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On Nikodym-type sets in high dimensions

by

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Abstract. We prove that the complement of a higher-dimensional Nikodym set must have full Hausdorff dimension.

1. Introduction. In [4] Nikodym constructed a subset F of the unit square in $\mathbb{R}^2$ such that F has planar measure 1, and for every point $x \in F$ there exists a line passing through x intersecting F in that single point. Such paradoxical sets are called *Nikodym sets*.

Falconer [3] extended Nikodym's result to higher dimensions. He proved that for every $n > 2$ there exists a set $F \subset \mathbb{R}^n$ such that the complement of F has Lebesgue measure zero, and for every $x \in F$ there is a hyperplane H so that $x \in H$ and $F \cap H = \{x\}$. We call such a set an *n-Nikodym set*.

The purpose of this paper is to show that the complement of an n-Nikodym set, even though small in terms of Lebesgue measure, must be large in terms of Hausdorff dimension. Namely, we use ideas from [1] and [2] to prove the following.

THEOREM. *The Hausdorff dimension of the complement of an n-Nikodym set is equal to n.*

A few remarks about our notation. $\mathcal{L}^k(\cdot)$ denotes k-dimensional Lebesgue measure and card$(\cdot)$ cardinality; $B(x,r)$ is the ball with center x and radius r; χ_A is the characteristic function of the set A; finally, $x \lesssim y$ means $x \le Cy$, where C is some positive constant not necessarily the same at each occurrence.

2. Proof of the Theorem. Let E be the complement of an n-Nikodym set in $\mathbb{R}^n$. Without loss of generality we may assume that there is a subset A of the unit cube with $\mathcal{L}^n(A) > 0$ such that for every $x \in A$ there exists a set H_x with the following properties:

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(P1) H_x is a rotated translation of $\underbrace{[0,1]\times\cdots\times[0,1]}_{n-1}\times\{0\}$.

(P2) The center of H_x is the point x.

(P3) The normal vector to H_x makes an angle less than $\pi/100$ with the unit vector $e_n=(0,\dots,0,1)$.

(P4) $H_x\cap E=H_x\setminus\{x\}$, so in particular $\mathfrak{L}^{n-1}(E\cap H_x)=1$.

We will show that for every $\varepsilon>0$ the $(n-\varepsilon)$-dimensional Hausdorff measure of E is not zero. Therefore, the Hausdorff dimension of E must equal n. To this end, fix a countable covering $\{B(x_j,r_j)\}$ of E, and for every integer k let

$$J_k=\{j:2^{-k}\le r_j\le 2^{-(k-1)}\},$$
$$E_k=E\cap\bigcup_{j\in J_k}B(x_j,r_j),\quad \widetilde{E}_k=\bigcup_{j\in J_k}B(x_j,2r_j).$$

We will bound $\sum_j r_j^{n-\varepsilon}$ from below by a constant depending only on ε.

Notice that for every $x\in A$ there exists an integer k_x such that

$$\mathfrak{L}^{n-1}(E_{k_x}\cap H_x)\ge\frac{1}{4k_x^2}.$$

Indeed, if this were not the case for some $x\in A$, we would have

$$1=\mathfrak{L}^{n-1}(E\cap H_x)\le\sum_k\mathfrak{L}^{n-1}(E_k\cap H_x)\le\sum_k\frac{1}{4k^2}<\frac{1}{2}.$$

Now let

$$A_k=\left\{x\in A:\mathfrak{L}^{n-1}(E_k\cap H_x)\ge\frac{1}{4k^2}\right\}. \tag{1}$$

Then

$$A=\bigcup_k A_k.$$

Therefore, there must be an integer N such that

$$\mathfrak{L}^n(A_N)\ge\frac{\mathfrak{L}^n(A)}{2N^2},$$

because otherwise we would have

$$\mathfrak{L}^n(A)\le\sum_k\mathfrak{L}^n(A_k)\le\sum_k\frac{\mathfrak{L}^n(A)}{2k^2}<\mathfrak{L}^n(A).$$

Next, we decompose the unit cube into a grid of small cubes, each of side 2^{-N}:

$$[0,1]^n=\bigcup_{i_1,\dots,i_n=1}^{2^N}\prod_{k=1}^{n}[(i_k-1)2^{-N},i_k2^{-N}]=\bigcup_{i_1,\dots,i_n=1}^{2^N}Q_{i_1\dots i_n}.$$

Let

$$I = \{(i_1, \dots, i_n) : Q_{i_1 \dots i_n} \cap A_N \neq \emptyset\}.$$

Notice that for each $(i_1, \dots, i_n) \in I$, property (P2) and (1) imply that there exists a rectangle $R_{i_1 \dots i_n}$ such that

- $R_{i_1 \dots i_n}$ has dimensions $\underbrace{1 \times \cdots \times 1}_{n-1} \times 2^{-N}$.
- $R_{i_1 \dots i_n}$ is parallel to H_x for some $x \in Q_{i_1 \dots i_n}$.
- $R_{i_1 \dots i_n} \cap Q_{i_1 \dots i_n} \neq \emptyset$.
- $\mathcal{L}^n(\widetilde{E}_N \cap R_{i_1 \dots i_n}) \gtrsim N^{-2} 2^{-N}$.

Now let

$$R'_{i_1 \dots i_n} = \begin{cases} R_{i_1 \dots i_n} & \text{if } (i_1, \dots, i_n) \in I, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
N^{-2}\mathcal{L}^n(A) &\lesssim \mathcal{L}^n(A_N) \le \sum_{(i_1,\dots,i_n)\in I} 2^{-nN} = 2^{-(n-1)N} N^2 \sum_{(i_1,\dots,i_n)\in I} N^{-2} 2^{-N} \\
&\lesssim 2^{-(n-1)N} N^2 \sum_{i_1,\dots,i_n=1}^{2^N} \mathcal{L}^n(\widetilde{E}_N \cap R'_{i_1 \dots i_n}) \\
&= 2^{-(n-1)N} N^2 \sum_{i_1,\dots,i_{n-1}=1}^{2^N} \Big(\int_{\widetilde{E}_N} \sum_{i_n=1}^{2^N} \chi_{R'_{i_1 \dots i_n}} \Big) \\
&\le 2^{-(n-1)N} N^2 \mathcal{L}^n(\widetilde{E}_N)^{1/2} \sum_{i_1,\dots,i_{n-1}=1}^{2^N} \Big(\int \Big(\sum_{i_n=1}^{2^N} \chi_{R'_{i_1 \dots i_n}} \Big)^2 \Big)^{1/2} \\
&= 2^{-(n-1)N} N^2 \mathcal{L}^n(\widetilde{E}_N)^{1/2} \sum_{i_1,\dots,i_{n-1}=1}^{2^N} \Big(\sum_{l,m=1}^{2^N} \int \chi_{R'_{i_1 \dots i_{n-1} l}} \chi_{R'_{i_1 \dots i_{n-1} m}} \Big)^{1/2} \\
&= 2^{-(n-1)N} N^2 \mathcal{L}^n(\widetilde{E}_N)^{1/2} \sum_{i_1,\dots,i_{n-1}=1}^{2^N} \Big(\sum_{l,m=1}^{2^N} \mathcal{L}^n(R'_{i_1 \dots i_{n-1} l} \cap R'_{i_1 \dots i_{n-1} m}) \Big)^{1/2}.
\end{aligned}$$

Now using property (P3), it is easy to show that for fixed $i_1, \dots, i_{n-1}$ we have

$$\mathcal{L}^n(R'_{i_1 \dots i_{n-1} l} \cap R'_{i_1 \dots i_{n-1} m}) \lesssim \frac{2^{-N}}{1 + |m - l|}.$$

Consequently,

$$\sum_{l,m=1}^{2^N} \mathcal{L}^n(R'_{i_1 \dots i_{n-1} l} \cap R'_{i_1 \dots i_{n-1} m}) \lesssim \log 2^N = N \log 2.$$

Therefore

$$N^{-2}\mathcal{L}^n(A) \lesssim 2^{-(n-1)N} N^2 \mathcal{L}^n(\widetilde{E}_N)^{1/2} 2^{(n-1)N} N^{1/2}$$

and so

$$\mathcal{L}^n(\widetilde{E}_N) \gtrsim N^{-9}\mathcal{L}^n(A)^2.$$

On the other hand, by the definition of $\widetilde{E}_N$ we have

$$\mathcal{L}^n(\widetilde{E}_N) \lesssim \operatorname{card}(J_N) 2^{-nN}.$$

Hence

$$\operatorname{card}(J_N) \gtrsim 2^{nN} N^{-9} \mathcal{L}^n(A)^2.$$

We conclude that

$$\sum_j r_j^{n-\varepsilon} \gtrsim \operatorname{card}(J_N)(2^{-N})^{n-\varepsilon} \gtrsim 2^{N\varepsilon} N^{-9} \mathcal{L}^n(A)^2 \gtrsim C_\varepsilon.$$

The proof is complete.

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