STUDIA MATHEMATICA 161 (3) (2004)

Strong summability of Ciesielski–Fourier series

by

FERENC WEISZ (Budapest)

Dedicated to Professor Zbigniew Ciesielski on his 70th birthday

Abstract. A strong summability result is proved for the Ciesielski–Fourier series of integrable functions. It is also shown that the strong maximal operator is of weak type $(1, 1)$.

1. Introduction. It was proved by Fejér [8] that the $(C, 1)$ or Fejér means of the trigonometric Fourier series of a continuous function converge uniformly to the function. The same problem for integrable functions was investigated by Lebesgue [11]. He proved that every integrable function is a.e. Fejér summable, i.e.

$$\frac{1}{n}\sum_{k=0}^{n-1}(s_k f(x) - f(x)) \to 0 \quad \text{as } n \to \infty$$

for a.e. $x \in [-\pi, \pi]$, where $s_k f$ denotes the kth partial sum of the Fourier series of f.

Strong summability, i.e. convergence of the strong means

$$\left(\frac{1}{n}\sum_{k=0}^{n-1}|s_k f(x) - f(x)|^r\right)^{1/r}$$

was first considered by Hardy and Littlewood [10]. They showed that these means tend to 0 a.e. as $n \to \infty$ whenever $f \in L_p$ $(1 < p < \infty)$. This result was generalized to L_1 functions and $r = 2$ by Marcinkiewicz [12] and to all $r > 0$ by Zygmund [25].

For Walsh–Fourier series and for integrable functions Fejér summability is due to Fine [9] (see also Schipp [17]), while strong summability was shown

2000 *Mathematics Subject Classification*: Primary 41A15, 40F05, 42A24; Secondary 42C10, 42B08.

Key words and phrases: strong summability, Fejér means, Walsh system, spline functions, Ciesielski system, weak type inequality.

This research was supported by the Hungarian Scientific Research Funds (OTKA) No T043769.

by Schipp [16] for $r = 2$ and by Rodin [15, 14] for $r > 0$ and for BMO means. Recently Schipp [19] gave a nice proof for $r > 0$ as well as for BMO means and characterized the points at which strong summability holds.

In this paper we consider a generalization of the Walsh system, the so called Ciesielski systems, which can be obtained from the spline systems of order (m, k) in the same way as the Walsh system arises from the Haar system (see Ciesielski [5, 2, 7]). In the special case $m = -1$ and $k = 0$ we obtain the Walsh system. Recently the author [23] extended the above result to the $(C, 1)$ means and proved that the Ciesielski–Fourier series of any integrable function is a.e. Fejér summable.

We will generalize the strong summability result to the Ciesielski–Fourier series of integrable functions and to $0 < r \leq 2$. We also show that the strong maximal operator is of weak type $(1, 1)$. The proof holds for all Ciesielski systems, so it can also be regarded as a new proof for the Walsh system.

2. Ciesielski systems. We consider the unit interval $[0, 1)$ and the Lebesgue measure λ on it. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write L_p instead of the real $L_p([0, 1), \lambda)$ space, and the norm (or quasi-norm) of this space is defined by $\|f\|_p := (\int_{[0,1)} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leq \infty)$. The space $L_{p,\infty} = L_{p,\infty}([0, 1), \lambda)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\varrho>0} \varrho\lambda(|f| > \varrho)^{1/p} < \infty,$$

and we set $L_{\infty,\infty} = L_\infty$. Note that $L_{p,\infty}$ is a quasi-normed space. It is easy to see that

$$L_p \subset L_{p,\infty} \quad \text{and} \quad \| \cdot \|_{p,\infty} \leq \| \cdot \|_p$$

for each $0 < p \leq \infty$.

First we define the Walsh system. Let

$$r(x) := \begin{cases} 1 & \text{if } x \in [0, 1/2), \\ -1 & \text{if } x \in [1/2, 1), \end{cases}$$

extended to $\mathbb{R}$ by periodicity with period 1. The *Rademacher system* $(r_n, n \in \mathbb{N})$ is defined by

$$r_n(x) := r(2^n x) \quad (x \in [0, 1), \, n \in \mathbb{N}).$$

The *Walsh functions* are given by

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k} \quad (x \in [0, 1), \, n \in \mathbb{N}),$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$ $(0 \leq n_k < 2)$. It is known that $w_n(t)w_n(x) = w_n(x \dot{+} t)$ $(n \in \mathbb{N}, t, x \in [0, 1))$, where the dyadic addition $\dot{+}$ is defined e.g. in Schipp, Wade, Simon and Pál [21].

Next we introduce the spline systems as in Ciesielski [5]. Denote by D the differentiation operator and define the integration operators

$$Gf(t) := \int_0^t f\,d\lambda, \qquad Hf(t) := \int_t^1 f\,d\lambda.$$

Define the *Haar system* χ_n, $n = 1, 2, \dots$, by $\chi_1 := 1$ and

$$\chi_{2^n+k}(x) := \begin{cases} 2^{n/2} & \text{if } x \in ((2k-2)2^{-n-1}, (2k-1)2^{-n-1}), \\ -2^{n/2} & \text{if } x \in ((2k-1)2^{-n-1}, (2k)2^{-n-1}), \\ 0 & \text{otherwise,} \end{cases}$$

for $n, k \in \mathbb{N}$, $0 < k \le 2^n$, $x \in [0, 1)$.

Let $m \ge -1$ be a fixed integer. Applying the Schmidt orthonormalization to the linearly independent functions

$$1, t, \dots, t^{m+1}, G^{m+1}\chi_n(t), \qquad n \ge 2,$$

we get the *unbounded Ciesielski system* or *spline system* $(f_n^{(m)}, n \ge -m)$ of order m. For $0 \le k \le m+1$ and $n \ge k-m$ define the splines

$$f_n^{(m,k)} := D^k f_n^{(m)}, \qquad g_n^{(m,k)} := H^k f_n^{(m)}$$

of order (m, k). Let us normalize these functions and introduce a more unified notation,

$$h_n^{(m,k)} := \begin{cases} f_n^{(m,k)} \|f_n^{(m,k)}\|_2^{-1} & \text{for } 0 \le k \le m+1, \\ g_n^{(m,-k)} \|f_n^{(m,-k)}\|_2 & \text{for } 0 \le -k \le m+1. \end{cases}$$

We get the Haar system if $m = -1$, $k = 0$ and the *Franklin system* if $m = 0$, $k = 0$. The systems $(h_i^{(m,k)}, i \ge |k| - m\})$ and $(h_j^{(m,-k)}, j \ge |k| - m\})$ are biorthogonal, i.e.

$$(h_i^{(m,k)}, h_j^{(m,-k)}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j, \end{cases}$$

where (f, g) denotes the usual scalar product $\int_{[0,1)} fg\,d\lambda$.

It is proved in Ciesielski [3, 4, 5] that

$$|D^N h_{2^\mu+\nu}^{(m,k)}(t)| \le C 2^{(N+1/2)\mu} q^{2^\mu |t-\nu/2^\mu|} \tag{1}$$

where $m \ge -1$, $|k| \le m+1$, $k + N \le m+1$, $\mu \in \mathbb{N}$ and $\nu = 1, \dots, 2^\mu$.

In this paper the constants C and q depend only on m, and the constants C_p depend only on p and m, and may be different in different contexts; q always denotes a constant for which $0 < q < 1$.

If $k+i \le m+1$ then the kernel function (see Ciesielski and Domsta [6] and also Ciesielski [5])

$$F_n^{(m,k)}(t,s) := \sum_{j=|k|-m}^{n} h_j^{(m,k)}(t) h_j^{(m,-k)}(s)$$

satisfies

$$|D_t^i F_n^{(m,k)}(t,s)| \le C n^{i+1} q^{n|t-s|} \qquad (i=0,1). \tag{2}$$

Starting with the spline system $(h_n^{(m,k)}, n \ge |k|-m)$ we define the *bounded Ciesielski system* $(c_n^{(m,k)}, n \ge |k|-m)$ in the same way as the Walsh system arises from the Haar system, namely,

$$c_n^{(m,k)} := h_n^{(m,k)} \qquad (n = |k|-m, \dots, 1),$$

and

$$c_{2^\nu+i}^{(m,k)} := \sum_{j=1}^{2^\nu} A_{i,j}^{(\nu)} h_{2^\nu+j}^{(m,k)} \qquad (1 \le i \le 2^\nu).$$

As mentioned before,

$$c_n^{(-1,0)} = w_{n-1} \qquad (n \ge 1)$$

is the usual Walsh system. One can show (see Schipp, Wade, Simon and Pál [21] or Ciesielski, Simon and Sjölin [7]) that

$$A_{i,j}^{(\nu)} = A_{j,i}^{(\nu)} = 2^{-\nu/2} w_{i-1}\left(\frac{2j-1}{2^{\nu+1}}\right). \tag{3}$$

The system $(c_n^{(m,k)})$ is uniformly bounded and it is biorthogonal to $(c_n^{(m,-k)})$ whenever $|k| \le m+1$.

Denote by $s_n^{(m,k)} f$ the nth partial sum of the Fourier series of $f \in L_1$, i.e.

$$s_n^{(m,k)} f := \sum_{j=|k|-m}^{n} (f, c_j^{(m,k)}) c_j^{(m,-k)} \qquad (n \in \mathbb{N}).$$

Obviously,

$$s_n^{(m,k)} f(x) = \int_0^1 f(t) D_n^{(m,k)}(t,x)\, dt,$$

where the *Dirichlet kernels* are defined by

$$D_n^{(m,k)}(t,x) := \sum_{j=|k|-m}^{n} c_j^{(m,k)}(t) c_j^{(m,-k)}(x) \qquad (n \in \mathbb{N},\ t,x \in [0,1)).$$

Since $w_j(t) w_j(x) = w_j(x \dot{+} t)$, for the Walsh–Dirichlet kernels, i.e. if $m=-1$, $k=0$, we use also the notation $D_n^{(-1,0)}(t \dot{+} x)$. So $D_n^{(-1,0)}(t \dot{+} x) := D_n^{(-1,0)}(t,x)$ and we can regard $D_n^{(-1,0)}$ as a function of one variable.

3. Strong summability of Ciesielski–Fourier series. The *Fejér means* and the *maximal Fejér operator* of an integrable function f are given by

$$\sigma_n^{(m,k)} f := \frac{1}{n}\sum_{j=1}^{n} s_j^{(m,k)} f, \qquad \sigma_*^{(m,k)} f := \sup_{n\ge 1} |\sigma_n^{(m,k)} f|.$$

The author proved in [23] that if $m \ge -1$ and $|k| \le m+1$ then

$$\sup_{\varrho>0} \varrho\, \lambda(\sigma_*^{(m,k)} f > \varrho) \le C\|f\|_1 \qquad (f \in L_1) \tag{4}$$

(for the Walsh system see also Schipp [17] and Weisz [22]). The Fejér summability of the Ciesielski–Fourier series of an integrable function follows from this, i.e.,

$$\sigma_n^{(m,k)} f \to f \qquad \text{a.e. as } n \to \infty \tag{5}$$

whenever $f \in L_1$ and $m \ge -1$, $|k| \le m+1$.

In this paper stronger results will be proved. We consider the *strong means*

$$S_n^{(m,k),(r)} f := \Big(\frac{1}{n}\sum_{j=1}^{n} |s_j^{(m,k)} f|^r\Big)^{1/r} \qquad (n \in \mathbb{N})$$

and the *strong maximal operator*

$$S_*^{(m,k),(r)} f := \sup_{n\ge 1} S_n^{(m,k),(r)} f,$$

where $0 < r < \infty$. We will prove that $S_*^{(m,k),(r)}$ $(0 < r \le 2)$ is of weak type $(1,1)$ and in case $f \in L_1$,

$$S_n^{(m,k),(r)}(f - f(x))(x) \to 0 \qquad \text{for a.e. } x \in [0,1) \text{ as } n \to \infty,$$

which generalizes (4) and (5).

We have also shown in [23] that $\sigma_*^{(m,k)}$ is bounded from the Hardy space H_p to L_p if $1/2 < p < \infty$. This is not true for the strong maximal operator (see Schipp and Simon [20]), so in the proof of the next theorem we have to use methods other than in [23].

In proving the weak type $(1,1)$ inequality the following lemmas will be used.

Lemma 1 (Calderón–Zygmund decomposition). *If $f \in L_1$ and $\varrho > \|f\|_1$ then there exist disjoint dyadic intervals $I_0, I_1, \ldots$ and functions g, b such that*

(i) $f = g + b$,
(ii) $\|g\|_\infty \le \varrho$,
(iii) $\operatorname{supp} b \subset \Omega$, *where* $\Omega := \bigcup_{k=0}^{\infty} I_k$,
(iv) $|\Omega| \le 2\|f\|_1/\varrho$,

(v) $\int_{I_k} b\,d\lambda = 0$ *and* $\int_{I_k} |b|\,d\lambda \le 2\varrho |I_k|$ $(k \in \mathbb{N})$,
(vi) $\|g\|_1 \le \|f\|_1$.

LEMMA 2. *Suppose that V is a bounded sublinear operator from L_∞ to L_∞ and*

$$\int_{[0,1)\setminus 2I} |Vf|\,d\lambda \le C\|f\|_1 \tag{6}$$

for all $f \in L_1$ and dyadic intervals I which satisfy

$$\operatorname{supp} f \subset I, \qquad \int_0^1 f\,d\lambda = 0, \tag{7}$$

where rI denotes the interval having the same center as I and length $r|I|$ $(r \in \mathbb{N})$. Then the operator V is of weak type $(1,1)$, *i.e.*

$$\sup_{\varrho>0} \varrho\lambda(|Vf| > \varrho) \le C\|f\|_1 \qquad (f \in L_1).$$

For the proofs of these lemmas see e.g. Schipp, Wade, Simon and Pál [21] or Weisz [24]. We also need another consequence of the Calderón–Zygmund decomposition.

LEMMA 3. *For $f \in L_1$ and $\varrho > \|f\|_1$ take the Calderón–Zygmund decomposition $f = g + b$ as in Lemma* 1. *Let $\Omega = \bigcup_{k=0}^\infty I_k$ and $r\Omega := \bigcup_{k=0}^\infty rI_k$ $(r \in \mathbb{N})$. If V is a bounded sublinear operator from L_∞ to L_∞ and*

$$\int_{[0,1)\setminus 8\Omega} |Vb|^2\,d\lambda \le C\varrho\|f\|_1$$

then

$$\varrho\lambda(|Vf| > \varrho) \le C\|f\|_1.$$

Proof. The boundedness of V on L_∞ and (ii) of Lemma 1 imply

$$|Vf(x)| \le |Vg(x)| + |Vb(x)| \le C_0\|g\|_\infty + |Vb(x)| \le C_0\varrho + |Vb(x)|,$$

where C_0 is a fixed constant. Hence

$$|\{x : |Vf(x)| > (C_0+1)\varrho\}| \le |\{x : |Vb(x)| > \varrho\}|.$$

By (iv) of Lemma 1 we have

$$|\{x \in 8\Omega : |Vb(x)| > \varrho\}| \le |8\Omega| \le 16\,\frac{\|f\|_1}{\varrho}.$$

On the other hand, using the assumption of the lemma, we obtain

$$|\{x \in (8\Omega)^{\mathrm{c}} : |Vb(x)| > \varrho\}| \le \frac{1}{\varrho^2} \int_{(8\Omega)^{\mathrm{c}}} |Vb(x)|^2\,dx \le C\,\frac{\|f\|_1}{\varrho},$$

which finishes the proof. ■

We now prove the main result of this paper.

THEOREM 1. *If* $m \ge -1$, $|k| \le m+1$ *and* $0 < r \le 2$ *then*

$$\|S_*^{(m,k),(r)} f\|_\infty \le C\|f\|_\infty \qquad (f \in L_\infty) \tag{8}$$

and

$$\sup_{\varrho>0} \varrho\, \lambda(S_*^{(m,k),(r)} f > \varrho) \le C\|f\|_1 \qquad (f \in L_1). \tag{9}$$

Proof. Set

$$G_\mu^{(m,k)}(t,s) := 2^{\mu/2} r_\mu(s) h_{2^\mu+\nu}^{(m,k)}(t) \quad \text{if } \frac{\nu-1}{2^\mu} \le s < \frac{\nu}{2^\mu} \tag{10}$$

$(1 \le \nu \le 2^\mu)$. Then, by (3), it is easy to see that

$$\begin{aligned} c_{2^\mu+\nu}^{(m,k)}(t) &= \int_0^1 w_{\nu-1}(s) r_\mu(s) G_\mu^{(m,k)}(t,s)\, ds \\ &= \int_0^1 w_{2^\mu+\nu-1}(s) G_\mu^{(m,k)}(t,s)\, ds, \end{aligned} \tag{11}$$

where $\mu \in \mathbb{N}$ and $1 \le \nu \le 2^\mu$ (see also Schipp [18] and Ciesielski, Simon and Sjölin [7]). Write $n \in \mathbb{N}$ in the form $n = 2^i + p$ with $2^i > p$. This implies that

$$\begin{aligned} D_n^{(m,k)}(t,x) &= \sum_{j=|k|-m}^{2^i+p} c_j^{(m,k)}(t) c_j^{(m,-k)}(x) \\ &= D_{2^i}^{(m,k)}(t,x) + \sum_{j=1}^{p} c_{2^i+j}^{(m,k)}(t) c_{2^i+j}^{(m,-k)}(x) \\ &= D_{2^i}^{(m,k)}(t,x) \\ &\quad + \sum_{j=1}^{p} \int_0^1\!\!\int_0^1 w_{j-1}(s) r_i(s) G_i^{(m,k)}(t,s) w_{j-1}(u) r_i(u) G_i^{(m,-k)}(x,u)\, ds\, du \\ &= D_{2^i}^{(m,k)}(t,x) \\ &\quad + \int_0^1\!\!\int_0^1 D_p^{(-1,0)}(s \dot{+} u) r_i(s \dot{+} u) G_i^{(m,k)}(t,s) G_i^{(m,-k)}(x,u)\, ds\, du. \end{aligned} \tag{12}$$

Let $(\tau_s h)(x) := h(x \dot{+} s)$ be the dyadic translation operator and $e_j := 2^{-j-1}$. Schipp [19] proved that

$$\begin{aligned} D_p^{(-1,0)} &= \sum_{l=0}^{i-1} 1_{[2^{-l-1},2^{-l})} \sum_{j=0}^{l} \varepsilon_{j,l} 2^{j-1} \tau_{e_j} w_p - w_p/2 + (p+1/2) 1_{[0,2^{-i})} \\ &= \sum_{j=0}^{i-1} \sum_{l=j}^{i-1} 1_{[2^{-l-1},2^{-l})} \varepsilon_{j,l} 2^{j-1} \tau_{e_j} w_p - w_p/2 + (p+1/2) 1_{[0,2^{-i})}, \end{aligned}$$

where $p < 2^i$ and

$$\varepsilon_{j,l} = \begin{cases} -1 & \text{if } j = 0, \dots, l-1, \\ 1 & \text{if } j = l. \end{cases}$$

Using this and (12) we get

$$\begin{aligned}
(13) \quad D_n^{(m,k)}(t,x) &= D_{2^i}^{(m,k)}(t,x) \\
&- \frac{1}{2} \int_0^1 \int_0^1 w_p(s \dot{+} u) r_i(s \dot{+} u) G_i^{(m,k)}(t,s) G_i^{(m,-k)}(x,u)\, ds\, du \\
&+ \int_0^1 \int_0^1 (p + 1/2) 1_{[0,2^{-i})}(s \dot{+} u) r_i(s \dot{+} u) G_i^{(m,k)}(t,s) G_i^{(m,-k)}(x,u)\, ds\, du \\
&+ \int_0^1 \int_0^1 \sum_{j=0}^{i-1} \sum_{l=j}^{i-1} 1_{[2^{-l-1},2^{-l})}(s \dot{+} u) \varepsilon_{j,l} 2^{j-1} \tau_{e_j} w_p(s \dot{+} u) r_i(s \dot{+} u) \\
&\times G_i^{(m,k)}(t,s) G_i^{(m,-k)}(x,u)\, ds\, du \\
&=: \sum_{l=1}^{4} E_{i,p}^{(l)}(t,x).
\end{aligned}$$

Since

$$S_n^{(m,k),(r)} \le S_n^{(m,k),(2)} \qquad (n \in \mathbb{N},\ 0 < r \le 2),$$

it is enough to prove the theorem for $r = 2$. Observe that

$$S_*^{(m,k),(2)} \le 2 \sup_{N \in \mathbb{N}} S_{2^N-1}^{(m,k),(2)}.$$

Then by (13),

$$\begin{aligned}
(14) \quad S_{2^N-1}^{(m,k),(2)} f(x) &= \left(\frac{1}{2^N - 1} \sum_{j=1}^{2^N-1} |s_j^{(m,k)} f(x)|^2 \right)^{1/2} \\
&= \left(\frac{1}{2^N - 1} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} |s_{2^i+p}^{(m,k)} f(x)|^2 \right)^{1/2} \\
&= \left(\frac{1}{2^N - 1} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} \left| \int_0^1 f(t) D_{2^i+p}^{(m,k)}(t,x)\, dt \right|^2 \right)^{1/2} \\
&\le \sum_{l=1}^{4} \left(\frac{1}{2^N - 1} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} \left| \int_0^1 f(t) E_{i,p}^{(l)}(t,x)\, dt \right|^2 \right)^{1/2} \\
&=: \sum_{l=1}^{4} V_{N,l} f(x).
\end{aligned}$$

We will verify (8) and (9) for the operators

$$V_{*,l} f := \sup_{N \geq 1} V_{N,l} f \qquad (l = 1, 2, 3, 4).$$

Consider the case $l = 1$:

$$|V_{N,1} f(x)|^2 = \frac{1}{2^N - 1} \sum_{i=0}^{N-1} 2^i \Big| \int_0^1 f(t) D_{2^i}^{(m,k)}(t, x)\, dt \Big|^2. \tag{15}$$

It follows from the fact $D_{2^i}^{(m,k)} = F_{2^i}^{(m,k)}$ and from (2) that

$$|V_{N,1} f(x)|^2 \leq C \|f\|_\infty^2 2^{-N} \sum_{i=0}^{N-1} 2^i \Big(\int_0^1 2^i q^{2^i |t-x|}\, dt \Big)^2 \leq C \|f\|_\infty^2 \tag{16}$$

whenever $f \in L_\infty$ and $x \in [0, 1)$, which shows (8) for $V_{*,1}$.

Now suppose that $f \in L_1$, the support of f is a dyadic interval I and $\int_0^1 f = 0$. Set

$$F(x) := \int_0^x f(t)\, dt \qquad (x \in [0, 1)).$$

In case $k \leq m$ we integrate by parts in (15) to obtain

$$|V_{N,1} f(x)|^2 = \frac{1}{2^N - 1} \sum_{i=0}^{N-1} 2^i \Big| \int_0^1 F(t) D_t D_{2^i}^{(m,k)}(t, x)\, dt \Big|^2.$$

If $x \notin 2I$ then (2) implies

$$\begin{aligned} |V_{N,1} f(x)|^2 &\leq C 2^{-N} \sum_{i=0}^{N-1} 2^{5i} \Big(\int_I |F(t)| q^{2^i |t-x|}\, dt \Big)^2 \\ &\leq C \|f\|_1^2 \sum_{i=0}^{\infty} 2^{4i} \Big(\int_I q^{2^i |t-x|}\, dt \Big)^2 \\ &\leq C \|f\|_1^2 |I|^2 \sum_{i=0}^{\infty} 2^{4i} q^{C 2^i |t_0 - x|} \leq C \|f\|_1^2 |I|^2 |t_0 - x|^{-4}, \end{aligned}$$

where t_0 denotes the center of I. In the last step we have used the inequality

$$\sum_{j=0}^{\infty} 2^{jM} q^{2^j |x-t|} \leq C_M |x - t|^{-M} \qquad (M > 0,\ x \neq t), \tag{17}$$

which is easy to show, or can be found in Ciesielski [4] and Weisz [24]. Thus

$$\int_{(2I)^c} V_{*,1} f(x)\, dx \leq C \|f\|_1 |I| \int_{(2I)^c} |t_0 - x|^{-2}\, dx \leq C \|f\|_1, \tag{18}$$

which shows (6), and consequently the operator $V_{*,1}$ is of weak type $(1, 1)$ whenever $k \leq m$ (cf. Lemma 2).

Let the length of the dyadic interval I be 2^{-K} $(K \in \mathbb{N})$. If $k = m+1$ and $i \le K$ then it is easy to see that, for a fixed x, $D_{2^i}^{(m,k)}(t,x)$ is constant on I and so $\int_0^1 f(t) D_{2^i}^{(m,k)}(t,x)\, dt = 0$. If $i > K$ then, by (2),

$$|D_{2^i}^{(m,k)}(t,x)| \le 2^{-K} 2^{2i} q^{2^i|t-x|}.$$

Taking into account (15) we conclude that

$$\begin{aligned}|V_{N,1}f(x)|^2 &\le C 2^{-N} \sum_{i=K+1}^{N-1} 2^{-2K} 2^{5i} \Big(\int_0^1 |f(t)| q^{2^i|t-x|}\, dt \Big)^2 \\ &\le C \|f\|_1^2 |I|^2 \sum_{i=0}^{\infty} 2^{4i} q^{2^i|t_0 - x|}\end{aligned}$$

and (18) also holds in this case. Hence $V_{*,1}$ is of weak type $(1,1)$ for all $|k| \le m+1$.

For the second term of (14) we have

$$\begin{aligned}|V_{N,2}f(x)|^2 \le C 2^{-N} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} \Big| \int_0^1 f(t) \\ \times \int_0^1\int_0^1 w_p(s \dot{+} u) r_i(s \dot{+} u) G_i^{(m,k)}(t,s) G_i^{(m,-k)}(x,u)\, ds\, du\, dt \Big|^2.\end{aligned}$$

By (1) and (10),

$$\begin{aligned}|V_{N,2}f(x)|^2 &\le C 2^{-N} \sum_{i=0}^{N-1} 2^i \Big(\int_0^1 |f(t)| \sum_{\nu=1}^{2^i} \sum_{\mu=1}^{2^i} 2^{2i} \\ &\quad \times \int_{(\nu-1)2^{-i}}^{\nu 2^{-i}} \int_{(\mu-1)2^{-i}}^{\mu 2^{-i}} q^{2^i|t-\nu 2^{-i}|} q^{2^i|x-\mu 2^{-i}|}\, ds\, du\, dt \Big)^2 \\ &\le C 2^{-N} \sum_{i=0}^{N-1} 2^i \Big(\int_0^1 |f(t)| \sum_{\nu=1}^{2^i} \sum_{\mu=1}^{2^i} q^{2^i|t-\nu 2^{-i}|} q^{2^i|x-\mu 2^{-i}|}\, dt \Big)^2 \\ &\le C \|f\|_1^2.\end{aligned}$$

Thus (8) and (9) are shown for $V_{*,2}$.

Consider the third term of (14):

$$\begin{aligned}|V_{N,3}f(x)|^2 \le C 2^{-N} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} p^2 \Big| \int_0^1 f(t) \int_0^1\int_0^1 1_{[0,2^{-i})}(s \dot{+} u) \\ \times r_i(s \dot{+} u) G_i^{(m,k)}(t,s) G_i^{(m,-k)}(x,u)\, ds\, du\, dt \Big|^2.\end{aligned} \tag{19}$$

Suppose that $f \in L_\infty$ and apply (10) to get

$$|V_{N,3}f(x)|^2 \le C2^{-N} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} p^2 \Big(\int_0^1 |f(t)| \sum_{\nu=1}^{2^i} \sum_{\mu=1}^{2^i} 2^{2i} \times \int_{(\nu-1)2^{-i}}^{\nu 2^{-i}} \int_{(\mu-1)2^{-i}}^{\mu 2^{-i}} 1_{[0,2^{-i})}(s \dot{+} u) q^{2^i|t-\nu 2^{-i}|} q^{2^i|x-\mu 2^{-i}|}\, ds\, du\, dt \Big)^2.$$

If $s \in [(\nu-1)2^{-i}, \nu 2^{-i})$ and $u \in [(\mu-1)2^{-i}, \mu 2^{-i})$, then

$$1_{[0,2^{-i})}(s \dot{+} u) = \begin{cases} 1 & \text{if } \nu = \mu, \\ 0 & \text{if } \nu \ne \mu. \end{cases}$$

Thus, for $x \in [0,1)$,

$$\begin{aligned} |V_{N,3}f(x)|^2 &\le C\|f\|_\infty^2 2^{-N} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} p^2 \Big(\sum_{\nu=1}^{2^i} 2^{2i} \times \int_0^1 \int_{(\nu-1)2^{-i}}^{\nu 2^{-i}} \int_{(\nu-1)2^{-i}}^{\nu 2^{-i}} q^{2^i|t-\nu 2^{-i}|} q^{2^i|x-\nu 2^{-i}|}\, ds\, du\, dt \Big)^2 \\ &\le C\|f\|_\infty^2 2^{-N} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} p^2 \Big(\sum_{\nu=1}^{2^i} 2^i \int_{(\nu-1)2^{-i}}^{\nu 2^{-i}} \int_{(\nu-1)2^{-i}}^{\nu 2^{-i}} q^{2^i|x-\nu 2^{-i}|}\, ds\, du \Big)^2 \\ &\le C\|f\|_\infty^2 2^{-N} \sum_{i=0}^{N-1} 2^{3i} \Big(\sum_{\nu=1}^{2^i} 2^{-i} q^{2^i|x-\nu 2^{-i}|} \Big)^2 \\ &\le C\|f\|_\infty^2 2^{-N} \sum_{i=0}^{N-1} 2^i \le C\|f\|_\infty^2. \end{aligned}$$

Assume that $f \in L_1$ satisfies (7). If $k \le m$ then integrating by parts in (19) we obtain

$$\begin{aligned} |V_{N,3}f(x)|^2 &\le C2^{-N} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} p^2 \Big| \int_0^1 F(t) \int_0^1 \int_0^1 \times 1_{[0,2^{-i})}(s \dot{+} u) r_i(s \dot{+} u) D_t G_i^{(m,k)}(t,s) G_i^{(m,-k)}(x,u)\, ds\, du\, dt \Big|^2 \\ &\le C2^{-N} \sum_{i=0}^{N-1} \sum_{p=0}^{2^i-1} p^2 \Big(\int_I |F(t)| \sum_{\nu=1}^{2^i} \sum_{\mu=1}^{2^i} 2^{3i} \int_{(\nu-1)2^{-i}}^{\nu 2^{-i}} \int_{(\mu-1)2^{-i}}^{\mu 2^{-i}} \\ &\qquad \times 1_{[0,2^{-i})}(s \dot{+} u) q^{2^i|t-\nu 2^{-i}|} q^{2^i|x-\mu 2^{-i}|}\, ds\, du\, dt \Big)^2 \\ &\le C\|f\|_1^2 2^{-N} \sum_{i=0}^{N-1} 2^{3i} \Big(\sum_{\nu=1}^{2^i} 2^i \int_I q^{2^i|t-\nu 2^{-i}|} q^{2^i|x-\nu 2^{-i}|}\, dt \Big)^2. \end{aligned}$$

Applying the inequality

$$\sum_{k=1}^{\infty} q^{|i-k|+|j-k|} \le C(q,r) r^{|i-j|} \qquad (q<r<1), \tag{20}$$

we conclude that

$$\begin{aligned}|V_{N,3}f(x)|^2 &\le C\|f\|_1^2 \sum_{i=0}^{N-1} 2^{4i}\Big(\int_I q^{2^i|x-t|}\,dt\Big)^2 \\ &\le C\|f\|_1^2|I|^2 \sum_{i=0}^{N-1} 2^{4i} q^{2^i|x-t_0|} \le C\|f\|_1^2|I|^2|x-t_0|^{-4}\end{aligned} \tag{21}$$

if $x \notin 2I$. The proof of (9) for $V_{*,3}$ can be finished as above.

Assume that $k=m+1$. It is easy to see that χ_{2^i+k} ($k=1,\dots,2^i$) is $\mathcal{F}_{i+1}$-measurable, where $\mathcal{F}_n$ denotes the σ-algebra

$$\mathcal{F}_n := \sigma\{[j2^{-n},(j+1)2^{-n}) : 0 \le j < 2^n\}.$$

Hence $h_{2^i+j}^{(m,m+1)}$ ($j=1,\dots,2^i$) and, for a fixed s, $G_i^{(m,m+1)}(\cdot,s)$ are $\mathcal{F}_{i+1}$-measurable as well. Denote by E_n the conditional expectation operator with respect to $\mathcal{F}_n$. Then

$$E_{i+1}f = 0 \qquad \text{if } 2^{-(i+1)} \ge |I|, \tag{22}$$

$$|E_{i+1}f(t)| \le 2^{i+1} 1_I(t)\int_I |f(x)|\,dx \qquad \text{if } 2^{-(i+1)} < |I|. \tag{23}$$

It follows from (19) that

$$\begin{aligned}|V_{N,3}f(x)|^2 &\le C2^{-N}\sum_{i=0}^{N-1}\sum_{p=0}^{2^i-1} p^2\Big|\int_0^1 (E_{i+1}f)(t) \\ &\quad\times \int_0^1\int_0^1 1_{[0,2^{-i})}(s\dot{+}u) r_i(s\dot{+}u) G_i^{(m,m+1)}(t,s) G_i^{(m,-m-1)}(x,u)\,ds\,du\,dt\Big|^2 \\ &\le C\|f\|_1^2 2^{-N}\sum_{i=0}^{N-1} 2^{3i}\Big(\sum_{\nu=1}^{2^i} 2^i \int_I q^{2^i|t-\nu 2^{-i}|} q^{2^i|x-\nu 2^{-i}|}\,dt\Big)^2.\end{aligned}$$

Similarly to (21) we can see that $V_{*,3}$ is of weak type $(1,1)$ for all $|k| \le m+1$.

The fourth term of (14) is much more complicated. Here we have to use another method. Obviously,

$$\begin{aligned}|V_{N,4}f(x)|^2 \le C2^{-N}\sum_{i=0}^{N-1}\sum_{p=0}^{2^i-1}\Big|\int_0^1 f(t)\int_0^1\int_0^1\sum_{j=0}^{i-1}\sum_{l=j}^{i-1} 1_{[2^{-l-1},2^{-l})}(s\dot{+}u)\varepsilon_{j,l}2^{j-1} \\ \times \tau_{e_j} w_p(s\dot{+}u) r_i(s\dot{+}u) G_i^{(m,k)}(t,s) G_i^{(m,-k)}(x,u)\,ds\,du\,dt\Big|^2.\end{aligned} \tag{24}$$

There is a vector

$$(a_{i,0}(x),\dots,a_{i,2^i-1}(x))\in\mathbb{R}^{2^i},\qquad \sum_{p=0}^{2^i-1}|a_{i,p}(x)|^2=1\qquad (x\in[0,1)),$$

such that

$$\begin{aligned}(25)\quad &\Big(\sum_{p=0}^{2^i-1}\Big|\int_0^1 f(t)\int_0^1\int_0^1\sum_{j=0}^{i-1}\sum_{l=j}^{i-1}1_{[2^{-l-1},2^{-l})}(s\dot+u)\varepsilon_{j,l}2^{j-1}\\ &\qquad\times\tau_{e_j}w_p(s\dot+u)r_i(s\dot+u)G_i^{(m,k)}(t,s)G_i^{(m,-k)}(x,u)\,ds\,du\,dt\Big|^2\Big)^{1/2}\\ &\quad=\sum_{p=0}^{2^i-1}a_{i,p}(x)\int_0^1\int_0^1\int_0^1 f(t)\sum_{j=0}^{i-1}\sum_{l=j}^{i-1}1_{[2^{-l-1},2^{-l})}(s\dot+u)\varepsilon_{j,l}2^{j-1}\\ &\qquad\times\tau_{e_j}w_p(s\dot+u)r_i(s\dot+u)G_i^{(m,k)}(t,s)G_i^{(m,-k)}(x,u)\,ds\,du\,dt.\end{aligned}$$

By Hölder's inequality the last expression can be written as

$$\begin{aligned}&\int_0^1\Big[\int_0^1\Big(\int_0^1 f(t)\sum_{j=0}^{i-1}\sum_{l=j}^{i-1}1_{[2^{-l-1},2^{-l})}(s\dot+u\dot+e_j)\varepsilon_{j,l}2^{j-1}r_i(s\dot+u\dot+e_j)\\ &\qquad\times G_i^{(m,k)}(t,s\dot+e_j)G_i^{(m,-k)}(x,u)\,dt\Big)\Big(\sum_{p=0}^{2^i-1}a_{i,p}(x)w_p(s\dot+u)\Big)\,ds\Big]\,du\\ &\le\int_0^1\Big[\Big(\int_0^1\Big|\int_0^1 f(t)\sum_{j=0}^{i-1}\sum_{l=j}^{i-1}1_{[2^{-l-1},2^{-l})}(s\dot+u\dot+e_j)\varepsilon_{j,l}2^{j-1}r_i(s\dot+u\dot+e_j)\\ &\qquad\times G_i^{(m,k)}(t,s\dot+e_j)G_i^{(m,-k)}(x,u)\,dt\Big|^2\,ds\Big)^{1/2}\\ &\qquad\times\Big(\int_0^1\Big|\sum_{p=0}^{2^i-1}a_{i,p}(x)w_p(s\dot+u)\Big|^2\,ds\Big)^{1/2}\Big]\,du.\end{aligned}$$

By Parseval's equality

$$\int_0^1\Big|\sum_{p=0}^{2^i-1}a_{i,p}(x)w_p(s\dot+u)\Big|^2\,ds=\sum_{p=0}^{2^i-1}|a_{i,p}(x)|^2=1.$$

Thus (25) can be estimated by

$$\begin{aligned}&\int_0^1\Big(\int_0^1\Big|\int_0^1 f(t)\sum_{j=0}^{i-1}\sum_{l=j}^{i-1}1_{[2^{-l-1},2^{-l})}(s\dot+u\dot+e_j)\varepsilon_{j,l}\\ &\qquad\times 2^{j-1}r_i(s\dot+u\dot+e_j)G_i^{(m,k)}(t,s\dot+e_j)G_i^{(m,-k)}(x,u)\,dt\Big|^2\,ds\Big)^{1/2}\,du.\end{aligned}$$

Substituting this into (24) we derive

$$(26)\qquad |V_{N,4}f(x)|^2 \le C2^{-N}\sum_{i=0}^{N-1}\Big[\int_0^1\Big(\int_0^1\Big|\int_0^1 f(t)\sum_{j=0}^{i-1}\sum_{l=j}^{i-1}1_{[2^{-l-1},2^{-l})}(s\dot+u\dot+e_j)$$
$$\times\, \varepsilon_{j,l}2^{j-1}r_i(s\dot+u\dot+e_j)G_i^{(m,k)}(t,s\dot+e_j)G_i^{(m,-k)}(x,u)\,dt\Big|^2\,ds\Big)^{1/2}\,du\Big]^2.$$

In case $f\in L_\infty$ apply (1) and (10) to obtain

$$|V_{N,4}f(x)|^2 \le C\|f\|_\infty^2 2^{-N}\sum_{i=0}^{N-1}\Big[\sum_{\mu=1}^{2^i}\int_{(\mu-1)2^{-i}}^{\mu 2^{-i}}\Big(\int_0^1\Big|\int_0^1\sum_{j=0}^{i-1}1_{[0,2^{-j})}(s\dot+u\dot+e_j)$$
$$\times\, 2^{j-1}2^{2i}\sum_{\nu=1}^{2^i}1_{[(\nu-1)2^{-i},\nu 2^{-i})\dot+e_j}(s)q^{2^i|t-\nu 2^{-i}|}q^{2^i|x-\mu 2^{-i}|}\,dt\Big|^2\,ds\Big)^{1/2}\,du\Big]^2.$$

Integrating by t we can see that

$$|V_{N,4}f(x)|^2 \le C\|f\|_\infty^2 2^{-N}\sum_{i=0}^{N-1}2^{2i}\Big[\sum_{\mu=1}^{2^i}\int_{(\mu-1)2^{-i}}^{\mu 2^{-i}}\Big(\int_0^1\Big|\sum_{j=0}^{i-1}1_{[0,2^{-j})}(s\dot+u\dot+e_j)$$
$$\times\, 2^{j}\sum_{\nu=1}^{2^i}1_{[(\nu-1)2^{-i},\nu 2^{-i})\dot+e_j}(s)q^{2^i|x-\mu 2^{-i}|}\Big|^2\,ds\Big)^{1/2}\,du\Big]^2$$
$$= C\|f\|_\infty^2 2^{-N}\sum_{i=0}^{N-1}2^{2i}\Big[\sum_{\mu=1}^{2^i}\int_{(\mu-1)2^{-i}}^{\mu 2^{-i}}\Big(\int_0^1 q^{2^{i+1}|x-\mu 2^{-i}|}$$
$$\times\sum_{j_1=0}^{i-1}1_{[0,2^{-j_1})}(s\dot+u\dot+e_{j_1})2^{j_1}\sum_{\nu_1=1}^{2^i}1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})\dot+e_{j_1}}(s)$$
$$\times\sum_{j_2=0}^{i-1}1_{[0,2^{-j_2})}(s\dot+u\dot+e_{j_2})2^{j_2}\sum_{\nu_2=1}^{2^i}1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})\dot+e_{j_2}}(s)\,ds\Big)^{1/2}du\Big]^2.$$

We may suppose that $j_2\le j_1$. Replacing $s\dot+e_{j_1}$ by s we conclude that

$$|V_{N,4}f(x)|^2 \le C\|f\|_\infty^2 2^{-N}\sum_{i=0}^{N-1}2^{2i}\Big[\sum_{\mu=1}^{2^i}\int_{(\mu-1)2^{-i}}^{\mu 2^{-i}}\Big(\int_0^1 q^{2^{i+1}|x-\mu 2^{-i}|}$$
$$\times\sum_{j_1=0}^{i-1}1_{[0,2^{-j_1})}(s\dot+u)2^{j_1}\sum_{\nu_1=1}^{2^i}1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})}(s)$$
$$\times\sum_{j_2=0}^{j_1}1_{[0,2^{-j_2})}(s\dot+u\dot+e_{j_1}\dot+e_{j_2})2^{j_2}$$
$$\times\sum_{\nu_2=1}^{2^i}1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})\dot+e_{j_1}\dot+e_{j_2}}(s)\,ds\Big)^{1/2}\,du\Big]^2.$$

Since for fixed i, j_1, j_2 the sets $[(\nu_2-1)2^{-i}, \nu_2 2^{-i}) \dot{+} e_{j_1} \dot{+} e_{j_2}$ are disjoint dyadic intervals,

$$\begin{aligned}
|V_{N,4}f(x)|^2 \le C\|f\|_\infty^2 2^{-N} &\sum_{i=0}^{N-1} 2^{2i} \Big[\sum_{\mu=1}^{2^i} \int\limits_{(\mu-1)2^{-i}}^{\mu 2^{-i}} \Big(\int\limits_0^1 q^{2^{i+1}|x-\mu 2^{-i}|} \\
&\times \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s \dot{+} u) 2^{j_1} \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i}, \nu_1 2^{-i})}(s) \\
&\times \sum_{j_2=0}^{j_1} 1_{[0,2^{-j_2})}(s \dot{+} u \dot{+} e_{j_1} \dot{+} e_{j_2}) 2^{j_2}\, ds \Big)^{1/2} du \Big]^2 \\
\le C\|f\|_\infty^2 2^{-N} &\sum_{i=0}^{N-1} 2^{2i} \Big[\sum_{\mu=1}^{2^i} \int\limits_{(\mu-1)2^{-i}}^{\mu 2^{-i}} \Big(\sum_{\nu=1}^{2^i} \int\limits_{(\nu-1)2^{-i}}^{\nu 2^{-i}} \\
&\times \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s \dot{+} u) 2^{2j_1} q^{2^{i+1}|x-\mu 2^{-i}|}\, ds \Big)^{1/2} du \Big]^2.
\end{aligned}$$

It is easy to see that for each μ there exists a set $S_{j,\mu}$ such that

$$1_{[0,2^{-j})}(s \dot{+} u) = \begin{cases} 1 & \text{if } \nu \in S_{j,\mu}, \\ 0 & \text{if } \nu \notin S_{j,\mu}, \end{cases} \tag{27}$$

where $s \in [(\nu-1)2^{-i}, \nu 2^{-i})$, $u \in [(\mu-1)2^{-i}, \mu 2^{-i})$. Moreover, $|S_{j,\mu}| = 2^{i-j}$ and

$$S_{j,\mu} \subset [\mu - 2^{i-j} + 1, \mu + 2^{i-j} - 1].$$

Then

$$\begin{aligned}
&|V_{N,4}f(x)|^2 \le C\|f\|_\infty^2 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \\
&\quad \times \Big[\sum_{\mu=1}^{2^i} \int\limits_{(\mu-1)2^{-i}}^{\mu 2^{-i}} \Big(\sum_{j=0}^{i-1} \sum_{\nu=\mu-2^{i-j}+1}^{\mu+2^{i-j}-1} \int\limits_{(\nu-1)2^{-i}}^{\nu 2^{-i}} 2^{2j} q^{2^{i+1}|x-\mu 2^{-i}|}\, ds \Big)^{1/2} du \Big]^2 \\
&\qquad \le C\|f\|_\infty^2 2^{-N} \sum_{i=0}^{N-1} 2^i \Big[\sum_{\mu=1}^{2^i} q^{2^i|x-\mu 2^{-i}|} \Big]^2 \le C\|f\|_\infty^2.
\end{aligned}$$

We now prove that $V_{*,4}$ is of weak type $(1,1)$. To this end, we apply Lemma 3. Suppose that $\varrho > \|f\|_1$ and take the Calderón–Zygmund decomposition $f = g + b$ as in Lemma 1. Let $\Omega = \bigcup_{k=0}^\infty I_k$ and suppose that $x \notin 8\Omega$. With the help of (26) we estimate $V_{N,4}b(x)$ as follows:

$$|V_{N,4}b(x)|^2 \le C2^{-N} \sum_{i=0}^{N-1} \Big[\int_0^1 \Big(\int_0^1 \Big| \int_0^1 b(t) \sum_{j=0}^{i-1} \sum_{l=j}^{i-1} 1_{[2^{-l-1},2^{-l})}(s \dot{+} u \dot{+} e_j) \varepsilon_{j,l}$$
$$\times 2^{j-1} r_i(s \dot{+} u \dot{+} e_j) G_i^{(m,k)}(t, s \dot{+} e_j) G_i^{(m,-k)}(x,u)\, dt \Big|^2 ds \Big)^{1/2} du \Big]^2.$$

Let $f_n := b 1_{I_n}$ and $2^{-\eta_n} := |I_n|$ $(n \in \mathbb{N})$. Then

$$|V_{N,4}b(x)|^2$$
$$\le C2^{-N} \sum_{i=0}^{N-1} \Big[\int_0^1 \Big(\int_0^1 \Big| \int_0^1 \sum_{n=0}^{\infty} f_n(t) \sum_{j=0}^{i-1} 1_{\{\eta_n \le j\}} \sum_{l=j}^{i-1} 1_{[2^{-l-1},2^{-l})}(s \dot{+} u \dot{+} e_j) \varepsilon_{j,l}$$
$$\times 2^{j-1} r_i(s \dot{+} u \dot{+} e_j) G_i^{(m,k)}(t, s \dot{+} e_j) G_i^{(m,-k)}(x,u)\, dt \Big|^2 ds \Big)^{1/2} du \Big]^2$$
$$+ C2^{-N} \sum_{i=0}^{N-1} \Big[\int_0^1 \Big(\int_0^1 \Big| \int_0^1 \sum_{n=0}^{\infty} f_n(t) \sum_{j=0}^{i-1} 1_{\{\eta_n > j\}} \sum_{l=j}^{i-1} 1_{[2^{-l-1},2^{-l})}(s \dot{+} u \dot{+} e_j) \varepsilon_{j,l}$$
$$\times 2^{j-1} r_i(s \dot{+} u \dot{+} e_j) G_i^{(m,k)}(t, s \dot{+} e_j) G_i^{(m,-k)}(x,u)\, dt \Big|^2 ds \Big)^{1/2} du \Big]^2.$$

Consider the following sets of vectors (n_1, n_2, j_1, j_2):

$$\begin{aligned}
J_1 &:= \{\eta_{n_1} \le j_1 \le \eta_{n_2} \le j_2\}, & J_2 &:= \{\eta_{n_1} \le \eta_{n_2} \le j_1 \le j_2\},\\
J_3 &:= \{\eta_{n_2} \le \eta_{n_1} \le j_1 \le j_2\}, & J_4 &:= \{\eta_{n_1} \le \eta_{n_2} \le j_2 \le j_1\},\\
J_5 &:= \{\eta_{n_2} \le \eta_{n_1} \le j_2 \le j_1\}, & J_6 &:= \{\eta_{n_2} \le j_2 \le \eta_{n_1} \le j_1\},\\
J_7 &:= \{j_1 < \eta_{n_1} < j_2 < \eta_{n_2}\}, & J_8 &:= \{j_1 < j_2 < \eta_{n_1} < \eta_{n_2}\},\\
J_9 &:= \{j_2 < j_1 < \eta_{n_1} < \eta_{n_2}\}, & J_{10} &:= \{j_1 < j_2 < \eta_{n_2} < \eta_{n_1}\},\\
J_{11} &:= \{j_2 < j_1 < \eta_{n_2} < \eta_{n_1}\}, & J_{12} &:= \{j_2 < \eta_{n_2} < j_1 < \eta_{n_1}\}.
\end{aligned}$$

Then

$$\Big| \int_0^1 \sum_{n=0}^{\infty} f_n(t) \sum_{j=0}^{i-1} 1_{\{\eta_n \le j\}} \sum_{l=j}^{i-1} 1_{[2^{-l-1},2^{-l})}(s \dot{+} u \dot{+} e_j) \varepsilon_{j,l}$$
$$\times 2^{j-1} r_i(s \dot{+} u \dot{+} e_j) G_i^{(m,k)}(t, s \dot{+} e_j) G_i^{(m,-k)}(x,u)\, dt \Big|^2$$
$$= \int_0^1 \sum_{n_1=0}^{\infty} f_{n_1}(t_1) \sum_{j_1=0}^{i-1} 1_{\{\eta_{n_1} \le j_1\}} \sum_{l_1=j_1}^{i-1} 1_{[2^{-l_1-1},2^{-l_1})}(s \dot{+} u \dot{+} e_{j_1}) \varepsilon_{j_1,l_1}$$
$$\times 2^{j_1-1} r_i(s \dot{+} u \dot{+} e_{j_1}) G_i^{(m,k)}(t_1, s \dot{+} e_{j_1}) G_i^{(m,-k)}(x,u)\, dt_1$$
$$\times \int_0^1 \sum_{n_2=0}^{\infty} f_{n_2}(t_2) \sum_{j_2=0}^{i-1} 1_{\{\eta_{n_2} \le j_2\}} \sum_{l_2=j_2}^{i-1} 1_{[2^{-l_2-1},2^{-l_2})}(s \dot{+} u \dot{+} e_{j_2}) \varepsilon_{j_2,l_2}$$

$$\times 2^{j_2-1} r_i(s \dot{+} u \dot{+} e_{j_2}) G_i^{(m,k)}(t_2, s \dot{+} e_{j_2}) G_i^{(m,-k)}(x,u)\, dt_2$$

$$= \sum_{l=1}^{6} \int_0^1\!\!\int_0^1 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} f_{n_1}(t_1) f_{n_2}(t_2) \sum_{j_1=0}^{i-1} \sum_{l_1=j_1}^{i-1} 1_{[2^{-l_1-1}, 2^{-l_1})}(s \dot{+} u \dot{+} e_{j_1}) \varepsilon_{j_1,l_1}$$

$$\times 2^{j_1-1} r_i(s \dot{+} u \dot{+} e_{j_1}) G_i^{(m,k)}(t_1, s \dot{+} e_{j_1}) G_i^{(m,-k)}(x,u)$$

$$\times \sum_{j_2=0}^{i-1} \sum_{l_2=j_2}^{i-1} 1_{[2^{-l_2-1}, 2^{-l_2})}(s \dot{+} u \dot{+} e_{j_2}) \varepsilon_{j_2,l_2} 2^{j_2-1} r_i(s \dot{+} u \dot{+} e_{j_2})$$

$$\times G_i^{(m,k)}(t_2, s \dot{+} e_{j_2}) G_i^{(m,-k)}(x,u) 1_{J_l}\, dt_1\, dt_2$$

$$=: \sum_{l=1}^{6} E_l(s,u).$$

If we define $E_l(s,u)$ in the same way for $l = 7, 8, \ldots, 12$, then we get immediately

$$|V_{N,4} b(x)|^2 \le \sum_{l=1}^{12} C 2^{-N} \sum_{i=0}^{N-1} \Big[\int_0^1 \Big(\int_0^1 \Big| E_l(s,u) \Big| \, ds \Big)^{1/2} du \Big]^2 =: \sum_{l=1}^{12} A_{N,l}(x). \tag{28}$$

Observe that the cases $l = 1, 2, 3$ are symmetric to $l = 6, 5, 4$ and the cases $l = 7, 8, 9$ are symmetric to $l = 12, 11, 10$. Therefore it is enough to consider $l = 1, 2, 3, 7, 8, 9$. Suppose first that $l = 2$. If $k \le m$ then we integrate by parts in t_2 to obtain

$$|E_2(s,u)|$$

$$= \Big| \int_0^1\!\!\int_0^1 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} f_{n_1}(t_1) F_{n_2}(t_2) \sum_{j_1=0}^{i-1} \sum_{l_1=j_1}^{i-1} 1_{[2^{-l_1-1}, 2^{-l_1})}(s \dot{+} u \dot{+} e_{j_1}) \varepsilon_{j_1,l_1}$$

$$\times 2^{j_1-1} r_i(s \dot{+} u \dot{+} e_{j_1}) G_i^{(m,k)}(t_1, s \dot{+} e_{j_1}) G_i^{(m,-k)}(x,u)$$

$$\times \sum_{j_2=0}^{i-1} \sum_{l_2=j_2}^{i-1} 1_{[2^{-l_2-1}, 2^{-l_2})}(s \dot{+} u \dot{+} e_{j_2}) \varepsilon_{j_2,l_2} 2^{j_2-1} r_i(s \dot{+} u \dot{+} e_{j_2})$$

$$\times D_{t_2} G_i^{(m,k)}(t_2, s \dot{+} e_{j_2}) G_i^{(m,-k)}(x,u) 1_{\{\eta_{n_1} \le \eta_{n_2} \le j_1 \le j_2\}}\, dt_1\, dt_2 \Big|.$$

Assume that $u \in [(\mu-1)2^{-i}, \mu 2^{-i})$ for some $\mu = 1, \ldots, 2^i$. Applying (1) and (10) we conclude that

$$|E_2(s,u)| \le C\int_0^1\int_0^1 \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} |f_{n_1}(t_1)|\,|F_{n_2}(t_2)| \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s\dot{+}u\dot{+}e_{j_1})2^{j_1}2^{2i}$$
$$\times \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})\dot{+}e_{j_1}}(s) q^{2^i|t_1-\nu_1 2^{-i}|} q^{2^i|x-\mu 2^{-i}|}$$
$$\times \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s\dot{+}u\dot{+}e_{j_2})2^{j_2}2^{3i} \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})\dot{+}e_{j_2}}(s)$$
$$\times q^{2^i|t_2-\nu_2 2^{-i}|} q^{2^i|x-\mu 2^{-i}|} 1_{\{\eta_{n_1}\le\eta_{n_2}\le j_1\le j_2\}}\, dt_1\, dt_2.$$

By (v) of Lemma 1,

$$|F_{n_2}(t_2)| \le 1_{I_{n_2}}(t_2) \int_{I_{n_2}} |f_{n_2}(x)|\, dx \le 2\varrho |I_{n_2}| 1_{I_{n_2}}(t_2) = 2\varrho 2^{-\eta_{n_2}} 1_{I_{n_2}}(t_2).$$

Since the dyadic intervals I_{k_2} are disjoint and $\eta_{n_1} \le \eta_{n_2}$ in case $l = 2$, we conclude

$$\sum_{n_2=0}^{\infty} |F_{n_2}(t_2)| 1_{\{\eta_{n_1}\le\eta_{n_2}\}} \le 2\varrho 2^{-\eta_{n_1}} \sum_{n_2=0}^{\infty} 1_{I_{n_2}}(t_2) \le 2\varrho 2^{-\eta_{n_1}}.$$

Hence

$$|E_2(s,u)| \tag{29}$$
$$\le C\varrho 2^{5i} \int_0^1\int_0^1 \sum_{n_1=0}^{\infty} 2^{-\eta_{n_1}} |f_{n_1}(t_1)| \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s\dot{+}u\dot{+}e_{j_1})2^{j_1}$$
$$\times \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})\dot{+}e_{j_1}}(s) q^{2^i|t_1-\nu_1 2^{-i}|} q^{2^i|x-\mu 2^{-i}|}$$
$$\times \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s\dot{+}u\dot{+}e_{j_2})2^{j_2} \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})\dot{+}e_{j_2}}(s)$$
$$\times q^{2^i|t_2-\nu_2 2^{-i}|} q^{2^i|x-\mu 2^{-i}|} 1_{\{\eta_{n_1}\le j_1\le j_2\}}\, dt_1\, dt_2.$$

If $k = m+1$ then

$$|E_2(s,u)|$$
$$= \Big|\int_0^1\int_0^1 \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} f_{n_1}(t_1)(E_{i+1}f_{n_2})(t_2) \sum_{j_1=0}^{i-1}\sum_{l_1=j_1}^{i-1} 1_{[2^{-l_1-1},2^{-l_1})}(s\dot{+}u\dot{+}e_{j_1})$$
$$\times \varepsilon_{j_1,l_1} 2^{j_1-1} r_i(s\dot{+}u\dot{+}e_{j_1}) G_i^{(m,m+1)}(t_1, s\dot{+}e_{j_1}) G_i^{(m,-m-1)}(x,u)$$

$$\times \sum_{j_2=0}^{i-1} \sum_{l_2=j_2}^{i-1} 1_{[2^{-l_2-1},2^{-l_2})}(s \dot{+} u \dot{+} e_{j_2}) \varepsilon_{j_2,l_2} 2^{j_2-1} r_i(s \dot{+} u \dot{+} e_{j_2})$$

$$\times G_i^{(m,m+1)}(t_2, s \dot{+} e_{j_2}) G_i^{(m,-m-1)}(x,u) 1_{\{\eta_{n_1} \le \eta_{n_2} \le j_1 \le j_2\}} \, dt_1 \, dt_2 \Big|.$$

Recall that $G_i^{(m,m+1)}(\cdot, s)$ is $\mathcal{F}_{i+1}$-measurable. Applying (22) and (23) we can obtain inequality (29) in the same way as above.

Integrating by t_2 in (29) we have

$$(30) \qquad |E_2(s,u)| \le C\varrho 2^{4i} \int_0^1 \sum_{n_1=0}^{\infty} 2^{-\eta_{n_1}} |f_{n_1}(t_1)| \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s \dot{+} u \dot{+} e_{j_1}) 2^{j_1}$$

$$\times \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i}, \nu_1 2^{-i}) \dot{+} e_{j_1}}(s) q^{2^i |t_1 - \nu_1 2^{-i}|} q^{2^{i+1}|x-\mu 2^{-i}|}$$

$$\times \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s \dot{+} u \dot{+} e_{j_2})$$

$$\times 2^{j_2} \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i}, \nu_2 2^{-i}) \dot{+} e_{j_2}}(s) 1_{\{\eta_{n_1} \le j_1 \le j_2\}} \, dt_1.$$

Substituting this into (28) we get

$$A_{N,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{4i} \Big[\sum_{\mu=1}^{2^i} \int_{(\mu-1)2^{-i}}^{\mu 2^{-i}} \Big(\int_0^1 \int_0^1 \sum_{n=0}^{\infty} 2^{-\eta_n} |f_n(t)|$$

$$\times \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s \dot{+} u \dot{+} e_{j_1}) 2^{j_1} \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i}, \nu_1 2^{-i}) \dot{+} e_{j_1}}(s)$$

$$\times q^{2^i|t-\nu_1 2^{-i}|} q^{2^{i+1}|x-\mu 2^{-i}|} \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s \dot{+} u \dot{+} e_{j_2}) 2^{j_2}$$

$$\times \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i}, \nu_2 2^{-i}) \dot{+} e_{j_2}}(s) 1_{\{\eta_n \le j_1 \le j_2\}} \, dt \, ds \Big)^{1/2} du \Big]^2$$

$$= C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{4i} \Big[\sum_{\mu=1}^{2^i} \int_{(\mu-1)2^{-i}}^{\mu 2^{-i}} \Big(\int_0^1 \int_0^1 \sum_{n=0}^{\infty} 2^{-\eta_n} |f_n(t)|$$

$$\times \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s \dot{+} u \dot{+} e_{j_1} \dot{+} e_{j_2}) 2^{j_1}$$

$$\times \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})\dot{+}e_{j_1}\dot{+}e_{j_2}}(s)$$

$$\times q^{2^i|t-\nu_1 2^{-i}|} q^{2^{i+1}|x-\mu 2^{-i}|} \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s \dot{+} u) 2^{j_2}$$

$$\times \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})}(s) 1_{\{\eta_n \le j_1 \le j_2\}}\, dt\, ds\Big)^{1/2} du\Big]^2.$$

It is easy to see that if $s \dot{+} u \in [0, 2^{-j_2})$, then $s \dot{+} u \dot{+} e_{j_1} \dot{+} e_{j_2} \in [0, 2^{-j_1})$ ($j_1 \le j_2$). As the sets $[(\nu_1-1)2^{-i}, \nu_1 2^{-i}) \dot{+} e_{j_1} \dot{+} e_{j_2}$ are disjoint dyadic intervals, if

$$[(\nu_1-1)2^{-i}, \nu_1 2^{-i}) \dot{+} e_{j_1} \dot{+} e_{j_2} = [(\nu_2-1)2^{-i}, \nu_2 2^{-i}),$$

then $\nu_1 2^{-i} = \nu_2 2^{-i} \dot{+} e_{j_1} \dot{+} e_{j_2}$. Thus

$$A_{N,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{4i} \Big[\sum_{\mu=1}^{2^i} \int_{(\mu-1)2^{-i}}^{\mu 2^{-i}} \Big(\int_0^1\int_0^1 \sum_{n=0}^{\infty} 2^{-\eta_n} |f_n(t)|$$

$$\times \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s \dot{+} u) 2^{j_2} \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})}(s)$$

$$\times q^{2^i|t-\nu_2 2^{-i}\dot{+}e_{j_1}\dot{+}e_{j_2}|} q^{2^{i+1}|x-\mu 2^{-i}|} 1_{\{\eta_n \le j_1 \le j_2\}}\, dt\, ds\Big)^{1/2} du\Big]^2.$$

By (27),

$$A_{N,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{4i} \Big[\sum_{\mu=1}^{2^i} \int_{(\mu-1)2^{-i}}^{\mu 2^{-i}} \Big(\int_0^1\int_0^1 \sum_{n=0}^{\infty} 2^{-\eta_n} |f_n(t)|$$

$$\times \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} \sum_{\nu=\mu-2^{i-j_2}+1}^{\mu+2^{i-j_2}-1} 1_{[(\nu-1)2^{-i},\nu 2^{-i})}(s)$$

$$\times q^{2^i|t-\nu 2^{-i}\dot{+}e_{j_1}\dot{+}e_{j_2}|} q^{2^{i+1}|x-\mu 2^{-i}|} 1_{\{\eta_n \le j_1 \le j_2\}}\, dt\, ds\Big)^{1/2} du\Big]^2$$

$$\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{i} \Big[\sum_{\mu=1}^{2^i} \Big(\int_0^1 \sum_{n=0}^{\infty} 2^{-\eta_n} |f_n(t)| \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2}$$

$$\times \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|t-(l+\mu)2^{-i}\dot{+}e_{j_1}\dot{+}e_{j_2}|} q^{2^{i+1}|x-\mu 2^{-i}|} 1_{\{\eta_n \le j_1 \le j_2\}}\, dt\Big)^{1/2}\Big]^2.$$

Hölder's inequality implies

$$\begin{aligned}
A_{N,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \Big[\sum_{\mu=1}^{2^i} q^{2^{i-1}|x-\mu 2^{-i}|} \\
\times \Big(\sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{\eta_n \le j_1 \le j_2\}} \\
\times \int_0^1 |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|t-(l+\mu)2^{-i} \dotplus e_{j_1} \dotplus e_{j_2}|} q^{2^i|x-\mu 2^{-i}|}\, dt \Big)^{1/2} \Big]^2 \\
\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \Big(\sum_{\mu=1}^{2^i} q^{2^i|x-\mu 2^{-i}|} \Big) \\
\times \Big(\sum_{\mu=1}^{2^i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{\eta_n \le j_1 \le j_2\}} \\
\times \int_0^1 |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|t-(l+\mu)2^{-i} \dotplus e_{j_1} \dotplus e_{j_2}|} q^{2^i|x-\mu 2^{-i}|}\, dt \Big) \\
\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{\mu=1}^{2^i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{\eta_n \le j_1 \le j_2\}} \\
\times \int_0^1 |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|t-(l+\mu)2^{-i} \dotplus 2^{-j_1-1} \dotplus 2^{-j_2-1}|} q^{2^i|x-\mu 2^{-i}|}\, dt.
\end{aligned}$$

Taking into account inequality (20), we derive

$$\begin{aligned}
A_{N,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{\eta_n \le j_1 \le j_2\}} \\
\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|x-t+l2^{-i} \dotplus 2^{-j_1-1} \dotplus 2^{-j_2-1}|}\, dt.
\end{aligned} \tag{31}$$

Note that $x \notin 8\Omega$ and so $x \notin 8I_n$ $(n \in \mathbb{N})$. It is easy to show that

$$|x - t + l2^{-i} \dotplus 2^{-j_1-1} \dotplus 2^{-j_2-1}| \ge C|x-t|,$$

because $\eta_n \le j_1 \le j_2$. From this it follows that

$$\begin{aligned}
A_{N,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{\eta_n \le j_1 \le j_2\}} \\
\times \int_{I_n} |f_n(t)| 2^{i-j_2} q^{C2^i|x-t|}\, dt
\end{aligned}$$

$$\leq C\varrho\sum_{i=0}^{N-1}2^i\sum_{n=0}^{\infty}2^{-\eta_n}\sum_{j_2=0}^{i-1}\sum_{j_1=0}^{j_2}2^{j_1}\int_{I_n}|f_n(t)|q^{C2^i|x-t|}\,dt$$
$$\leq C\varrho\sum_{i=0}^{\infty}2^{2i}\sum_{n=0}^{\infty}2^{-\eta_n}\int_{I_n}|f_n(t)|q^{C2^i|x-t|}\,dt$$
$$\leq C\varrho\sum_{n=0}^{\infty}2^{-\eta_n}\int_{I_n}|f_n(t)|\,|x-t|^{-2}\,dt.$$

Then by Lemma 1,

$$\int_{(8\Omega)^c}\sup_{N\in\mathbb{N}}A_{N,2}(x)\,dx\leq C\varrho\sum_{n=0}^{\infty}2^{-\eta_n}\int_{I_n}|f_n(t)|\int_{(8I_n)^c}|x-t|^{-2}\,dx\,dt$$
$$\leq C\varrho\sum_{n=0}^{\infty}\int_{I_n}|f_n(t)|\,dt\leq C\varrho\|f\|_1.$$

The expression $A_{N,1}$ can be estimated exactly in the same way.

If we replace η_{n_1} and η_{n_2} and integrate over t_1 in (29), we deduce similarly to (30) and (31) that

$$|E_3(s,u)|\leq C\varrho 2^{4i}\int_0^1\sum_{n_2=0}^{\infty}2^{-\eta_{n_2}}|f_{n_2}(t_2)|\sum_{j_1=0}^{i-1}1_{[0,2^{-j_1})}(s\dotplus u\dotplus e_{j_1})2^{j_1}$$
$$\times\sum_{\nu_1=1}^{2^i}1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})\dotplus e_{j_1}}(s)q^{2^{i+1}|x-\mu 2^{-i}|}\sum_{j_2=0}^{i-1}1_{[0,2^{-j_2})}(s\dotplus u\dotplus e_{j_2})$$
$$\times 2^{j_2}\sum_{\nu_2=1}^{2^i}1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})\dotplus e_{j_2}}(s)q^{2^i|t_2-\nu_2 2^{-i}|}1_{\{\eta_{n_2}\leq j_1\leq j_2\}}\,dt_2$$

and

$$A_{N,3}(x)\leq C\varrho 2^{-N}\sum_{i=0}^{N-1}2^i\sum_{n=0}^{\infty}2^{-\eta_n}\sum_{j_1=0}^{i-1}2^{j_1}\sum_{j_2=0}^{i-1}2^{j_2}1_{\{\eta_n\leq j_1\leq j_2\}}$$
$$\times\int_{I_n}|f_n(t)|\sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1}q^{2^i|x-t+l2^{-i}|}\,dt.$$

The inequality

$$\int_{(8\Omega)^c}\sup_{N\in\mathbb{N}}A_{N,3}(x)\,dx\leq C\varrho\|f\|_1$$

can be shown as above for $l=2$.

Suppose now that $l = 7$. Similarly to (30) and (31) we obtain

$$
\begin{aligned}
|E_7(s,u)| \le C\varrho 2^{4i} \int_0^1 \sum_{n_1=0}^{\infty} |f_{n_1}(t_1)| \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s \dotplus u \dotplus e_{j_1}) 2^{j_1} \\
\times \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})\dotplus e_{j_1}}(s) q^{2^i|t_1-\nu_1 2^{-i}|} q^{2^{i+1}|x-\mu 2^{-i}|} \\
\times \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s \dotplus u \dotplus e_{j_2}) \\
\times \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})\dotplus e_{j_2}}(s) 1_{\{j_1<\eta_{n_1}<j_2\}}\, dt_1
\end{aligned}
$$

and

$$
\begin{aligned}
A_{N,7}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{\{j_1<\eta_n<j_2\}} \\
\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|x-t+l2^{-i}\dotplus 2^{-j_1-1}\dotplus 2^{-j_2-1}|}\, dt.
\end{aligned}
$$

Obviously,

$$
A_{N,7}(x) = A_{N,7,1}(x) + A_{N,7,2}(x),
$$

where

$$
\begin{aligned}
A_{N,7,1}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{\{j_1<\eta_n<j_2\}} \\
\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|x-t+l2^{-i}\dotplus 2^{-j_1-1}\dotplus 2^{-j_2-1}|} \\
\times 1_{\{|x-t\dotplus 2^{-j_1-1}|\le 8\cdot 2^{-j_2}\}}\, dt
\end{aligned}
$$

and

$$
\begin{aligned}
A_{N,7,2}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{\{j_1<\eta_n<j_2\}} \\
\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|x-t+l2^{-i}\dotplus 2^{-j_1-1}\dotplus 2^{-j_2-1}|} \\
\times 1_{\{|x-t\dotplus 2^{-j_1-1}|>8\cdot 2^{-j_2}\}}\, dt.
\end{aligned}
$$

Therefore,

$$A_{N,7,1}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{\{j_1<\eta_n<j_2\}} \times \int_{I_n} |f_n(t)| 1_{\{|x-t\dot{+}2^{-j_1-1}|\le 8\cdot 2^{-j_2}\}}\,dt$$
$$\le C\varrho \sum_{n=0}^{\infty} \sum_{j_1=0}^{\eta_n-1} 2^{j_1} \sum_{j_2=\eta_n+1}^{\infty} \int_{I_n} |f_n(t)| 1_{\{|x-t\dot{+}2^{-j_1-1}|\le 8\cdot 2^{-j_2}\}}\,dt$$

and so

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,7,1}(x)\,dx \le C\varrho \sum_{n=0}^{\infty} \sum_{j_1=0}^{\eta_n-1} 2^{j_1} \sum_{j_2=\eta_n+1}^{\infty} 2^{-j_2} \int_{I_n} |f_n(t)|\,dt \le C\varrho \|f\|_1.$$

Since

$$|x-t+l2^{-i} \dot{+} 2^{-j_1-1} \dot{+} 2^{-j_2-1}| \ge C|x-t \dot{+} 2^{-j_1-1}|$$

in $A_{N,7,2}(x)$, we have

$$A_{N,7,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{-j_2} 1_{\{j_1<\eta_n<j_2\}} \times \int_{I_n} |f_n(t)| q^{C2^i|x-t\dot{+}2^{-j_1-1}|} 1_{\{|x-t\dot{+}2^{-j_1-1}|>8\cdot 2^{-j_2}\}}\,dt.$$

We may suppose that $x \ge t$. As $x \notin 8I_n$ $(n\in\mathbb{N})$, $A_{N,7,2}(x)$ can estimated by the sum of the terms

$$A_{N,7,2,1}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{-j_2} 1_{\{j_1<\eta_n<j_2\}} 1_{\{j_1\ge l\}} \times \int_{I_n} |f_n(t)| q^{C2^i|x-t\dot{+}2^{-j_1-1}|} 1_{\{|x-t\dot{+}2^{-j_1-1}|>8\cdot 2^{-j_2}\}} 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt,$$

$$A_{N,7,2,2}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{-j_2} 1_{\{j_1<\eta_n<j_2\}} 1_{\{j_1<l-1\}} \times \int_{I_n} |f_n(t)| q^{C2^i|x-t\dot{+}2^{-j_1-1}|} 1_{\{|x-t\dot{+}2^{-j_1-1}|>8\cdot 2^{-j_2}\}} 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt$$

and

$$A_{N,7,2,3}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{-j_2} 1_{\{j_1<\eta_n<j_2\}} 1_{\{j_1=l-1\}} \times \int_{I_n} |f_n(t)| q^{C2^i|x-t\dot{+}2^{-j_1-1}|} 1_{\{|x-t\dot{+}2^{-j_1-1}|>8\cdot 2^{-j_2}\}} 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt.$$

It is easy to see that

$$A_{N,7,2,1}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{-j_2} 1_{\{j_1<\eta_n<j_2\}} 1_{\{j_1\ge l\}}$$

$$\times \int_{I_n} |f_n(t)| q^{C2^i|x-t|} 1_{\{|x-t\dot{+}2^{-j_1-1}|>8\cdot 2^{-j_2}\}} 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt$$

$$\le C\varrho \sum_{i=0}^{N-1} 2^{i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_2=0}^{i-1} \sum_{j_1=0}^{j_2} 2^{j_1}$$

$$\times \int_{I_n} |f_n(t)| q^{C2^i|x-t|} 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt$$

$$\le C\varrho \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \int_{I_n} |f_n(t)| q^{C2^i|x-t|}\,dt$$

$$\le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \int_{I_n} |f_n(t)|\, |x-t|^{-2}\,dt.$$

This implies that

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,7,2,1}(x)\,dx \le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \int_{I_n} |f_n(t)| \int_{(8I_n)^c} |x-t|^{-2}\,dx\,dt$$

$$\le C\varrho \|f\|_1.$$

For $A_{N,7,2,2}(x)$ we obtain

$$A_{N,7,2,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{-j_2} 1_{\{j_1<\eta_n<j_2\}} 1_{\{j_1<l-1\}}$$

$$\times \int_{I_n} |f_n(t)| q^{C2^{i-j_1-1}} 1_{\{|x-t\dot{+}2^{-j_1-1}|>8\cdot 2^{-j_2}\}} 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt$$

$$\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{l-2} 2^{j_1} \sum_{j_2=\eta_n+1}^{i-1} 2^{-j_2}$$

$$\times \int_{I_n} |f_n(t)| q^{C2^{i-j_1-1}} 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt.$$

Since $q^x \le C/x$,

$$A_{N,7,2,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{l-2} 2^{2j_1}$$

$$\times \int_{I_n} |f_n(t)| 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt$$

$$\le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^{2l} \int_{I_n} |f_n(t)| 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt.$$

Integrating over x, we conclude that

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,7,2,2}(x)\,dx \le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^l \int_{I_n} |f_n(t)|\,dt \le C\varrho \|f\|_1.$$

The term $A_{N,7,2,3}(x)$ can be written as

$$A_{N,7,2,3}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=\eta_n+1}^{i-1} 2^{-j_2} \\ \times \int_{I_n} |f_n(t)| q^{C2^i|x-t\dot{+}2^{-l}|} 1_{\{|x-t\dot{+}2^{-l}|>8\cdot 2^{-j_2}\}} 1_{\{x-t\in[2^{-l},2^{-l+1})\}}\,dt.$$

Obviously, $x-t\dot{+}2^{-l}\in[2^{-j_2+3},2^{-l})$. Thus

$$\begin{aligned} A_{N,7,2,3}(x) &\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=\eta_n+1}^{i-1} 2^{-j_2} \\ &\qquad \times \sum_{s=l+1}^{j_2-3} \int_{I_n} |f_n(t)| q^{C2^i|x-t\dot{+}2^{-l}|} 1_{\{x-t\dot{+}2^{-l}\in[2^{-s},2^{-s+1})\}}\,dt \\ &\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=\eta_n+1}^{i-1} 2^{-j_2} \\ &\qquad \times \sum_{s=l+1}^{j_2-3} \int_{I_n} |f_n(t)| q^{C2^{i-s}} 1_{\{x-t\dot{+}2^{-l}\in[2^{-s},2^{-s+1})\}}\,dt \\ &\le C\varrho \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=\eta_n+1}^{\infty} 2^{-j_2} \\ &\qquad \times \sum_{s=l+1}^{j_2-3} 2^s \int_{I_n} |f_n(t)| 1_{\{x-t\dot{+}2^{-l}\in[2^{-s},2^{-s+1})\}}\,dt. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,7,2,3}(x)\,dx &\le C\varrho \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=\eta_n+1}^{\infty} 2^{-j_2} \sum_{s=l+1}^{j_2-3} \int_{I_n} |f_n(t)|\,dt \\ &\le C\varrho \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} \sum_{j_2=\eta_n+1}^{\infty} 2^{l-j_2}(j_2-l) \int_{I_n} |f_n(t)|\,dt \\ &\le C\varrho \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} 2^{l-\eta_n}(\eta_n-l) \int_{I_n} |f_n(t)|\,dt \\ &\le C\varrho \sum_{n=0}^{\infty} \int_{I_n} |f_n(t)|\,dt \le C\varrho \|f\|_1. \end{aligned}$$

For $l = 8$ we can see that

$$|E_8(s,u)| \le C\varrho 2^{4i} \int_0^1 \sum_{n_1=0}^{\infty} 2^{-\eta_{n_1}} |f_{n_1}(t_1)| \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s \dot{+} u \dot{+} e_{j_1}) 2^{j_1}$$
$$\times \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})\dot{+}e_{j_1}}(s) q^{2^i|t_1-\nu_1 2^{-i}|} q^{2^{i+1}|x-\mu 2^{-i}|}$$
$$\times \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s \dot{+} u \dot{+} e_{j_2})$$
$$\times 2^{j_2} \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})\dot{+}e_{j_2}}(s) 1_{\{j_1<j_2<\eta_{n_1}\}} \, dt_1$$

and

$$A_{N,8}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{j_1<j_2<\eta_n\}}$$
$$\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|x-t+l2^{-i}\dot{+}2^{-j_1-1}\dot{+}2^{-j_2-1}|} \, dt.$$

The right hand side can be split into the sum of

$$A_{N,8,1}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{j_1<j_2<\eta_n\}}$$
$$\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|x-t+l2^{-i}\dot{+}2^{-j_1-1}\dot{+}2^{-j_2-1}|} \, dt$$
$$\times 1_{\{8\cdot 2^{\eta_n-j_2} I_n \dot{+} 2^{-j_1-1}\}}(x)$$

and

$$A_{N,8,2}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{j_1<j_2<\eta_n\}}$$
$$\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_2}+1}^{2^{i-j_2}-1} q^{2^i|x-t+l2^{-i}\dot{+}2^{-j_1-1}\dot{+}2^{-j_2-1}|} \, dt$$
$$\times 1_{\{(8\cdot 2^{\eta_n-j_2} I_n \dot{+} 2^{-j_1-1})^c\}}(x).$$

One can see as above that

$$A_{N,8,1}(x) \le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_2=0}^{\eta_n-1} 2^{j_2} \\ \times \sum_{j_1=0}^{j_2-1} 2^{j_1} \int_{I_n} |f_n(t)|\, dt\, 1_{\{8\cdot 2^{\eta_n-j_2} I_n \dot{+} 2^{-j_1-1}\}}(x)$$

and

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,8,1}(x)\, dx \le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_2=0}^{\eta_n-1} \sum_{j_1=0}^{j_2-1} 2^{j_1} \int_{I_n} |f_n(t)|\, dt \le C\varrho \|f\|_1.$$

Moreover,

$$A_{N,8,2}(x) \le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{\{j_1<j_2<\eta_n\}} \\ \times \int_{I_n} |f_n(t)| q^{C2^i|x-t_n \dot{+} 2^{-j_1-1}|}\, dt\, 1_{\{(8\cdot 2^{\eta_n-j_2} I_n \dot{+} 2^{-j_1-1})^c\}}(x),$$

where t_n denotes the center of I_n. If we suppose again that $x \ge t$ then $A_{N,8,2}(x)$ can be estimated by the sum of

$$A_{N,8,2,1}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{\{j_1<j_2<\eta_n\}} 1_{\{j_1\ge l\}} \\ \times \int_{I_n} |f_n(t)| q^{C2^i|x-t_n \dot{+} 2^{-j_1-1}|}\, dt\, 1_{\{(8\cdot 2^{\eta_n-j_2} I_n \dot{+} 2^{-j_1-1})^c\}}(x) 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}},$$

$$A_{N,8,2,2}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{\{j_1<j_2<\eta_n\}} 1_{\{j_1< l-1\}} \\ \times \int_{I_n} |f_n(t)| q^{C2^i|x-t_n \dot{+} 2^{-j_1-1}|}\, dt\, 1_{\{(8\cdot 2^{\eta_n-j_2} I_n \dot{+} 2^{-j_1-1})^c\}}(x) 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}},$$

and

$$A_{N,8,2,3}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 1_{\{j_1<j_2<\eta_n\}} 1_{\{j_1= l-1\}} \\ \times \int_{I_n} |f_n(t)| q^{C2^i|x-t_n \dot{+} 2^{-j_1-1}|}\, dt\, 1_{\{(8\cdot 2^{\eta_n-j_2} I_n \dot{+} 2^{-j_1-1})^c\}}(x) 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}}.$$

Then

$$A_{N,8,2,1}(x) \le C\varrho \sum_{i=0}^{N-1} 2^{i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_2=0}^{i-1} \sum_{j_1=0}^{j_2-1} 2^{j_1} \\ \times \int_{I_n} |f_n(t)| q^{C2^i|x-t_n|}\, dt\, 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}}$$

$$\leq C\varrho \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \int_{I_n} |f_n(t)| q^{C2^i|x-t_n|}\, dt$$

$$\leq C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \int_{I_n} |f_n(t)|\, |x-t_n|^{-2}\, dt$$

and so

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,8,2,1}(x)\, dx \leq C\varrho \|f\|_1.$$

As before, we obtain

$$\begin{aligned}
&A_{N,8,2,2}(x)\\
&\quad\leq C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_2=0}^{i-1} \sum_{j_1=0}^{i-1} 2^{j_1} 1_{\{j_1<j_2<\eta_n\}} 1_{\{j_1\leq l-2\}}\\
&\qquad \times \int_{I_n} |f_n(t)| q^{C2^{i-j_1-1}}\, dt\, 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}}\\
&\quad\leq C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_2=0}^{\eta_n-1} \sum_{j_1=0}^{j_2-1} 2^{2j_1} 1_{\{j_1\leq l-2\}}\\
&\qquad \times \int_{I_n} |f_n(t)|\, dt\, 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}}\\
&\quad\leq C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_2=0}^{\eta_n-1} \sum_{j_1=0}^{(j_2-1)\wedge(l-2)} 2^{2j_1} \int_{I_n} |f_n(t)|\, dt\, 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}}\\
&\quad\leq C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_2=0}^{\eta_n-1} \sum_{j_1=0}^{j_2-1} 2^{2j_1} 1_{\{j_2\leq l-1\}} \int_{I_n} |f_n(t)|\, dt\, 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}}\\
&\qquad + C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} \sum_{j_2=0}^{\eta_n-1} \sum_{j_1=0}^{l-2} 2^{2j_1} 1_{\{j_2> l-1\}} \int_{I_n} |f_n(t)|\, dt\, 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}}\\
&\quad =: A_{N,8,2,2,1}(x) + A_{N,8,2,2,2}(x).
\end{aligned}$$

It is easy to see that

$$A_{N,8,2,2,1}(x) \leq C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^{2l} \int_{I_n} |f_n(t)|\, dt\, 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}}$$

and

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,8,2,2,1}(x)\, dx \leq C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^{l} \int_{I_n} |f_n(t)|\, dt \leq C\varrho \|f\|_1.$$

Similarly,

$$A_{N,8,2,2,2}(x) \le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^{2l}(\eta_n - l) \int_{I_n} |f_n(t)|\, dt\, 1_{\{x-t_n \in [2^{-l}, 2^{-l+1})\}},$$

which implies that

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,8,2,2,2}(x)\, dx \le C\varrho \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} 2^{l-\eta_n}(\eta_n - l) \int_{I_n} |f_n(t)|\, dt \le C\varrho \|f\|_1.$$

It follows easily that

$$\begin{aligned} A_{N,8,2,3}(x) \le{}& C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^l \\ & \times \sum_{j_2=l}^{\eta_n-1} \int_{I_n} |f_n(t)| q^{C2^i|x-t_n \dot{+} 2^{-l}|}\, dt \\ & \times 1_{\{(8\cdot 2^{\eta_n-j_2} I_n \dot{+} 2^{-l})^c\}}(x) 1_{\{x-t_n\in[2^{-l},2^{-l+1})\}} \\ \le{}& C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=l}^{\eta_n-1} \sum_{s=l+1}^{j_2-2} \\ & \times \int_{I_n} |f_n(t)| q^{C2^i|x-t_n \dot{+} 2^{-l}|}\, dt 1_{\{x-t_n \dot{+} 2^{-l} \in [2^{-s}, 2^{-s+1})\}} \\ \le{}& C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=l}^{\eta_n-1} \sum_{s=l+1}^{j_2-2} \\ & \times \int_{I_n} |f_n(t)| q^{C2^{i-s}}\, dt\, 1_{\{x-t_n \dot{+} 2^{-l} \in [2^{-s}, 2^{-s+1})\}} \\ \le{}& C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=l}^{\eta_n-1} \sum_{s=l+1}^{j_2-2} 2^s \int_{I_n} |f_n(t)|\, dt \\ & \times 1_{\{x-t_n \dot{+} 2^{-l} \in [2^{-s}, 2^{-s+1})\}}. \end{aligned}$$

Hence

$$\begin{aligned} \int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,8,2,3}(x)\, dx &\le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{l=0}^{\eta_n-1} 2^l \sum_{j_2=l}^{\eta_n-1} \sum_{s=l+1}^{j_2-2} \int_{I_n} |f_n(t)|\, dt \\ &\le C\varrho \sum_{n=0}^{\infty} \sum_{l=0}^{\eta_n-1} 2^{l-\eta_n}(\eta_n - l)^2 \int_{I_n} |f_n(t)|\, dt \\ &\le C\varrho \|f\|_1. \end{aligned}$$

Let us investigate the case $l = 9$:

$$
\begin{aligned}
|E_9(s,u)| \le C\varrho 2^{4i} \int_0^1 \sum_{n_1=0}^{\infty} 2^{-\eta_{n_1}} |f_{n_1}(t_1)| \sum_{j_1=0}^{i-1} 1_{[0,2^{-j_1})}(s \dotplus u \dotplus e_{j_1}) 2^{j_1} \\
\times \sum_{\nu_1=1}^{2^i} 1_{[(\nu_1-1)2^{-i},\nu_1 2^{-i})\dotplus e_{j_1}}(s) q^{2^i|t_1-\nu_1 2^{-i}|} q^{2^{i+1}|x-\mu 2^{-i}|} \\
\times \sum_{j_2=0}^{i-1} 1_{[0,2^{-j_2})}(s \dotplus u \dotplus e_{j_2}) \\
\times 2^{j_2} \sum_{\nu_2=1}^{2^i} 1_{[(\nu_2-1)2^{-i},\nu_2 2^{-i})\dotplus e_{j_2}}(s) 1_{\{j_2<j_1<\eta_{n_1}\}} \, dt_1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
A_{N,9}(x) &\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \sum_{j_2=0}^{i-1} 2^{j_2} 1_{\{j_2<j_1<\eta_n\}} \\
&\quad \times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_1}+1}^{2^{i-j_1}-1} q^{2^i|x-t+l2^{-i}|} \, dt \\
&\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{(i-1)\wedge(\eta_n-1)} 2^{2j_1} \\
&\quad \times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_1}+1}^{2^{i-j_1}-1} q^{2^i|x-t+l2^{-i}|} \, dt.
\end{aligned}
$$

The last expression can be split into the sum of

$$
\begin{aligned}
A_{N,9,1}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{(i-1)\wedge(\eta_n-1)} 2^{2j_1} \\
\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_1}+1}^{2^{i-j_1}-1} q^{2^i|x-t+l2^{-i}|} \, dt \, 1_{\{8\cdot 2^{\eta_n-j_1} I_n\}}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
A_{N,9,2}(x) := C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^i \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{(i-1)\wedge(\eta_n-1)} 2^{2j_1} \\
\times \int_{I_n} |f_n(t)| \sum_{l=-2^{i-j_1}+1}^{2^{i-j_1}-1} q^{2^i|x-t+l2^{-i}|} \, dt \, 1_{\{(8\cdot 2^{\eta_n-j_1} I_n)^c\}}(x).
\end{aligned}
$$

For the first term we have

$$A_{N,9,1}(x) \le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{\eta_n-1} 2^{2j_1} \int_{I_n} |f_n(t)|\,dt\, 1_{\{8\cdot 2^{\eta_n-j_1} I_n\}}(x)$$

and

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,9,1}(x)\,dx \le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{\eta_n-1} 2^{j_1} \int_{I_n} |f_n(t)|\,dt \le C\varrho \|f\|_1.$$

Finally,

$$\begin{aligned} A_{N,9,2}(x) &\le C\varrho 2^{-N} \sum_{i=0}^{N-1} 2^{2i} \sum_{n=0}^{\infty} 2^{-\eta_n} \sum_{j_1=0}^{i-1} 2^{j_1} \\ &\quad \times \int_{I_n} |f_n(t)| q^{2^i|x-t|}\,dt\, 1_{\{(8\cdot 2^{\eta_n-j_1} I_n)^c\}}(x) \\ &\le C\varrho \sum_{n=0}^{\infty} 2^{-\eta_n} \int_{I_n} |f_n(t)|\,|x-t|^{-2}\,dt \end{aligned}$$

and consequently,

$$\int_{(8\Omega)^c} \sup_{N\in\mathbb{N}} A_{N,9,2}(x)\,dx \le C\varrho \|f\|_1.$$

Taking into account (28), we conclude that

$$\int_{(8\Omega)^c} |V_{*,4} b(x)|^2\,dx \le C\varrho \|f\|_1.$$

Now Lemma 3 implies that

$$\sup_{\varrho>0} \varrho\lambda(|V_{*,4}f| > \varrho) \le C\|f\|_1.$$

This completes the proof of Theorem 1. ∎

The next corollary follows easily by interpolation.

Corollary 1. *If* $m \ge -1$, $|k| \le m+1$, $0 < r \le 2$ *and* $1 < p \le \infty$ *then*

$$\|S_*^{(m,k),(r)} f\|_p \le C_p \|f\|_p \qquad (f \in L_p).$$

The weak type $(1,1)$ inequality in Theorem 1 and the usual density argument of Marcinkiewicz and Zygmund [13] imply

Corollary 2. *If* $m \ge -1$, $|k| \le m+1$ *and* $0 < r \le 2$ *then* $f \in L_1$ *implies*

$$\left(\frac{1}{n}\sum_{j=1}^{n} |s_j^{(m,k)} f(x) - f(x)|^r\right)^{1/r} \to 0 \quad \textit{for a.e. } x \in [0,1) \textit{ as } n \to \infty.$$

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Department of Numerical Analysis
Eötvös L. University
Pázmány P. sétány 1/C
H-1117 Budapest, Hungary
E-mail: weisz@ludens.elte.hu

Received February 10, 2003
Revised version September 15, 2003 (5141)