STUDIA MATHEMATICA 161 (1) (2004)

Factorization of unbounded operators on Köthe spaces

by

T. Terzioğlu (Istanbul), M. Yurdakul (Ankara)
and V. Zahariuta (Istanbul)

Abstract. The main result is that the existence of an unbounded continuous linear operator T between Köthe spaces $\lambda(A)$ and $\lambda(C)$ which factors through a third Köthe space $\lambda(B)$ causes the existence of an unbounded continuous quasidiagonal operator from $\lambda(A)$ into $\lambda(C)$ factoring through $\lambda(B)$ as a product of two continuous quasidiagonal operators. This fact is a factorized analogue of the Dragilev theorem [3, 6, 7, 2] about the quasidiagonal characterization of the relation $(\lambda(A), \lambda(B)) \in \mathcal{B}$ (which means that all continuous linear operators from $\lambda(A)$ to $\lambda(B)$ are bounded). The proof is based on the results of [9] where the bounded factorization property $\mathcal{BF}$ is characterized in the spirit of Vogt's [10] characterization of $\mathcal{B}$. As an application, it is shown that the existence of an unbounded factorized operator for a triple of Köthe spaces, under some additonal asumptions, causes the existence of a common basic subspace at least for two of the spaces (this is a factorized analogue of the results for pairs [8, 2]).

1. Introduction. We denote by $\lambda(A)$ the Köthe space defined by the matrix $A = (a_i^p)$, and by (e_n) the canonical basis of $\lambda(A)$. For a mapping $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence (t_n) of scalars the operator $D : \lambda(A) \to \lambda(B)$ defined by $D(e_n) = t_n\, e_{\sigma(n)}$, $n \in \mathbb{N}$, is called *quasidiagonal*. Dragilev [3] proved that the existence of an unbounded continuous linear operator from $\lambda(A)$ to $\lambda(B)$, where both spaces are assumed to be nuclear, implies the existence of a continuous unbounded quasidiagonal operator from $\lambda(A)$ to $\lambda(B)$ (cf. [6, 7]). This result has recently been generalized by Djakov and Ramanujan [2] by omitting the nuclearity assumption.

We recall that the closed linear span of a subbasis (e_{i_n}) is called a *basic subspace* of a Köthe space. If $\lambda(A)$ and $\lambda(B)$ have a common basic subspace, then it is easy to construct a continuous linear operator mapping $\lambda(A)$ into $\lambda(B)$, which is unbounded unless the common basic subspace is a Banach space. Under certain conditions on $\lambda(A)$ and $\lambda(B)$ the converse of this trivial fact is also true. Namely, if both spaces are nuclear, Nurlu and Terzioğlu [8]

2000 *Mathematics Subject Classification*: 46A03, 46A13, 46A04, 46A20, 46A32, 47L05.
Key words and phrases: locally convex spaces, unbounded operators, Köthe spaces, bounded factorization property.

proved that the existence of an unbounded continuous linear operator $T : \lambda(A) \to \lambda(B)$ implies, under some additional conditions, the existence of a common basic subspace of $\lambda(A)$ and $\lambda(B)$; this result was generalized by Djakov and Ramanujan in [2] to the non-nuclear case. In these works Dragilev's theorem plays a crucial role.

It was discovered in [13, 14] that if the matrices A and B satisfy the conditions d_2, d_1, respectively, then every continuous linear operator from $\lambda(A)$ into $\lambda(B)$ is bounded. This phenomenon was studied extensively by many authors; the most comprehensive result is due to Vogt [10], where all pairs of Fréchet spaces with this property are characterized. Terzioğlu and Zahariuta [9] characterized those triples (X, Y, Z) of Fréchet spaces such that each continuous linear operator $T : X \to Z$ which factors through Y is automatically bounded.

The aim of the present work is to prove a factorization analogue of Dragilev's theorem [3] and its generalization [2]. Namely, we prove that if there is an unbounded continuous linear operator $T : \lambda(A) \to \lambda(C)$ which factors through $\lambda(B)$, then, in fact, there exists an unbounded continuous quasidiagonal operator $D : \lambda(A) \to \lambda(C)$ that factors through $\lambda(B)$ as a product of two continuous quasidiagonal operators. As an application, similarly to [8, 2], we show that the existence of an unbounded factorized operator for a triple of Köthe spaces causes that, under some additional conditions, these spaces (or at least two of them) have a common basic subspace.

2. Bounded factorization property and quasidiagonal operators. We denote by $L(X, Y)$ and $LB(X, Y)$ the spaces of all continuous linear operators and of all bounded linear operators from the locally convex space X into the locally convex space Y. If for each $S \in L(X, Y)$ and $R \in L(Y, Z)$ we have $T = RS \in LB(X, Z)$, we say (X, Y, Z) has the *bounded factorization property* and write $(X, Y, Z) \in \mathcal{BF}$ ([9]).

Dealing with several Fréchet spaces we always use the same notation $\{|\cdot|_p : p \in \mathbb{N}\}$ for a system of seminorms defining their topologies and $\{|\cdot|_p^* : p \in \mathbb{N}\}$ for the corresponding system of polar norms in the dual spaces. For any operator $T \in L(E, F)$ we consider the operator seminorms

$$|T|_{p,q} = \sup\{|Tx|_p : |x|_q \le 1\}, \quad p, q \in \mathbb{N},$$

which may take the value $+\infty$. In particular, for any one-dimensional operator $T = x' \otimes y$, $x' \in E'$, $y \in F$, we have $|T|_{p,q} = |x'|_q^* \cdot |y|_p$.

Dealing with a Köthe space $\lambda(A)$ we always assume that the matrix $A = (a_i^p)$ satisfies the condition

$$a_i^p \le a_i^{p+1}, \quad i, p \in \mathbb{N}. \tag{1}$$

An operator $T \in L(\lambda(A), \lambda(B))$ is *quasidiagonal* if $T(e_i) = t_i e_{\tau(i)}$, $i \in \mathbb{N}$, for some map $\tau : \mathbb{N} \to \mathbb{N}$ and scalar sequence (t_i). We denote by $Q(A,B)$ the set of all quasidiagonal operators and by $Q_\tau(A,B)$ its subset corresponding to the map τ. We note that $Q_\tau(A,B)$ is a subspace of $L(\lambda(A),\lambda(B))$ whereas $Q(A,B)$ is only a subset.

Our aim is to prove the following characterization of the bounded factorization property for triples of Köthe spaces in terms of quasidiagonal operators, which is a natural generalization of Dragilev's theorem ([3, 2]).

THEOREM 1. *We have $(\lambda(A),\lambda(B),\lambda(C)) \in \mathcal{BF}$ if and only if for each $S \in Q(A,B)$ and $R \in Q(B,C)$ the quasidiagonal operator $T = RS$ is bounded.*

The proof will be given in Section 3 after some intermediate results. In what follows we will use the following result from [9].

PROPOSITION 2. *We have $(\lambda(A),\lambda(B),\lambda(C)) \in \mathcal{BF}$ if and only if for each non-decreasing map $\pi : \mathbb{N} \to \mathbb{N}$ there is $r \in \mathbb{N}$ such that for every $q \in \mathbb{N}$ there exists $n = n(q) \in \mathbb{N}$ so that the inequality*

$$\frac{c_i^q}{a_j^r} \le n \max_{p=1,\dots,n} \left\{ \frac{b_\nu^p}{a_j^{\pi(p)}} \right\} \cdot \max_{p=1,\dots,n} \left\{ \frac{c_i^p}{b_\nu^{\pi(p)}} \right\} \tag{2}$$

holds for all $(i,j,\nu) \in \mathbb{N}^3$.

Given two Fréchet spaces E and F and a map $\pi : \mathbb{N} \to \mathbb{N}$, we consider the Fréchet space

$$L_\pi(E,F) := \{T \in L(E,F) : |T|_{p,\pi(p)} < \infty,\ p \in \mathbb{N}\}$$

with the topology generated by the system of seminorms $\{|\cdot|_{p,\pi(p)} : p \in \mathbb{N}\}$.

We note that, in the case of Köthe spaces, the intersection

$$Q_\sigma^\pi(A,B) := Q_\sigma(A,B) \cap L_\pi(\lambda(A),\lambda(B))$$

is a closed subspace of $L_\pi(\lambda(A),\lambda(B))$. Fix σ, ϱ, and π and assume that for each $S \in L_\sigma(A,B)$, $R \in L_\varrho(B,C)$ the composition RS is bounded. If we apply Lemma 2.1 from [9] to the bilinear map

$$\theta : Q_\sigma^\pi(A,B) \times Q_\varrho^\pi(B,C) \to LB(\lambda(A),\lambda(C))$$

which simply sends each (S,R) to RS, we obtain the following result.

PROPOSITION 3. *Let σ and ϱ be two maps of $\mathbb{N}$ into $\mathbb{N}$. If for each $S \in Q_\sigma(A,B)$ and $R \in Q_\varrho(B,C)$ the composition RS is bounded, then for each $\pi : \mathbb{N} \to \mathbb{N}$ there is $r \in \mathbb{N}$ such that for every $q \in \mathbb{N}$ there exists $n = n(q) \in \mathbb{N}$ such that the inequality*

$$\frac{c_{\varrho(\sigma(j))}^q}{a_j^r} \le n \max_{p=1,\dots,n} \left\{ \frac{b_{\sigma(j)}^p}{a_j^{\pi(p)}} \right\} \cdot \max_{p=1,\dots,n} \left\{ \frac{c_{\varrho(\sigma(j))}^p}{b_{\sigma(j)}^{\pi(p)}} \right\} \tag{3}$$

holds for every $j \in \mathbb{N}$.

We note that here both r and n depend not only on π and q but also on our choice of σ and ϱ. This is an obstacle to deriving Theorem 1 immediately from Proposition 3. On the other hand, the methods of [9] cannot be applied directly to $Q(A,B)$, since it is not a subspace. So we need some other considerations.

3. Proof of Theorem 1. Suppose $(\lambda(A),\lambda(B),\lambda(C))\notin\mathcal{BF}$. Then, by Proposition 2, there is a non-decreasing map $\pi:\mathbb{N}\to\mathbb{N}$ such that for each $r\in\mathbb{N}$ there exists $q=q(r)\in\mathbb{N}$ such that for any $n\in\mathbb{N}$ there are $i_n=i_n(r)$, $j_n=j_n(r)$, $\nu_n=\nu_n(r)$ with

$$\frac{c_{i_n}^q}{a_{j_n}^r}>n\max_{p=1,\dots,n}\left\{\frac{b_{\nu_n}^p}{a_{j_n}^{\pi(p)}}\right\}\cdot\max_{p=1,\dots,n}\left\{\frac{c_{i_n}^p}{b_{\nu_n}^{\pi(p)}}\right\}. \tag{4}$$

With this notation we have the following technical result, which is crucial for our proof.

LEMMA 4. *For any $r\ge r_0=\pi(\pi(1))$ the sequences $(i_n)_n$, $(j_n)_n$, $(\nu_n)_n$ diverge to $+\infty$.*

Proof. First we notice that (4) is equivalent to the system of inequalities

$$\frac{c_{i_n}^q}{a_{j_n}^r}>n\,\frac{b_{\nu_n}^p\cdot c_{i_n}^s}{a_{j_n}^{\pi(p)}\cdot b_{\nu_n}^{\pi(s)}},\qquad 1\le p,s\le n. \tag{5}$$

Suppose that j_n does not tend to $+\infty$, that is, $j_{n_k}=j=\text{const}$ for some subsequence n_k. This contradicts (5): take $s=q$, $p=\pi(q)$, $n=n_k>\pi(q)$.

Analogously, assuming that $i_{n_k}=i=\text{const}$ for some subsequence n_k, we get a contradiction by putting $s=1$, $p=\pi(1)$, $n=n_k>\pi(1)$ in (5) and taking into account the assumption $r\ge\pi(\pi(1))$.

Finally, the assumption $\nu_{n_k}=\nu=\text{const}$ also leads to a contradiction: consider (5) with $s=q$, $p=1$, $n=n_k>q$, remembering that $r\ge r_0\ge\pi(1)$. ∎

We are now ready to prove a result which is somewhat stronger than Theorem 1.

PROPOSITION 5. *If $(\lambda(A),\lambda(B),\lambda(C))\notin\mathcal{BF}$ then there are bijections σ and ϱ on $\mathbb{N}$ and operators $S\in Q_\sigma(A,B)$ and $R\in Q_\varrho(B,C)$ such that the operator $T=RS$ is unbounded.*

Proof. From our assumption we have (4) with the same notation. Passing to subsequences three times and using Lemma 4, for any fixed $r\ge r_0:=\pi(\pi(1))$ we construct a subsequence $L_r=\{n_k(r)\}$ of $\mathbb{N}$ such that each coordinate of $(j_{n_k(r)},\nu_{n_k(r)},i_{n_k(r)})$ takes different values for different k. Let us represent each infinite set L_r as a disjoint union of infinite subsets

$$L_r = \bigcup_{\mu=0}^{\infty} L_{r,\mu}.$$

Let us now construct a new sequence of infinite disjoint sets

$$\widetilde{L}_r = \{l_\mu(r) : \mu \in \mathbb{N}\} \subset L_r, \qquad r \geq r_0,$$

in the following inductive way. We form $\widetilde{L}_{r_0}$ by taking precisely one element $l_\mu(r_0)$ from each $L_{r_0,\mu}$, $\mu \in \mathbb{N}$. Assume we have already constructed pairwise disjoint sets $\widetilde{L}_s$ for $r_0 \leq s \leq r$, so that each $\widetilde{L}_s$ contains exactly one element from $L_{s,\mu}$ and is disjoint from $L_{s,0}$. We then construct $\widetilde{L}_{r+1}$ by taking from each $L_{r+1,\mu}$, $\mu \in \mathbb{N}$, one element different from every $l_\mu(s)$, $r_0 \leq s \leq r$. By induction this concludes the construction of $\widetilde{L}_r$, $r \geq r_0$. The set $I_0 := \mathbb{N} \setminus \bigcup_{r=r_0}^{\infty} I_{\widetilde{L}_r}$ is infinite since it contains $I_{L_{r,0}}$ for each $r \geq r_0$. By the same token the sets

$$J_0 := \mathbb{N} \setminus \bigcup_{r=r_0}^{\infty} J_{\widetilde{L}_r}, \qquad N_0 := \mathbb{N} \setminus \bigcup_{r=r_0}^{\infty} N_{\widetilde{L}_r}$$

are also infinite.

Let $\alpha : J_0 \to N_0$ and $\beta : N_0 \to I_0$ be arbitrary bijections. Consider the maps $\varrho : \mathbb{N} \to \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathbb{N}$ defined by

$$\sigma(j) := \begin{cases} \alpha(j) & \text{if } j \in J_0, \\ \nu_{l_\mu(r)} & \text{if } j = j_{l_\mu(r)} \in J_{\widetilde{L}_r},\ r \geq r_0, \end{cases}$$

$$\varrho(\nu) := \begin{cases} \beta(\nu) & \text{if } \nu \in N_0, \\ i_{l_\mu(r)} & \text{if } \nu = \nu_{l_\mu(r)} \in N_{\widetilde{L}_r},\ r \geq r_0. \end{cases}$$

For each r we have

$$\frac{c^{q(r)}_{\varrho(\sigma(j))}}{a^r_j} > n \max_{p=1,\dots,n} \left\{ \frac{b^p_{\sigma(j)}}{a^{\pi(p)}_j} \right\} \cdot \max_{p=1,\dots,n} \left\{ \frac{c^p_{\varrho(\sigma(j))}}{b^{\pi(p)}_{\sigma(j)}} \right\}$$

for all $j = j_n$, where $n \in \widetilde{L}_r$. Hence by Proposition 3, there exist $S \in Q_\sigma(A,B)$ and $R \in Q_\varrho(B,C)$ with RS unbounded. ■

4. Some consequences. Nurlu and Terzioğlu [8] studied the consequences of the existence of an unbounded operator between nuclear Köthe spaces. They showed, in particular, that if the spaces satisfy a splitting condition of Apiola type [1], then the existence of an unbounded operator implies the existence of a common basic subspace. Djakov and Ramanujan [2] obtain the same result without the assumption of nuclearity and assuming the weaker splitting condition of Krone and Vogt [5].

Before dealing with the main result of this section (see Theorem 10 below) we discuss certain modifications and factorized analogues of some properties, important for studying the relation $\mathrm{Ext}^1(F, E) = 0$ (see, e.g., [11, 12, 4]). A pair (F, E) of Fréchet spaces satisfies the *condition* $\mathcal{S}$ if there is a mapping $\tau : \mathbb{N} \to \mathbb{N}$ such that for all p, $r \in \mathbb{N}$ there exists a constant $C = C(p, r)$ such that the estimate

$$|T|_{r,\tau(p)} \le C \max\{|T|_{\tau(p),p}, |T|_{\tau(r),r}\} \tag{6}$$

holds for any one-dimensional operator

$$T = e' \otimes f, \qquad e' \in E',\ f \in F.$$

It is easy to check that the condition $\mathcal{S}$ is an equivalent slight variation of Vogt's condition S_2^* ([11]). It is known that the property $\mathrm{Ext}^1(F, E) = 0$ is characterized by $(F, E) \in \mathcal{S}$ whenever both spaces are either Köthe spaces ([5]) or nuclear ([4]). A pair of Köthe spaces $E = \lambda(A)$ and $F = \lambda(B)$ satisfies the condition $\mathcal{S}$ if and only if the condition (6) holds for the operators $T = e_i' \otimes e_j$, $i, j \in \mathbb{N}$ ([5]).

If the estimate (6) is true for arbitrary operators $T \in L(E, F)$ (with an obvious meaning if some of the operator norms equals $+\infty$) then we write $(F, E) \in \overline{\mathcal{S}}$ (in fact, one can see that this condition is reasonable only for bounded operators T). It is easy to check that the condition $(F, E) \in \overline{\mathcal{S}}$ coincides with the condition on $LB(E, F)$ considered by Dierolf, Frerick, Mangino, and Wengenroth (see, e.g., [4, the proof of Theorem 2.2]); moreover, by Vogt [12], this condition coincides with the condition (wQ) for the natural representation of $LB(E, F)$ as an (LF)-space.

In what follows we shall denote by $\lambda(A)_L$ the basic subspace of a Köthe space $\lambda(A)$ which is the closed linear envelope of $\{e_n : n \in L\}$, $L \subset \mathbb{N}$.

Suppose now $(\lambda(A), \lambda(B), \lambda(C)) \notin \mathcal{BF}$ and $(\lambda(C), \lambda(A)) \in \mathcal{S}$. By Theorem 1 we know that there are $S \in Q_\sigma(A, B)$ and $R \in Q_\varrho(B, C)$ with some bijective maps σ and ϱ on $\mathbb{N}$ such that $T = RS$ is an unbounded quasidiagonal operator. The theorem of Djakov and Ramanujan [2] implies the existence of infinite subsets J and I of $\mathbb{N}$ such that T maps $\lambda(A)_J$ isomorphically onto $\lambda(C)_I$. Then one can easily check that for $N := \sigma(J) = \varrho^{-1}(I)$ both $S : \lambda(A)_J \to \lambda(B)_N$ and $R : \lambda(B)_N \to \lambda(C)_J$ are also isomorphisms. We have therefore proved the following result.

PROPOSITION 6. *Let $E = \lambda(A)$, $G = \lambda(B)$, and $F = \lambda(C)$. Suppose that $(E, G, F) \notin \mathcal{BF}$ and $(F, E) \in \mathcal{S}$. Then there is a common basic subspace for all three spaces.*

Now we consider a factorized analogue of the condition $\mathcal{S}$. A triple of Fréchet spaces (F, G, E) satisfies the *condition* $\mathcal{SF}$ (we write $(F, G, E) \in \mathcal{SF}$) if for any one-dimensional operator $T = RS$, with both $S \in L(E, G)$

and $R \in L(G, F)$ also one-dimensional, the inequality

$$(7) \qquad |T|_{r,\tau(p)} \le C \max\{|R|_{\tau(p),p}, |R|_{\tau(r),r}\} \cdot \max\{|S|_{\tau(p),p}, |S|_{\tau(r),r}\}$$

holds with the same requisites as in (6).

If the condition (7) holds for an arbitrary operator $T = RS$ with $S \in L(E, G)$ and $R \in L(G, F)$ we will write $(F, G, E) \in (\overline{\mathcal{SF}})$ (with the evident meaning when some of the operator norms equals $+\infty$; in fact, this condition is reasonable only for bounded operators T).

We note that if $E = G$ or $G = F$ the condition $(F, G, E) \in \mathcal{SF}$ reduces simply to $(F, E) \in \mathcal{S}$, and $(F, G, E) \in \overline{\mathcal{SF}}$ reduces to $(F, E) \in \overline{\mathcal{S}}$.

Proposition 7. *Let E, G, F be arbitrary Fréchet spaces. If $(E, G, F) \in \mathcal{BF}$, then $(F, G, E) \in \overline{\mathcal{SF}}$.*

Proof. Suppose that $(E, G, F) \in \mathcal{BF}$. Denote by $\Pi(p)$ the set of all strictly increasing mappings $\pi \in \mathbb{N}^{\mathbb{N}}$ such that $\pi(1) = p$. By Theorem 2.2 from [9], for any $\pi \in \Pi(p)$ there are $q \in \mathbb{N}$ and $\mu \in \mathbb{N}^{\mathbb{N}}$ such that for every $T = RS$ with $S \in L(E, G)$ and $R \in L(G, F)$ the inequality

$$(8) \qquad |T|_{r,q} \le \mu(r) \max_{l=1}^{\mu(r)} \{|R|_{l,\pi(l)}\} \cdot \max_{l=1}^{\mu(r)} \{|S|_{l,\pi(l)}\}$$

holds for each $r \in \mathbb{N}$. We denote by $\Pi_q(p)$ the set of all $\pi \in \Pi(p)$ satisfying (8) with a given $q \in \mathbb{N}$. It is obvious that $\Pi(p) = \bigcup_{q=1}^{\infty} \Pi_q(p)$ and $\Pi_q(p) \subset \Pi_{q+1}(p)$, $q \in \mathbb{N}$. Therefore for each $p \in \mathbb{N}$ there is $q = \varrho(p)$ such that $\sup\{\pi(q) : \pi \in \Pi_q(p)\} = \infty$. Now we fix an arbitrary $r \in \mathbb{N}$ and apply (8) with $q = \varrho(p)$ and $\pi \in \Pi_q(p)$ such that $\pi(q) \ge r$. Taking into account that

$$|R|_{l,\pi(l)} \le \begin{cases} |R|_{q,p} & \text{if } 1 \le l \le q, \\ |R|_{\mu(r),r} & \text{if } q < l \le \mu(r), \end{cases}$$

and that the same holds for S, we derive from (8) that

$$|T|_{r,\varrho(p)} \le \mu(r) \max\{|R|_{\varrho(p),p}, |R|_{\mu(r),r}\} \cdot \max\{|S|_{\varrho(p),p}, |S|_{\mu(r),r}\}.$$

Hence one can easily conclude that there are $\tau \in \mathbb{N}^{\mathbb{N}}$ and $C = C(p, r)$ such that (7) holds. Thus $(F, G, E) \in \overline{\mathcal{SF}}$. ∎

In particular, if $F = G$ or $G = E$, we get the following

Corollary 8. *Let E and F be Fréchet spaces. Then $(E, F) \in \mathcal{B}$ implies $(F, E) \in \overline{\mathcal{S}}$.*

This is a generalization of Proposition 3.4 from [5], where the case of Köthe spaces was considered (for Köthe spaces the conditions $\mathcal{S}$ and $\overline{\mathcal{S}}$ coincide): basically, our proof of Proposition 7 is a generalized direct version of the proof ad absurdum from [5]).

Now we compare the conditions $\mathcal{S}$ and $\overline{\mathcal{S}}$ with their factorized versions.

PROPOSITION 9. *Let E, G, and F be arbitrary Fréchet spaces. If the couple (F,E) satisfies $\overline{\mathcal{S}}$ (or $\mathcal{S}$), then the triple (F,G,E) satisfies $\overline{\mathcal{SF}}$ (respectively, $\mathcal{SF}$).*

Proof. Because of complete similarity we consider only the case $\overline{\mathcal{S}}$. Suppose that $(F,E)\in\overline{\mathcal{S}}$. Then there is a function $\tau:\mathbb{N}\to\mathbb{N}$ such that for each $T\in L(E,F)$ the estimate

$$|T|_{r,\tau(p)}\le C\max\{|T|_{\tau(p),p},|T|_{\tau(r),r}\} \tag{9}$$

holds for all $p,r\in\mathbb{N}$ with some constant $C=C(p,r)$. Without loss of generality we assume $\tau(p)\ge p$ for every $p\in\mathbb{N}$. Using now the evident estimate

$$|T|_{\tau(p),p}\le|S|_{p,p}\cdot|R|_{\tau(p),p}\le|S|_{\tau(p),p}\cdot|R|_{\tau(p),p},\qquad p\in\mathbb{N},$$

for any operator $T=RS$, we obtain the estimate (7), which means that $(F,G,E)\in\overline{\mathcal{SF}}$. ∎

The following example shows that $\mathcal{SF}$ is strictly weaker than $\mathcal{S}$. Here we use the notation $\Lambda_\alpha(a):=K(\exp(\alpha_p a_i))$ with $\alpha_p\uparrow\alpha\le\infty$, $a=(a_i)$.

EXAMPLE. Let $a=(a_i)$ be a positive sequence increasing to infinity. Since $(\Lambda_1(a),\Lambda_\infty(a))\in\mathcal{B}$ ([14]), we have $(\Lambda_1(a),\Lambda_\infty(a),\Lambda_1(a))\in\mathcal{BF}$ trivially. Hence $(\Lambda_1(a),\Lambda_\infty(a),\Lambda_1(a))\in\mathcal{SF}$ by Proposition 7. However $(\Lambda_1(a),\Lambda_\infty(a))\notin\mathcal{S}$.

We conclude with a generalizaton of Djakov–Ramanujan's result ([2, Proposition 3]) in the context of factorization.

THEOREM 10. *Suppose $(\lambda(A),\lambda(B),\lambda(C))\notin\mathcal{BF}$ and $(\lambda(C),\lambda(B),\lambda(A))\in\mathcal{SF}$. Then one of the pairs $(\lambda(A),\lambda(B))$ or $(\lambda(B),\lambda(C))$ has a common basic subspace.*

Proof. By Theorem 1 there exist quasidiagonal operators $S\in Q_\sigma(A,B)$ and $R\in Q_\varrho(B,C)$ with σ and ϱ bijective such that $T=RS$ is unbounded. Without loss of generality we assume in what follows that all three operators are identity embeddings, since otherwise we can get this property by considering a new triple of Köthe spaces obtained from the original one by some permutations and normalizations of their canonical bases (note that the property $\mathcal{SF}$ is preserved under such reconstruction). When applied to the above embeddings, the condition $\mathcal{SF}$ gives the following: there is a map $\tau:\mathbb{N}\to\mathbb{N}$ such that

$$\frac{c_i^r}{a_i^{\tau(p)}}\le C\max\left\{\frac{b_i^{\tau(p)}}{a_i^p},\frac{b_i^{\tau(r)}}{a_i^r}\right\}\cdot\max\left\{\frac{c_i^{\tau(p)}}{b_i^p},\frac{c_i^{\tau(r)}}{b_i^r}\right\} \tag{10}$$

for all $(p,r,i)\in\mathbb{N}^3$ with some constant $C=C(p,r)$.

It now suffices to prove that there is an infinite set $I \subset \mathbb{N}$ such that $\lambda(A)_I = \lambda(B)_I$ or $\lambda(B)_I = \lambda(C)_I$. Suppose that this assertion is false. Then for each infinite set $I \subset \mathbb{N}$ and $m \in \mathbb{N}$ there is $r \geq m$ such that

$$\liminf_{i\in I} \frac{b_i^{\tau(r)}}{a_i^r} = \liminf_{i\in I} \frac{c_i^{\tau(r)}}{b_i^r} = 0. \tag{11}$$

We define inductively the sets $N_0 \supset N_1 \supset \ldots$ by

$$N_0 := \mathbb{N}, \qquad N_p := \left\{ i \in N_{p-1} : \max\left\{ \frac{b_i^{\tau(p)}}{a_i^p}, \frac{c_i^{\tau(p)}}{b_i^p} \right\} \geq 1 \right\}, \qquad p \in \mathbb{N}, \tag{12}$$

with τ from (10).

We claim that for each $p \in \mathbb{N}$ the embedding T is unbounded on the basic subspace X_p of $\lambda(A)$ spanned by $\{e_i : i \in N_{p-1} \setminus N_p\}$. If that is not so, then for each $q \in \mathbb{N}$ there is an infinite subset $I_q \subset N_{p-1} \setminus N_p$ and $m(q) \in \mathbb{N}$ with

$$\lim_{i\in I_q} \frac{c_i^{m(q)}}{a_i^q} = \infty. \tag{13}$$

For $I = I_q$ we find $r \geq m(q)$ such that (11) holds. Then there is an infinite set $J_q \subset I_q$ with

$$\max\left\{ \frac{c_i^{\tau(r)}}{b_i^r}, \frac{c_i^{\tau(r)}}{b_i^r} \right\} < 1, \qquad i \in J_q. \tag{14}$$

On the other hand, by (12), we have

$$\max\left\{ \frac{c_i^{\tau(p)}}{b_i^p}, \frac{c_i^{\tau(p)}}{b_i^p} \right\} < 1, \qquad i \in I_q. \tag{15}$$

Applying now (10) with $q = \tau(p)$ and r chosen above and taking into account the estimates (14) and (15), we obtain

$$\frac{c_i^r}{a_i^q} \leq C$$

for all $i \in J_q$, which contradicts (13). This proves our claim that the embedding T is bounded on each X_p. Hence, for every $p \in \mathbb{N}$, the operator T must be unbounded on the basic subspace Y_p generated by $\{e_i : i \in N_p\}$, which, in particular, implies that N_p is an infinite set.

Now we construct a sequence $I = \{i_p\}$ so that $i_p \in N_p$, $i_{p+1} \neq i_p$, $p \in \mathbb{N}$. Then, due to (12), there is an infinite set $J \subset I$ such that at least one of the inequalities $a_i^p \leq b_i^{\varrho(p)}$ or $b_i^p \leq c_i^{\varrho(p)}$ holds for all $p \in \mathbb{N}$ and all $i \in J$ such that $i \geq p$, which contradicts the assumption (11). This completes the proof. ■

Acknowledgements. The authors would like to thank the referee for his/her suggestion how to simplify essentially the proof of Lemma 4.

References

[1] H. Apiola, *Characterization of subspaces and quotients of nuclear $L_f(a,\infty)$-spaces*, Compositio Math. 50 (1983), 65–81.
[2] P. B. Djakov and M. S. Ramanujan, *Bounded and unbounded operators between Köthe spaces*, Studia Math. 152 (2002), 11–31.
[3] M. M. Dragilev, *Riesz classes and multiple regular bases*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 15 (1972), 65–77 (in Russian).
[4] L. Frerick, *A splitting condition for nuclear Fréchet spaces*, in: Functional Analysis (Trier, 1994), de Gruyter, Berlin, 1996, 163–167.
[5] J. Krone and D. Vogt, *The splitting relation for Köthe spaces*, Math. Z. 190 (1985), 387–400.
[6] Z. Nurlu, *On basic sequences in some Köthe spaces and existence of non-compact operators,* Ph.D. thesis, Clarkson College of Technology, Potsdam, NY, 1981.
[7] —, *On pairs of Köthe spaces between which all operators are compact*, Math. Nachr. 122 (1985), 277–287.
[8] Z. Nurlu and T. Terzioğlu, *Consequences of the existence of a non-compact operator between nuclear Köthe spaces*, Manuscripta Math. 47 (1984), 1–12.
[9] T. Terzioğlu and V. Zahariuta, *Bounded factorization property for Fréchet spaces,* Math. Nachr. 253 (2003), 1–11.
[10] D. Vogt, *Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.
[11] —, *On the functor* $\mathrm{Ext}^1(E,F)$ *for Fréchet spaces*, Studia Math. 85 (1987), 163–197.
[12] —, *Regularity properties of* (LF)*-spaces*, in: Progress in Functional Analysis, North-Holland Math. Stud. 170, North-Holland, Amsterdam, 1992, 57–84.
[13] V. Zahariuta, *On isomorphisms of Cartesian products of linear topological spaces*, Funktsional. Anal. i Prilozhen. 4 (1970), no. 2, 87–88 (in Russian).
[14] —, *On the isomorphism of cartesian products of locally convex spaces,* Studia Math. 46 (1973), 201–221.

T. Terzioğlu
Sabancı University
81474 Tuzla–Istanbul, Turkey
E-mail: tosun@sabanciuniv.edu.tr

M. Yurdakul
Department of Mathematics
Middle East Technical University
06531 Ankara, Turkey
E-mail: myur@metu.edu.tr

V. P. Zahariuta
Department of Mathematics
Middle East Technical University
&
Faculty of Engineering
and Natural Sciences
Sabancı University
81474 Tuzla–Istanbul, Turkey
E-mail: zaha@sabanciuniv.edu.tr

Received August 12, 2002
Revised version March 6, 2003 (5014)