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Intersection properties for cones of nondecreasing concave functions

by

INNA KOZLOV (Haifa)

Abstract. We prove that the basic facts of the real interpolation method remain true for couples of cones obtained by intersection of the cone of concave functions with rearrangement invariant spaces.

1. Introduction. The interpolation theory of operators acting in Banach cones is a new branch of Interpolation Space Theory whose development has been partly motivated by inner requirements of the theory and partially by application to Harmonic Analysis, Operator Theory and Approximation Theory; see, in particular, [14], [12], [15] and references therein. Unfortunately, the basic facts of the real and complex methods are untrue or hard to achieve in the cone setting (in the case of the complex method, even a judicious definition is unknown for the Banach cone case). Therefore it is important for applications to single out classes of cones for which the basic results of the real method are true. It is easily seen that the first fundamental result of the theory, the *interpolation theorem*, holds in this situation (for linear operators preserving the cone structure). Unfortunately, the second fundamental result, the *reiteration theorem*, does not generally hold. This leads to the study of a certain subclass of triples $(\overline{X}, Q)$ having the so-called *intersection property*. It can be shown (see, e.g., [9]) that the reiteration theorem does hold for triples with this property. The corresponding notion was first introduced by Y. Sagher [13] in the case of the couple $(f_m \cap l_p, f_m \cap l_q)$, where f_m denotes the cone generated by the Fourier coefficients of 2π-periodic integrable functions with nonincreasing Fourier coefficients. In general, the property introduced in [13] can be defined as follows.

DEFINITION 1.1. A cone Q has the *intersection property* (IP) with respect to a Banach couple $\overline{X} = (X_0, X_1)$ if for all $t > 0$,

$$(X_0 + tX_1) \cap Q = (X_0 \cap Q) + t(X_1 \cap Q), \tag{1.1}$$

where the norms are equivalent up to constants independent of t.

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Here the norm of $(X_0 + tX_1) \cap Q$ is the restriction to Q of the natural norm (K-functional) on $X_0 + tX_1$, and the norm on $(X_0 \cap Q) + t(X_1 \cap Q)$ is taken to be

$$K(f, t; \overline{X} \cap Q) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1,\ f_i \in X_i \cap Q\},$$

i.e., it is the K-functional of the couple of cones $\overline{X} \cap Q := (X_0 \cap Q, X_1 \cap Q)$.

Hence the intersection property (1.1) is equivalent to the two-sided inequality

$$K(f, t; \overline{X} \cap Q) \approx K(f, t; \overline{X}) \quad (f \in Q,\ t > 0). \tag{1.2}$$

Since the right hand side is evidently majorized by the left one, the main point is to prove the inequality

$$K(t, f; X_0 \cap Q, X_1 \cap Q) \le cK(t, f; \overline{X}) \quad (f \in Q,\ t > 0) \tag{1.3}$$

with c independent of f and t.

Several couples of Banach cones with the IP, important in applications, were discovered in [13], [14], [1], [5], [6], [8], [9]. Applications of these results to several problems of analysis were presented, in particular, in [8], [3] and [7].

In this paper we investigate this property for a couple of rearrangement invariant (r.i.) spaces of a special kind (see definition below) and the cone C of nonnegative nondecreasing concave functions of $\mathbb{R}_+$; for the role of this cone in Interpolation Space Theory see [4, Ch. 3] and [11, Ch. 2]. Since C contains functions on an unbounded interval that do not belong to any $L_p(\mathbb{R}_+)$ with $0 < p < \infty$, it is natural to use a modified definition of r.i. spaces.

Definition 1.2. A Banach lattice X of (classes of) measurable functions on $\mathbb{R}_+$ is said to be a *generalized rearrangement invariant space* (*g.r.i.*) if

(a) X has the Fatou property;
(b) any two compactly supported equimeasurable functions have equal norms.

Let us recall that X has the *Fatou property* if each nondecreasing sequence $\{f_j\} \subset X$ uniformly bounded in X satisfies

$$\|\sup_j f_j\|_X = \sup_j \|f_j\|_X.$$

From this it immediately follows that

$$\|f\|_X = \sup_{N>0} \|f_N^*\|_X, \tag{1.4}$$

where $f_N := f\chi_{(0,N)}$ and h^* is a nonincreasing rearrangement of h.

In order to formulate our main result, let us recall

DEFINITION 1.3. The cone C consists of nonnegative nondecreasing concave functions defined on $\mathbb{R}_+$.

THEOREM 1.4. *Let* $\overline{X} := (X_0, X_1)$ *be a couple of g.r.i. spaces on* $\mathbb{R}_+$. *Then the cone* C *has the* IP *with respect to this couple.*

REMARK 1.5. Note that the result is trivial if the extreme functions $f_s(t) := \min(1, t/s)$, $t \in \mathbb{R}_+$, $s > 0$, of the cone C do not belong to $X_0 \cap X_1$, since in this case $X_i \cap C = \{0\}$ for $i = 0$ or 1. Nevertheless, our proof does not use this membership.

REMARK 1.6. A typical example of a space of Definition 1.2 is a generalization $L^\alpha_{pq}(\mathbb{R}_+)$ of the Lorentz space defined by the quasinorm

$$\|f\|_{L^\alpha_{pq}} := \sup_{N>0} \left\{ \int_0^\infty |t^{1/p} f^*_N|^q \frac{dt}{t(1+t)^\alpha} \right\}^{1/q}, \tag{1.5}$$

where $\alpha \ge 0$ and $1 \le p, q \le \infty$. By the Hardy inequality, (1.5) is equivalent to a norm if $p > 1$. Note that $L^0_{pq} = L_{pq}$, and the extreme functions f_s are in L^α_{pq} iff $\alpha \ge q/p$, but they do not belong to L_{pq}.

2. Proof of the main result. Let (X_0, X_1) be a couple of g.r.i. spaces. It is well known (see, for example, [4, p. 599]) that for any function $f \in \Sigma(\overline{X})$ and for each $t > 0$ there exists a measurable subset A_t such that

$$\|f\chi_{A_t}\|_{X_0} + t\|f\chi_{A_t^c}\|_{X_1} \le \gamma K(f, t; \overline{X}). \tag{2.6}$$

Here A_t^c is the complement to A_t in $\mathbb{R}_+$ and we can take, e.g., $\gamma = 11$.

It is sufficient to prove that for $f \in C$ and every $t > 0$,

$$K(f, t; \overline{X} \cap C) \le c(\|f\chi_{A_t}\|_{X_0} + t\|f\chi_{A_t^c}\|_{X_1}), \tag{2.7}$$

where c is an absolute constant.

It is clear that if $\mu(A_t) < \infty$ then there exists a subset A of A_t^c with $\mu(A) = \mu(A_t)$, which lies to the right of A_t. Since f is nondecreasing and X_1 is a Banach lattice we have

$$\|f\chi_{A_t}\|_{X_1} \le \|f\chi_A\|_{X_1} \le \|f\chi_{A_t^c}\|_{X_1}.$$

From (2.6) we then get

$$K(f, t; \overline{X} \cap C) \le t\|f\|_{X_1} \le 2t\|f\chi_{A_t^c}\|_{X_1} \le 2\gamma K(f, t; \overline{X}). \tag{2.8}$$

If $\mu(A_t^c) < \infty$, then in the same way we first obtain

$$\|f\chi_{A_t^c}\|_{X_0} \le \|f\chi_{A_t}\|_{X_0}$$

and this leads to

$$K(f, t; \overline{X} \cap C) \le \|f\|_{X_0} \le 2\|f\chi_{A_t}\|_{X_0} \le 2\gamma K(f, t; \overline{X}). \tag{2.9}$$

It remains to consider the case $\mu(A_t) = \infty$ and $\mu(A_t^c) = \infty$. As follows from the proof of (2.6), given in [4], the sets A_t and A_t^c can be represented in the following form:

$$A_t = \bigcup_{i=0}^{\infty}[x_{2i}, x_{2i+1}), \qquad A_t^{\mathrm{c}} = \bigcup_{i=0}^{\infty}[x_{2i+1}, x_{2i+2}),$$

where $x_0 = 0$ and $x_i < x_{i+1}$. Since $\sum_{i=1}^{\infty}(x_{2i-1} - x_{2i-2}) = \infty$ we can find for every $x > 0$ an index $i_0 = i_0(x)$ and a number $a = a(x)$, $0 \leq a < x_{2i_0+1} - x_{2i_0}$, such that

$$x = \sum_{i=1}^{i_0}(x_{2i-1} - x_{2i-2}) + a = x_{2i_0} + a - \sum_{j=1}^{i_0}(x_{2j} - x_{2j-1}). \tag{2.10}$$

Similarly we can represent x as

$$x = \sum_{i=1}^{i_1}(x_{2i} - x_{2i-1}) + b = x_{2i_1+1} + b - \sum_{j=0}^{i_1}(x_{2j+1} - x_{2j}) \tag{2.11}$$

for some $i_1 = i_1(x)$ and $b = b(x)$, where $0 \leq b < x_{2i_1+2} - x_{2i_1+1}$. For every $x \in [x_i, x_{i+1})$ we define the functions $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$g(x) = f(x_{2i_0} + a) - \sum_{j=1}^{i_0}(f(x_{2j}) - f(x_{2j-1})), \tag{2.12}$$

$$h(x) = f(x_{2i_1+1} + b) - \sum_{j=1}^{i_1}(f(x_{2j+1}) - f(x_{2j})), \tag{2.13}$$

where $i_0 = i_0(x)$, $i_1 = i_1(x)$ and $a = a(x)$, $b = b(x)$.

Let us check that g and h belong to C. We will show this for g; the case of h is similar.

We prove, first, that g is continuous. It suffices to check that g is continuous at every point $\overline{x}_i$, where

$$\overline{x}_i := \sum_{j=1}^{i}(x_{2j-1} - x_{2j-2}).$$

Note that if x is close to $\overline{x}_i$ then $i_0 = i - 1$ if $x < \overline{x}_i$ and $i_0 = i$ if $x \geq \overline{x}_i$. Moreover,

$$\lim_{x \to \overline{x}_i - 0} a(x) = x_{2i-1} - x_{2i-2}, \qquad \lim_{x \to \overline{x}_i + 0} a(x) = 0.$$

So, we have

$$\begin{aligned}\lim_{x \to \overline{x}_i - 0} g(x) &= \lim_{x \to \overline{x}_i - 0} \Big\{ f(x_{2i-2} + a) - \sum_{j=1}^{i-1}(f(x_{2j}) - f(x_{2j-1})) \Big\} \\ &= f(x_{2i-1}) - \sum_{j=1}^{i-1}(f(x_{2j}) - f(x_{2j-1})).\end{aligned}$$

On the other hand,

$$\lim_{x\to\overline{x}_i+0} g(x) = \sum_{j=1}^{i}(f(x_{2j-1}) - f(x_{2j-2})) = \lim_{x\to\overline{x}_i-0} g(x).$$

So, g is continuous.

To prove that g is concave, first note that according to the definition of g, there are constants a_i and b_i such that for $\overline{x}_i < x < \overline{x}_{i+1}$,

$$g(x) = f(x - a_i) - b_i.$$

Thus g is concave on $(\overline{x}_i, \overline{x}_{i+1})$. It remains to check that

$$g'_{\text{left}}(\overline{x}_i - 0) \geq g'_{\text{right}}(\overline{x}_i + 0). \tag{2.14}$$

Since g is concave on every interval $(\overline{x}_i, \overline{x}_{i+1})$, the one-sided derivatives exist. But by the definition

$$g'_{\text{left}}(\overline{x}_i - 0) = f'(x_{2i-1} - 0), \qquad g'_{\text{right}}(\overline{x}_i + 0) = f'(x_{2i} + 0).$$

Since $x_{2i-1} < x_{2i}$ and f is concave, $f'(x_{2i-1} - 0) \geq f'(x_{2i} + 0)$. Thus, (2.14) is proved.

We now check that

$$g(x) + h(x) \geq f(x) \quad \text{for every } x \in \mathbb{R}_+. \tag{2.15}$$

By (2.12) and (2.13) we get

$$\begin{aligned} g(x) + h(x) = f(x_{2i_0} + a) &- \sum_{j=1}^{i_0}(f(x_{2j}) - f(x_{2j-1})) \\ &+ f(x_{2i_1+1} + b) - \sum_{j=1}^{i_1}(f(x_{2j+1}) - f(x_{2j})). \end{aligned} \tag{2.16}$$

Suppose first that $2i_0 < 2i_1 + 1$. Since f is nondecreasing, the intervals $[f(x_{2j+1}), f(x_{2j}))$, $i_0 < j \leq i_1$, are pairwise disjoint, and are contained in $[f(x_{2i_0+1}) - f(0), f(x_{2i_1+1}) + b]$. Hence

$$\begin{aligned} &f(x_{2i_1+1} + b) - \sum_{j=1}^{i_1}(f(x_{2j+1}) - f(x_{2j})) - \sum_{j=1}^{i_0}(f(x_{2j}) - f(x_{2j-1})) \\ &= f(x_{2i_1+1} + b) - (f(x_{2i_0+1}) - f(0)) - \sum_{j=i_0+1}^{i_1}(f(x_{2j+1}) - f(x_{2j})) \geq 0. \end{aligned} \tag{2.17}$$

By (2.16), (2.17), and (2.10), we conclude that

$$g(x) + h(x) \geq f(x_{2i_0} + a) \geq f(x).$$

In the same way we get (2.15) for $2i_0 > 2i_1 + 1$.

Thus we have constructed two concave functions satisfying (2.15).

Let us now prove that

$$\|g\|_{X_0} \le \|f\chi_{A_t}\|_{X_0}, \quad s\|h\|_{X_1} \le \|f\chi_{A_t^{\mathrm{c}}}\|_{X_1}. \tag{2.18}$$

Fix $i > 0$ and set

$$N := \sum_{j=0}^{i} (x_{2j+1} - x_{2j}), \quad M := x_{2j+1}.$$

Then from the definition of g it follows that

$$(g_N)^* \le ((f\chi_{A_t})_M)^*.$$

Recall that $h_N := h\chi_{(0,N)}$. The monotonicity of the norm leads to

$$\|(g_N)^*\|_{X_0} \le \|((f\chi_{A_t})_M)^*\|_{X_0} \le \|f\chi_{A_t}\|_{X_0}$$

(see (1.4)). Taking the supremum over $i > 0$ (and therefore N), one gets by (1.4) the first inequality of (2.18). The proof of the second is similar.

To complete the proof we need the following decomposition lemma:

LEMMA 2.1 (Asekritova [1], see also [4, p. 316]). *Let $f, g, h \in C$ satisfy $f \le g + h$. Then there exists a decomposition $f = f_0 + f_1$ with $f_j \in C$ such that $f_0 \le g$ and $f_1 \le h$.*

Applying this lemma to our functions f, g, h we conclude that

$$\begin{aligned} K(f,t;\overline{X}\cap C) &\le \|f_0\|_{X_0} + t\|f_1\|_{X_1} \le \|g\|_{X_0} + t\|h\|_{X_1} \\ &\le \|f\chi_{A_t}\|_{X_0} + t\|f\chi_{A_t^{\mathrm{c}}}\|_{X_1} \le \gamma K(f,t;\overline{X}). \ \blacksquare \end{aligned} \tag{2.19}$$

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Electro-Optics R&D Ltd, Technion
Haifa, 32000, Israel
E-mail: eorddik@techunix.technion.ac.il

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