

Calkin algebras for Banach spaces with finitely decomposable quotients

by

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Abstract. For a Banach space X such that all quotients only admit direct decompositions with a number of summands smaller than or equal to n , we show that every operator T on X can be identified with an $n \times n$ scalar matrix modulo the strictly cosingular operators $SC(X)$. More precisely, we obtain an algebra isomorphism from the Calkin algebra $L(X)/SC(X)$ onto a subalgebra of the algebra of $n \times n$ scalar matrices which is triangularizable when X is indecomposable. From this fact we get some information on the class of all semi-Fredholm operators on X and on the essential spectrum of an operator acting on X .

1. Introduction. A Banach space is *indecomposable* if each complemented subspace of X is finite-dimensional or finite-codimensional. Gowers and Maurey [7] constructed a Banach space X_{GM} which is *hereditarily indecomposable*, i.e., every subspace of X_{GM} is indecomposable. This example gives a negative answer to several long-standing open problems; for example, X_{GM} contains no unconditional basic sequences. We refer to [1, 8] for other examples of hereditarily indecomposable spaces. We say that a Banach space is *n-decomposable* if it can be written as a direct sum of n infinite-dimensional closed subspaces.

Here we study the structure of the Calkin algebra $L(X)/SC(X)$, where $SC(X)$ stands for the *strictly cosingular* operators on an (infinite-dimensional) Banach space X which has n -decomposable quotients but has no $(n+1)$ -decomposable quotients. We denote by QD_n the class of such spaces and we call the class QD_1 the *quotient indecomposable* spaces. The space X_{GM} is quotient indecomposable [3]. If the dual space X^* is hereditarily decomposable, then X is quotient indecomposable. Moreover, the product of n copies of QD_1 spaces is a QD_n space [6, Theorem 1]. However, these examples do not exhaust the class QD_n (see Remark 3.3(c)).

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We show that for each complex space $X \in QD_n$, there exists an algebra isomorphism from $L(X)/SC(X)$ onto a subalgebra of the algebra $M_n(\mathbb{C})$ of $n \times n$ scalar matrices. Since the class $\Phi(X)$ of Fredholm operators on X coincides with the class of those $T \in L(X)$ such that the corresponding class in $L(X)/SC(X)$ is invertible, we identify $\Phi(X)$ with the set of all invertible elements of a subalgebra of $M_n(\mathbb{C})$ and we prove that the essential spectrum of an operator $T \in L(X)$ coincides with the spectrum of an $n \times n$ matrix.

We also show that a complex Banach space is n -decomposable if and only if there is an operator T on it whose essential spectrum admits a partition into n non-empty compact subsets. Applying this result to the case where X is an indecomposable QD_n space, we deduce that the Calkin algebra $L(X)/SC(X)$ can be identified with a subalgebra of the algebra of all upper triangular $n \times n$ matrices with constant diagonal.

For this purpose, we study the quotient spaces $L(X, Y)/SC(X, Y)$ when $X \in QD_n$ and $Y \in QD_m$. First we consider the case in which X and Y are products of QD_1 spaces. Then we apply the results to the general case. Whenever $Y \in QD_m$ is isomorphic to a quotient of $X \in QD_n$ we show that $L(X, Y)/SC(X, Y)$ can be identified with a subspace of the space of $m \times n$ scalar matrices. From this result we obtain a representation of the Calkin algebra $L(X)/SC(X)$ for X a QD_n space.

Some of the results and ideas behind this paper are a dual version of the results of Ferenczi [2], who considers hereditarily indecomposable spaces and strictly singular operators. However, our development is more closely related to operator theory. In this way our proofs are shorter and our presentation is more transparent. Moreover, we obtain a representation of $L(X)/SC(X)$ as an algebra of matrices, which gives additional information on the Calkin algebra. We observe that our scheme could be applied to the case considered in [2], improving in this way the results contained there.

Along the paper X, Y, Z, \dots will denote Banach spaces over the field of real or complex numbers. All statements are valid in both cases unless explicitly mentioned otherwise. X^* will stand for the dual space of X and $L(X, Y)$ for the (continuous linear) operators from X into Y . We set $L(X) = L(X, X)$, and I is the identity map. Given operators $S \in L(Z, X)$ and $T \in L(Z, Y)$, we define their product $S \times T \in L(Z, X \times Y)$ by $(S \times T)(z) := (S(z), T(z))$.

Subspaces will always be closed and infinite-dimensional, and quotients will be infinite-dimensional. We denote the quotient map from a Banach space X onto a quotient Y by Q_Y .

Given subsets $A \subseteq X$, $B \subseteq X^*$, we denote by A^\perp , ${}_\perp B$, their respective annihilators in X^* and X .

2. Quasi-maximal quotients. In this section we describe the quasi-maximal quotients of a Banach space. We also give some results that we will need later.

DEFINITION 2.1. We say that a quotient Y of X is *quasi-maximal* if there is no quotient Z of X such that $Q_Y \times Q_Z$ is surjective.

We say that the space X is *quotient indecomposable* if no quotient of X can be written as the direct sum of two subspaces.

Clearly, X is quotient indecomposable if and only if every quotient of X is quasi-maximal.

DEFINITION 2.2. An operator $T \in L(X, Y)$ is said to be *strictly cosingular*, $T \in SC(X, Y)$, if there is no quotient Z of Y so that $Q_Z T$ is surjective.

An operator $T \in L(X, Y)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X, Y)$, if $R(T)$ is closed and $Y/R(T)$ is finite-dimensional.

The following two results will be useful.

PROPOSITION 2.3 [5, Corollary 1]. *Suppose that Y is quotient indecomposable. Then for every X ,*

$$L(X, Y) = \Phi_-(X, Y) \cup SC(X, Y).$$

LEMMA 2.4. *Let M and N be subspaces of X . Suppose that $M + N$ is infinite-codimensional. Let $0 < \varepsilon < 1$. Then there exists a compact operator $K \in L(X)$ with $\|K\| < \varepsilon$ such that*

- (a) $I + K$ induces an isomorphism from a quotient of X/N onto a quotient of X/M ,
- (b) $I + K^*$ induces an isomorphism from a weak* closed subspace of M^\perp into N^\perp .

Proof. If $M + N$ is closed, then $(M + N)^\perp$ is a weak* closed subspace contained in both M^\perp and N^\perp , and $X/(M + N)$ is a quotient of both X/M and X/N . Thus, we get the result by setting $K = 0$.

Suppose that $M + N$ is not closed; then $M^\perp + N^\perp$ is also not closed [10, Theorem IV.4.8]. In particular, $M^\perp + N^\perp$ is not a direct sum. Thus

$$(1) \quad \inf\{\|f - g\| : f \in M^\perp, g \in N^\perp, \|f\| = \|g\| = 1\} = 0.$$

By (1) we can find $f_1 \in M^\perp, g_1 \in N^\perp$ such that $\|f_1\| = \|g_1\| = 1$ and $\|f_1 - g_1\| < \varepsilon/2^2$. Then we choose $x_1 \in X$ such that $\|x_1\| < 2$ and $f_1(x_1) = 1$.

Assume that we have selected $f_i \in M^\perp, g_i \in N^\perp$ and $x_i \in X$ such that

$$\|f_i\| = \|g_i\| = 1, \quad f_i(x_j) = \delta_{ij}, \quad \|x_i\| \|f_i - g_i\| < \frac{\varepsilon}{2^i}$$

for $i, j \leq n$. We set $F_n = \text{span}\{x_1, \dots, x_n\}$ and $G_n = \text{span}\{f_1, \dots, f_n\}$. Then

$$X = F_n \oplus {}_\perp G_n, \quad X^* = F_n^\perp \oplus G_n.$$

Let P_n be the projection on X with $\text{Ker } P_n = F_n$ and $R(P_n) = {}_{\perp}G_n$. Note that $(M+F_n)+N$ is not closed. Again by (1) we can choose $f_{n+1} \in (M+F_n)^{\perp}$ and $g_{n+1} \in N^{\perp}$ such that $\|f_{n+1}\| = \|g_{n+1}\| = 1$ and

$$\|f_{n+1} - g_{n+1}\| < \frac{\varepsilon}{2^{n+2}\|P_n\|}.$$

We take $y_{n+1} \in X$ such that $\|y_{n+1}\| < 2$ and $f_{n+1}(y_{n+1}) = 1$, and set $x_{n+1} = P_n(y_{n+1})$. Clearly $f_{n+1}(x_i) = \delta_{n+1,i}$ for $i = 1, \dots, n + 1$. Moreover,

$$\|x_{n+1}\| \|f_{n+1} - g_{n+1}\| < \frac{\varepsilon}{2^{n+1}}.$$

We define an operator $K: X \rightarrow X$ by $K(x) = \sum_{n=1}^{\infty} (g_n - f_n)(x)x_n$. Note that $\|K\| \leq \sum_{n=1}^{\infty} \|g_n - f_n\| \|x_n\| < \varepsilon$. Therefore $I + K$ is an isomorphism on X .

Set $M_0 = \bigcap_{n=1}^{\infty} \text{Ker } f_n$ and $N_0 = \bigcap_{n=1}^{\infty} \text{Ker } g_n$. Then we have $M \subseteq M_0$ and $N \subseteq N_0$, so $M_0^{\perp} \subseteq M^{\perp}$ and $N_0^{\perp} \subseteq N^{\perp}$. It is not difficult to check that $(I + K)^* f_n = g_n$, whence $\text{Ker } f_n = (I + K)(\text{Ker } g_n)$. Since $I + K$ is bijective, $(I + K)(\bigcap_{n=1}^{\infty} \text{Ker } g_n) = \bigcap_{n=1}^{\infty} (I + K)(\text{Ker } g_n)$ and $(I + K)N_0 = M_0$. Therefore $I + K$ induces an isomorphism from X/N_0 onto X/M_0 , so part (a) follows. Moreover $(I + K)^* M_0^{\perp} = N_0^{\perp}$ and part (b) follows as well. ■

DEFINITION 2.5. Let Y and Z be quotients of X . We say that Y and Z are $(I + K)$ -isomorphic if there is an operator $K: X \rightarrow X$ such that $I + K$ induces an isomorphism from Y onto Z .

PROPOSITION 2.6. *Let Y and Z be quotients of X and let $\varepsilon > 0$. Suppose that there is no quotient W of Z such that the map $Q_Y \times Q_W$ is surjective. Then Y and Z have $(I + K)$ -isomorphic quotients with $\|K\| < \varepsilon$. This is the case, in particular, when Y is quasi-maximal.*

Proof. Let $Y = X/M$ and $Z = X/N$. By the hypothesis, for any infinite-codimensional subspace L of X such that $L \supseteq N$, we have $M + L \neq X$. Thus $M + N$ is infinite-codimensional and Lemma 2.4 applies. ■

COROLLARY 2.7. *Let X be a quotient indecomposable space and let Y and Z be quotients of X . Then Y has a quotient isomorphic to a quotient of Z .*

PROPOSITION 2.8. *Let Y be a quasi-maximal quotient of X .*

(a) *An operator $S \in L(Z, X)$ is strictly cosingular if and only if $Q_Y S$ is. Therefore*

$$\dim L(Z, X)/SC(Z, X) \leq \dim L(Z, Y)/SC(Z, Y).$$

(b) *For each $T \in L(X, Z)$ there is a quotient Z_1 of Z and $T_1: Y \rightarrow Z_1$ such that $Q_{Z_1} T - T_1 Q_Y$ is strictly cosingular.*

Proof. (a) If $S: Z \rightarrow X$ is not strictly cosingular, then we have a quotient X_1 of X such that $Q_{X_1}S$ is surjective. As Y is a quasi-maximal quotient, Proposition 2.6 applies so that there exist $K \in L(X)$ with $\|K\| < 1/2$ and $(I + K)$ -isomorphic quotients Y_1 of Y and Z_1 of X_1 . Let $U \in L(Y_1, Z_1)$ be the isomorphism induced by $I + K$ and let Q denote the quotient map from Y onto Y_1 . Since $\|K\| < 1/2$, $I + K$ is an isomorphism. Thus $QQ_Y S = U^{-1}Q_{Z_1}(I + K)S$ is surjective and $Q_Y S$ is not strictly cosingular. For the second part, it is enough to observe that the correspondence $[S] \mapsto [Q_Y S]$ defines a one-to-one map from $L(Z, X)/SC(Z, X)$ into $L(Z, Y)/SC(Z, Y)$.

(b) Let $T \in L(X, Z)$. If T is strictly cosingular, then we take $Z_1 = Z$ and $T_1 = 0$. Suppose that T is not strictly cosingular. Then there is a quotient W of Z such that $Q_W T$ is surjective. Now Y is a quasi-maximal quotient, so Proposition 2.6 applies to $Q_W T$ and Q_Y , and we get quotients Y_1 of Y and Z_1 of W , a compact operator $K: X \rightarrow X$ and an isomorphism $\phi: Y_1 \rightarrow Z_1$ so that $Q_{Z_1}T(I + K) = \phi Q_{Y_1}$. Let q denote the quotient map from Y onto Y_1 . Set $T_1 = \phi q$. Then $Q_{Z_1}T - T_1 Q_Y = -Q_{Z_1}TK \in SC(X, Z_1)$. ■

3. Operators on fundamental QD_n spaces. Recall that X and Y are *quotient incomparable* if no quotient of X is isomorphic to a quotient of Y , and X is said to be *n-decomposable* if $X = X_1 \oplus \dots \oplus X_n$, with X_1, \dots, X_n subspaces of X .

DEFINITION 3.1. We say that a Banach space X is a *QD_n space*, $X \in QD_n$, if X has an n -decomposable quotient, but it has no $(n + 1)$ -decomposable quotient.

We say that X is *quotient indecomposable* if it is a QD_1 space.

DEFINITION 3.2. A QD_n space X is *fundamental* if it is a product $X = \prod_{i=1}^n X_i$ of quotient indecomposable spaces such that, for every $i, j = 1, \dots, n$, either $X_i = X_j$, or X_i, X_j are quotient incomparable.

For a fundamental QD_n space $X = \prod_{i=1}^n X_i$ including exactly k different factors, we define the *characteristic* of X as

$$\chi(X) := (n_1, \dots, n_k),$$

where $n_1 \leq \dots \leq n_k$ are the times that each space appears in the product, arranged in increasing order.

REMARK 3.3. (a) The definition of a fundamental QD_n space looks very restrictive, since we require some spaces to be equal, not just isomorphic. However, it will allow us to simplify some arguments.

(b) It follows readily from the definition that $X \in QD_n$ if and only if it has a quasi-maximal quotient isomorphic to a fundamental QD_n space.

(c) The product of n quotient indecomposable spaces is a QD_n space [6, Theorem 1]. However, these products do not exhaust the class of all QD_n

spaces: for each n there exists a hereditarily indecomposable space \widehat{X} which is a QD_n space.

Indeed, it was proved by Ferenczi [3, Appendix] that for each $n \in \mathbb{N}$, there exist complex spaces X_1, \dots, X_n such that every quotient of each X_i is hereditarily indecomposable, and a subspace Z of $X_1 \times \dots \times X_n$ so that

$$\widehat{X} := \frac{X_1 \times \dots \times X_n}{Z}$$

is hereditarily indecomposable. Moreover, \widehat{X} has a quotient which is n -decomposable. Thus \widehat{X} is a QD_n space.

Let $X = \prod_{j=1}^n X_j \in QD_n$ and $Y = \prod_{i=1}^m Y_i \in QD_m$ be fundamental spaces and let $\Psi: X \rightarrow Y$ be an operator. We will identify Ψ with the $m \times n$ matrix (Ψ_{ij}) , where $\Psi_{ij}: X_j \rightarrow Y_i$ is the i th coordinate of the restriction of Ψ to X_j . The operator Ψ is strictly cosingular if and only if every Ψ_{ij} is strictly cosingular; in this case we will say that (Ψ_{ij}) is a *strictly cosingular matrix*. In the case $n = m$, we say that Ψ is *diagonal* when $\Psi_{ij} = 0$ for $i \neq j$.

We denote by $M_{X,Y}$ the set of all $m \times n$ scalar matrices $A = (a_{ij}) \in M_{m,n}$ such that $a_{ij} = 0$ when X_j and Y_i are quotient incomparable. Sometimes we will write $M_{m,n}(\mathbb{C})$ to emphasize that we are considering the field of complex numbers.

If $\chi(X) = (n_1, \dots, n_k)$, then it is clear that

$$\dim M_{X,X} = n_1^2 + \dots + n_k^2.$$

Given a product space $X = \prod_{j=1}^n X_j$ and a permutation σ of $\{1, \dots, n\}$ such that $X_{\sigma(i)} = X_i$ for every i , we denote by V_σ the operator defined by $V_\sigma(x_1, \dots, x_n) := (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, and we say that V_σ is an *allowable permutation* of the factors of X .

If $A \in M_{m,p}$ we denote by $(A|0) \in M_{m,p+q}$ the matrix whose first p columns are those of A and the rest of the entries are 0.

PROPOSITION 3.4. *Let $X = \prod_{j=1}^n X_j \in QD_n$ and $Y = \prod_{i=1}^m Y_i \in QD_m$ be fundamental spaces. Suppose that Y is isomorphic to a quotient of X . Then there exists a quotient Z of Y which is a fundamental QD_m space and a surjective operator from X onto Z with matrix $(D|0)V_\sigma$, where D is an $m \times m$ diagonal surjective matrix and V_σ is an allowable permutation of the factors of X .*

Proof. Let $\Psi = (\Psi_{ij}): X \rightarrow Y$ be a surjective operator, where $\Psi_{ij} \in L(X_j, Y_i)$. Clearly, for each i there is at least one j so that Ψ_{ij} is not strictly cosingular. Hence Ψ_{ij} is lower semi-Fredholm, by Proposition 2.3. Thus, if we pass to a finite-codimensional subspace of Y_i , which we also denote by Y_i , then there exists a surjective map $q_i: X_j \rightarrow Y_i$. We take as Z the product of these new spaces Y_i .

Suppose that Ψ_{ij} and Ψ_{kl} are not strictly cosingular. Then the definition of a fundamental QD_n space and Corollary 2.7 imply that

$$(2) \quad Y_i = Y_k \quad \text{if and only if} \quad X_j = X_l.$$

Since the sum of a lower semi-Fredholm operator and a strictly cosingular operator is lower semi-Fredholm [11, Chap. V, Theorem 3.4], the operator from the product of copies of X_j into the product of copies of Y_i induced by Ψ is lower semi-Fredholm. Thus, taking into account that the product of k quotient indecomposable spaces is a QD_k space [6, Theorem 1], we conclude that the number of copies of Y_i in Y is smaller than or equal to the number of copies of X_j in X .

It is enough to select a suitable V_σ and take as D a diagonal matrix with a number of copies of q_i on the diagonal as necessary. ■

REMARK 3.5. Similarly to [2, Lemma 5], we could have proved a result stronger than Proposition 3.4. Namely, for each couple of fundamental spaces $X \in QD_n, Y \in QD_m$ and each surjective operator $\Psi: X \rightarrow Y$, there exist a quotient Z of Y which is a fundamental QD_m space, an automorphism U of Z and a permutation V_σ of the factors of X so that

$$UQ_Z\Psi V_\sigma = (D + K|B),$$

where D is a diagonal surjective $m \times m$ matrix, K is a strictly cosingular $m \times m$ matrix and B is an $m \times (n - m)$ matrix.

However, we do not need this result in its full generality.

REMARK 3.6. Let $Y = \prod_{i=1}^m Y_i \in QD_m$ and $X = \prod_{j=1}^n X_j \in QD_n$ be fundamental spaces. Suppose that there exists a surjective operator from X onto Y determined by a matrix $(D|0)$, where $D = \text{diag}(q_1, \dots, q_m)$ is an $m \times m$ diagonal matrix. Let $A \in M_{X,Y}$. Then $a_{ij} = 0$ whenever $X_i \neq X_j$. We will denote by DA the operator from X into Y with entries $a_{ij}q_i$.

Moreover, if $\chi(X) = (n_1, \dots, n_k)$ and $\chi(Y) = (m_1, \dots, m_l)$, then

$$\dim M_{X,Y} = m_1n_1 + \dots + m_l n_l.$$

The following result was essentially proved in [5]:

PROPOSITION 3.7. *Let Z be a complex quotient indecomposable space. Then for every surjective operator $q: Z \rightarrow X$ we can write*

$$L(Z, X) = \mathbb{C}q \oplus SC(Z, X).$$

Proof. Observe that there exists an isomorphism Ψ from a quotient Y of Z onto X so that $q = \Psi Q_Y$. By [5, Theorem 4], $L(Z, Y) = \mathbb{C}Q_Y \oplus SC(Z, Y)$. Let $T \in L(Z, X)$. Then $\Psi^{-1}T = \lambda Q_Y + K$ for some $\lambda \in \mathbb{C}$ and some $K \in SC(Z, Y)$. Thus $T = \lambda q + \Psi K$ with $\Psi K \in SC(Z, X)$. ■

PROPOSITION 3.8. *Let $X \in QD_n$ and $Y \in QD_m$ be complex fundamental spaces. Suppose that there exists a surjective operator from X onto Y with*

matrix $(D|0)$, where D is an $m \times m$ diagonal matrix. Then for every $T \in L(X, Y)$ there exist a matrix $A \in M_{X,Y}(\mathbb{C})$ and an operator $K \in SC(X, Y)$ such that

$$T = DA + K.$$

Proof. Let (T_{ij}) be the matrix associated to T . Set $D = \text{diag}(q_1, \dots, q_m)$. Then by Proposition 3.7, $T_{ij} = a_{ij}q_i + K_{ij}$, where $a_{ij} \in \mathbb{C}$ and $K_{ij} \in SC(X_j, Y_i)$. Moreover, by Remark 3.6, $a_{ij} = 0$ for $X_i \neq X_j$. In matrix terms, this is equivalent to the equality

$$(T_{ij}) = \text{diag}(q_1, \dots, q_m)(a_{ij}) + (K_{ij}),$$

where $A = (a_{ij}) \in M_{X,Y}(\mathbb{C})$ and $K = (K_{ij}) \in SC(X, Y)$. ■

4. Operators on QD_n spaces. Here we show that some properties of the operators on a QD_n space X can be derived from those obtained for a fundamental QD_n space. The key will be Theorem 4.8, in which we use a filter defined in the class of all fundamental quotients of X (see Definition 4.1 below) to identify matrices and operators, modulo the strictly cosingular operators.

DEFINITION 4.1. Let $X \in QD_n$. A *fundamental quotient* of X is a pair (\tilde{X}, U) , where \tilde{X} is a quotient of X and U is an isomorphism from \tilde{X} onto a fundamental QD_n space $\prod_{i=1}^n X_i$. We usually write $\tilde{X} \simeq \prod_{i=1}^n X_i$ or simply \tilde{X} .

Let $\tilde{X}_1 \simeq \prod_{i=1}^n X_i$ and $\tilde{X}_2 \simeq \prod_{i=1}^n Y_i$ be two fundamental quotients of X . We say that \tilde{X}_1 is a *diagonal quotient* of \tilde{X}_2 if there exists a diagonal surjective matrix from $\prod_{i=1}^n Y_i$ onto $\prod_{i=1}^n X_i$.

We write $\tilde{X}_1 \leq \tilde{X}_2$ when $\prod_{i=1}^n X_i$ is a diagonal quotient of $\prod_{i=1}^n Y_i$ up to a permutation of the factors.

REMARK 4.2. Every fundamental quotient (\tilde{X}, U) of X is quasi-maximal.

PROPOSITION 4.3. *Every $X \in QD_n$ has a fundamental quotient.*

Proof. By definition, X has a quotient $\prod_{i=1}^n X_i$ with X_i quotient indecomposable for every i . If X_i and X_j are not quotient incomparable, then passing to further quotients, we can suppose that they are isomorphic. If X_i and X_j are quotient incomparable, then so are their quotients. By an iterative process of passing to further quotients we get a product with factors either isomorphic or quotient incomparable. Applying suitable isomorphisms in the factors we get a fundamental QD_n space. ■

For the convenience of the reader we recall some basic facts about product spaces.

LEMMA 4.4. *Let M and $M_i, i = 1, \dots, n$, be subspaces of X . Let $q_i: X \rightarrow X/M_i$ be the quotient maps and $q = q_1 \times \dots \times q_n$. Then*

(a) *The map q is surjective if and only if $M_1^\perp + \dots + M_n^\perp$ is a direct sum.*

(b) *Suppose that q is surjective and that $M + M_i$ is infinite-codimensional for every i . Then there exist infinite-codimensional subspaces $L_i \supseteq M_i$ and a surjective map from X/M onto $\prod_{i=1}^n (X/L_i)$.*

Proof. (a) Observe that, via the natural identifications $(X/M_i)^* = M_i^\perp$, the dual map $q^*: M_1^\perp \times \dots \times M_n^\perp \rightarrow X^*$ is given by $q^*(f_1, \dots, f_n) = f_1 + \dots + f_n$. Now, q is surjective if and only if q^* is injective with closed range, and this amounts to $M_1^\perp + \dots + M_n^\perp$ being a direct sum.

(b) Since $q: X \rightarrow \prod_{i=1}^n X/M_i$ is surjective, $M_1^\perp + \dots + M_n^\perp$ is a direct sum. Thus there exists $0 < c < 1$ such that for every $f_i \in M_i^\perp, i = 1, \dots, n$, we have $\|\sum_{i=1}^n f_i\| \geq c \sum_{i=1}^n \|f_i\|$.

The sums $M + M_i$ are infinite-codimensional. Thus by Lemma 2.4 there exist subspaces $N_i \supseteq M$ and $L_i \supseteq M_i$, and compact operators $K_i: X \rightarrow X$ with $\|K_i\| < c/2$ so that $(I + K_i)^* L_i^\perp = N_i^\perp$.

Let $f_i \in L_i^\perp, i = 1, \dots, n$. Then

$$\left\| \sum_{i=1}^n (I + K_i^*)(f_i) \right\| \geq \left\| \sum_{i=1}^n f_i \right\| - \left\| \sum_{i=1}^n K_i^*(f_i) \right\| \geq (c/2) \sum_{i=1}^n \|f_i\|.$$

Thus the subspaces N_i^\perp form a direct sum. Hence $p: X \rightarrow \prod_{i=1}^n (X/N_i)$ is onto, by (a). As $N_i^\perp \subseteq M^\perp$, p admits a factorization $p = p_0 Q_{X/M}$ through X/M . Since $I + K_i$ induces an isomorphism U_i from X/N_i onto X/L_i and p is surjective, composing p_0 and $U_1 \times \dots \times U_n$ we get a surjective map from X/M onto $\prod_{i=1}^n (X/L_i)$. ■

PROPOSITION 4.5. *Let $X \in QD_n$ and $Y \in QD_m$. Suppose that Y is isomorphic to a quotient of X (so that $m \leq n$). Let \tilde{X} and \tilde{Y} be fundamental quotients of X and Y , respectively. Then Y has a fundamental quotient \tilde{Z} such that $\tilde{Z} \leq \tilde{Y}$ and there exists a surjective operator from \tilde{X} onto \tilde{Z} determined by a diagonal surjective matrix.*

Proof. Set $\tilde{X} \simeq \prod_{j=1}^n X_j$ and $\tilde{Y} \simeq \prod_{i=1}^m Y_i$. All spaces in the statement are isomorphic to quotients of X , so we can write $\tilde{X} = X/M$ and $Y_i \simeq X/N_i \simeq Y/L_i$ for suitable subspaces. Since $\tilde{X} = X/M$ is quasi-maximal, $M + N_i$ is infinite-codimensional for every i . So we can apply Lemma 4.4 to the onto operator $X \rightarrow \prod_{i=1}^m X/N_i$, and find a surjection $X/M \rightarrow \prod_{i=1}^m (X/V_i)$ for some $V_i \supseteq N_i$. Now $X/V_i \simeq Y/W_i = Z_i$ for some $W_i \supseteq L_i$, and we have a surjective operator $X/M \rightarrow \prod_{i=1}^m Z_i$. The surjective operator $\Lambda: Y \rightarrow \prod_{i=1}^m Y_i \rightarrow \prod_{i=1}^m Z_i$ factorizes through an isomorphism $U: \tilde{Z} = Y/\text{Ker } \Lambda \simeq \prod_{i=1}^m Z_i$ so that (\tilde{Z}, U) is a fundamental quotient of Y

and $\tilde{Z} \leq \tilde{Y}$. The map $q: \tilde{X} = X/M \rightarrow \prod_{i=1}^m Z_i \simeq \tilde{Z}$ is surjective and it is enough to apply Proposition 3.4. ■

COROLLARY 4.6. *Let $X \in QD_n$. Then the relation \leq defines a filter on the set of all fundamental quotients of X ; i.e., given two fundamental quotients \tilde{X}_1, \tilde{X}_2 , there exists a fundamental quotient \tilde{X}_3 such that $\tilde{X}_3 \leq \tilde{X}_1$ and $\tilde{X}_3 \leq \tilde{X}_2$.*

Proof. Taking $Y = X, \tilde{X} = \tilde{X}_1$ and $\tilde{Y} = \tilde{X}_2$ in Proposition 4.5, we get a fundamental quotient $\tilde{X}_3 \leq \tilde{X}_2$ and a diagonal surjective matrix from \tilde{X}_1 onto \tilde{X}_3 . Thus $\tilde{X}_3 \leq \tilde{X}_1$. ■

For a fundamental quotient $\tilde{X} \simeq \prod_{j=1}^n X_j$ we set $\chi(\tilde{X}) = \chi(\prod_{j=1}^n X_j)$.

COROLLARY 4.7. *Let X be a QD_n space and let \tilde{X}_1 and \tilde{X}_2 be fundamental quotients of X . Then $\chi(\tilde{X}_1) = \chi(\tilde{X}_2)$.*

Proof. By Corollary 4.6, it is enough to consider the case $\tilde{X}_1 \leq \tilde{X}_2$. After a permutation of the factors, we can suppose that there is a diagonal surjective matrix from $\tilde{X}_2 \simeq \prod_{j=1}^n X_j$ into $\tilde{X}_1 \simeq \prod_{j=1}^n Y_j$. Then, by (2), $X_i = X_j$ if and only if $Y_i = Y_j$, which yields the assertion. ■

THEOREM 4.8. *Let $X \in QD_n$ and $Y \in QD_m$ be complex spaces. Suppose that Y is isomorphic to a quotient of X . Let $\tilde{X} \simeq \prod_{j=1}^n X_j$ and $\tilde{Y} \simeq \prod_{i=1}^m Y_i$ be fundamental quotients of X and Y , respectively, so that there exists a surjective operator from \tilde{X} onto \tilde{Y} with matrix $(D|0)$, where D is a diagonal matrix. Then the map*

$$\psi: M_{\tilde{X}, \tilde{Y}}(\mathbb{C}) \rightarrow L(X, \tilde{Y})/SC(X, \tilde{Y})$$

defined by $\psi(A) = [DAQ_{\tilde{X}}]$ is bijective. In particular,

$$\dim L(X, Y)/SC(X, Y) \leq \dim M_{\tilde{X}, \tilde{Y}}.$$

Proof. Let $T \in L(X, Y)$. Since \tilde{X} is quasi-maximal, we can apply Proposition 2.8(b) with $Y = Y_i$. Then we obtain quotients $Q_{Z_i}: Y_i \rightarrow Z_i$ and operators $S_i: \tilde{X} \rightarrow Z_i$ such that $S_i Q_{\tilde{X}} - Q_{Z_i} T \in SC(X, Z_i)$. Passing to further quotients of Z_i , we can suppose that $\prod_{i=1}^m Z_i$ is a fundamental QD_m space. Let $\tilde{Z} \simeq \prod_{i=1}^m Z_i$ be the corresponding fundamental quotient. Taking $S = \prod_{j=1}^m S_j$ and $Q_{\tilde{Z}} \in L(\tilde{Y}, \tilde{Z})$, where $Q_{\tilde{Z}}(y_1, \dots, y_m) = (Q_{Z_1}(y_1), \dots, Q_{Z_m}(y_m))$, we obtain

$$(3) \quad SQ_{\tilde{X}} - Q_{\tilde{Z}}T \in SC(X, \tilde{Z}).$$

Applying Proposition 3.8 to \tilde{X}, \tilde{Z} and $D_1 = Q_{\tilde{Z}}D$, we have $S = D_1A + K_1$ with $A \in M_{\tilde{X}, \tilde{Y}}(\mathbb{C})$ and $K_1 \in SC(\tilde{X}, \tilde{Z})$. Putting this in (3), we get $D_1AQ_{\tilde{X}} - Q_{\tilde{Z}}T = Q_{\tilde{Z}}(DAQ_{\tilde{X}} - T) \in SC(X, \tilde{Z})$. As \tilde{Z} is quasi-maximal, $DAQ_{\tilde{X}} - T \in SC(X, \tilde{Y})$ by Proposition 2.8 and ψ is onto.

Now, if $DAQ_{\tilde{X}} \in SC(X, \tilde{Y})$, then $DA \in SC(\tilde{X}, \tilde{Y})$, i.e., $\phi_i a_{ij}$ is strictly cosingular for every i, j . As every ϕ_i is onto, it follows that $a_{ij} = 0$ for every i, j and $A = 0$, so ψ is injective. Hence

$$\dim L(X, Y)/SC(X, Y) \leq \dim L(X, \tilde{Y})/SC(X, \tilde{Y}) = \dim M_{\tilde{X}, \tilde{Y}}(\mathbb{C}),$$

by Proposition 3.8. ■

REMARK 4.9. Given complex spaces $X \in QD_n$ and $Y \in QD_m$, with Y isomorphic to a quotient of X , Proposition 4.5 ensures the existence of fundamental quotients \tilde{X}, \tilde{Y} under the hypothesis of Theorem 4.8.

REMARK 4.10. The map $\psi: M_{\tilde{X}, \tilde{Y}}(\mathbb{C}) \rightarrow L(X, \tilde{Y})/SC(X, \tilde{Y})$ that appears in Theorem 4.8 depends on the choice of the fundamental quotients \tilde{X} and \tilde{Y} . However, for different choices of fundamental quotients \tilde{X}_1, \tilde{X}_2 and \tilde{Y}_1, \tilde{Y}_2 , the subspaces of $m \times n$ matrices $M_{\tilde{X}_1, \tilde{Y}_1}(\mathbb{C})$ and $M_{\tilde{X}_2, \tilde{Y}_2}(\mathbb{C})$ coincide up to permutation.

Denote by $\psi_{\tilde{X}}$ the isomorphism ψ of Theorem 4.8 when $X = Y, \tilde{X} = \tilde{Y}$ and $D = I$.

THEOREM 4.11. Let $X \in QD_n$ be a complex space. Let $\tilde{X} \simeq \prod_{j=1}^n X_j$ be a fundamental quotient of X . Then the map

$$\Theta: L(X)/SC(X) \rightarrow M_{\tilde{X}, \tilde{X}}$$

defined by $\Theta[T] = \psi_{\tilde{X}}^{-1}[Q_{\tilde{X}}T]$ is an algebra isomorphism from $L(X)/SC(X)$ onto a subalgebra of $M_{\tilde{X}, \tilde{X}}$.

Proof. First of all, Θ is injective by Proposition 2.8(a) and Theorem 4.8. In view of Theorem 4.8, it just remains to prove the multiplicativity.

Let $[T_1], [T_2] \in L(X)/SC(X)$ with $\Theta[T_1] = A_1$ and $\Theta[T_2] = A_2$. Then $Q_{\tilde{X}}T_1 - A_1Q_{\tilde{X}} = S_1$ and $Q_{\tilde{X}}T_2 - A_2Q_{\tilde{X}} = S_2$ with $S_1, S_2 \in SC(X, \tilde{X})$. Thus

$$Q_{\tilde{X}}T_1T_2 - A_1A_2Q_{\tilde{X}} = A_1S_2 + S_1T_2 \in SC(X, \tilde{X})$$

and $\Theta([T_1][T_2]) = \Theta([T_1T_2]) = A_1A_2 = \Theta[T_1]\Theta[T_2]$. ■

5. Spectral theory on QD_n spaces. Recall that an operator $T \in L(X, Y)$ is said to be *semi-Fredholm* if $R(T)$ is closed and either $\text{Ker}(T)$ or $\dim Y/R(T)$ is finite. For a semi-Fredholm operator T we define the *index* by

$$\text{ind}(T) := \dim \text{Ker}(T) - \dim Y/R(T) \in \mathbb{Z} \cup \{\pm\infty\}.$$

It is well known that the index is a continuous map [10, Theorem IV.5.17].

The operator $T \in L(X, Y)$ is said to be *Fredholm* if it is semi-Fredholm with finite index. The *essential spectrum* of $T \in L(X)$ is defined as follows:

$$\sigma_e(T) = \{\lambda : \lambda I - T \text{ is not Fredholm}\}.$$

The following result follows from the fact that the strictly cosingular operators SC form a perturbation ideal in the sense of Heuser [9, Section 51].

PROPOSITION 5.1. *An operator $T \in L(X)$ is Fredholm if and only if $[T]$ is invertible in $L(X)/SC(X)$. In particular, $\sigma_e(T) = \sigma([T])$ for every $T \in L(X)$.*

PROPOSITION 5.2. *Let X be a QD_n space. Then $\text{ind}(T) = 0$ for every semi-Fredholm operator T on X .*

Proof. We suppose first that X is complex. In this case it is enough to observe that $\text{ind}(T - \lambda I)$ is a continuous discrete function in λ , that $\mathbb{C} \setminus \sigma_e(T)$ is connected by Proposition 5.1 and Theorem 4.11, and that $\text{ind}(T - \lambda I) = 0$ for $|\lambda| > \|T\|$.

Suppose now that X is a real space and that $X_{\mathbb{C}}$ denotes its complexification. Then $X_{\mathbb{C}}$ is isomorphic to $X \oplus X$ as a real space, thus it is a real QD_{2n} space [6, Theorem 2]. As every complex quotient of $X_{\mathbb{C}}$ is also a real quotient, $X_{\mathbb{C}}$ is a complex QD_m space for some $m \leq 2n$. If $T_{\mathbb{C}}$ is the complexification of a semi-Fredholm operator T , then $T_{\mathbb{C}}$ is also semi-Fredholm and $\text{ind}(T_{\mathbb{C}}) = \text{ind}(T)$. By the first part $\text{ind}(T_{\mathbb{C}}) = 0$, so $\text{ind}(T) = 0$. ■

COROLLARY 5.3. *Suppose that X is a QD_n space for some n . Then X is isomorphic neither to any of its proper subspaces nor to any of its proper quotients.*

Proof. Let $Q: X \rightarrow Y$ be a quotient and let $U: Y \rightarrow X$ be an isomorphism. Then UQ is a semi-Fredholm operator on X . By Proposition 5.2, UQ is Fredholm of index 0. As UQ is surjective, $\text{Ker } Q = 0$ and $Q = I$. The proof for the case of subspaces is analogous. ■

The following results are a consequence of Theorem 4.11.

PROPOSITION 5.4. *Let X be a complex QD_n space and let $T \in L(X)$. Then*

- (a) $|\sigma_e(T)| \leq n$.
- (b) T is not Fredholm if and only if there exists a non-strictly cosingular operator $U \in L(X)$ such that $TU \in SC(X)$.

Proof. (a) It is enough to observe that $\sigma_e(T)$ coincides with $\sigma([T])$ by Proposition 5.1, which is the set of eigenvalues of the scalar matrix $\Theta[T]$ by Theorem 4.11.

(b) If T is Fredholm, then $[T] \in L(X)/SC(X)$ is invertible, and $TU \in SC(X)$ implies $U \in SC(X)$.

Conversely, suppose that T is not Fredholm and $[T] \neq 0$. Let p be the minimal polynomial of $[T]$. Since $[T]$ is not invertible, it follows that $p(0) = 0$ and there exists a polynomial q with $\text{deg } q = \text{deg } p - 1$ such that $Tq(T) =$

$p(T)$. By the definition of a minimal polynomial, $U = q(T)$ cannot be strictly cosingular. Since $TU = p(T) \in SC(X)$, the result is proved. ■

For a complex QD_n space X , Theorem 4.11 allows us to identify the Calkin algebra $L(X)/SC(X)$ with a subalgebra of $M_{\tilde{X}, \tilde{X}}$. The next result provides additional information on such subalgebras.

THEOREM 5.5. *Let $n \in \mathbb{N}$. A complex space X is n -decomposable if and only if there exists an operator $T \in L(X)$ such that $|\sigma_e(T)| = n$.*

Proof. Suppose that $X = X_1 \oplus \dots \oplus X_n$. Then

$$T(x_1, x_2, \dots, x_n) := (x_1, 2x_2, \dots, nx_n)$$

defines an operator on $T \in L(X)$ such that $\sigma_e(T) = \{1, 2, \dots, n\}$.

Conversely, let $T \in L(X)$ be such that $\sigma_e(T) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. By [4, Theorem V.1.8], $\lambda I - T$ is invertible on $\mathbb{C} \setminus \{\lambda_1, \dots, \lambda_n\}$ with the possible exceptions of isolated points.

We select closed simple curves C_1, \dots, C_n on $\mathbb{C} \setminus \sigma(T)$ which do not intersect so that each λ_i is in the interior of C_i and every point in $\sigma(T)$ is contained in the interior of C_j for some j .

The analytic operational calculus [12, Section V.8] allows us to define

$$P_i := \int_{C_i} (\lambda I - T)^{-1} d\lambda, \quad i = 1, \dots, n.$$

Then each P_i is a projection and $X = R(P_1) \oplus \dots \oplus R(P_n)$ [12, Theorem V.9.1]. Thus, X is n -decomposable. ■

REMARK 5.6. Clearly, the proof of Theorem 5.5 shows that a complex Banach space is n -decomposable if and only if there exists an operator $T \in L(X)$ such that $\sigma_e(T)$ has n components, i.e., $\sigma_e(T)$ admits a partition into n non-empty compact subsets.

The following example shows that in Proposition 5.4 it is not enough to suppose X indecomposable in order to get $|\sigma_e(T)| = 1$ for every $T \in L(X)$.

EXAMPLE [8, (4.2)]. There exists a complex indecomposable space X and an operator $S \in L(X)$ such that $\sigma_e(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Moreover $\text{ind}(S) = -1$.

An interesting question is to describe the subalgebras of $M_n(\mathbb{C})$ which can be identified with the Calkin algebra $L(X)/SC(X)$ for some QD_n space X , as in Theorem 4.11.

In the case of an indecomposable QD_n space X , Theorem 5.5 implies that $|\sigma_e(T)| = 1$ for every $T \in L(X)$. Therefore, $L(X)/SC(X)$ can be identified by Theorem 4.11 with a subalgebra \mathcal{A} of $M_n(\mathbb{C})$ such that each matrix in \mathcal{A} has only one eigenvalue. A trivial example of such a subalgebra is $\mathcal{A} = \{\lambda I : \lambda \in \mathbb{C}\}$. A second example is the algebra of all upper triangular

matrices in $M_n(\mathbb{C})$ with constant diagonal. We denote this algebra by \mathcal{U}_n . Note that the first example is a subalgebra of \mathcal{U}_n . Let us see that the Kolchin Theorem on unipotent algebraic groups allows us to show that up to a change of basis all the subalgebras \mathcal{A} are of this kind.

We recall that a matrix $A \in M_n(\mathbb{C})$ is said to be *unipotent* if its sole eigenvalue is 1, i.e. if its characteristic polynomial is $q_A(x) = (x - 1)^n$.

THEOREM 5.7. *Let X be an indecomposable QD_n space. Then the Calkin algebra $L(X)/SC(X)$ can be identified with a subalgebra of \mathcal{U}_n .*

Proof. Let \mathcal{A} be the subalgebra of $M_n(\mathbb{C})$ identified with $L(X)/SC(X)$ by Theorem 4.11 and let G be the invertible elements in \mathcal{A} . We claim that the set G_u of all unipotent elements of G is a subgroup.

Each matrix $A \in \mathcal{A}$ has a unique eigenvalue, which we denote by $\alpha(A)$. Thus, $q_A(x) = (x - \alpha(A))^n$ is the characteristic polynomial of A and it is easy to see that

$$\alpha(A) = \frac{1}{n} \text{trace}(A), \quad \alpha(A)^n = \det(A).$$

Therefore the assignment $A \mapsto \alpha(A)$ defines a continuous map α from G into \mathbb{C} satisfying $(\alpha(A)\alpha(B)\alpha(AB)^{-1})^n = \det(A)\det(B)\det(AB)^{-1} = 1$. This implies that the function $f: G \times G \rightarrow \mathbb{C}$, defined by

$$f(A, B) = \alpha(A)\alpha(B)\alpha(AB)^{-1},$$

takes its values in the finite set of n th roots of unity. The function f is continuous and $G \times G$ is connected, so $f(G \times G)$ must be a point. As $f(I, I) = 1$, it follows that $\alpha(A)\alpha(B) = \alpha(AB)$ for every A, B in G , and the claim is proved.

Since G_u is a subgroup of G , we can apply the Kolchin Theorem [13, Theorem 8.2] to G_u , and we deduce that there exists an invertible matrix $B \in M_n(\mathbb{C})$ such that $B^{-1}G_uB$ is a subgroup of the group U_n of all upper triangular unipotent matrices in $M_n(\mathbb{C})$. The algebras generated by G_u and U_n in $M_n(\mathbb{C})$ are \mathcal{A} and \mathcal{U}_n , respectively. Thus, $B^{-1}\mathcal{A}B \subseteq \mathcal{U}_n$, as we wanted to prove. ■

References

- [1] S. A. Argyros and V. Felouzis, *Interpolating hereditarily indecomposable Banach spaces*, J. Amer. Math. Soc. 13 (2000), 243–294.
- [2] V. Ferenczi, *Hereditarily finitely decomposable spaces*, Studia Math. 123 (1997), 135–149.
- [3] —, *Quotient hereditarily indecomposable Banach spaces*, Canad. J. Math. 51 (1999), 566–584.
- [4] S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, 1966.

- [5] M. González and J. M. Herrera, *Operators on quotient indecomposable spaces*, Publ. Math. Debrecen 59 (2001), 271–288.
- [6] —, —, *Finitely decomposable Banach spaces and the three-space property*, Arch. Math. (Basel), to appear.
- [7] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.
- [8] —, —, *Banach spaces with small spaces of operators*, Math. Ann. 307 (1997), 543–568.
- [9] H. G. Heuser, *Functional Analysis*, Wiley, 1982.
- [10] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1976.
- [11] D. Przeworska-Rolewicz and S. Rolewicz, *Equations in Linear Spaces*, PWN–Polish Sci. Publ., 1968.
- [12] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, Wiley, 1980.
- [13] W. C. Waterhouse, *Introduction to Affine Group Schemes*, Springer, 1979.

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