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Multiplier operators on product spaces

by

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Abstract. The author proves the boundedness for a class of multiplier operators on product spaces. This extends a result obtained by Lung-Kee Chen in 1994.

1. Introduction. Fourier analysis and H^p-theory on product domains have recently been developed by many authors (for more detailed and complete treatment of these matters, the reader may consult e.g. S.-Y. Chang and R. Fefferman [2] and R. Fefferman [4]).

In 1987, Robert Fefferman gave a powerful theorem (see [4]), which effectively helps us study the boundedness of linear operators on the product spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Thanks to this theorem, it suffices to check the boundedness of the linear operators acting on H^p rectangle atoms, even though such atoms are not dense in the product domains (see counterexample of L. Carleson [1]). For convenience, we state Fefferman's theorem below:

THEOREM 1 (see [4]). *Let T be a bounded linear operator on $L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Suppose that if a is an $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ rectangle atom* $(0<p\le 1)$ *supported on R, then*

$$\int_{{}^{\mathrm{c}}R_r} |T(a)|^p(x_1,x_2)\,dx_1\,dx_2 \le Cr^{-\sigma} \quad \textit{for all } r\ge 2$$

and some fixed $\sigma>0$, where ${}^{\mathrm{c}}R_r$ denotes the complement of the r-fold enlargement of R. Then T is a bounded operator from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

Based on Fefferman's Theorem and some ideas from [6], Lung-Kee Chen proved the following theorem.

THEOREM 2 (see [3]). *Let $k=\lfloor n_1(1/p-1/2)\rfloor+1$, $l=\lfloor n_2(1/p-1/2)\rfloor+1$, $0<p\le 1$. Suppose $m\in C^k(\mathbb{R}^{n_1})\times C^l(\mathbb{R}^{n_2})$ and*

$$|\partial_\zeta^\alpha\partial_\eta^\beta m(\zeta,\eta)|\le C|\zeta|^{-|\alpha|}|\eta|^{-|\beta|} \quad \textit{for } |\alpha|\le k,\ |\beta|\le l.$$

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Then the multiplier operator T_m, defined on the Fourier transform side by $\widehat{T_m f}(\zeta,\eta) = m(\zeta,\eta)\widehat{f}(\zeta,\eta)$, maps $H^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ boundedly to $L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for $p \le q \le 2$.

REMARKS. (a) R. Fefferman and K. C. Lin [5] have obtained the above result for the case $p = 1$ under a weaker hypothesis:

$$\int\limits_{s_1<|\zeta|<2s_1}\ \int\limits_{s_2<|\eta|<2s_2} |\partial_\zeta^\alpha \partial_\eta^\beta m(\zeta,\eta)|^2\, d\zeta\, d\eta \le C s_1^{-2|\alpha|+n_1} s_2^{-2|\beta|+n_2}.$$

(b) It is worth to mention the theorem below, obtained by A. Nilsson [8].

THEOREM 3 (see [8]). *Assume that $m \in C^k(\mathbb{R}^n \setminus \{0\})$ and*

$$\int\limits_{\Delta_j} \sum_{|\alpha|\le k} |2^{j|\alpha|}\partial^\alpha m(\zeta)|^2\, d\zeta \le 2^{nj(2-q)/q}, \qquad j \in \mathbb{Z},$$

where k is the least integer $> n(2-q)/(2q)$, $1 \le q \le 2$, and $\Delta_j = \{\zeta \in \mathbb{R}^n : 2^{j-1} \le |\eta| \le 2^{j+1}\}$. Then the convolution by $K = \widehat{m}$ maps $H^1(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

The purpose of this note is to obtain the same result as in Theorem 2, but with a different hypothesis.

THEOREM 4. *Let m be a bounded function in $C^k(\mathbb{R}^{n_1}) \times C^l(\mathbb{R}^{n_2})$, where*

$$k = \lfloor n_1(1/p - 1/2)\rfloor + 1, \qquad l = \lfloor n_2(1/p-1/2)\rfloor + 1, \qquad 0 < p \le 1.$$

Suppose that

$$\int\limits_{\Delta_i\times\Delta_j} \sum_{|\alpha|\le k} \sum_{|\beta|\le l} |2^{i|\alpha|} 2^{j|\beta|} \partial_\zeta^\alpha \partial_\eta^\beta (\zeta^u \eta^v m(\zeta,\eta))|^2\, d\zeta\, d\eta \le C 2^{n_1 i} 2^{n_2 j} 2^{2i|u|} 2^{2j|v|},$$

where $\Delta_i = \{\zeta \in \mathbb{R}^{n_1} : 2^i \le |\zeta| \le 2^{i+1}\}$, and a similar definition for Δ_j,

$$\sup_{\zeta\in\mathbb{R}^{n_1}} \Big\{ \int\limits_{\Delta_j} \sum_{|\beta|\le l} |2^{j|\beta|} \partial_\eta^\beta (\eta^v m(\zeta,\eta))|^2\, d\eta \Big\} \le C 2^{n_2 j} 2^{2j|v|},$$

and

$$\sup_{\eta\in\mathbb{R}^{n_2}} \Big\{ \int\limits_{\Delta_i} \sum_{|\alpha|\le k} |2^{i|\alpha|} \partial_\zeta^\alpha (\zeta^u m(\zeta,\eta))|^2\, d\zeta \Big\} \le C 2^{n_1 i} 2^{2i|u|}.$$

Then the linear operator T defined by $\widehat{Tf}(\zeta,\eta) = m(\zeta,\eta)\widehat{f}(\zeta,\eta)$ maps $H^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ boundedly to $L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for $p \le q \le 2$.

Proof. Throughout the proof, C denotes a constant which is not necessarily the same each time it appears. The proof is an application of Theorem 1, and the idea of the proof is closely related to that of [3].

Let $a(x,y)$ denote an $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ $(0 < p \le 1)$ rectangle atom such that

(a) $a(x,y)$ is supported on a rectangle $R = I\times J$, where I and J are cubes in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively,

(b) $\|a\|_{L^2} \le R^{1/2-1/p} = |I|^{1/2-1/p}|J|^{1/2-1/p}$,

(c) $\int_I x^\alpha a(x,y)\,dx = 0$ for all $y\in J$ and $|\alpha|\le k$, and

(d) $\int_J y^\beta a(x,y)\,dy = 0$ for all $x\in I$ and $|\beta|\le l$,

where k and l are defined in Theorem 4.

By Theorem 1, it suffices to show that

$$\int\limits_{{}^{\mathrm{c}}R_r} |T(a)|^p(x_1,x_2)\,dx_1\,dx_2 \le Cr^\sigma \quad \text{for all } r\ge 2,$$

and some fixed $\sigma>0$.

Let ϕ be a smooth function on $\mathbb{R}$ such that $\widehat{\phi}(t)$ has compact support $\{t\in\mathbb{R} : 1/2\le|t|\le 2\}$ and $\sum_{j\in\mathbb{Z}}\widehat{\phi}(2^{-j}|t|)=1$ for all $t\ne 0$.

Let m_{ij} be defined by $m_{ij}(\zeta,\eta) = m(\zeta,\eta)\widehat{\phi}(2^{-i}|\zeta|)\widehat{\phi}(2^{-j}|\eta|)$ and let $\widehat{T_{ij}f}(\zeta,\eta) = m_{ij}(\zeta,\eta)\widehat{f}(\zeta,\eta) \equiv (K_{ij}*f)^\wedge(\zeta,\eta)$. It is clear that $Tf = \sum_{i,j}T_{ij}f$.

By Leibniz's formula and the hypothesis of Theorem 4, we have

$$\int \sum_{|\alpha|\le k}\sum_{|\beta|\le l} |2^{i|\alpha|}2^{j|\beta|}\partial_\zeta^\alpha\partial_\eta^\beta(\zeta^u\eta^v m_{ij}(\zeta,\eta))|^2\,d\zeta\,d\eta \le C2^{n_1 i}2^{n_2 j}2^{2i|u|}2^{2j|v|}, \tag{1}$$

$$\sup_{\zeta\in\mathbb{R}^{n_1}}\Big\{\int\limits_{\mathbb{R}^{n_2}}\sum_{|\beta|\le l}|2^{j|\beta|}\partial_\eta^\beta(\eta^v m_j(\zeta,\eta))|^2\,d\eta\Big\} \le C2^{n_2 j}2^{2j|v|}, \tag{2}$$

$$\sup_{\eta\in\mathbb{R}^{n_2}}\Big\{\int\limits_{\mathbb{R}^{n_1}}\sum_{|\alpha|\le k}|2^{i|\alpha|}\partial_\zeta^\alpha(\zeta^u m_i(\zeta,\eta))|^2\,d\zeta\Big\} \le C2^{n_1 i}2^{2i|u|}, \tag{3}$$

where $m_j(\zeta,\eta) = \sum_{i\in\mathbb{Z}} m_{ij}(\zeta,\eta)$ and $m_i(\zeta,\eta) = \sum_{j\in\mathbb{Z}} m_{ij}(\zeta,\eta)$.

We now partition ${}^{\mathrm{c}}R_r$ into three regions:

$${}^{\mathrm{c}}R_r^1 = \{(\zeta,\eta) : \zeta\in{}^{\mathrm{c}}I_r,\ \eta\in J_2\},$$
$${}^{\mathrm{c}}R_r^2 = \{(\zeta,\eta) : \zeta\in I_2,\ \eta\in{}^{\mathrm{c}}J_r\}, \qquad {}^{\mathrm{c}}R_r^3 = {}^{\mathrm{c}}R_r\setminus({}^{\mathrm{c}}R_r^1\cup{}^{\mathrm{c}}R_r^2).$$

We may assume that the center of the rectangle $R=I\times J$ is at the origin $(0,0)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. We then write

$$\begin{aligned} T_{ij}a(x,y) &= \int\limits_{I\times J} K_{ij}(x-x',y-y')a(x',y')\,dx'\,dy' \\ &= \int\limits_{I\times J}\Big\{K_{ij}(x-x',y-y') \\ &\qquad - \sum_{|\alpha|\le\lambda_1-1}\frac{1}{\alpha!}\,\partial_x^\alpha K_{ij}(x,y-y')(-x')^\alpha\Big\}a(x',y')\,dx'\,dy' \end{aligned} \tag{4}$$

$$= \lambda_1 \sum_{|\alpha|=\lambda_1} \frac{1}{\alpha!} \int_{I\times J}\int_0^1 (1-t)^{\lambda_1-1} \partial_x^\alpha K_{ij}(x-tx', y-y')$$

$$\times (-x')^\alpha a(x',y')\,dt\,dx'\,dy'$$

$$= \lambda_1 \sum_{|\alpha|=\lambda_1} \frac{1}{\alpha!} \int_0^1 \int_{I\times J} (1-t)^{\lambda_1-1} \Big\{ \partial_x^\alpha K_{ij}(x-tx', y-y')$$

$$- \sum_{|\beta|\le\lambda_2-1} \frac{1}{\beta!}\, \partial_y^\beta \partial_x^\alpha K_{ij}(x-tx', y)(-y')^\beta \Big\} (-x')^\alpha a(x',y')\,dy'\,dx'\,dt$$

$$= \lambda_1\lambda_2 \sum_{|\alpha|=\lambda_1} \frac{1}{\alpha!} \sum_{|\beta|=\lambda_2} \frac{1}{\beta!} \Big\{ \int_{I\times J} \int_0^1\int_0^1 (1-t)^{\lambda_1-1}\,(1-s)^{\lambda_2-1}$$

$$\times (-x')^\alpha(-y')^\beta \partial_y^\beta\partial_x^\alpha K_{ij}(x-tx', y-sy') a(x',y')\,ds\,dt\,dx'\,dy' \Big\}$$

where $0 \le \lambda_1 \le k$ and $0 \le \lambda_2 \le l$.

Notice that if $\lambda_1 = 0$ or $\lambda_2 = 0$, then we do not subtract the Taylor polynomials appearing in (4) above.

The integral on the right-hand side of (4) is dominated by

$$(5)\qquad \int_{I\times J}\int_0^1\int_0^1 |(-x')^\alpha(-y')^\beta \partial_y^\beta\partial_x^\alpha K_{ij}(x-tx', y-sy') a(x',y')|\,ds\,dt\,dx'\,dy'$$

$$\le \|a\|_{L^2} \Big\{ \int_{I\times J}\int_0^1\int_0^1 |(-x')^\alpha(-y')^\beta \partial_y^\beta\partial_x^\alpha K_{ij}(x-tx', y-sy')|^2\,ds\,dt\,dx'\,dy' \Big\}^{1/2}$$

$$\le |I|^{1/2-1/p+\lambda_1/n_1} |J|^{1/2-1/p+\lambda_2/n_2}$$

$$\times \Big\{ \int_{I\times J}\int_0^1\int_0^1 |\partial_y^\beta\partial_x^\alpha K_{ij}(x-tx', y-sy')|^2\,ds\,dt\,dx'\,dy' \Big\}^{1/2}$$

$$\equiv |I|^{1/2-1/p+\lambda_1/n_1} |J|^{1/2-1/p+\lambda_2/n_2} L_{ij}(x,y).$$

We now estimate $\int_{{}^{c}R_r^3} |L_{ij}(x,y)|^p\,dx\,dy$. By Hölder's inequality, we have

$$(6)\qquad \int_{{}^{c}R_r^3} |L_{ij}(x,y)|^p dx\,dy$$

$$= \int_{{}^{c}R_r^3} \{A(x)^{-kp/2}B(y)^{-lp/2}\}\{A(x)^{k/2}B(y)^{l/2}L_{ij}(x,y)\}^p\,dx\,dy$$

$$\le \Big\{ \int_{{}^{c}R_r^3} A(x)^{-kp/(2-p)} B(y)^{-lp/(2-p)} \Big\}^{(2-p)/2}$$

$$\times \Big\{ \int_{{}^{c}R_r^3} A(x)^k B(y)^l |L_{ij}(x,y)|^2 \, dx\, dy \Big\}^{p/2}.$$

Here $A(x) = 1 + 2^{2(i-1)}|x|^2$ and $B(y) = 1 + 2^{2(j-1)}|y|^2$. Observe that

$$\Big\{ \int_{{}^{c}R_r^3} A(x)^{-kp/(2-p)} B(y)^{-lp/(2-p)} \, dx\, dy \Big\}^{(2-p)/2} \tag{7}$$

$$\le C 2^{-ikp} 2^{-jlp} |I|^{-kp/n_1 + (2-p)/2} |J|^{-lp/n_2 + (2-p)/2} r^{-kp - lp + (n_1+n_2)(2-p)/2}$$

Meanwhile,

$$\int_{{}^{c}R_r^3} A(x)^k B(y)^l |L_{ij}(x,y)|^2 \, dx\, dy$$

$$= \int_{{}^{c}R_r^3} (1 + 2^{2(i-1)}|x|^2)^k (1 + 2^{2(j-1)}|y|^2)^l$$

$$\times \Big\{ \int_{I\times J} \int_0^1 \int_0^1 |\partial_y^\beta \partial_x^\alpha K_{ij}(x - tx', y - sy')|^2 \, ds\, dt\, dx'\, dy' \Big\} \, dx\, dy.$$

Since $(x', y') \in I \times J$, $(x,y) \in {}^{c}R_r^3$, and $0 \le s, t \le 1$, we have

$$|x| \le \frac{|x - tx'|}{|x| - |I|^{1/n_1}/2} \cdot |x| \le 2|x - tx'|.$$

Similarly, $|y| \le 2|y - sy'|$. Therefore, the above integral is dominated by

$$\int_{{}^{c}R_r^3} (1 + 2^{2i}|x - tx'|^2)^k (1 + 2^{2j}|y - sy'|^2)^l \tag{8}$$

$$\times \Big\{ \int_{I\times J} \int_0^1 \int_0^1 |\partial_y^\beta \partial_x^\alpha K_{ij}(x - tx', y - sy')|^2 \, ds\, dt\, dx'\, dy' \Big\} \, dx\, dy$$

$$\le \int_0^1 \int_0^1 \int_{I\times J} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} (1 + 2^{2i}|x - tx'|^2)^k (1 + 2^{2j}|y - sy'|^2)^l$$

$$\times |\partial_y^\beta \partial_x^\alpha K_{ij}(x - tx', y - sy')|^2 \, dx\, dy\, dx'\, dy'\, ds\, dt$$

$$= \int_0^1 \int_0^1 \int_{I\times J} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} (1 + 2^{2i}|x|^2)^k (1 + 2^{2j}|y|^2)^l$$

$$\times |\partial_y^\beta \partial_x^\alpha K_{ij}(x,y)|^2 \, dx\, dy\, dx'\, dy'\, ds\, dt$$

$$
\begin{aligned}
&\le |I|\cdot|J| \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} (1+2^{2i}|x|^2)^k(1+2^{2j}|y|^2)^l|\partial_y^\beta\partial_x^\alpha K_{ij}(x,y)|^2\,dx\,dy\\
&= |I|\cdot|J| \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \Bigg(\sum_{|u|\le k}\frac{k!}{(k-|u|)!u!}\,|2^{i|u|}x^u|^2\Bigg)\\
&\qquad\times\Bigg(\sum_{|v|\le l}\frac{l!}{(l-|v|)!v!}\,|2^{j|v|}y^v|^2\Bigg)|\partial_y^\beta\partial_x^\alpha K_{ij}(x,y)|^2\,dx\,dy\\
&\le C|I|\cdot|J|\sum_{|u|\le k}\sum_{|v|\le l}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} |2^{i|u|}x^u2^{j|v|}y^v\partial_y^\beta\partial_x^\alpha K_{ij}(x,y)|^2\,dx\,dy\\
&= C|I|\cdot|J|\sum_{|u|\le k}\sum_{|v|\le l}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \{2^{i|u|}2^{j|v|}\partial_\zeta^u\partial_\eta^v((\partial_\zeta^\alpha\partial_\eta^\beta K_{ij})^\wedge(\zeta,\eta))\}^2\,d\zeta\,d\eta\\
&= C|I|\cdot|J|\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\sum_{|u|\le k}\sum_{|v|\le l}|2^{i|u|}2^{j|v|}\partial_\zeta^u\partial_\eta^v(\zeta^\alpha\eta^\beta m_{ij}(\zeta,\eta))|^2\,d\zeta\,d\eta\\
&\le C|I|\cdot|J|2^{n_1i}2^{n_2j}2^{2i|\alpha|}2^{2j|\beta|} = C|I|\cdot|J|2^{n_1i}2^{n_2j}2^{2i\lambda_1}2^{2j\lambda_2}.
\end{aligned}
$$

Here the last inequality follows from (1), and the last equality from $|\alpha|=\lambda_1$ and $|\beta|=\lambda_2$ (see (4)). Combining inequalities (6)–(8) yields

$$
\begin{aligned}
\int_{{}^{\mathrm{c}}R_r^3}|L_{ij}(x,y)|^p\,dx\,dy &\le C|I|^{1-kp/n_1}|J|^{1-lp/n_2}2^{ip(n_1/2+\lambda_1-k)}\\
&\qquad\times 2^{jp(n_2/2+\lambda_2-l)}r^{-kp-lp+(n_1+n_2)(2-p)/2}.
\end{aligned}\tag{9}
$$

Consequently, if we combine (4)–(9), we have

$$
\begin{aligned}
&\int_{{}^{\mathrm{c}}R_r^3}|T_{ij}a|^p\,dx\,dy\\
&\qquad\le C\int\{|I|^{1/2-1/p+\lambda_1/n_1}|J|^{1/2-1/p+\lambda_2/n_2}|L_{ij}(x,y)|\}^p\,dx\,dy\\
&\qquad\le C|I|^{p/2+\lambda_1p/n_1-kp/n_1}|J|^{p/2+\lambda_2p/n_2-lp/n_2}\\
&\qquad\quad\times 2^{-ip(k-n_1/2-\lambda_1)}2^{-jp(l-n_2/2-\lambda_2)}r^{-kp-lp+(n_1+n_2)(2-p)/2}.
\end{aligned}\tag{10}
$$

Now if we set $\lambda_1=0=\lambda_2$; $\lambda_1=k$ and $\lambda_2=0$; $\lambda_1=0$ and $\lambda_2=l$; and $\lambda_1=k$, $\lambda_2=l$ in inequality (10), we obtain

$$
\begin{aligned}
\int_{{}^{\mathrm{c}}R_r^3}|T_{ij}a|^p dx\,dy &\le Cr^{-kp-lp+(n_1+n_2)((2-p)/2)}\\
&\quad\times\min\{|I|^{p/2-kp/n_1}|J|^{p/2-lp/n_2}\,2^{-ip(k-n_1/2)}2^{-jp(l-n_2/2)},\\
&\qquad |I|^{p/2}|J|^{p/2-lp/n_2}\,2^{ipn_1/2}\,2^{-jp(l-n_2/2)},\\
&\qquad |I|^{p/2-kp/n_1}\,|J|^{p/2}2^{-ip(k-n_1/2)}2^{jpn_2/2},\\
&\qquad |I|^{p/2}\,|J|^{p/2}2^{ipn_1/2}2^{jpn_2/2}\}.
\end{aligned}\tag{11}
$$

We now consider $\int_{{}^cR_r^2} |T_{ij}a|^p\,dx\,dy$. Let $T_j a = \sum_i T_{ij}a$. Observe that

$$\widehat{T}_j a(\zeta,\eta) = m(\zeta,\eta)\widehat{\phi}(2^{-j}|\eta|)\widehat{a}(\zeta,\eta) \equiv \widehat{K}_j(\zeta,\eta)\widehat{a}(\zeta,\eta).$$

We write

$$\begin{aligned}(12)\qquad \int_{{}^cR_r^2}\Big|\sum_i T_{ij}a\Big|^p\,dx\,dy &= \int_{{}^cR_r^2}|T_j a|^p\,dx\,dy\\ &\le \int_{{}^cR_r^2} B(y)^{-lp/2}[B(y)^{l/2}|T_j a|]^p\,dx\,dy\\ &\le \Big\{\int_{{}^cR_r^2} B(y)^{-lp/(2-p)}\,dx\,dy\Big\}^{(2-p)/2}\Big\{\int_{{}^cR_r^2} B(y)^l|T_j a|^2\,dx\,dy\Big\}^{p/2},\end{aligned}$$

where again $B(y) = 1 + 2^{2(j-1)}|y|^2$. It is clear that

$$\begin{aligned}(13)\qquad &\Big\{\int_{{}^cR_r^2} B(y)^{-lp/(2-p)}\,dx\,dy\Big\}^{(2-p)/2}\\ &\qquad\le C|I|^{(2-p)/2}\,|J|^{-lp/n_2+(2-p)/2}2^{-jlp}r^{n_2(2-p)/2-lp}.\end{aligned}$$

For the estimate of $\int_{{}^cR_r^2} B(y)^l|T_j a|^2\,dx\,dy$, we write

$$\begin{aligned}T_j a(x,y) &= K_j * a(x,y) = \int_{I\times J} K_j(x-x',y-y')a(x',y')\,dx'\,dy'\\ &= \int_{I\times J}\Big\{K_j(x-x',y-y')\\ &\qquad - \sum_{|\beta|\le\lambda_2-1}\frac{1}{\beta!}\,\partial_y^\beta K_j(x-x',y)(-y')^\beta\Big\}a(x',y')\,dx'\,dy'\\ &= \lambda_2\sum_{|\beta|=\lambda_2}\frac{1}{\beta!}\int_{I\times J}\int_0^1(1-s)^{\lambda_2-1}\partial_y^\beta K_j(x-x',y-sy')(-y')^\beta a(x',y')\,ds\,dx'\,dy'.\end{aligned}$$

Thus

$$\begin{aligned}(14)\qquad \int_{{}^cR_r^2} B(y)^l|T_j a|^2\,dx\,dy &= \int_{{}^cR_r^2} B(y)^l\Big|\lambda_2\sum_{|\beta|=\lambda_2}\frac{1}{\beta!}\int_0^1\int_{I\times J}(1-s)^{\lambda_2-1}\\ &\qquad\times\partial_y^\beta K_j(x-x',y-sy')(-y')^\beta a(x',y')\,dx'\,dy'\,ds\Big|^2\,dx\,dy\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{|\beta|=\lambda_2}\int_{{}^{c}R_r^2} B(y)^l \\
&\quad\times\Big|\int_0^1\int_{I\times J}(-y')^\beta\,\partial_y^\beta K_j(x-x',y-sy')a(x',y')\,dx'\,dy'\,ds\Big|^2\,dx\,dy \\
&\le C\sum_{|\beta|=\lambda_2}\Big\{\int_{|y|\ge r|J|^{1/n_2}/2} B(y)^l\Big[\int_0^1\int_J|(-y')^\beta| \\
&\quad\times\Big(\int_{|x|\le|I|^{1/n_1}/2}\Big|\int_I\partial_y^\beta K_j(x-x',y-sy')a(x',y')dx'\Big|^2\,dx\Big)^{1/2}\,dy'\,ds\Big]^2\,dy\Big\}
\end{aligned}
$$

(by Minkowski's inequality for integrals)

$$
\begin{aligned}
&\le C\sum_{|\beta|=\lambda_2}\Big\{\int_{|y|\ge r|J|^{1/n_2}/2} B(y)^l\Big[\int_0^1\int_J|(-y')^\beta| \\
&\quad\times\Big(\int_{\mathbb{R}^{n_1}}|\widehat{\partial_y^\beta K_j^{(1)}}(\zeta,y-sy')\widehat{a}^{(1)}(\zeta,y')|^2\,d\zeta\Big)^{1/2}\,dy'\,ds\Big]^2\,dy\Big\}
\end{aligned}
$$

(here $\widehat{a}^{(1)}$ means the Fourier Transform of a with respect to the first variable, and a similar definition for $\widehat{\partial_y^\beta K_j^{(1)}}$)

$$
\begin{aligned}
&\le C|J|^{2\lambda_2/n_2}\sum_{|\beta|=\lambda_2}\Big\{\int_{|y|\ge r|J|^{1/n_2}/2} B(y)^l\Big[\int_0^1\int_J\Big(\int_{\mathbb{R}^{n_1}}|\widehat{\partial_y^\beta K_j^{(1)}}(\zeta,y-sy') \\
&\qquad\qquad\times\widehat{a}^{(1)}(\zeta,y')|^2\,d\zeta\Big)^{1/2}\,dy'\,ds\Big]^2\,dy\Big\} \\
&\le C|J|^{2\lambda_2/n_2+1}\sum_{|\beta|=\lambda_2}\int_{|y|\ge r|J|^{1/n_2}/2} B(y)^l\int_0^1\int_J\int_{\mathbb{R}^{n_1}}|\widehat{\partial_y^\beta K_j^{(1)}}(\zeta,y-sy') \\
&\qquad\qquad\times\widehat{a}^{(1)}(\zeta,y')|^2\,d\zeta\,dy'\,ds\,dy \\
&\quad= C|J|^{2\lambda_2/n_2+1}\sum_{|\beta|=\lambda_2}\int_{|y|\ge r|J|^{1/n_2}/2}\int_0^1\int_J\int_{\mathbb{R}^{n_1}}|B(y)^{l/2}\widehat{\partial_y^\beta K_j^{(1)}}(\zeta,y-sy') \\
&\qquad\qquad\times\widehat{a}^{(1)}(\zeta,y')|^2\,d\zeta\,dy'\,ds\,dy \\
&\le C|J|^{2\lambda_2/n_2+1}\sum_{|\beta|=\lambda_2}\int_{|y|\ge r|J|^{1/n_2}/2}\int_0^1\int_J\int_{\mathbb{R}^{n_1}}|(1+2^{2j}|y-sy'|^2)^{l/2} \\
&\qquad\qquad\times\widehat{\partial_y^\beta K_j^{(1)}}(\zeta,y-sy')\widehat{a}^{(1)}(\zeta,y')|^2\,d\zeta\,dy'\,ds\,dy
\end{aligned}
$$

$$\leq C|J|^{2\lambda_2/n_2+1} \sum_{|\beta|=\lambda_2} \int_J \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} |(1+2^{2j}|y|^2)^l \widehat{\partial_y^\beta K_j^{(1)}}(\zeta,y) \times \widehat{a}^{(1)}(\zeta,y')|^2 \, dy \, d\zeta \, dy'.$$

But

$$\int_{\mathbb{R}^{n_2}} |(1+2^{2j}|y|^2)^l \widehat{\partial_y^\beta K_j^{(1)}}(\zeta,y)|^2 \, dy \tag{15}$$

$$\leq C \int_{\mathbb{R}^{n_2}} \sum_{|v|\leq l} |2^{j|v|} y^v \partial_y^\beta \widehat{K_j^{(1)}}(\zeta,y)|^2 \, dy$$

$$= C \int_{\mathbb{R}^{n_2}} \sum_{|v|\leq l} |2^{j|v|} \partial_\eta^v (\eta^\beta \widehat{K}_j(\zeta,\eta))|^2 \, d\eta$$

$$= C \int_{\mathbb{R}^{n_2}} \sum_{|v|\leq l} |2^{j|v|} \partial_\eta^v (\eta^\beta m_j(\zeta,\eta))|^2 \, d\eta.$$

Combining (14) and (15), we have

$$\int_{{}^cR_r^2} B(y)^l |T_j a|^2 \, dx \, dy$$

$$\leq C|J|^{2\lambda_2/n_2+1} \sum_{|\beta|=\lambda_2} \int_J \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \sum_{|v|\leq l} |2^{j|v|} \partial_\eta^v (\eta^\beta m_j(\zeta,\eta))|^2 \, d\eta \times |\widehat{a}^{(1)}(\zeta,y')|^2 \, d\zeta \, dy'$$

$$\leq C|J|^{2\lambda_2/n_2+1} \sum_{|\beta|=\lambda_2} 2^{n_2 j} 2^{2j|\beta|} \int_J \int_{\mathbb{R}^{n_1}} |\widehat{a}^{(1)}(\zeta,y')|^2 \, d\zeta \, dy'$$

$$\leq C|J|^{2\lambda_2/n_2+1} 2^{n_2 j} 2^{2j\lambda_2} \|a\|_{L^2}^2.$$

The next to last inequality follows from (2). So

$$\Big\{ \int_{{}^cR_r^2} B(y)^l |T_j a|^2 \, dx \, dy \Big\}^{p/2} \leq C|J|^{\lambda_2 p/n_2 + p/2} 2^{j n_2 p/2} 2^{j\lambda_2 p} \|a\|_{L^2}^p. \tag{16}$$

Combining (12), (13), and (16) yields

$$\int_{{}^cR_r^2} |T_j a|^p \, dx \, dy \leq C|J|^{-lp/n_2+p/2+\lambda_2 p/n_2} 2^{-jp(l-n_2/2+\lambda_2)} r^{n_2((2-p)/2)-lp}. \tag{17}$$

If we set $\lambda_2 = 0$ and $\lambda_2 = l$ in (17), we have

$$\int_{{}^cR_r^2} |T_j a|^p \, dx \, dy \leq C r^{n_2((2-p)/2)-lp} \min\{|J|^{p/2-lp/n_2} \, 2^{-jp(l-n_2/2)}, \ |J|^{p/2} \, 2^{jpn_2/2}\}. \tag{18}$$

Observe that if we let $T_i a = \sum_j T_{ij} a$, where

$$\widehat{T_i}a(\zeta,\eta) = m(\zeta,\eta)\widehat{\phi}(2^{-i}|\zeta|)\widehat{a}(\zeta,\eta) \equiv \widehat{K}_i(\zeta,\eta)\widehat{a}(\zeta,\eta)$$

and write

$$\int_{{}^{\mathrm{c}}R_r^1} \Big|\sum_j T_{ij} a\Big|^p \, dx\, dy = \int_{{}^{\mathrm{c}}R_r^1} |T_i a|^p \, dx\, dy,$$

then by symmetry, we obtain

$$\begin{aligned}(19)\qquad &\int_{{}^{\mathrm{c}}R_r^1} |T_i a|^p \, dx\, dy \\ &\qquad \le C r^{n_1((2-p)/2)-kp} \min\{|I|^{p/2-kp/n_1} 2^{-ip(k-n_1/2)},\ |I|^{p/2} 2^{ipn_1/2}\}.\end{aligned}$$

Finally, there exist $i_0, j_0 \in \mathbb{Z}$ such that $2^{i_0-1} \le |I|^{1/n_1} \le 2^{i_0}$ and $2^{j_0-1} \le |J|^{1/n_2} \le 2^{j_0}$. Because $k = \lfloor n_1(1/p - 1/2)\rfloor + 1$ and $l = \lfloor n_2(1/p - 1/2)\rfloor + 1$, there is a $\sigma > 0$ such that

$$\sigma \le \min\{kp - n_1(2-p)/2,\ lp - n_2(2-p)/2\}.$$

Thus inequalities (11), (18), and (19) respectively become

$$\begin{aligned}(20)\qquad \int_{{}^{\mathrm{c}}R_r^3} |T_{ij} a|^p \, dx\, dy \le C r^{-2\sigma} \min\{&2^{-(i+i_0)(kp-n_1p/2)}\, 2^{-(j+j_0)(lp-n_2p/2)}, \\ &2^{(i+i_0)n_1p/2}\, 2^{-(j+j_0)(lp-n_2p/2)},\ 2^{-(i+i_0)(kp-n_1p/2)}\, 2^{(j+j_0)n_2p/2}, \\ &2^{(i+i_0)n_1p/2}\, 2^{(j+j_0)n_2p/2}\}\end{aligned}$$

$$(21)\qquad \int_{{}^{\mathrm{c}}R_r^2} |T_j a|^p \, dx\, dy \le C r^{-\sigma} \min\{2^{-(j+j_0)(lp-n_2p/2)},\ 2^{(j+j_0)n_2p/2}\},$$

$$(22)\qquad \int_{{}^{\mathrm{c}}R_r^1} |T_i a|^p \, dx\, dy \le C r^{-\sigma} \min\{2^{-(i+i_0)(kp-n_1p/2)},\ 2^{(i+i_0)n_1p/2}\}.$$

Consequently, for $0 < p \le 1$ we have

$$\begin{aligned}\int_{{}^{\mathrm{c}}R_r} |Ta|^p \, dx\, dy &= \int_{{}^{\mathrm{c}}R_r^3} |Ta|^p \, dx\, dy + \int_{{}^{\mathrm{c}}R_r^2} |Ta|^p \, dx\, dy + \int_{{}^{\mathrm{c}}R_r^1} |Ta|^p \, dx\, dy \\ &\le \sum_{ij} \int_{{}^{\mathrm{c}}R_r^3} |T_{ij} a|^p \, dx\, dy + \sum_j \int_{{}^{\mathrm{c}}R_r^2} |T_j a|^p \, dx\, dy \\ &\quad + \sum_i \int_{{}^{\mathrm{c}}R_r^1} |T_i a|^p \, dx\, dy \\ &\le C r^{-\sigma},\end{aligned}$$

where the last inequality follows from (20)–(22). The proof is finished.

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