FUNDAMENTA
MATHEMATICAE
231 (2015)

Density of the set of symbolic dynamics with all ergodic measures supported on periodic orbits

by

Tatiane Cardoso Batista (Guarapuava),
Juliano dos Santos Gonschorowski (Guarapuava)
and **Fabio Armando Tal** (São Paulo)

Abstract. Let K be the Cantor set. We prove that arbitrarily close to a homeomorphism $T : K \to K$ there exists a homeomorphism $\widetilde{T} : K \to K$ such that the ω-limit of every orbit is a periodic orbit. We also prove that arbitrarily close to an endomorphism $T : K \to K$ there exists an endomorphism $\widetilde{T} : K \to K$ with every orbit finally periodic.

1. Introduction. Given a topological space X, a dynamics $T : X \to X$ and an observable $f : X \to \mathbb{R}$, both continuous functions, the fundamental question in ergodic optimization is to determine, in the set $M_T(X)$ of all T-invariant Borel probability measures, which measures maximize the functional $F_f : M_T(X) \to \mathbb{R}$ defined by

$$F_f(\mu) = \int_X f\, d\mu.$$

A measure maximizing this functional is usually called an *f-maximizing measure.*

This relatively new field of study has seen a fast development in the last decade, and several interesting lines of research have been pursued (see for instance [J] and references therein). Among the different lines of research, one that has received great attention is to determine what is the typical support of maximizing measures, where "typical" is of course context dependent. For instance, one of the most relevant conjectures, later proved by Contreras [C], considered the case where there existed a fixed expansive T, and asked if, in the space of Lipschitz functions from X to $\mathbb{R}$, the set of observables for which there exists a single f-maximizing measure, with this measure supported on a periodic orbit, is a G_δ.

2010 *Mathematics Subject Classification*: Primary 37B10; Secondary 37A05, 37D35.
Key words and phrases: ergodic measures, Cantor set, periodic orbits.

In the same spirit but in a different context, some works search to determine the expected behaviour of maximizing measures when the observable f is fixed, but the dynamics varies in a given space. This has been done for instance when X is a compact Riemanian manifold, where in [TA2, AT] the typical behaviour when T belongs to the set of homeomorphisms of X is studied, and in [TA1, BGT] this is done with T varying in the space of all continuous surjections of X. This work was born out of a similar question, to study this behaviour when X is the Cantor set K, but the investigation led to a result which is somewhat more general, dealing only with the dynamics in K and bypassing discussions on observable functions.

The Cantor set K is defined as a totally disconnected, perfect and compact metric space, and a classical result states that any two such sets are homeomorphic. Since all of our results are topological, we can consider any realization of the Cantor set, and we will usually work with the set Σ_N of one-sided or two-sided sequences of N symbols with the usual metric. We consider the set $\mathrm{End}(K)$ of endomorphisms of K, that is, all continuous surjections of K, with the metric

$$D(T,\widetilde{T}) = \max_{x\in K} d(T(x),\widetilde{T}(x)),$$

and also the subset $\mathrm{Hom}(K)$ of all homeomorphisms of K, endowed with the induced metric. Our main results are:

THEOREM 1.1. *Given $T \in \mathrm{End}(K)$ and $\varepsilon > 0$, there exists $\widetilde{T} \in \mathrm{End}(K)$ such that $D(T,\widetilde{T}) < \varepsilon$ and every orbit of $\widetilde{T}$ is finally periodic* ([1]).

THEOREM 1.2. *Given $T \in \mathrm{Hom}(K)$ and $\varepsilon > 0$, there exists $\widetilde{T} \in \mathrm{Hom}(K)$ such that $D(T,\widetilde{T}) < \varepsilon$ and the ω-limit of every orbit of $\widetilde{T}$ is a periodic orbit.*

As a direct consequence of these theorems we are able to obtain a positive answer for the ergodic optimization problem:

COROLLARY 1.3. *Let K be a Cantor set. Given an endomorphism (respectively homeomorphism) $T : K \to K$, a continuous function $f : K \to \mathbb{R}$ and $\varepsilon > 0$, there exists an endomorphism (respectively homeomorphism) $\widetilde{T} : K \to K$ with*

$$D(T,\widetilde{T}) = \max_{x\in K} d(T(x),\widetilde{T}(x)) < \varepsilon$$

and such that $\widetilde{T}$ has an f-maximizing measure supported on a periodic orbit.

This short note is designed to be self-contained. The study of typical dynamics in the Cantor set apart from any ergodic optimization result already has some important literature. For instance, the works [AGW, AHK, BD]

([1]) We say that the orbit of a point $x \in K$ under T is *finally periodic* if there exist $j, N > 0$ such that $T^{N+j}(x) = T^j(x)$.

provide a much stronger result, showing the existence of a conjugacy class in $\mathrm{Hom}(K)$ which itself is generic, and describing this conjugacy class. Parts of Theorems 1.1 and 1.2 could be derived more directly from those works, but the proofs therein are more involved.

2. Preliminaries. We begin by listing some simple and immediate properties of the Cantor set:

PROPOSITION 2.1. *A Cantor set can be partitioned into N disjoint non-empty Cantor sets.*

PROPOSITION 2.2. *Given $\varepsilon > 0$, there exists $M > 0$ and disjoint subsets $K_j \subset K$, $0 \le j \le M$, such that*

(i) *each K_j is a Cantor set;*
(ii) $K = \bigcup_{j=1}^{M} K_j$;
(iii) $\mathrm{diam}(K_j) < \varepsilon$ *for each $j = 1, \dots, M$.*

PROPOSITION 2.3. *There exists a disjoint sequence $(K_m)_{m\in\mathbb{N}}$, $K_m \subset K$, and a point $p \in K \setminus \bigcup_{m\in\mathbb{N}} K_m$ such that*

(i) *K_m is a Cantor set for any $m \in \mathbb{N}$;*
(ii) $K = (\bigcup_{m=1}^{\infty} K_m) \cup \{p\}$;
(iii) *given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that* $\mathrm{diam}(K_m) < \varepsilon$ *for $m > m_0$.*

3. Endomorphisms. Let Σ_N^+ be the space of all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \{1, \dots, N\}$, and consider the usual metric

$$d(x, y) = 2^{-\min\{i : x_i \neq y_i\}} \quad \text{for } x, y \in \Sigma_N^+.$$

Define

$$W_i = \{x : x = (i, x_2, x_3, \dots) \in \Sigma_N^+\}.$$

LEMMA 3.1. *Let $T \in \mathrm{End}(\Sigma_N^+)$ be such that for all $1 \le i \le N$ there exists $x^i \in W_i$ with $T(x^i) \in W_i$. Then there exists $\widetilde{T} \in \mathrm{End}(\Sigma_N^+)$ such that*

$$\widetilde{T}(x) \in W_i \ \Leftrightarrow\ T(x) \in W_i,$$

and every orbit of $\widetilde{T}$ is finally periodic.

Proof. By the continuity of T, for all $i = 1, \dots, N$, there exists l_i such that, if $R_i = x_2^i, x_3^i, \dots, x_{l_i}^i$, then for all x of the form $(i, R_i, x_{l_i+1}, x_{l_i+2}, \dots)$, we have $T(x) \in W_i$.

For each $i = 1, \dots, N$, let $W_{i,i} = \{x : x = (i, R_i, x_{l_i+1}, x_{l_i+2}, \dots) \in \Sigma_N^+\}$. Let

$$\widetilde{T}(x) = \begin{cases} (i, R_i, R_i, \dots) & \text{if } T(x) \in W_i \text{ but } x \notin W_{i,i}, \\ (i, x_{l_i+1}, \dots) & \text{if } x \in W_{i,i}. \end{cases}$$

For all $x \in K$, if $\widetilde{T}^n(x) \in W_{x_1,x_1}$ for all positive n, then $x = (x_1, R_{x_1}, R_{x_1}, \dots)$

and x is fixed by $\widetilde{T}$. Otherwise there exists some $n_0 \geq 0$ such that $\widetilde{T}^{n_0+1}(x)$ is not in W_{x_1,x_1}, in which case $\widetilde{T}^{n_0+1}(x)$ is also fixed by $\widetilde{T}$. ∎

Theorem 1.1 is a direct consequence of the following theorem and of Proposition 2.2.

THEOREM 3.2. *Given $T: K \to K$ and $K_1, \ldots, K_N$ disjoint Cantor sets with $K = \bigcup_{i=1}^{N} K_i$, there exists $\widetilde{T}: K \to K$ such that*

$$\widetilde{T}(x) \in K_i \;\Leftrightarrow\; T(x) \in K_i,$$

and every orbit of $\widetilde{T}$ is finally periodic.

Proof. The proof is by induction on N. For $N = 1$ it suffices to take $\widetilde{T}(x)$ to be the identity. Suppose now that the assertion is valid for $N-1$: If for all $i = 1, \ldots, N$ there exists $x \in K_i$ such that $T(x) \in K_i$, then by the previous lemma the theorem is valid. Assume then that there exists some $1 \leq j \leq N$ such that $T(x) \notin K_j$ for all $x \in K_j$. Let $V_l, 1 \leq l \leq N-1$, be disjoint Cantor sets satisfying $\bigcup_{l=1}^{N-1} V_l = K_j$, and assume that, for each l, there exists i_l such that $V_l \subset K_j \cap T^{-1}(K_{i_l})$. Since we allow i_{l_1} to be equal to i_{l_2} even if $l_1 \neq l_2$, by Proposition 2.1 we can further assume that each V_l is nonempty.

Since T is surjective, the sets $T^{-1}(K_j)$ are nonempty. Let $U_l, 1 \leq l \leq N-1$, be again disjoint nonempty Cantor sets satisfying $\bigcup_{l=1}^{N-1} U_l = T^{-1}(K_j)$ and such that, for each l, there exists k_l with $U_l \subset K_{k_l} \cap T^{-1}(K_j)$.

For each $1 \leq l \leq N-1$, let h_l be a homeomorphism between U_l and V_l. Define $\widehat{T}: K \to K$ by

$$\widehat{T}(x) = \begin{cases} h_l(x) & \text{if } x \in U_l, \\ T(x) & \text{if } x \notin T^{-1}(K_j). \end{cases}$$

Let $g: K \setminus K_j \to K \setminus K_j$ be

$$g(x) = \begin{cases} T(x) & \text{if } x \notin U_l, \\ \widehat{T}^2(x) & \text{if } x \in U_l. \end{cases}$$

By the induction hypothesis, there exists $\widetilde{g}: K \setminus K_j \to K \setminus K_j$ such that, for $i = 1, \ldots, j-1, j+1, \ldots, N$,

$$\widetilde{g}(x) \in K_i \;\Leftrightarrow\; g(x) \in K_i,$$

and every orbit is finally periodic.

Finally, define $\widetilde{T}: K \to K$ by

$$\widetilde{T}(x) = \begin{cases} \widetilde{g}(x) & \text{if } x \notin K_j \cup T^{-1}(K_j), \\ \widehat{T}(x) & \text{if } x \in U_l, \\ \widetilde{g}(h_l^{-1}(x)) & \text{if } x \in V_l. \end{cases}$$

Since for all $x \notin K_j$, $\widetilde{g}(x)$ is equal to either $\widetilde{T}(x)$ or $\widetilde{T}^2(x)$, and all $\widetilde{g}$-orbits are finally periodic, we conclude that all $\widetilde{T}$-orbits are also finally periodic. ∎

4. Homeomorphisms. Theorem 1.2 is again a direct consequence of Proposition 2.2 and the following result:

THEOREM 4.1. *Let $T : K \to K$ be a homeomorphism, and let $K_1, \ldots, K_N$ be disjoint Cantor sets with $K = \bigcup_{i=1}^{N} K_i$. Then there exists a homeomorphism $\widetilde{T} : K \to K$ such that*

$$\widetilde{T}(x) \in K_i \;\Leftrightarrow\; T(x) \in K_i,$$

and the ω-limit of every orbit of $\widetilde{T}$ is a periodic orbit.

The proof of this theorem is also by induction and, similarly to Theorem 3.2, we analyze the next particular case (Lemma 4.2) in order to use it in the first case of the proof.

LEMMA 4.2. *Let $T \in \mathrm{Hom}(K)$ and $K_1, \ldots, K_N$ be disjoint Cantor sets with $K = \bigcup_{i=1}^{N} K_i$. Suppose that for every $1 \le i \le N$, at least one of the following properties is satisfied:*

(i) $T^{-1}(y) \in K_i$ *for any* $y \in K_i$,
(ii) $T(x) \in K_i$ *for any* $x \in K_i$.

Then there exists a homeomorphism $\widetilde{T} : K \to K$ such that

$$\widetilde{T}(x) \in K_i \;\Leftrightarrow\; T(x) \in K_i,$$

and every orbit of $\widetilde{T}$ converges to a fixed point.

Proof. Let K_{R_l} be the Cantor sets that do not satisfy property (i), with $l = 1, \ldots, r$ for $1 \le r \le N$, and K_{A_q} be the Cantor sets that do not satisfy property (ii), with $q = 1, \ldots, n$ for $1 \le n \le N$; let $\overline{K} = K \setminus ((\bigcup_{l=1}^{r} K_{R_l}) \cup (\bigcup_{q=1}^{n} K_{A_q}))$. Note that $N \ge q + r$, and if $K_i \subset \overline{K}$, then $T(K_i) = K_i$. Furthermore, since $\overline{K}$ is invariant, and $\bigcup_{l=1}^{r} K_{R_l}$ properly contains its image and $\bigcup_{q=1}^{n} K_{A_q}$ is properly contained in its image, we see that r is null if and only if so is n.

(1) For each l, q, define

$$W_{A_q}^{R_l} = \{x \in K_{R_l} : T(x) \in K_{A_q}\} \ne \emptyset, \qquad V_{R_l} = \{x \in K_{R_l} : T(x) \in K_{R_l}\}.$$

Note that $(\bigcup_{q=1}^{n} W_{A_q}^{R_l}) \cup V_{R_l} = K_{R_l}$, and that $T(V_{R_l}) = K_{R_l}$. By 2.3, for each $l \le r$, there exists a sequence of Cantor sets $\widetilde{K}_m^l \subset V_{R_l}$ and a point $p_l \in V_{R_l}$ such that $V_{R_l} = \bigcup_{m=1}^{\infty} \widetilde{K}_m^l \cup \{p_l\}$ with $\mathrm{diam}(\widetilde{K}_m^l) \to 0$.

Consider, for each $m \ge 2$, the homeomorphisms $h_m^l : \widetilde{K}_m^l \to \widetilde{K}_{m-1}^l$ and $\overline{h}^l : \widetilde{K}_1^l \to \bigcup_{q=1}^{n} W_{A_q}^{R_l}$.

Define $\widehat{T_{R_l}} : V_{R_l} \to K_{R_l}$ by

$$\widehat{T_{R_l}}(x) = \begin{cases} h_m^l(x) & \text{if } x \in \widetilde{K}_m^l,\ m \geq 2, \\ \overline{h}^l(x) & \text{if } x \in \widetilde{K}_1^l, \\ p_l & \text{if } x = p_l. \end{cases}$$

Note that $\widehat{T_{R_l}}^{m+1}(x) \notin V_{R_l}$ for $x \in \widetilde{K}_m^l$ with $x \neq p_l$.

(2) For each K_{A_q}, there exists a sequence of Cantor sets $\widehat{K}_m^q \subset K_{A_q}$ and a point $p_q \in K_{A_q}$ such that $K_{A_q} = \bigcup_{m=1}^{\infty} \widehat{K}_m^q \cup \{p_q\}$ with $\operatorname{diam}(\widehat{K}_m^q) \to 0$.

Consider, for each $m \geq 2$, the homeomorphism $h_m^q : \widehat{K}_{m-1}^q \to \widehat{K}_m^q$.

Define $\widehat{T_{A_q}} : K_{A_q} \to K_{A_q}$ by

$$\widehat{T_{A_q}}(x) = \begin{cases} h_m^q(x) & \text{if } x \in \widehat{K}_{m-1}^q,\ m \geq 2, \\ p_q & \text{if } x = p_q. \end{cases}$$

Note that, for all $x \in K_{A_q}$, $\widehat{T}_{A_q}(x)$ converges to the fixed point $p_q \in K_{A_q}$.

To define the homeomorphism in K we use the fact that, for each A_q with $q = 1, \ldots, n$ and $1 \leq n \leq N$, there exists a homeomorphism $h_{A_q} : W_{A_q}^{R_1} \cup W_{A_q}^{R_2} \cup \cdots \cup W_{A_q}^{R_l} \to \widetilde{K}_1^l \subset K_{A_q}$.

Finally, using (1) and (2), we can define $\widetilde{T} : K \to K$ by

$$\widetilde{T}(x) = \begin{cases} \widehat{T_{R_l}}(x) & \text{if } x \in V_{R_l},\ l = 1, \ldots, r, \\ \widehat{T_{A_q}}(x) & \text{if } x \in K_{A_q},\ q = 1, \ldots, n, \\ h_{A_q}(x) & \text{if } x \in K_{R_l} \setminus V_{R_l},\ l = 1, \ldots, r, \\ \widehat{T}(x) = x & \text{if } x \in \overline{K}. \ \blacksquare \end{cases}$$

Proof of Theorem 4.1. The proof is by induction on N. The case where $N = 1$ is trivial: take $\widetilde{T}$ to be the identity. Assume the result is true for $N - 1$. There are two possibilities: First, for each i, K_i is either forward or backward invariant. Second, there exists some j such that $T^{-1}(K_j) \neq K_j$ and $T(K_j) \neq K_j$. In the first possibility, the result follows from Lemma 4.2. So consider now the second possibility.

Let $U_0 = K_j \cup T^{-1}(K_j)$, and let U_l, $1 \leq l \leq N - 1$, be again disjoint nonempty Cantor sets satisfying $\bigcup_{l=0}^{N-1} U_l = T^{-1}(K_j)$ and such that, for each $l > 0$, there exists $k_l \neq j$ with $U_l \subset K_{k_l} \cap T^{-1}(K_j)$. Let $V_0 = U_0$, and let V_l, $1 \leq l \leq N - 1$, be disjoint Cantor sets satisfying $\bigcup_{l=0}^{N-1} V_l = K_j$ and such that, for each $l > 0$, there exists $i_l \neq 0$ with $V_l \subset K_j \cap T^{-1}(K_{i_l})$. Again, by Proposition 2.1 we can further assume that each U_l, V_l, $l > 0$, is nonempty, and V_0 is empty only if so is U_0.

For each $0 \leq l \leq N - 1$, let h_l be a homeomorphism between U_l and V_l. Define $\widehat{T} : K \to K$ by

$$\widehat{T}(x) = \begin{cases} h_l(x) & \text{if } \ x \in U_l, \\ T(x) & \text{if } x \notin T^{-1}(K_j). \end{cases}$$

The rest of the proof is identical to the proof of Theorem 3.2. ∎

Proof of Corollary 1.3. It is a well known fact in ergodic maximization theory that given an endomorphism T of a compact space and a continuous observable f, there is always at least one ergodic T-invariant measure which is f-maximizing. But if $\widetilde{T}$ is a transformation such that for every x either the orbit by $\widetilde{T}$ is finally periodic or the ω-limit of x is a periodic orbit, then it follows that any $\widetilde{T}$-invariant ergodic measure is supported on a periodic orbit. ∎

Acknowledgements. The first author was supported by Capes, the second author was supported by CNPq and the third author was partially supported by CNPq and FAPESP.

References

[AT] S. Addas-Zanata and F. Tal, *Support of maximizing measures for typical $\mathcal{C}^0$ dynamics on compact manifolds*, Discrete Contin. Dynam. Systems 26 (2010), 795–804.

[AGW] E. Akin, E. Glasner and B. Weiss, *Generically there is but one self homeomorphism of the Cantor set*, Trans. Amer. Math. Soc. 360 (2008), 3613–3630.

[AHK] E. Akin, M. Hurley and J. Kennedy, *Dynamics of topologically generic homeomorphisms*, Mem. Amer. Math. Soc. 164 (2003), no. 783.

[BGT] T. Batista, J. Gonschorowski and F. Tal, *Density of the set of endomorphisms with a maximizing measure supported on a periodic orbit*, Discrete Contin. Dynam. Systems 35 (2015), 3315–3326.

[BD] N. C. Bernardes Jr. and U. B. Darji, *Graph theoretic structure of maps of the Cantor space*, Adv. Math. 231 (2012), 1655–1680.

[C] G. Contreras, *Ground satates are generically a periodic orbit*, arXiv:1307.0559v2 (2014).

[J] O. Jenkinson, *Ergodic optimization*, Discrete Contin. Dynam. Systems 15 (2006), 197–224.

[TA1] F. Tal and S. Addas-Zanata, *Maximizing measures for endomorphisms of the circle*, Nonlinearity 21 (2008), 2347–2359.

[TA2] F. Tal and S. Addas-Zanata, *On maximizing measures of homeomorphisms on compact manifolds*, Fund. Math. 200 (2008), 145–159.

Tatiane Cardoso Batista,
Juliano dos Santos Gonschorowski
Universidade Tecnológica Federal do Paraná
85053-525 Guarapuava, Brazil
E-mail: tatianebatista@utfpr.edu.br
julianod@utfpr.edu.br

Fabio Armando Tal
Instituto de Matemática e Estatística
Universidade de São Paulo
05508-090 São Paulo, Brazil
E-mail: fabiotal@ime.usp.br

Received 14 October 2014;
in revised form 15 January 2015