x_n) = \sum_{i=2}^{n} (-1)^i [(x_1, x_2, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_n) \\ - (x_1\, \overline{x_i}\rceil, x_2\, \overline{x_i}\rceil, \dots, x_{i-1}\, \overline{x_i}\rceil, x_{i+1}, \dots, x_n)]. \tag{2}$$

The groups $(C_*^R(Q), \partial)$ form a chain complex. Following [2], an n-simplex which has consecutive terms $x_i = x_{i+1}$, where $1 \le i \le n-1$, is *degenerate*. Let $C_n^D(Q)$ be the nth *degenerate chain group*. As a consequence of axiom (Q1), $\partial : C_n^D(Q) \to C_{n-1}^D(Q)$, and thus the chain complex of degenerate chains forms a subcomplex of the rack chain complex, $(C_*^R(Q), \partial)$. The *quandle chain complex* is then defined as the quotient

$$C_*(Q) = C_*^R(Q)/C_*^D(Q),$$

where degenerate simplices are considered to be 0. By abuse of notation, we use ∂, defined as above, as the boundary operator in the quandle chain complex as well.

For the remainder of this article, we will assume that all the groups in question, in particular the quandle chain groups $C_n(Q)$, cycle groups $Z_n(Q)$, boundary groups $B_n(Q)$, and homology groups $H_n(Q)$, are taken with integer, $\mathbb{Z}$, coefficients unless otherwise stated.

We will make use of this in the last section. There, we give an application extending a theorem relating Dehn quandle homology to Lefschetz fibrations, so as to encompass Anosov (non-periodic irreducible) monodromy homeomorphisms on $\mathbb{T}^2$. We also use the homology theory to show existence of factorizations of certain linear transformations having specified fixed points.

2. The Dehn quandle Q on the torus $\mathbb{T}^2$, and an extension. We will consider the meridian m and longitude l of a torus to be oriented as shown in Figure 1.

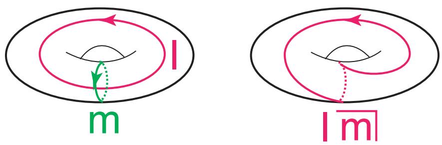

Fig. 1

The meridian and longitude shown form a basis for all simple closed curves on the torus, and any essential circle on the torus $\mathbb{T}^2$ can be expressed as an integral linear combination $al+bm$ with $\gcd(a,b)=1$. It will be useful to be able to describe such circles as vectors in terms of these basis vectors. We let

$$l=\langle 1,0\rangle \quad \text{and} \quad m=\langle 0,1\rangle.$$

Then a general circle may be expressed as a vector $\langle x,y\rangle$. It is often also useful to consider such a vector or circle as corresponding to a slope on the torus, represented as a square before identifying edges. Thus, as a slope, we have

$$\langle x,y\rangle \leftrightarrow y/x.$$

Yetter [11] gives a formula relating the components of a slope or vector y/x, corresponding to a circle on $\mathbb{T}^2$ to the Dehn twists (right or left) about that circle. Figure 1 shows the result of the twist $l\,\overline{m}|$. Dehn twists correspond to matrices in $\mathrm{PSL}_2(\mathbb{Z})$ with trace 2 and determinant 1. To agree with our convention that quandle actions in general, and Dehn twists in particular, are written on the *right* of the object which is being acted upon, we modify Yetter's formula appropriately, taking the transpose of the matrix he gets. We then have, in general,

$$\overline{\langle x,y\rangle}| \leftrightarrow M_{y/x}=M_{\langle x,y\rangle}=\begin{pmatrix} xy+1 & y^2 \\ -x^2 & 1-xy \end{pmatrix}. \tag{3}$$

The matrices corresponding to the inverses, left Dehn twists $|\overline{\langle x,y\rangle}$, may be similarly computed in a straightforward manner.

In what follows, we will generalize the quandle action of the original Dehn quandle Q on the torus, for circles on circles. We now work exclusively on the flat torus $\mathbb{T}^2$ and the objects now under consideration will be measured geodesic foliations, defined below, on $\mathbb{T}^2$. The quandle operation will generalize full Dehn twists about circles to the notion of "shears" along such foliations.

DEFINITION 2.1. A *measured geodesic foliation* $(\mathcal{F},\mu)$ on a surface F is a foliation of F by geodesics, the leaves, together with an invariant transversal measure μ.

For a torus obtained by identifications of opposite sides of a parallelogram, foliation of the plane by parallel lines in any given direction descends to a geodesic foliation of the torus. On such a torus, an invariant measure of an arc transverse to the foliation can be described by a fixed multiple of the euclidean width of the strip of leaves of the foliation that intersect the arc. The note below suggests such a foliation by true circles. From this point on, we will assume all foliations under discussion are measured geodesic foliations with constant density on $\mathbb{T}^2$, and denote the set of such by $\mathcal{MGF}(\mathbb{T}^2)$.

Note. We associate to each element of the original Dehn quandle Q, a circle on $\mathbb{T}^2$, a geodesic foliation on $\mathbb{T}^2$ by foliating the torus by parallel copies of the given circle. Then we associate to this foliation a measure with constant density.

There is a 1-1 correspondence between elements of $\mathcal{MGF}(\mathbb{T}^2)$ and vectors $\langle a,b\rangle \in \mathbb{R}^2 - \{(0,0)\}$, up to the equivalence $\langle a,b\rangle \sim \langle -a,-b\rangle$. Under this correspondence, the slope $m = b/a$ of the vector determines the leaves of the foliation, and the square of its length, $\|\langle a,b\rangle\|^2$, gives the density of transversal measure. The fact that the density δ of the measure determines the shearing factor for a measured foliation will come out of the geometric description below, and from the computation in the proofs of Lemma 2.4 and the main Theorem 2.5.

In trying to extend the original Dehn quandle on the torus, we are basically asking if a more general notion of "shearing" can be defined on elements of $\mathcal{MGF}(\mathbb{T}^2)$; in particular, we require that in the action of one such operator foliation, upon an operand foliation, if the measures on the operator and operand are specified, the defined action results in a quandle structure on elements of $\mathcal{MGF}(\mathbb{T}^2)$.

We first show that the action of the bracket operator for vectors, given in (3), may be expressed in another congenial form. Recall for a vector $\langle x,y\rangle \in \mathbb{Z}^2$ we defined our bracket actions via matrix multiplication on the right:

$$\overline{\langle x,y\rangle\rvert} \leftrightarrow \begin{pmatrix} xy+1 & y^2 \\ -x^2 & 1-xy \end{pmatrix} \quad \text{and} \quad \overline{\lvert\langle x,y\rangle} \leftrightarrow \begin{pmatrix} xy+1 & y^2 \\ -x^2 & 1-xy \end{pmatrix}^{-1}.$$

We now take the same formulas to define the action by vectors $\langle x,y\rangle \in \mathbb{R}^2$ on other vectors in $\mathbb{R}^2$.

Proposition 2.2. *Let measured geodesic foliations γ, ζ on $\mathbb{T}^2$ be represented by vectors $\langle a,b\rangle$ and $\langle x,y\rangle$, respectively. Then*

$$\langle x,y\rangle\, \overline{\langle a,b\rangle\rvert} = \langle x,y\rangle + \det\begin{pmatrix} x & y \\ a & b \end{pmatrix} \cdot \langle a,b\rangle, \tag{4}$$

$$\langle x,y\rangle\, \overline{\lvert\langle a,b\rangle} = \langle x,y\rangle - \det\begin{pmatrix} x & y \\ a & b \end{pmatrix} \cdot \langle a,b\rangle. \tag{5}$$

Proof. It is immediate that e.g.

$$\langle x,y\rangle\, \overline{\langle a,b\rangle\rvert} = \langle x,y\rangle \begin{pmatrix} 1+ab & b^2 \\ -a^2 & 1-ab \end{pmatrix} = \langle x+abx-a^2y, b^2x+y-aby\rangle,$$

while

$$\langle x, y\rangle + \det\begin{pmatrix} x & y \\ a & b \end{pmatrix}\langle a, b\rangle = \langle x, y\rangle + (xb - ya)\langle a, b\rangle$$
$$= \langle x + abx - a^2y, y + b^2x - aby\rangle,$$

and similarly for the left bracket. Also note that these formulae are well defined with respect to the equivalence of vectors describing elements of $\mathcal{MGF}(\mathbb{T}^2)$, up to sign, immediately following the note after Definition 2.1. Consider equation (4), and let $\mathbf{v} = \langle x, y\rangle$ and $\mathbf{w} = \langle a, b\rangle$. Replacing the operand $\mathbf{v}$ by $-\mathbf{v}$ yields

$$\overline{\mathbf{w}}\rceil = -\mathbf{v} + \det\begin{pmatrix} -\mathbf{v} \\ \mathbf{w} \end{pmatrix}\mathbf{w} = -\mathbf{v} - \det\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix}\mathbf{w} = -(\,\mathbf{v}\,\overline{\mathbf{w}}\rceil) \sim \mathbf{v}\,\overline{\mathbf{w}}\rceil.$$

On the other hand, replacing the operator $\mathbf{w}$ by $-\mathbf{w}$ yields

$$\mathbf{v}\,\overline{-\mathbf{w}}\rceil = \mathbf{v} + \det\begin{pmatrix} \mathbf{v} \\ -\mathbf{w} \end{pmatrix}(-\mathbf{w}) = \mathbf{v} + \det\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \mathbf{v}\,\overline{\mathbf{w}}\rceil.$$

One makes similar checks in the case of equation (5). ∎

Remark. This exhibits the action of the brackets in a manner consistent with the definition of the alternating quandle in Example 1.2.5. Here the anti-symmetric bilinear form is the determinant which gives the intersection number for pairs of imbedded circles on $\mathbb{T}^2$. See e.g. [10].

Proposition 2.3. *The action given in either of the equations* (4) *and* (5) *is independent of basis, under an area-preserving change of basis.*

Proof. We assume the change of basis is given by an area-preserving linear map with matrix A. It suffices to show, when applying A (on the right) to the vectors $\mathbf{v} = \langle x, y\rangle$ and $\mathbf{w} = \langle a, b\rangle$, that

$$\left(\mathbf{v} + \det\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix}\mathbf{w}\right)A = \mathbf{v}A + \det\begin{pmatrix} \mathbf{v}A \\ \mathbf{w}A \end{pmatrix}\mathbf{w}A.$$

The equality occurs since applying A to the parallelogram formed by the vectors $\mathbf{v}$ and $\mathbf{w}$ yields a new parallelogram with the same area, and hence the same determinant as the original one. Thus, the value of the multiplicative factor and the form of the given formulae are preserved. ∎

For general elements $\mu, \nu \in \mathcal{MGF}(\mathbb{T}^2)$, we shall describe a shearing action of μ on ν, given by $\nu\,\overline{\mu}\rceil = \sigma$. We need to account for the effect of the shear on the slope of the operand, ν, and also on its density, δ_ν. The shear will take parallel lines in ν to parallel lines in σ. Viewed in the universal cover, $\mathbb{R}^2$, each point of a pair of points on parallel leaves of μ, at distance d apart, will see the other point move to the right through a distance $t = d\delta_\mu$, where δ_μ is the constant density of μ. Figure 2C shows this and the effect of a shear along a general element $\mu \in \mathcal{MGF}(\mathbb{T}^2)$ upon a segment (vector $\mathbf{v}_1$) in another element $\nu \in \mathcal{MGF}(\mathbb{T}^2)$.

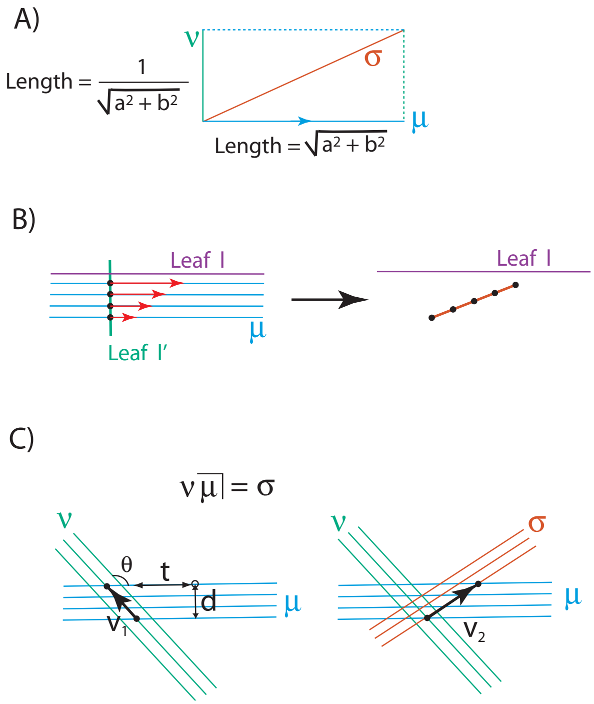

Fig. 2. Shearing, A) full twist along a closed curve, B) locally, C) for general elements of $\mathcal{MGF}(T^2)$, viewed in the universal cover $\widetilde{\mathbb{T}}^2 = \mathbb{R}^2$ with operator, operand, and result.

This yields a new element σ, shown with the image (vector $\mathbf{v}_2$) of the original segment. The density, δ_σ, of the resulting foliation $\sigma \in \mathcal{MGF}(\mathbb{T}^2)$ is related to the density of the operand ν via

$$\delta_\sigma = \delta_\nu \frac{\|\mathbf{v}_2\|^2}{\|\mathbf{v}_1\|^2}. \tag{6}$$

Lemma 2.4. *Let the vector $\langle a, b\rangle$ represent a closed circle μ on $\mathbb{T}^2$. (So a, b are relatively prime integers.) Then a shear corresponding to a full right twist $\overline{\mu|} = \overline{\langle a, b\rangle|}$ has its constant density $\delta_\mu = t/d$ given by $\delta_\mu = a^2 + b^2$.*

Proof. A full twist about μ is a shear along its entire length, $\|\langle a, b\rangle\| = \sqrt{a^2 + b^2}$, the length of a closed leaf l of μ. We assume this twist is "distributed evenly" over the torus. Since the area of the torus is 1, the length of a closed leaf l' perpendicular to μ is $1/\sqrt{a^2 + b^2}$. So if we move a distance $1/\sqrt{a^2 + b^2}$ perpendicularly off of μ, we reach the same circle again. In the course of the shear, a point at distance d from l is moved to the right, through a distance t, which is proportional to d. Thus,

$$\delta_\mu = \frac{t}{d} = \frac{\sqrt{a^2+b^2}}{1/\sqrt{a^2+b^2}} = a^2+b^2.$$

This is shown schematically in Figures 2A, 2B. Following the interpretation of δ_μ as the density of the measured foliation μ, integration along d gives the measure, which then corresponds to the total amount, t, of shearing. ∎

REMARK. Here is an algebraic way to compute the density of the measure of the geodesic foliation determined by a non-zero vector $\langle a,b\rangle$. Given such an $\langle a,b\rangle$, and $\langle x,y\rangle \in \mathbb{R}^2$, consider the map

$$\langle x,y\rangle \,\overline{\langle a,b\rangle\rceil} = \langle x,y\rangle + \det\begin{pmatrix} x & y \\ a & b \end{pmatrix} \cdot \langle a,b\rangle$$

as a linear map A, acting on $\mathbb{R}^2$. One easily sees that $\langle a,b\rangle$ is an eigenvector of the map A, having eigenvalue 1. The action of A on an orthogonal vector $\langle b,-a\rangle$ is expressed via

$$\begin{aligned} \langle b,-a\rangle \,\overline{\langle a,b\rangle\rceil} &= \langle b,-a\rangle + \det\begin{pmatrix} b & -a \\ a & b \end{pmatrix} \cdot \langle a,b\rangle \\ &= \langle b,-a\rangle + (a^2+b^2)\cdot\langle a,b\rangle. \end{aligned}$$

In fact, in the new basis

$$(\langle a/\sqrt{a^2+b^2}, b/\sqrt{a^2+b^2}\rangle, \langle b/\sqrt{a^2+b^2}, -a/\sqrt{a^2+b^2}\rangle),$$

the map A may be expressed as a parabolic shearing matrix with translation factor a^2+b^2,

$$A = \begin{pmatrix} 1 & 0 \\ a^2+b^2 & 1 \end{pmatrix}.$$

We can now state and prove the main theorem.

THEOREM 2.5. *There is a quandle structure, $\hat{Q}$, on the collection of measured geodesic foliations (with constant density) on $\mathbb{T}^2$. The action is given by shearing along such foliations. Moreover:*

1. *$\hat{Q}$ is a quotient of the alternating quandle on vectors in $\mathbb{R}^2$, with the anti-symmetric form given by the determinant (generalizing intersection number of circles on the torus), via a 2-to-1 map of quandles. The map takes vectors $\mathbf{v}, -\mathbf{v}$ to the same element of $\hat{Q}$.*
2. *The original Dehn quandle Q, for circles on $\mathbb{T}^2$, is imbedded as a subquandle of this new quandle.*

Proof. Let $\mu,\nu \in \mathcal{MGF}(\mathbb{T}^2)$ be represented by vectors $\langle a,b\rangle$ and $\langle x,y\rangle$ respectively. Using the notation from Lemma 2.4 and placing $\langle a,b\rangle$ and $\langle x,y\rangle$ tail to tail, a right shear along $\langle a,b\rangle$ applied to $\langle x,y\rangle$ moves the head of $\langle x,y\rangle$

by a distance t along a line l parallel to $\langle a, b\rangle$, i.e. it adds a multiple of $\langle a, b\rangle$ to $\langle x, y\rangle$. So its effects may be described via the map

$$\langle x, y\rangle \mapsto \langle x, y\rangle + t\frac{\langle a, b\rangle}{\sqrt{a^2+b^2}}. \tag{7}$$

Again, let d be the length of the projection of $\langle x, y\rangle$ onto a line perpendicular to $\langle a, b\rangle$. As in Lemma 2.4, we have

$$\frac{t}{d} = a^2 + b^2.$$

Let θ be the angle between $\langle a, b\rangle$ and $\langle x, y\rangle$ (see Fig. 2C). Then

$$d = \sqrt{x^2+y^2} \cdot \sin\theta.$$

Furthermore, we may write

$$\det\begin{pmatrix} x & y \\ a & b \end{pmatrix} = \sin\theta \cdot \sqrt{x^2+y^2} \cdot \sqrt{a^2+b^2}, \tag{8}$$

so

$$d = \sqrt{x^2+y^2} \cdot \sin\theta = \frac{\det\begin{pmatrix} x & y \\ a & b \end{pmatrix}}{\sqrt{a^2+b^2}}. \tag{9}$$

Thus

$$t = (a^2+b^2)d = \sqrt{a^2+b^2} \cdot \det\begin{pmatrix} x & y \\ a & b \end{pmatrix}. \tag{10}$$

Substituting this for t in (7), we get

$$\langle x, y\rangle \mapsto \langle x, y\rangle + \det\begin{pmatrix} x & y \\ a & b \end{pmatrix}\langle a, b\rangle,$$

which is precisely equation (4) expressing the action of vectors on vectors, yielding a symplectic quandle structure as in Example 1.2.5, with the determinant giving the alternating bilinear form. By the proof of Proposition 2.2, the action is well defined on vectors up to sign.

Thus we get a quandle structure, $\hat{Q}$, on elements $\sigma \in \mathcal{MGF}(\mathbb{T}^2)$, where the effect of the quandle action (right shear) on slopes of foliations is given by (4), and the effect on the densities is given by (6). It is clear from Lemma 2.4 and the foregoing discussion that this quandle generalizes the original Dehn quandle for circles on $\mathbb{T}^2$. In particular, the original Dehn quandle Q is a subquandle of $\hat{Q}$. ■

In light of Theorem 2.5 above, it seems reasonable to ask:

QUESTION 1. *For a surface F of genus $g > 1$, does there exist a quandle structure on the measured geodesic foliations on F, with action given by shearing, which extends the original Dehn quandle on F?*

In the genus $g > 1$ case, we do not have an immediate translation to a vector/matrix correspondence, of the type we exploited in the proofs above for the $\mathbb{T}^2$ case. However, we note for genus $g \geq 2$ that there is a representation of $\mathrm{MCG}(F)$ into the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$, with an action on vectors representing cycles in $H_1(F, \mathbb{Z})$. Under this action, vectors in $H_1(F)$ form an alternating quandle with the anti-symmetric bilinear form given by the homology intersection form on $H_1(F)$. However, this does not extend, as in the $\mathbb{T}^2$ case, to all circles on F, since now there exist nontrivial separating curves which are homologically trivial. These do not have well defined matrix actions. This alternating quandle was implicit in [11].

3. Some applications. The basic definitions and background information may be found in [4] or [5]. We also recall here some results from [13], for which we find analogs and extensions in this section.

DEFINITION 3.1. A *Lefschetz fibration* of a smooth, compact, connected, oriented 4-manifold X, over a smooth compact oriented 2-manifold B (possibly with boundary), is a smooth map $\pi : X \to B$ such that, at each critical point of B, there is an orientation-preserving chart on which $\pi : \mathbb{C}^2 \to \mathbb{C}$ is given by $\pi(w, z) = wz$.

Examples 3.2 and 3.6 below give explicit instances of Lefschetz fibrations over D^2.

There are a finite number of critical values, say $\{b_1, \ldots, b_n\}$, which lie in the interior of B. Preimages of regular values are fibers F, all diffeomorphic to a model surface Σ_g of fixed genus g. We say the genus of the fibration is g. The singular fibers are immersed surfaces with a single transverse self-intersection, here assumed to be *relatively minimal*, i.e. containing no imbedded sphere of self-intersection -1. In the discussions below, we will consider Lefschetz fibrations mainly with $B = D^2$, the 2-disk.

The preimage $\pi^{-1}(b_i)$ of a singular value is a singular fiber of the Lefschetz fibration. It is a copy of the fibering surface F, in which a specified simple closed curve (circle) $c_i \in F$, the *vanishing cycle*, has been collapsed to a point. One may consider the Lefschetz fibration $\pi : X \to B$ as formed from $B \times \Sigma_g$ by attaching 2-handles along the curves corresponding to the vanishing cycles of the singular fibers, with appropriate framing.

Over a small disk neighborhood D_i^2 of a critical value b_i, we have a Σ_g bundle, with a single singular fiber F_i, and a nonsingular surface bundle over the circle $S^1 = \partial D_i^2$. This is the mapping torus, $\Sigma_g \times I/\sim$, whose boundary surfaces are identified via the homeomorphism of Σ_g determined by a right handed Dehn twist $\overline{c_i\rvert}$, about the circle c_i, the vanishing cycle of the singular fiber F_i. The homeomorphism given by $\overline{c_i\rvert}$ is the *local monodromy* about ∂D_i^2.

For a Lefschetz fibration over a disk D^2, with critical values $\{b_1, \ldots, b_n\}$, we may think of a regular value $b \in \partial D^2$, and a series of mutually disjoint (except at b) arcs γ_i joining it to the critical values, arrayed in a "fan". We order the arcs and critical values, moving counterclockwise from ∂D^2. Over ∂D^2, we get a mapping torus, $\partial D^2 \times I/\!\sim$. The boundary surfaces are now identified via the homeomorphism $\overline{c_1\rvert}\,\overline{c_2\rvert}\cdots\overline{c_n\rvert}$, the composition of the ordered sequence of Dehn twists about the vanishing cycles of all of the singular surfaces. This homeomorphism is the *global monodromy* of the fibration. The Lefschetz fibration is determined up to isomorphism by this ordered sequence of Dehn twists, modulo the operations of conjugating each one by the same fixed homeomorphism, and certain rearranging moves on the sequence, called *elementary transformations*. These change the local monodromy but preserve the global monodromy. See [4] or [5] for further details.

EXAMPLE 3.2. A genus 1 Lefschetz fibration over D^2, with a single critical point yielding a "fishtail" fiber, has vanishing cycle given by the circle l in Fig. 1, and monodromy given by $\overline{\langle 1, 0\rangle\rvert}$, corresponding to $\left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$. See e.g. [5].

Certain connections between the 2-homology of the Dehn quandle of a closed orientable surface F and the monodromy of Lefschetz fibrations over a disk D^2, having copies of F as its nonsingular fibers, are described in [13]. There, a given fibration over D^2 with an ordered monodromy n-tuple $[c_1, \ldots, c_n]$ was considered. It was assumed that the monodromy n-tuple corresponded to a *reducible* homeomorphism (one which fixes a closed submanifold of F) $\psi = \overline{c_1\rvert}\cdots\overline{c_n\rvert}$, the ordered product of the Dehn twists around the vanishing cycles, and that a was a circle fixed by ψ. Let $C_2(Q)$ be the collection of Dehn quandle 2-chains.

For a specified circle $x \subset F$, a map $f_x : \{\text{monodromy } n\text{-tuples}\} \to C_2(Q)$ was defined by

$$[c_1, \ldots, c_n] \overset{f_x}{\mapsto} (x, c_1) + (x\,\overline{c_1\rvert}, c_2) + \cdots + (x\,\overline{c_1\rvert}\,\overline{c_2\rvert}\cdots\overline{c_{n-1}\rvert}, c_n).$$

When x is fixed by the monodromy homeomorphism $\psi = \overline{c_1\rvert}\cdots\overline{c_n\rvert}$, it was shown that the image 2-chain is a 2-cycle in $Z_2(Q)$ which represents a homology class in $H_2(Q)$. The following result concerning Lefschetz fibrations with reducible monodromy is from [13].

THEOREM 3.3. *Let Σ_g be a fixed oriented surface of genus g, with a circle $a \subset \Sigma_g$, and consider Lefschetz fibrations over the disk, $\pi : X \to D^2$, with fiber $F \cong \Sigma_g$.*

1. *Two monodromy n-tuples for a reducible homeomorphism ψ, with $(a)\psi = a$, which represent isomorphic Lefschetz fibrations with fiber $F \cong \Sigma_g$, yield homologous 2-cycles under f_a, and represent the same generator in $H_2(Q)$.*
2. *For a specified monodromy n-tuple, and fixed element a, there is a 1-1 correspondence between the images under f_a and incompressible tori in the mapping torus*

$$\partial X = \frac{\Sigma_g \times I}{(p, 0) \sim ((p)\psi, 1)}, \qquad p \in \Sigma_g.$$

The proof of the first part hinges on showing that the two "moves" governing equivalence of Lefschetz fibrations yield homologous 2-cycles; i.e. applying a specified conjugating homeomorphism to each of the vanishing cycles in a monodromy n-tuple $[c_1, \ldots, c_n]$, or applying an elementary transformation to the tuple, on the left side of f_a gives rise to homologous 2-cycles on the right side.

We wish to construct an analog and extension of the theorem above, which held for Lefschetz fibrations with reducible monodromy. We now allow Lefschetz fibrations which have global monodromy given by an Anosov (nontrivial, irreducible, nonperiodic) homeomorphism, where the fibering surface is a torus. These are known as *elliptic fibrations.* In this case, each such homeomorphism comes equipped with a pair of measured geodesic foliations, the stable and unstable foliations λ^s, λ^u, whose underlying geometric foliations are preserved by the homeomorphism.

To carry out a program analogous to the one above, we need to be able to consider foliations as the fixed objects of Anosov homeomorphisms, just as 1-manifolds on a surface F were the fixed objects for reducible homeomorphisms. However, an Anosov homeomorphism of $\mathbb{T}^2$ does not fix a measured geodesic foliation on the nose, and its representative linear map will not *fix* the associated vector. The vector instead behaves as an eigenvector of the associated linear transformation, with non-unit eigenvalue.

To address this, we consider a particular *relative chain complex* for our extended Dehn quandle $\hat{Q}$, to ensure that the 2-chains we consider will actually "close up" to 2-cycles. For a quandle X and $A \subset X$ we will have chain groups

$$C_n(X, A) = C_n(X)/C_n(A).$$

To get a chain complex

$$\cdots \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \cdots$$

which is well defined, we need the restriction of the original boundary map to chains on A to yield a chain subcomplex, i.e. we need $\partial : C_n(A) \to C_{n-1}(A)$. Homology is given by $H_n(X, A) = \operatorname{Ker} \partial / \operatorname{Im} \partial$.

Let ψ be an Anosov homeomorphism of $\mathbb{T}^2$ given by a product of right Dehn twists along circles on $\mathbb{T}^2$. By alphabetical interpolation, let λ^t stand for either of the fixed foliations λ^s or λ^u of ψ. Let $A_t = \{k \cdot \lambda^t \mid k \in \mathbb{R}\}$, so A_t is the set of elements of $\mathcal{MGF}(\mathbb{T}^2)$ which are in the same projective equivalence class as the geodesic measured foliation λ^t.

LEMMA 3.4. *Take X to be $\hat{Q}$, the extended Dehn quandle of the torus, and let $A = A_t$. Then the chain groups $C_*(A_t)$ form a chain subcomplex of $C_*(\hat{Q})$.*

Proof. We need to show that $\partial C_n(A_t) \subset C_{n-1}(A_t)$. Note that any simplex $(x_1, \ldots, x_n) \in C_n(A_t)$ has as all of its entries foliations in A_t, i.e. multiples of λ^t. Recalling the definition, (2), we have

$$\begin{aligned}\partial_n(x_1, \ldots, x_n) = \sum_{i=2}^{n} &(-1)^i[(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n) \\ &- (x_1 \overline{x_i}\rceil, x_2 \overline{x_i}\rceil, \ldots, x_{i-1} \overline{x_i}\rceil, x_{i+1}, \ldots, x_n)].\end{aligned}$$

If all the entries of the original simplex were in A_t, all the resulting entries in the first set of terms of the sum are also in A_t. The second set of terms involve application of an $\overline{x_i}\rceil$ to all of the previous entries. But from the formulae given in Proposition 2.2, it is clear that applying $\overline{x_i}\rceil$ to a member of A_t, where $x_i \in A_t$, simply amounts to adding multiples of the same underlying vector, yielding again a multiple of this vector, i.e. another element of A_t. Thus $\partial C_n(A_t) \subset C_{n-1}(A_t)$, and $C_*(A_t)$ is a well defined chain subcomplex of $C_*(Q)$. ∎

What we have incidentally shown here is that any such projective class A_t of a foliation $\lambda^t \subset \mathbb{T}^2$ is closed under the quandle operations and forms a subquandle of the generalized Dehn quandle $\hat{Q}$ on $\mathbb{T}^2$.

In the proof of the extension of Theorem 3.3, we will make use of an analog of the map f used above. For a specified "fixed" foliation λ, we will take

$$\hat{f}_{\lambda^t} : (\text{monodromy } n\text{-tuples}) \to C_2(\hat{Q}, A_t).$$

THEOREM 3.5. *Consider elliptic fibrations over the disk, $\pi : X \to D^2$, and let $\hat{Q}$ be the extended Dehn quandle of the fiber, $\mathbb{T}^2$.*

1. *Suppose two monodromy n-tuples for an irreducible nonperiodic homeomorphism ψ of $\mathbb{T}^2$ represent isomorphic elliptic fibrations with fiber $F \cong \mathbb{T}^2$. Also suppose there is a measured geodesic foliation $\lambda \subset \mathbb{T}^2$ with $(\lambda)\psi = \lambda$ (i.e. a fixed foliation for ψ). Let $A_\lambda \subset Q$ be the projective class of λ. Then the monodromy n-tuples are mapped, by $\hat{f}_\lambda$, to homologous 2-cycles in the relative chain groups $C_2(\hat{Q}, A_\lambda)$.*
2. *For a specified monodromy n-tuple for an irreducible nonperiodic homeomorphism, having a fixed foliation λ, there is a 1-1 correspon-*

dence between the images under $\hat{f}_\lambda$ and incompressible surfaces $\lambda \times_\psi S^1$ in the mapping torus

$$\partial X = \frac{\Sigma_g \times I}{(p, 0) \sim ((p)\psi, 1)}, \qquad p \in \mathbb{T}^2.$$

Proof. The proof is very similar to the one for Theorem 3.3 above, given in [13].

1. We do not need the full generality of Theorem 2.5, i.e. we do not need to apply shears along arbitrary elements in $\mathcal{MGF}(\mathbb{T}^2)$ to other such arbitrary elements. In this context we still only need to apply standard Dehn twists about circles. Lemma 3.4 ensures that the relative chain complex $C_*(\hat{Q}, A_\lambda)$ and, in particular, relative 2-chains are well defined, and we may apply the definitions and ideas of Section 1.2.

Suppose that $[c_1, \ldots, c_n]$ and $[c'_1, \ldots, c'_n]$ are the monodromy n-tuples for the irreducible nonperiodic homeomorphism ψ yielding equivalent Lefschetz fibrations. From [1], there is a foliation λ fixed by ψ (actually there are two: λ^u, λ^s). The map $\hat{f}_\lambda$ is defined, and the fixed foliation λ now provides the "anchor" for the image 2-cycles, just as the fixed circle played the "anchor" in the reducible case. Again, $\hat{f}_\lambda$ now takes monodromy n-tuples to relative 2-chains of the extended Dehn quandle:

$$[c_1, \ldots, c_n] \overset{\hat{f}_\lambda}{\mapsto} (\lambda, c_1) + (\lambda\, \overline{c_1}|, c_2) + \cdots + (\lambda\, \overline{c_1}|\, \overline{c_2}| \cdots \overline{c_{n-1}}|, c_n),$$

$$[c'_1, \ldots, c'_n] \overset{\hat{f}_\lambda}{\mapsto} (\lambda, c'_1) + (\lambda\, \overline{c'_1}|, c'_2) + \cdots + (\lambda\, \overline{c'_1}|\, \overline{c'_2}| \cdots \overline{c'_{n-1}}|, c'_n).$$

These images are relative cycles since the first entry (foliation) in the first 2-simplex and its final image under the composition of Dehn twists in the factorization of ψ are both multiples of λ, and thus lie in A_λ. Successive simplices just record the successive images in its trajectory under the given factorizations of ψ. The entries in the n-tuples are *standard circles* corresponding to the vanishing cycles of the singular fibers, but are now considered as foliations, i.e. vectors (see the Note after Definition 2.1). Thus, the same argument as was given in [13] holds, to show that the operations governing equivalence of Lefschetz fibrations, namely conjugation by a specified homeomorphism, and elementary transformations, apply to these standard circles, and yield homologous 2-cycles.

2. For the second statement, the argument follows essentially verbatim the proof of the second portion of the theorem from [13] cited above. The only changes are to replace the circle a, fixed by the originally reducible homeomorphism, by the foliation λ, fixed by the now irreducible nonperiodic homeomorphism ψ. We get a mapping torus $S^1 \times_\psi \lambda$, lying in $\partial X = S^1 \times_\psi F$, which the argument shows is incompressible in ∂X. ■

We now give an example illustrating this theorem. We take $F \cong \mathbb{T}^2$. The fixed foliation is easily located and described using matrices from $\mathrm{SL}_2(\mathbb{Z})$.

EXAMPLE 3.6. Again we take the standard longitude and meridian,

$$l \leftrightarrow \langle 1,0\rangle \quad \text{and} \quad m \leftrightarrow \langle 0,1\rangle,$$

as a basis for circles on the torus T^2 (recall Figure 1) and represent arbitrary circles as vectors. Let

$$u = \langle 1,2\rangle, \quad \text{so} \quad \overline{u}\rceil \leftrightarrow \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix},$$

$$v = \langle 3,1\rangle, \quad \text{so} \quad \overline{v}\rceil \leftrightarrow \begin{pmatrix} 4 & 1 \\ -9 & -2 \end{pmatrix}.$$

Then

$$\overline{u}\rceil\ \overline{v}\rceil \leftrightarrow \begin{pmatrix} -24 & -5 \\ 5 & 1 \end{pmatrix}.$$

The characteristic polynomial of this latter matrix is

$$\mathrm{CharPoly}\left(\begin{pmatrix} -24 & -5 \\ 5 & 1 \end{pmatrix}\right) = x^2 + 23x + 1,$$

and has the following properties:

1. It is irreducible over $\mathbb{Z}$.
2. Its roots are $-23/2 \pm 5\sqrt{21}/2$, with norm $\neq 1$, hence are not roots of unity.
3. It is not a polynomial in x^n for any n.

Thus by Lemma 5.1 of [1], the homeomorphism $\psi = \overline{u}\rceil\ \overline{v}\rceil : \mathbb{T}^2 \to \mathbb{T}^2$ is irreducible and nonperiodic. Consider a genus 1 Lefschetz fibration over the disk, $\pi : X \to D^2$, having global monodromy given by the factorization $\psi = \overline{u}\rceil\ \overline{v}\rceil$. So the circles u, v are vanishing cycles for the singular fibers of $\pi : X \to D^2$. The matrix for $\overline{u}\rceil\ \overline{v}\rceil$ has two eigenvectors

$$\alpha = \left\langle \frac{-5}{2} - \frac{\sqrt{21}}{2}, 1 \right\rangle \quad \text{and} \quad \beta = \left\langle \frac{-5}{2} + \frac{\sqrt{21}}{2}, 1 \right\rangle.$$

These yield irrational slopes on $\mathbb{T}^2$ and so correspond to non-circle foliations $\lambda_\alpha, \lambda_\beta \subset \mathbb{T}^2$ which are fixed by $\overline{u}\rceil\ \overline{v}\rceil$.

From part 1 of Theorem 3.5, via the map $\hat{f}$, we have a 2-cycle associated to (anchored at) each foliation:

$$[u,v] \xmapsto{\hat{f}_{\lambda_\alpha}} (\lambda_\alpha, u) + (\lambda_\alpha\, \overline{u}\rceil, v) \quad \text{and} \quad [u,v] \xmapsto{\hat{f}_{\lambda_\beta}} (\lambda_\beta, u) + (\lambda_\beta\, \overline{u}\rceil, v).$$

From part 2 of the theorem, we also get the associated incompressible surfaces $S^1 \times \lambda_\alpha$ and $S^1 \times \lambda_\beta$ in

$$\partial X = \frac{\mathbb{T}^2 \times I}{(p,0) \sim ((p)\psi, 1)}, \qquad p \in \mathbb{T}^2.$$

In light of Theorem 3.5 for elliptic fibrations with nonperiodic irreducible monodromy, and following Question 1, we have:

Remark. If there is a positive answer to Question 1., i.e. if for surfaces F of genus $g > 1$, there exists a quandle structure on $\mathcal{MGF}(F)$ extending the original Dehn quandle on F, then an appropriate analog of Theorem 3.5 should hold for Lefschetz fibrations having fiber of genus $g > 1$, where the fibering surfaces are copies of F and have hyperbolic structures, and the monodromy maps are pseudo-Anosov.

We now look at another application of some of the ideas discussed above.

Background. The following basic relations for elements (circles) a, c of the Dehn quandle of a surface F were proved in [12]:

(F0) $a\,\overline{c}| = a = a\,|\overline{c}$ iff $|a \cap c| = 0$.
(F1) $a\,\overline{c}| = c\,|\overline{a}$, or equivalently, $a\,\overline{c}|\,\overline{a}| = c$ iff $|a \cap c| = 1$,

where $|a \cap c|$ denotes the geometric intersection number for circles a, c. Using this, [14] describes elements having "intersection numbers 0 and 1" in *general quandles.*

Definition 3.7. For a rack or quandle X, elements $a, c \in X$ have *intersection number* 0 or 1 respectively, via

$$|a \cap c| = \begin{cases} 0 & \text{iff } a\,\overline{c}| = a = a\,|\overline{c}, \\ 1 & \text{iff } a\,\overline{c}| = c\,|\overline{a}\,. \end{cases}$$

Note. If $a = c$ the intersection number 1 criterion holds. For simplicity and in keeping with the properties of intersections of circles on surfaces, we make the convention that for any $a \in X$, the intersection number of a with itself is taken to be 0, not 1.

The next examples of 2-cycles, based on pairs of elements with intersection number 1, are from [14].

Example 3.8. For any quandle X having sufficiently many elements with intersection numbers 0 and 1, consider elements a_i, $i = 1, 2, 3$, a, and c, with $|c \cap a_i| = 1$, $|c \cap a| = 1$, and $|a_i \cap a_j| = 0$. We then have the following nontrivial 2-cycles:

- $\beta = (c\,\overline{a}|, c) + (a, c\,\overline{a}|\,\overline{a}|)$,
- $\gamma = (c, a) + (c\,\overline{a}|, c) + (a, c\,\overline{a}|)$,
- $\zeta_2 = (c, a_1) + (c\,\overline{a_1}|, a_2) + (c\,\overline{a_1}|\,\overline{a_2}|, c) + (c\,\overline{a_1}|\,\overline{a_2}|\,\overline{c}|, a_1)$
  $+ (c\,\overline{a_1}|\,\overline{a_2}|\,\overline{c}|\,\overline{a_1}|, a_2)$,
- $\zeta_3 = (c, a_1) + (c\,\overline{a_1}|, a_2) + (c\,\overline{a_1}|\,\overline{a_2}|, a_3) + (c\,\overline{a_1}|\,\overline{a_2}|\,\overline{a_3}|, c)$
  $+ (c\,\overline{a_1}|\,\overline{a_2}|\,\overline{a_3}|\,\overline{c}|, a_1) + (c\,\overline{a_1}|\,\overline{a_2}|\,\overline{a_3}|\,\overline{c}|\,\overline{a_1}|, a_2)$

$$
\begin{aligned}
&+ (c\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}\,\overline{c|}\,\overline{a_1|}\,\overline{a_2|}, a_3) + (c\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}\,\overline{c|}\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}, c)\\
&+ (c\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}\,\overline{c|}\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}\,\overline{c|}, a_1) + (c\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}\,\overline{c|}\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}\,\overline{c|}\,\overline{a_1|}, a_2)\\
&+ (c\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}\,\overline{c|}\,\overline{a_1|}\,\overline{a_2|}\,\overline{a_3|}\,\overline{c|}\,\overline{a_1|}\,\overline{a_2|}, a_3).
\end{aligned}
$$

The theorem we shall prove below will be valid for these 2-cycles or for any other 2-cycle similarly based on elements with intersection number 1. The proof will hold in general, but will be given for γ in particular. Thus it does not hurt to show how the intersection number 1 property comes into play in the formation of such a 2-cycle. We have

$$
\begin{aligned}
\partial\gamma &= c - c\,\overline{a|} + c\,\overline{a|} - c\,\overline{a|}\,\overline{c|} + a - a\,\overline{c\,\overline{a|}|} \\
&= c - c\,\overline{a|} + c\,\overline{a|} - a + a - c = 0.
\end{aligned}
\tag{11}
$$

where

1. $c\,\overline{a|}\,\overline{c|} = a$, by relation (F1), since $|a \cap c| = 1$;
2. $a\,\overline{c\,\overline{a|}|} = a\,\overline{|a}\,\overline{c|}\,\overline{a|} = a\,\overline{c|}\,\overline{a|} = c$ by axioms (Q3), (Q1) and relation (F1) respectively.

We now interpret the intersection number 1 criterion in the context of the generalized Dehn quandle $\hat{Q}$, for measured geodesic foliations on $\mathbb{T}^2$, from Section 2. Recall it was exhibited as a quandle on vectors (see Proposition 2.2) via the formulae

$$
\langle x, y\rangle\,\overline{\langle a, b\rangle|} = \langle x, y\rangle + \det\begin{pmatrix} x & y \\ a & b \end{pmatrix} \cdot \langle a, b\rangle,
$$

$$
\langle x, y\rangle\,\overline{|\langle a, b\rangle} = \langle x, y\rangle - \det\begin{pmatrix} x & y \\ a & b \end{pmatrix} \cdot \langle a, b\rangle.
$$

Recall the comments on equivalence of vectors up to sign, following Definition 2.1. Taking vectors $\mathbf{a} = \langle a, b\rangle$ and $\mathbf{c} = \langle x, y\rangle$, the criterion becomes

$$
\mathbf{c}\,\overline{\mathbf{a}|} = \mathbf{a}\,\overline{|\mathbf{c}} \;\Leftrightarrow\; \left.\begin{array}{l} \mathbf{c} + k\mathbf{a} = (\mathbf{a} + k\mathbf{c}) \text{ or} \\ \mathbf{c} + k\mathbf{a} = -(\mathbf{a} + k\mathbf{c}) \end{array}\right\} \text{so} \left\{\begin{array}{l} (k-1)(\mathbf{c} - \mathbf{a}) = \mathbf{0} \text{ or} \\ (k+1)(\mathbf{c} + \mathbf{a}) = \mathbf{0} \end{array}\right.
\tag{12}
$$

where $k = \det(M)$ for the appropriate matrix M. So in the nontrivial case, where $\mathbf{c} \neq \pm\mathbf{a}$, the determinant must be $k = \pm 1$. We have

THEOREM 3.9. *Take $\hat{Q}$ as above, now considered as the quandle for the action of $\mathrm{SL}_2(\mathbb{R})$ on vectors in $\mathbb{R}^2$. Let ζ be a quandle 2-cycle based on intersection number 1 elements, for instance $\zeta = \beta$, γ, ζ_2, or ζ_3 above.*

1. *For any vector $\mathbf{c} = \langle x, y\rangle \in \mathbb{R}^2$, there exists a 1-parameter family of vectors $\mathbf{a}_t$ with $|\mathbf{a}_t \cap \mathbf{c}| = 1$, and a 1-parameter family of linear transformations M_t arising from $\mathbf{a}_t$ and ζ, having $\mathbf{c}$ as a fixed point.*

2. *For each such linear transformation fixing* $\mathbf{c}$*, given by a matrix* $M \in \mathrm{SL}_2(\mathbb{R})$*, the choice of* ζ *determines a factorization of* M*, up to sign, by matrices in* $\mathrm{SL}_2(\mathbb{R})$ *with* $\mathrm{Tr} = 2$.

Proof. We prove the theorem for a 2-cycle of type γ. The proof is analogous in the case of other 2-cycles of the type mentioned above. Choose a vector $\mathbf{c} = \langle x, y\rangle$ which will be fixed by the linear transformation we construct. We require vectors $\mathbf{a}_t = \langle s, t\rangle$ with $|\mathbf{a}_t \cap \mathbf{c}| = 1$. From (12), assuming positivity, this means we need

$$\det\begin{pmatrix} x & y \\ s & t \end{pmatrix} = 1,$$

but any specified value t then determines s, and this gives our family $\mathbf{a}_t$. For any such specified value of t, form the 2-cycle

$$\gamma = (\mathbf{c}, \mathbf{a}) + (\mathbf{c}\,\overline{\mathbf{a}}\rvert, \mathbf{c}) + (\mathbf{a}, \mathbf{c}\ \overline{\mathbf{a}}\rvert),$$

and let $\mathbf{c} = \langle x, y\rangle$ and $\mathbf{a} = \langle a, b\rangle$. Successive applications of shears along the right hand term in each 2-simplex, to the left term in the simplex, yield the left term of the next simplex. Specifically, using (4), we compute

$$\begin{aligned} \mathbf{c}\,\overline{\mathbf{a}}\rvert &= \langle x, y\rangle\ \overline{\langle a, b\rangle}\rvert = \langle x, y\rangle + \det\begin{pmatrix} x & y \\ a & b \end{pmatrix}\langle a, b\rangle \\ &= \langle x, y\rangle + 1\langle a, b\rangle = \langle x + a, y + b\rangle. \end{aligned} \tag{13}$$

Using this as input for the operator $\overline{\mathbf{c}}\rvert$, we have

$$\begin{aligned} \mathbf{c}\,\overline{\mathbf{a}}\rvert\,\overline{\mathbf{c}}\rvert &= \langle x + a, y + b\rangle + \det\begin{pmatrix} x + a & y + b \\ x & y \end{pmatrix}\langle x, y\rangle \\ &= \langle x + a, y + b\rangle - 1\langle x, y\rangle = \langle a, b\rangle = \mathbf{a} \end{aligned} \tag{14}$$

and then

$$\begin{aligned} \mathbf{a}\,\overline{\mathbf{c}\,\overline{\mathbf{a}}\rvert}\rvert &= \langle a, b\rangle\ \overline{\langle x + a, y + b\rangle}\rvert \\ &= \langle a, b\rangle + \det\begin{pmatrix} a & b \\ x + a & y + b \end{pmatrix}\langle x + a, y + b\rangle \\ &= \langle a, b\rangle - 1\langle x + a, y + b\rangle = \langle -x, -y\rangle. \end{aligned} \tag{15}$$

But $\langle x, y\rangle$ and $\langle -x, -y\rangle$ represent the same measured geodesic foliation, since they yield the same slope and have the same length. Using formula (3), we may convert the second entry of each simplex into a matrix in $\mathrm{SL}_2(\mathbb{R})$. This is just the process of turning a quandle element, e.g. $\mathbf{a}$, into an operator, $\overline{\mathbf{a}}\rvert$. In each case, this yields a matrix $M_{\mathbf{a}}$ with trace $\mathrm{Tr}(M_{\mathbf{a}}) = 2$. Then from the fact that γ is indeed a quandle 2-cycle, the measured foliation represented by $\mathbf{c}$ is fixed by the linear transformation whose matrix is

$$M = -M_{\mathbf{a}}\, M_{\mathbf{c}}\, M_{\mathbf{c}\,\overline{\mathbf{a}|}},$$

corresponding to the composition

$$\overline{\mathbf{a}|}\,\overline{\mathbf{c}|}\,\overline{\mathbf{c}\,\overline{\mathbf{a}|}\,|}$$

of operators associated to the second entries of the 2-simplices in γ. The negative sign arises from the computation above. ∎

From the discussion at the end of Section 1, we may now consider the quandle homology theory associated to the new extended measured geodesic foliation quandle $\hat{Q}$. The existence of purely positive cycles of the type arising in Theorem 3.5 and in [14] suggests the existence of a slightly more exotic form for a purely positive 2-cycle in the 2-chains of $\hat{Q}$. Theorem 3.5 involved relative positive 2-cycles, "anchored" at a measured geodesic foliation λ, which was fixed by a factorization of an Anosov homeomorphism into a product of standard Dehn twists. Using Theorem 3.9, and the 2-cycle γ as a template, we now construct an example of a purely positive 2-cycle, in $C_2(\hat{Q})$, whose "anchor" is a generalized (non-circle) foliation in $\mathcal{MGF}(\mathbb{T}^2)$ and *all* of whose other entries are similar elements in $\mathcal{MGF}(\mathbb{T}^2)$. In fact, we could use any 2-cycle of the types mentioned above in Example 3.8, and different choices of the parameter t would yield a family of such generalized foliation 2-cycles.

Example 3.10. Let $\mathbf{c} = \langle 2, \sqrt{2}\rangle$. For $\mathbf{a} = \langle s, t\rangle$, suppose $t = 1$. Then $|\mathbf{c} \cap \mathbf{a}| = 1$ forces $s = 1/\sqrt{2}$, and $\mathbf{a} = \langle 1/\sqrt{2}, 1\rangle$. So both vectors $\mathbf{a}$ and $\mathbf{c}$ correspond to noncompact measured geodesic foliations on $\mathbb{T}^2$. Using (3) to convert vectors to matrices, we have

$$\begin{aligned} M = -M_{\mathbf{a}}\, M_{\mathbf{c}}\, M_{\mathbf{c}\,\overline{\mathbf{a}|}} &= -\begin{pmatrix} 1+1/\sqrt{2} & 1 \\ -1/2 & 1-1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1+2\sqrt{2} & 2 \\ -4 & 1-2\sqrt{2} \end{pmatrix} \\ &\qquad \times \begin{pmatrix} (4\sqrt{2}+5)/\sqrt{2} & 3+2\sqrt{2} \\ (-9-4\sqrt{2})/2 & (-2\sqrt{2}-5)/\sqrt{2} \end{pmatrix}, \end{aligned}$$

and the 2-cycle

$$\begin{aligned} \gamma &= (\mathbf{c}, \mathbf{a}) + (\mathbf{c}\,\overline{\mathbf{a}|}, \mathbf{c}) + (\mathbf{a}, \mathbf{c}\,\overline{\mathbf{a}|}) \\ &= (\langle 2, \sqrt{2}\rangle, \langle 1/\sqrt{2}, 1\rangle) + (\langle 2+1/\sqrt{2}, 1+\sqrt{2}\rangle\,, \langle 2, \sqrt{2}\rangle) \\ &\quad + (\langle 1/\sqrt{2}, 1\rangle, \langle 2+1/\sqrt{2}, 1+\sqrt{2}\rangle). \end{aligned}$$

So a non-identity composition of shears along measured geodesic foliations can carry a nontrivial foliation back to itself. Perhaps somewhat more surprisingly, a similar composition of such shears, applied to a *circle*, can disassemble and then reassemble the given circle.

We also have the following result.

COROLLARY 3.11. *The* 2*-cycle* γ *here represents a nontrivial homology class in* $H_2(\hat{Q})$*. Moreover, all the homology groups* $H_i(\hat{Q})$ *are nontrivial for* $i = 1, 2, \ldots$.

Proof. Any singleton 1-simplex is a 1-cycle, and Corollary 3.9 of [13] tells us that it is not a boundary. 2-cycles of the type represented by γ are purely positive, i.e. all coefficients of simplices are positive. Theorem 3.7 of [13] guarantees that such a cycle represents a nontrivial homology class in $H_2(\hat{Q})$. The discussion preceding Lemma 3.4 gives us the projective class of $\mathbf{c}$, denoted $A_{\mathbf{c}}$. It is an infinite family of elements having the property that any pair of distinct vectors from this class $\mathbf{c}_i, \mathbf{c}_j$ have intersection number 0. To see this, let $\mathbf{c}_i = \langle x, y\rangle$ and let $\mathbf{c}_j = \langle kx, ky\rangle$ for some positive $k \in \mathbb{R}$. The quandle action $\mathbf{c}_i \,\overline{\mathbf{c}_j}\rceil$ is given by (4), yielding

$$\langle x, y\rangle \,\overline{\langle kx, ky\rangle}\rceil = \langle x, y\rangle + \det\begin{pmatrix} x & y \\ kx & ky \end{pmatrix} \cdot \langle kx, ky\rangle.$$

But then the determinant is 0, so

$$\langle x, y\rangle \,\overline{\langle kx, ky\rangle}\rceil = \langle x, y\rangle,$$

and similarly for the left bracket. Then by Definition 3.7, $|\mathbf{c}_i \cap \mathbf{c}_j| = 0$. So we get an infinite collection of intersection number 0 vectors.

Theorem 3.1 of [14] now tells us that in the presence of an infinite family of such elements $\mathbf{c}_i$ with $i = 1, 2, \ldots$, we can promote the 2-cycle γ to cycles of the same form in any dimension. These are all purely positive, so by the same argument as that given in part (2) of the proof of Corollary 3.2 of [14], these promoted cycles, of "shape" γ, all represent nontrivial elements in homology in their respective dimensions. ∎

References

[1] S. Bleiler and A. Casson, *Automorphisms of Surfaces after Nielsen and Thurston*, London Math. Soc. Student Texts 9, Cambridge Univ. Press, 1988.

[2] J. S. Carter, S. Kamada and M. Saito, *Geometric interpretations of quandle homology*, J. Knot Theory Ramif. 10 (2001), 345–386.

[3] J. S. Carter, S. Kamada and M. Saito, *Surfaces in 4-space*, Encyclopaedia Math. Sci. 142, Springer, 2004.

[4] T. Fuller, *Lefschetz fibrations of 4-dimensional manifolds*, Cubo 5 (2003), 275–294.

[5] R. Gompf and A. Stipsicz, *4-Manifolds and Kirby Calculus*, Grad. Stud. Math. 20, Amer. Math. Soc., 1999.

[6] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982), 37–65.

[7] L. Kauffman, *Knots and Physics*, 2nd ed., World Sci., 1993.

[8] E. Navas and S. Nelson, *On symplectic quandles*, Osaka J. Math. 45 (2008), 973–985.

[9] M. Niebrzydowski and J. Przytycki, *The quandle of the trefoil knot as the Dehn quandle of the torus*, Osaka J. Math. 46 (2009), 645–659.

[10] D. Rolfsen, *Knots and Links*, Publish or Perish, 1976.

[11] D. Yetter, *Quandles and monodromy*, J. Knot Theory Ramif. 12 (2003), 523–541.

[12] J. Zablow, *Loops and disks in surfaces and handlebodies*, J. Knot Theory Ramif. 12 (2003), 203–223.

[13] J. Zablow, *On relations and omology of the Dehn quandle*, Algebr. Geom. Topol. 8 (2008), 101–133.

[14] J. Zablow, *Some consequences of small intersection numbers in the Dehn quandle and beyond*, J. Knot Theory Ramif. 20 (2011), 1741–1768.

Reza Chamanara
Department of Mathematics
Brooklyn College
Brooklyn, NY 11210, U.S.A.
E-mail: RChamanara@brooklyn.cuny.edu

Joel Zablow
Department of Mathematics
Long Island University, Brooklyn Campus
1 University Plaza
Brooklyn, NY 11201, U.S.A.
E-mail: dquandle@netscape.net
jzablow@gmail.com

Jun Hu
Department of Mathematics
Brooklyn College
Brooklyn, NY 11210, U.S.A.
and
Ph.D. Program in Mathematics
Graduate Center, CUNY
365 5th Ave.
New York, NY 10016, U.S.A.
E-mail: junhu@brooklyn.cuny.edu

Received 27 December 2011;
in revised form 24 March 2013