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Classes of measures closed under mixing and convolution. Weak stability

by

J. K. Misiewicz (Zielona Góra), K. Oleszkiewicz (Warszawa)
and K. Urbanik (Wrocław)

Abstract. For a random vector X with a fixed distribution μ we construct a class of distributions $\mathcal{M}(\mu) = \{\mu \circ \lambda : \lambda \in \mathcal{P}\}$, which is the class of all distributions of random vectors $X\Theta$, where Θ is independent of X and has distribution λ. The problem is to characterize the distributions μ for which $\mathcal{M}(\mu)$ is closed under convolution. This is equivalent to the characterization of the random vectors X such that for all random variables Θ_1, Θ_2 independent of X, X' there exists a random variable Θ independent of X such that

$$X\Theta_1 + X'\Theta_2 \stackrel{d}{=} X\Theta.$$

We show that for every X this property is equivalent to the following condition:

$$\forall a, b \in \mathbb{R}\ \exists \Theta \text{ independent of } X, \qquad aX + bX' \stackrel{d}{=} X\Theta.$$

This condition reminds the characterizing condition for symmetric stable random vectors, except that Θ is here a random variable, instead of a constant.

The above problem has a direct connection with the concept of generalized convolutions and with the characterization of the extreme points for the set of pseudo-isotropic distributions.

1. Introduction. Let $\mathbb{E}$ be a separable real Banach space. By $\mathcal{P}(\mathbb{E})$ we denote the set of all Borel probability measures on $\mathbb{E}$. For $\mathbb{E} = \mathbb{R}$ we will use the simplified notation $\mathcal{P}(\mathbb{R}) = \mathcal{P}$, and the set of all probability measures on $[0, \infty)$ will be denoted by $\mathcal{P}_+$. For every $a \in \mathbb{R}$ and every probability measure μ, we define the rescaling operator $\mu \mapsto T_a\mu$ by the formula $(T_a\mu)(A) = \mu(A/a)$ when $a \neq 0$, and $T_0(\mu) = \delta_0$. This means that $T_a\mu$ is the distribution of the random vector aX if μ is the distribution of the vector X. For every $\mu \in \mathcal{P}(\mathbb{E})$ and $\lambda \in \mathcal{P}$ we define a scale mixture $\mu \circ \lambda$

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of the measure μ with respect to the measure λ by the formula

$$(\mu \circ \lambda)(A) = \int_{\mathbb{R}} (T_a \mu)(A)\, \lambda(da).$$

It is easy to see that $\mu \circ \lambda$ is the distribution of the random vector $X\Theta$ if X and Θ are independent, X has distribution μ, and Θ has distribution λ.

We consider the set of all mixtures of the measure μ, i.e.

$$\mathcal{M}(\mu) = \{\mu \circ \lambda : \lambda \in \mathcal{P}\} = \mathcal{P} \circ \mu.$$

When it is more convenient we will write $\mathcal{M}(\widehat{\mu})$ instead of $\mathcal{M}(\mu)$. The corresponding set of characteristic functions is denoted by

$$\Phi(\mu) = \{\widehat{\nu} : \nu = \mu \circ \lambda,\, \lambda \in \mathcal{P}\} = \Big\{\varphi : \varphi(\xi) = \int \widehat{\mu}(\xi t)\, \lambda(dt),\, \lambda \in \mathcal{P},\, \xi \in \mathbb{E}^*\Big\}.$$

The problem discussed here has a very elementary formulation: characterize those probability measures μ on $\mathbb{E}$ for which the set $\mathcal{M}(\mu)$ is closed under convolution, i.e.

$$\forall \nu_1, \nu_2 \in \mathcal{M}(\mu), \qquad \nu_1 * \nu_2 \in \mathcal{M}(\mu). \tag{A}$$

In the language of random vectors, this condition looks even simpler: Let $X, X', \Theta_1, \Theta_2$ be independent, where X and X' have distribution μ. If condition (A) holds, then there exists a random variable Θ independent of X such that

$$X\Theta_1 + X'\Theta_2 \stackrel{d}{=} X\Theta.$$

In particular, under the previous assumptions,

$$\forall a, b \in \mathbb{R}\ \exists \Theta = \Theta(a,b), \qquad X \text{ and } \Theta \text{ independent and } aX + bX' \stackrel{d}{=} X\Theta.$$

The main result of this paper states that condition (A) is equivalent to

$$\forall a, b \in \mathbb{R}, \qquad T_a\mu * T_b\mu \in \mathcal{M}(\mu). \tag{B}$$

Example 1. The class of symmetric distributions on $\mathbb{R}$ is closed under mixing and under convolution. It is easy to see that this class can be written as $\mathcal{M}(\tau)$ for $\tau = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Checking (B) in this case is especially simple. In the language of characteristic functions we have

$$\begin{aligned}\widehat{\tau}(at)\widehat{\tau}(bt) = \cos(at)\cos(bt) &= \frac{1}{2}\cos((a+b)t) + \frac{1}{2}\cos((a-b)t) \\ &= \int_{\mathbb{R}} \cos(ts)\left(\frac{1}{2}\,\delta_{a+b} + \frac{1}{2}\,\delta_{a-b}\right)(ds),\end{aligned}$$

which means that we can take $\frac{1}{2}\delta_{a+b} + \frac{1}{2}\delta_{a-b}$ for λ. But there are many other possibilities, since if X is a symmetric random vector, and X and Θ are independent, then $X\Theta \stackrel{d}{=} X|\Theta|$. Thus the measure λ is not uniquely determined and condition (B) holds for every λ_{pq}, $p, q \in [0, 1/2]$, where

$$\lambda_{pq} := p\delta_{a+b} + \left(\frac{1}{2} - p\right)\delta_{-a-b} + q\delta_{a-b} + \left(\frac{1}{2} - q\right)\delta_{b-a}.$$

It is easy to see that the set $K(\delta_a, \delta_b) = \{\lambda_{pq} : p, q \in [0, 1/2]\}$ is closed and convex. This property turns out to be general.

In [3, 10–13] Kucharczak, Urbanik and Vol'kovich considered a very similar problem. They studied the properties of *weakly stable* random variables and measures, where a random variable $X \geq 0$ with distribution μ on $[0, \infty)$ is said to be *weakly stable* if for any $a, b \in \mathbb{R}_+$ there exists a nonnegative random variable Q with distribution λ such that

$$\text{(C)} \qquad T_a\mu * T_b\mu = \mu \circ \lambda.$$

From now on we will say that a distribution μ for which (C) holds is $\mathbb{R}_+$-*weakly stable*, and that μ is *weakly stable* when (B) is satisfied. The next example shows that these two conditions are not equivalent.

Example 2. Assume that a random vector X has a symmetric α-stable distribution μ with $\alpha \in (0, 2]$. This means that for every $a, b \in \mathbb{R}$ we have $aX + bX' \stackrel{d}{=} cX$, where $c^\alpha = |a|^\alpha + |b|^\alpha$, so condition (B) holds for $\lambda = \delta_c$. It is easy to see that the opposite implication also holds, i.e. if for every $a, b \in \mathbb{R}$ there exists a Dirac measure satisfying condition (B), then μ is symmetric stable. This is a little different from the usual condition, where the assumption

$$\text{(D)} \qquad \forall a, b > 0\ \exists c > 0, \qquad aX + bX' \stackrel{d}{=} cX.$$

is equivalent to X having a strictly stable distribution. Thus, a strictly stable distribution is $\mathbb{R}_+$-weakly stable, but it may not be weakly stable. A symmetric stable distribution is both $\mathbb{R}_+$-weakly stable and weakly stable.

Example 3. Consider the random vector $X_{k,n} = (U_1, \ldots, U_k)$ for $k \leq n$ which is the k-dimensional projection of $U^n = (U_1, \ldots, U_n)$ with the uniform distribution on the unit sphere $S_{n-1} \subset \mathbb{R}^n$. The distribution $\mu_{k,n}$ of $X_{k,n}$ for $k < n$ is absolutely continuous with respect to the Lebesgue measure with density

$$f(x_1, \ldots, x_k) = c(n, k)\Big(1 - \sum_{i=1}^{k} x_i^2\Big)_+^{(n-k)/2-1},$$

where $c(n, k)$ is a normalizing constant. The set $\mathcal{M}(\mu_{n,n})$ is well known, being the set of all rotationally invariant distributions on $\mathbb{R}^n$. The set $\mathcal{M}(\mu_{k,n})$ is a convex and closed subset of $\mathcal{M}(\mu_{k,k})$. If $n = k + 2$, then $\mu_{k,n}$ is the uniform distribution on the unit ball $B_k \subset \mathbb{R}^k$. In particular, $\mathcal{M}(\mu_{1,3})$ is the set of symmetric unimodal probability measures on $\mathbb{R}$.

In order to show that all these classes are also closed under convolution, we need to use the following characterization:

μ is rotationally invariant on $\mathbb{R}^k$

$\Leftrightarrow \widehat{\mu}(\xi)$ depends only on $\|\xi\|_2 = (|\xi_1|^2 + \cdots + |\xi_k|^2)^{1/2}$, i.e. $\widehat{\mu}(\xi) = \varphi(\|\xi\|_2)$ for some function φ

$\Leftrightarrow \mu$ is the distribution of $U^n\Theta$, where $\Theta \geq 0$ is independent of U^n.

Now, let $\nu_1, \nu_2 \in \mathcal{M}(\mu_{k,n})$. This means that there exist independent rotationally invariant random vectors X_1 and X_2 on $\mathbb{R}^n$ such that ν_1 and ν_2 are the distributions of the k-dimensional projections of X_1 and X_2. For every $a, b \in \mathbb{R}$, the random vector $aX_1 + bX_2$ is also rotationally invariant on $\mathbb{R}^n$ since

$$\begin{aligned}&\mathbf{E}\exp\{i\langle aX_1,\xi\rangle + i\langle bX_2,\xi\rangle\}\\ &\qquad = \mathbf{E}\exp\{i\langle aX_1,\xi\rangle\}\mathbf{E}\exp\{i\langle bX_2,\xi\rangle\} = f_1(|a|\|\xi\|_2)f_2(|b|\|\xi\|_2),\end{aligned}$$

so the right hand side is a function depending only on $\|\xi\|_2$ (a, b are just some parameters here). This means that there exists a random variable $Q = Q_{a,b}$ such that $aX_1 + bX_2 \stackrel{d}{=} U^nQ$. It is easy to see now that $T_a\nu_1 + T_b\nu_2$ is the distribution of a k-dimensional projection of U^nQ, which was to be shown. It is interesting that the variable $Q_{a,b}$ for the measure $\mu_{k,n}$ does not depend on k; in fact $Q_{a,b}$ has the same distribution as $\|aX_1 + bX_2\|_2$.

2. Conditions (A) and (B) are equivalent

Lemma 1. *Assume that a measure μ has property* (B). *Then, for any discrete measures $\nu_1 = \sum_i p_i\delta_{a_i}$ and $\nu_2 = \sum_i q_i\delta_{b_i}$, the measure $(\mu\circ\nu_1) * (\mu\circ\nu_2)$ belongs to $\mathcal{M}(\mu)$.*

Proof. Let λ_{ij} be such that $T_{a_i}\mu * T_{b_j}\mu = \mu\circ\lambda_{ij}$. Then

$$(\mu\circ\nu_1)*(\mu\circ\nu_2) = \sum_{i,j} p_iq_jT_{a_i}\mu * T_{b_j}\mu = \sum_{i,j} p_iq_j\mu\circ\lambda_{ij} = \mu\circ\Big(\sum_{i,j} p_iq_j\lambda_{ij}\Big). \blacksquare$$

Lemma 2. *Let $\mu \neq \delta_0$ be a probability measure on a separable Banach space $\mathbb{E}$ and let $\mathcal{A} \subset \mathcal{P}$. If the set $\mathcal{B} = \{\mu\circ\lambda : \lambda\in\mathcal{A}\}$ is tight, then so is $\mathcal{A}$.*

Proof. Let $\mu = \mathcal{L}(X)$ and $\lambda = \mathcal{L}(Q_\lambda)$ for X and Q_λ independent, $\lambda\in\mathcal{A}$. Let $\varepsilon > 0$. Since $\mathcal{B}$ is tight there exists a compact set $L \subset \mathbb{E}$ such that

$$\mathbf{P}(Q_\lambda X \in L) \geq 1 - \varepsilon\mathbf{P}(X \neq 0).$$

Put $L_n = [-1/n, 1/n]\cdot L = \{sx : s\in[-1/n,1/n],\ x\in L\}$. Since L is bounded we have

$$\liminf_{n\to\infty} \mathbf{P}(X \notin L_n) \geq \mathbf{P}(X \neq 0).$$

Choose n such that $\mathbf{P}(X \notin L_n) \geq \mathbf{P}(X \neq 0)/2$. Then

$$\begin{aligned}\varepsilon\mathbf{P}(X \neq 0) &\geq \mathbf{P}(Q_\lambda X \notin L) \geq \mathbf{P}(|Q_\lambda| > n,\, X \notin L_n)\\ &= \mathbf{P}(|Q_\lambda| > n)\mathbf{P}(X \notin L_n) \geq \mathbf{P}(|Q_\lambda| > n)\mathbf{P}(X \neq 0)/2,\end{aligned}$$

so that $\mathbf{P}(|Q_\lambda| > n) \leq 2\varepsilon$ for all $\lambda \in \mathcal{A}$. This implies tightness of $\mathcal{A}$. ∎

LEMMA 3. *The set $\mathcal{M}(\mu)$ is closed in the topology of weak convergence and the set of extreme points of $\mathcal{M}(\mu)$ is $\{T_a\mu : a \in \mathbb{R}\}$.*

Proof. If $\mu = \delta_0$ then the assertion follows immediately, so we assume that $\mu \neq \delta_0$. Assume that $\mu \circ \lambda_n \Rightarrow \nu$. Then the set $\{\mu \circ \lambda_n : n \in \mathbb{N}\}$ is tight, and, by Lemma 2 the set $\{\lambda_n : n \in \mathbb{N}\}$ is also tight. Thus it contains a subsequence λ_{n_k} converging weakly to a probability measure λ on $\mathbb{R}$. Since the function $\widehat{\mu}(t)$ is bounded and continuous, we obtain

$$\int \widehat{\mu}(ts)\,\lambda_{n_k}(ds) \to \int \widehat{\mu}(ts)\,\lambda(ds).$$

On the other hand, we have

$$\int \widehat{\mu}(ts)\,\lambda_n(ds) \to \widehat{\nu}(t).$$

This means that $\nu = \mu \circ \lambda$ and consequently $\nu \in \mathcal{M}(\mu)$.

If $a = 0$, then $T_a\mu = \delta_0$ and it is easy to check that δ_0 is an extreme point in $\mathcal{M}(\mu)$. Assume that for some $a \in \mathbb{R}$, $a \neq 0$, there exist $\lambda_1, \lambda_2 \in \mathcal{P}$ and $p \in (0,1)$ such that

$$T_a\mu = p\mu \circ \lambda_1 + (1-p)\mu \circ \lambda_2 = \mu \circ (p\lambda_1 + (1-p)\lambda_2).$$

This means that $aX \stackrel{d}{=} X\Theta$ for some random variable Θ independent of X with distribution $p\lambda_1 + (1-p)\lambda_2$. The result of Mazurkiewicz (see [5]) implies that $\mathbf{P}\{\Theta = a\} = 1$ if the distribution of X is not symmetric, and $\mathbf{P}\{|\Theta| = |a|\} = 1$ otherwise. In the first situation we would have

$$\delta_a = p\lambda_1 + (1-p)\lambda_2,$$

so $\lambda_1 = \lambda_2 = \delta_a$ since δ_a is an extreme point in $\mathcal{P}$. If X has symmetric distribution we obtain

$$\begin{aligned}\delta_{|a|}(A) &= p\lambda_1(A) + (1-p)\lambda_2(A) + p\lambda_1(-A) + (1-p)\lambda_2(-A)\\ &=: p|\lambda_1|(A) + (1-p)|\lambda_2|(A)\end{aligned}$$

for every Borel set $A \subset (0,\infty)$. Since $\delta_{|a|}$ is an extreme point in $\mathcal{P}_+$, we have $\delta_{|a|} = |\lambda_1| = |\lambda_2|$. Now, it is enough to notice that for a symmetric distribution μ, the equality $\mu \circ \lambda = \mu \circ |\lambda|$ holds for every probability measure λ. Consequently, we obtain

$$T_a\mu = \mu \circ |\lambda_1| = \mu \circ \lambda_1 = \mu \circ |\lambda_2| = \mu \circ \lambda_2.$$

The above reasoning works for $\mu \in \mathcal{P}$. For $\mu \in \mathcal{P}(\mathbb{E})$ the following two situations are possible. If μ is nonsymmetric then one can choose $\xi \in \mathbb{E}^*$ such that $\xi(X)$ is nonsymmetric and use the result of Mazurkiewicz as before. If

μ is symmetric then there exists $\xi \in \mathbb{E}^*$ such that $\xi(X) \not\equiv 0$ since $\mu \neq \delta_0$, so that $\delta_{|a|} = |\lambda_1| = |\lambda_2|$, as before. The rest of the reasoning does not need any change.

Assume now that the probability measure ν is an extreme point of $\mathcal{M}(\mu)$. Then there exists a probability measure λ such that $\nu = \mu \circ \lambda$. If $\lambda \neq \delta_a$ for any $a \in \mathbb{R}$ then we could divide $\mathbb{R}$ into two Borel sets A and $A' = \mathbb{R} \setminus A$ such that $\lambda(A) = \alpha \in (0,1)$. Then

$$\mu = \alpha\mu \circ (\alpha^{-1}\lambda|_A) + (1-\alpha)\mu \circ ((1-\alpha)^{-1}\lambda|_{A'}),$$

in contradiction with the assumption that ν is extremal. ∎

LEMMA 4. *Assume that for a probability measure $\mu \neq \delta_0$ and some $\nu_1, \nu_2 \in \mathcal{P}$ the set*

$$K_\mu(\nu_1, \nu_2) := \{\lambda : (\mu \circ \nu_1) * (\mu \circ \nu_2) = \mu \circ \lambda\}$$

is not empty. Then it is convex and weakly compact.

Proof. Notice that

$$\{(\mu \circ \nu_1) * (\mu \circ \nu_2)\} = \{\mu \circ \lambda : \lambda \in K_\mu(\nu_1, \nu_2)\},$$

and the set $\{(\mu \circ \nu_1) * (\mu \circ \nu_2)\}$ contains only one point. Then the weak compactness of $K_\mu(\nu_1, \nu_2)$ follows from Lemma 2. The convexity is trivial. ∎

LEMMA 5. *Assume that $\mu \neq \delta_0$ is a probability measure and $K_\mu(\nu_n^1, \nu_n^2) \neq \emptyset$ for every $n \in \mathbb{N}$, where $\nu_n^1 \to \nu_1$ weakly, $\nu_n^2 \to \nu_2$ weakly, and $\nu_n^i, \nu_i \in \mathcal{P}$. Then $K_\mu(\nu_1, \nu_2) \neq \emptyset$.*

Proof. Let $\mathcal{A} = \bigcup_{n=1}^\infty K_\mu(\nu_n^1, \nu_n^2)$ and

$$\mathcal{B} = \{\mu \circ \lambda : \lambda \in \mathcal{A}\} = \{(\mu \circ \nu_n^1) * (\mu \circ \nu_n^2) : n \in \mathbb{N}\}.$$

Since $\mathcal{B}$ is tight, so is $\mathcal{A}$ by Lemma 2. Choosing now $\lambda_n \in K_\mu(\nu_n^1, \nu_n^2)$ for every $n \in \mathbb{N}$, we can find a subsequence λ_{n_k} converging weakly to a probability measure λ. Since

$$(\mu \circ \nu_{n_k}^1) * (\mu \circ \nu_{n_k}^2) = \mu \circ \lambda_{n_k},$$

we also have

$$(\mu \circ \nu_1) * (\mu \circ \nu_2) = \mu \circ \lambda,$$

and consequently $\lambda \in K_\mu(\nu_1, \nu_2) \neq \emptyset$. ∎

THEOREM 1. *For every probability distribution μ properties* (A) *and* (B) *are equivalent.*

Proof. The implication (A)⇒(B) is trivial. Assume that $\mu \neq \delta_0$ and (B) holds. This means that $K_\mu(\delta_a, \delta_b) \neq \emptyset$ for any $a, b \in \mathbb{R}$. It follows from Lemma 1 that $K_\mu(\nu_1, \nu_2) \neq \emptyset$ for any discrete measures ν_1, ν_2. Let now $\lambda_1, \lambda_2 \in \mathcal{P}$. We can find two sequences of discrete measures $\nu_{1,n}$ and $\nu_{2,n}$ converging weakly to λ_1 and λ_2 respectively. Since $K_\mu(\nu_{1,n}, \nu_{2,n}) \neq \emptyset$ for every $n \in \mathbb{N}$, Lemma 5 shows that also $K_\mu(\lambda_1, \lambda_2) \neq \emptyset$, which implies (A). ∎

PROPOSITION 1. *Let $X = (X_1, \dots, X_n)$ be a symmetric α-stable random vector, and let Θ be a random variable independent of X. Then $Y = X\Theta$ is weakly stable iff $|\Theta|^\alpha$ is $\mathbb{R}_+$-weakly stable.*

Proof. Notice that

$$aX\Theta + bX'\Theta' \stackrel{d}{=} (|a\Theta|^\alpha + |b\Theta'|^\alpha)^{1/\alpha} X,$$

where X', Θ' are independent copies of X, Θ such that X, X', Θ, Θ' are independent. Assume that Y is weakly stable. Since $X\Theta \stackrel{d}{=} X|\Theta|$ we obtain

$$(|a\Theta|^\alpha + |b\Theta'|^\alpha)^{1/\alpha} X \stackrel{d}{=} X \cdot |\Theta| \cdot Q$$

for some random variable Q. Without loss of generality we can assume that $Q \geq 0$. A symmetric stable distribution is cancellable (see [3, Prop. 1.1]), thus we obtain

$$|a|^\alpha |\Theta|^\alpha + |b|^\alpha |\Theta'|^\alpha \stackrel{d}{=} |\Theta|^\alpha Q^\alpha.$$

This implies that $|\Theta|^\alpha$ is $\mathbb{R}_+$-weakly stable. The converse is trivial. ■

3. Symmetrizations of mixing measures are uniquely determined. Assume that a measure $\mu \neq \delta_0$ on $\mathbb{R}$ is weakly stable. We have seen before that $K_\mu(\nu_1, \nu_2)$ is a nonempty convex and weakly compact set in $\mathcal{P}$ for all $\nu_1, \nu_2 \in \mathcal{P}$. In this section we discuss further properties of $K_\mu(\nu_1, \nu_2)$.

For a weakly stable measure μ we define

$$\Phi(\mu) = \{\widehat{\nu} : \nu = \mu \circ \lambda,\ \lambda \in \mathcal{P}\},$$

and let $L(\mu)$ denote the complex linear space generated by $\Phi(\mu)$. Weak stability of μ implies that for any $f, g \in L(\mu)$ we have $fg, \overline{f} \in L(\mu)$. Since $\mu \circ \delta_0 = \delta_0$ the space $L(\mu)$ contains the constants.

We denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\Delta\}$ the one-point compactification of the real line, and by $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ the one-point compactification of the nonnegative half-line. Let $C(Y)$ denote the space of continuous real functions on the topological space Y. Then $C(\overline{\mathbb{R}}_+)$ can be identified with the set of even (symmetric) functions from $C(\overline{\mathbb{R}})$.

Now, for a probability measure μ, we define

$$A(\mu) = \{f \in L(\mu) : f = \overline{f},\ \lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x)\}.$$

If μ is weakly stable then $A(\mu)$ is an algebra (over the reals).

LEMMA 6. *If a probability measure μ on $\mathbb{R}$ is not symmetric, then the set $A(\mu)$ separates points of $\overline{\mathbb{R}}$.*

Proof. Let γ be a symmetric Cauchy distribution with Fourier transform $\widehat{\gamma}(t) = e^{-|t|}$. For every $c \in \mathbb{R}$, we define

$$h_c(t) = (\mu \circ (\gamma * \delta_c))^\wedge(t) \in \Phi(\mu).$$

First we show that there exists $a \in \mathbb{R}$ such that $\Im m(h_a) \neq 0$. Assume the opposite, i.e. $\Im m(h_c) \equiv 0$ for every $c \in \mathbb{R}$. This means that

$$\Im m\big(h_c(t)\big) = \int_{-\infty}^{\infty} e^{-|tx|} \sin(ctx)\, \mu(dx) = 0$$

for all $c, t \in \mathbb{R}$. Substituting $u = ct$, we obtain

$$\int_{-\infty}^{\infty} e^{-|ux|/|c|} \sin(ux)\, \mu(dx) = 0$$

for $u \in \mathbb{R}$ and $c \neq 0$. This implies that

$$\lim_{c\to\infty} \int_{-\infty}^{\infty} e^{-|ux|/|c|} \sin(ux)\, \mu(dx) = \int_{-\infty}^{\infty} \sin(ux)\, \mu(dx) = 0,$$

which means that the characteristic function $\widehat{\mu}$ is real, which contradicts our assumption.

Now let $a, t_0 \in \mathbb{R}$ be such that $\Im m(h_a(t_0)) \neq 0$. For every $s \neq 0$, we define

$$g_s(t) = \Im m\bigg(h_a\bigg(\frac{t \cdot t_0}{s}\bigg)\bigg).$$

It is easy to see that $g_s(t) \in A(\mu)$, and $g_s(t) = -g_s(-t)$. We can now see that for every $r \in \mathbb{R}$, $r \neq 0$, the function $g_r(t)$ separates the points r and $-r$ since $g_r(r) = h_a(t_0) \neq g_r(-r)$. To finish the proof, it is enough to notice that the function

$$h_0(t) = \int_{-\infty}^{\infty} e^{-|tx|}\, \mu(dx)$$

separates points $t_1, t_2 \in \mathbb{R}$ if $|t_1| \neq |t_2|$, including the case $t_i = \Delta$. ∎

LEMMA 7. *If a probability measure μ on $\mathbb{R}$ is symmetric and $\mu \neq \delta_0$, then $A(\mu)$ separates points of $\overline{\mathbb{R}}_+$.*

Proof. It is enough to notice that the function $h_0(t) = \int e^{-|tx|}\, \mu(dx)$ separates points of $\overline{\mathbb{R}}_+$. ∎

THEOREM 2. *If a weakly stable measure $\mu \neq \delta_0$ on $\mathbb{R}$ is not symmetric, then for any $\nu_1, \nu_2 \in \mathcal{P}$ the set $K_\mu(\nu_1, \nu_2)$ contains only one measure.*

Proof. Assume that $\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2)$. This means that $\mu \circ \lambda_1 = \mu \circ \lambda_2$, and consequently, for every $\lambda \in \mathcal{P}$,

$$(\mu \circ \lambda) \circ \lambda_1 = (\mu \circ \lambda) \circ \lambda_2.$$

Hence, for every $\lambda \in \mathcal{P}$,

$$\int_{-\infty}^{\infty} (\mu \circ \lambda)^\wedge(tx)\, \lambda_1(dx) = \int_{-\infty}^{\infty} (\mu \circ \lambda)^\wedge(tx)\, \lambda_2(dx).$$

This implies that for every $f \in A(\mu)$,

$$(*) \qquad \int_{-\infty}^{\infty} f(x)\,\lambda_1(dx) = \int_{-\infty}^{\infty} f(x)\,\lambda_2(dx).$$

From Lemma 6 we know that the algebra $A(\mu)$ separates points of $\overline{\mathbb{R}}$, so by the Stone–Weierstrass Theorem (see Theorem 4E in [4]), it is dense in $C(\overline{\mathbb{R}})$ in the topology of uniform convergence. This means that $(*)$ holds for every $f \in C(\overline{\mathbb{R}})$, and consequently $\lambda_1 = \lambda_2$. ∎

Let $\tau = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. The *symmetrization* of a measure $\lambda \in \mathcal{P}$ is defined to be the measure $\lambda \circ \tau$. Notice that λ is symmetric if and only if $\lambda = \lambda \circ \tau$.

THEOREM 3. *If a weakly stable measure $\mu \neq \delta_0$ on $\mathbb{R}$ is symmetric and $\nu_1, \nu_2 \in \mathcal{P}$, then*

$$\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2) \Rightarrow \lambda_1 \circ \tau = \lambda_2 \circ \tau.$$

If $\lambda_1 \circ \tau = \lambda_2 \circ \tau$ and $\lambda_1 \in K_\mu(\nu_1, \nu_2)$ then $\lambda_2 \in K_\mu(\nu_1, \nu_2)$.

Proof. The second implication is trivial because for every symmetric measure μ we have $\mu \circ \lambda = \mu \circ (\lambda \circ \tau)$. To prove the first implication assume that $\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2)$. This implies that $\mu \circ \lambda_1 = \mu \circ \lambda_2$, and consequently $(\mu \circ \lambda) \circ \lambda_1 = (\mu \circ \lambda) \circ \lambda_2$ for every $\lambda \in \mathcal{P}$. This means that for every even function $f \in A(\mu)$ the following equality holds:

$$(**) \qquad \int_{0}^{\infty} f(x)\,(\tau \circ \lambda_1)(dx) = \int_{0}^{\infty} f(x)\,(\tau \circ \lambda_2)(dx).$$

It follows from the proof of Lemma 7 that the even functions from $A(\mu)$ separate points in $\overline{\mathbb{R}}_+$. Applying the Stone–Weierstrass Theorem again we conclude that the set of even functions from $A(\mu)$ is dense in $C(\overline{\mathbb{R}}_+)$ in the topology of uniform convergence. This means that $(**)$ holds for every $f \in C(\overline{\mathbb{R}}_+)$, so the measures $\tau \circ \lambda_1$ and $\tau \circ \lambda_2$ coincide on $\mathbb{R}_+$, and, by symmetry, also on $\mathbb{R}$. ∎

REMARK 1. Notice that it follows from the proofs of Theorems 2 and 3 that weakly stable distributions are reducible in the sense that:

• If X, Y, Z are independent real random variables and X is nonsymmetric and weakly stable then the equality $XY \stackrel{d}{=} XZ$ implies $\mathcal{L}(Y) = \mathcal{L}(Z)$.

• If X, Y, Z are independent, Y, Z are real, and X is a nonsymmetric weakly stable random vector taking values in a separable Banach space $\mathbb{E}$, then $XY \stackrel{d}{=} XZ$ implies $\mathcal{L}(Y) = \mathcal{L}(Z)$. To see this, apply the previous remark to the random variable $\xi(X)$, where $\xi \in \mathbb{E}^*$ is such that $\xi(X)$ is not symmetric.

• If X, Y, Z are independent, Y, Z take values in $\mathbb{E}$, and X is a nonsymmetric weakly stable real random variable, then $XY \stackrel{d}{=} XZ$ implies

$\mathcal{L}(Y)=\mathcal{L}(Z)$. To see this, note that it suffices to prove $\xi(Y)\stackrel{d}{=}\xi(Z)$ for all $\xi\in\mathbb{E}^*$.

• If X,Y,Z are independent and $X\not\equiv 0$ is symmetric weakly stable, then $XY\stackrel{d}{=}XZ$ implies $\mathcal{L}(Y)\circ\tau=\mathcal{L}(Z)\circ\tau$.

REMARK 2. Notice that if μ is weakly stable then so is $\mu\circ\tau$. Indeed, if $T_a\mu * T_b\mu=\nu_1\circ\mu$ and $T_a\mu * T_{-b}\mu=\nu_2\circ\tau$ then

$$T_a(\mu\circ\tau)*T_b(\mu\circ\tau)=\left(\frac{1}{2}\,\nu_1+\frac{1}{2}\,\nu_2\right)\circ(\mu\circ\tau).$$

4. Some general properties of weakly stable distributions

LEMMA 8. *If a measure μ on $\mathbb{R}$ is weakly stable then $\mu(\{0\})=0$ or 1.*

Proof. Let X be a weakly stable variable such that $\mathcal{L}(X)=\mu$, $\mathbf{P}\{X=0\}=p<1$, and let X' be its independent copy. We define the random variable Y with distribution $\mathcal{L}(X\,|\,X\neq 0)$ and Y' its independent copy. The random variable Y/Y' has at most countably many atoms, so there exists $a\in\mathbb{R}$, $a\neq 0$, such that $\mathbf{P}\{Y=aY'\}=0$. Now let Θ be the random variable independent of X such that

$$X-aX'\stackrel{d}{=}X\Theta.$$

Then we have

$$p\le\mathbf{P}\{X\Theta=0\}=\mathbf{P}\{X-aX'=0\}=p^2+(1-p)^2\mathbf{P}\{Y-aY'=0\}=p^2.$$

This holds only if $p=0$, which ends the proof. ∎

LEMMA 9. *Assume that a weakly stable probability measure $\mu\neq\delta_0$ on $\mathbb{R}$ has at least one atom. Then the discrete part of μ (normalized to be a probability measure) is also weakly stable.*

Proof. Let $\mu=\alpha\mu_1+(1-\alpha)\mu_2$, $\alpha\in(0,1)$, where $\alpha\mu_1$ is the discrete part of μ, $\mu_1(\mathbb{R})=1$, and μ_2 is such that $\mu_2(\mathbb{R})=1$ and $\mu_2(\{x\})=0$ for every $x\in\mathbb{R}$. If μ is weakly stable, then for every $a\in\mathbb{R}$ there exists a probability measure λ such that $\mu*T_a\mu=\mu\circ\lambda$. Now we have

$$\begin{aligned}\mu*T_a\mu=\alpha^2\mu_1*T_a\mu_1&+\alpha(1-\alpha)\mu_1*T_a\mu_2\\&+\alpha(1-\alpha)\mu_2*T_a\mu_1+(1-\alpha)^2\mu_2*T_a\mu_2.\end{aligned}$$

Clearly for $a\neq 0$ the discrete part of $\mu*T_a\mu$ is equal to $\alpha^2\mu_1*T_a\mu_1$. On the other hand, we have

$$\mu\circ\lambda=(1-\beta)\mu\circ\lambda_2+\alpha\beta\mu_1\circ\lambda_1+(1-\alpha)\beta\mu_2\circ\lambda_1,$$

where $\lambda_1(\mathbb{R})=\lambda_2(\mathbb{R})=1$, λ_1 is a discrete measure, $\lambda_2(\{x\})=0$ for every $x\in\mathbb{R}$ and $\lambda=\beta\lambda_1+(1-\beta)\lambda_2$.

Let $S = \{a \in \mathbb{R} : \mu * T_a\mu(\{0\}) = 0\}$. If $a \in S$, $a \neq 0$, then $\lambda(\{0\}) = 0$ and $\mu \circ \lambda_2(\{x\}) = \beta\mu_2 \circ \lambda_1(\{x\}) = 0$ for every $x \in \mathbb{R}$, so

$$\alpha^2\mu_1 * T_a\mu_1 = \alpha\beta\mu_1 \circ \lambda_1.$$

This means that $\alpha = \beta$ and $\mu_1 * T_a\mu_1 = \mu_1 \circ \lambda_1$.

If $a \notin S$ then there exists a sequence $a_n \in S \setminus \{0\}$, $n \in \mathbb{N}$, such that $\lim_n a_n = a$. Then $\mu * T_{a_n}\mu \Rightarrow \mu * T_a\mu$ and $\mu_1 * T_{a_n}\mu_1 \Rightarrow \mu_1 * T_a\mu_1$. For every $n \in \mathbb{N}$ there exists λ_n such that $\mu_1 * T_{a_n}\mu_1 = \mu_1 \circ \lambda_n$, i.e. $\lambda_n \in K_{\mu_1}(\delta_1, \delta_{a_n})$. In view of Lemma 5 there exists $\lambda \in K_{\mu_1}(\delta_1, \delta_a)$, which ends the proof. ∎

THEOREM 4. *Assume that a random vector X taking values in a separable Banach space $\mathbb{E}$ and having distribution μ is such that $\mathbf{E}\|X\| < \infty$ and $\mathbf{E}X = a \neq 0$. Then μ is weakly stable if and only if $\mu = \delta_a$.*

Proof. Assume first that $\mathbb{E} = \mathbb{R}$. If $\mu = \delta_a$ for some $a \neq 0$, then it is weakly stable. Conversely, let μ be weakly stable and $\mathbf{E}X = a \neq 0$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with distribution μ. The Weak Law of Large Numbers implies that

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to a$$

weakly as $n \to \infty$. The measure μ is weakly stable, thus for every $n \in \mathbb{N}$ there exists a measure ν_n such that

$$\mu_n = \mathcal{L}\bigg(\frac{1}{n}\sum_{k=1}^{n} X_k\bigg) = (T_{1/n}\mu)^{*n} = \mu \circ \nu_n.$$

Since $\mu_n \Rightarrow \delta_a$, it follows from Lemma 2 that the family $\{\nu_n\}$ is tight and it contains a sequence ν_{n_k} such that $\nu_{n_k} \Rightarrow \nu$ for some probability measure ν. Now, we obtain

$$\delta_a = \lim_{n\to\infty} \mu_n = \lim_{k\to\infty} \mu \circ \nu_{n_k} = \mu \circ \nu.$$

Since $a \neq 0$ the last equality is possible only if $\mu = \delta_x$ and $\nu = \delta_y$ for some $x, y \in \mathbb{R}$ with $xy = a$. Since $\mathbf{E}X = a$, we conclude that $\mu = \delta_a$.

If X is a random vector in a separable Banach space $\mathbb{E}$ with $\mathbf{E}X = a \neq 0$ then the previous considerations yield $\mathbf{P}\{\xi(X) = \xi(a)\} = 1$ for each $\xi \in \mathbb{E}^*$ with $\xi(a) \neq 0$. Such ξ's form a dense subset in $\mathbb{E}^*$. Consequently, $\mathbf{P}\{X = a\} = 1$. ∎

THEOREM 5. *Assume that for a weakly stable measure $\mu \neq \delta_0$ on a separable Banach space $\mathbb{E}$ there exists $\varepsilon \in (0, 1]$ such that for every $\xi \in \mathbb{E}^*$ and every $p \in (0, \varepsilon)$,*

$$\int_{\mathbb{E}} |\xi(x)|^p \,\mu(dx) < \infty.$$

Then there exists $\alpha_0 \in [\varepsilon, 2]$ such that $\mathcal{M}(\mu)$ contains a strictly α-stable measure for every $\alpha \in (0, \alpha_0)$.

Proof. Let $p \in (0, \varepsilon)$. Since $\mathcal{M}(\mu)$ is closed under scale mixing, $\mu \circ m_n \in \mathcal{M}(\mu)$ for every $n \in \mathbb{N}$, where

$$m_n(dx) = c(n) x^{-p-1} \mathbf{1}_{(1/n,\infty)} dx, \qquad c(n) = pn^{-p}.$$

As $\mathcal{M}(\mu)$ is also closed under convolution and under taking convex linear combinations, and weakly closed, for every $n \in \mathbb{N}$ we have

$$\nu_n = \exp\{c(n)^{-1}(\mu \circ m_n)\} \in \mathcal{M}(\mu),$$

where $\exp(\kappa) := e^{-\kappa(\mathbb{E})} \sum_{k=0}^{\infty} \kappa^{*k}/k!$ for every finite measure κ on $\mathbb{E}$. Notice that for every $\xi \in \mathbb{E}^*$ we have

$$\begin{aligned}\widehat{\nu}_n(\xi) &= \exp\Big\{-\int_{\mathbb{E}} \int_{1/n}^{\infty} (1 - e^{i\xi(sx)}) s^{-p-1}\, ds\, \mu(dx)\Big\} \\ &= \exp\Big\{-\int_{\mathbb{E}} |\xi(x)|^p \int_{|\xi(x)|/n}^{\infty} (1 - e^{iu \operatorname{sgn}(\xi(x))}) u^{-p-1}\, du\, \mu(dx)\Big\}.\end{aligned}$$

Let $h(u) = (1 - e^{iu \operatorname{sgn}(\xi(x))}) u^{-p-1}$. Then h is integrable on $[0, \infty)$ since $p \in (0,1)$, and $|h(u)| = 2|\sin(u/2)| u^{-p-1}$, thus $|h(u)| \le u^{-p}$ for $u < 1$ and $|h(u)| \le 2u^{-p-1}$ for $u \ge 1$. This implies that the function

$$H_p(\xi(x)) = \int_0^{\infty} (1 - e^{iu \operatorname{sgn}(\xi(x))}) u^{-p-1}\, du$$

is well defined and bounded on $\mathbb{E}$, thus

$$\widehat{\nu}_n(\xi) \to \exp\Big\{-\int_{\mathbb{E}} |\xi(x)|^p H_p(\xi(x))\, \mu(dx)\Big\} =: \widehat{\gamma}_p(\xi).$$

It is easy to see that $\widehat{\gamma}_p$ is the characteristic function of a strictly p-stable random variable and the corresponding measure γ_p belongs to $\mathcal{M}(\mu)$ since this class is weakly closed. Now we define

$$\alpha_0 = \sup\{\alpha \in (0, 2] : \mathcal{M}(\mu) \text{ contains a strictly } \alpha\text{-stable measure}\}.$$

To end the proof it is enough to recall that for every $0 < \beta < \alpha \le 2$ and every strictly α-stable measure γ_α the measure $\gamma_\alpha \circ \lambda_{\beta/\alpha}$ is strictly β-stable, where $\lambda_{\beta/\alpha}$ is the distribution of the random variable $\Theta_{\beta/\alpha}^{1/\alpha}$, and $\Theta_{\beta/\alpha} \ge 0$ is such that $\mathbb{E} \exp\{-t\Theta_{\beta/\alpha}\} = \exp\{-t^{\beta/\alpha}\}$. ∎

REMARK 3. Notice that if a weakly stable measure $\mu \ne \delta_0$ on $\mathbb{E}$ is such that $\int |\xi(x)|^p\, \mu(dx) < \infty$ for every $\xi \in \mathbb{E}^*$ and $p \in (0, \varepsilon)$ for some $\varepsilon \in (0, 2]$ then $\mathcal{M}(\mu)$ contains a symmetric p-stable measure for every $p \in (0, \varepsilon)$.

To see this it is enough to notice that if μ is symmetric, then so is the measure ν_n constructed in the proof of Theorem 5. Consequently,

$$\widehat{\nu}_n(\xi) = \exp\Big\{-\int_{\mathbb{E}} |\xi(x)|^p \int_{|\xi(x)|/n}^{\infty} (1-\cos u)u^{-p-1}\,du\,\mu(dx)\Big\}.$$

Let $h(u) = (1-\cos u)u^{-p-1}$. Then $|h(u)| < u^{1-p}$ for $u < 1$, and $|h(u)| < 2u^{-p-1}$ for $u > 1$, so h is integrable on $[0,\infty)$ for every $p \in (0,2)$. For the constants

$$H_p = \int_0^{\infty} (1-\cos u)u^{-p-1}\,du$$

we obtain

$$\widehat{\nu}_n(\xi) \to \exp\Big\{-H_p \int_{\mathbb{E}} |\xi(x)|^p\,\mu(dx)\Big\},$$

which is the characteristic function of a symmetric p-stable random vector.

If μ is not symmetric then we replace μ by $\mu\circ\tau$ in this construction. This is possible since $\mu\circ\tau$ is symmetric, belongs to $\mathcal{M}(\mu)$, and has the same moments as μ.

Remark 4. In the situation described in Remark 3, if $\mathbb{E} = \mathbb{R}$ then $\mathcal{M}(\mu)$ also contains a symmetric ε-stable random variable. Indeed, it follows from Remark 3 that

$$\exp\Big\{-H_p \int_{\mathbb{R}} |tx|^p\,\mu(dx)\Big\} = \exp\Big\{-|t|^p H_p \int_{\mathbb{R}} |x|^p\,\mu(dx)\Big\}$$

is the characteristic function of some measure from $\mathcal{M}(\mu)$. Since rescaling is admissible, $\exp\{-|t|^p\}$ is also the characteristic function of some measure from $\mathcal{M}(\mu)$. Now it is enough to notice that

$$\lim_{p\nearrow\varepsilon} \exp\{-|t|^p\} = \exp\{-|t|^\varepsilon\},$$

and use Lemma 3.

Remark 5. There exist measures μ such that $\mu\circ\nu$ is symmetric α-stable for some probability measure ν, but μ is not weakly stable. Any measure of the form $\mu = q\delta_{-1} + (1-q)\delta_1$ for $q \in (0,1)\setminus\{1/2\}$ can serve as an example.

Lemma 10. *Let X be a real random variable with distribution μ. If μ is weakly stable and supported on a finite set then either there exists $a \in \mathbb{R}$ such that $\mu = \delta_a$ or there exists $a \neq 0$ such that $\mu = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$.*

Proof. Let X' be an independent copy of X. Assume that $\mu \neq \delta_a$ for all $a \in \mathbb{R}$. Theorem 4 implies that X must take on both negative and positive values with positive probability. Let $V = \{x \in \mathbb{R} : \mu(\{x\}) > 0\}$. By Lemma 8 we have $0 \notin V$. Let b be the greatest and $-a$ the least element of V. Clearly, $a, b > 0$. We will prove first that $a = b$.

Assume that $b > a$. For $\lambda \in \mathbb{R}$ consider the set of values taken on by $X - \lambda X'$ with positive probability: $V_\lambda = \{v - \lambda w : v, w \in V\}$. Clearly, for $\lambda \in (0,1)$ the greatest element of V_λ is $b + \lambda a$, and the least is $-(a + \lambda b)$. Moreover $a + \lambda b < b + \lambda a$ (hence $b + \lambda a$ has strictly the greatest absolute value among all elements of V_λ). Since μ is weakly stable there exists a real random variable Y_λ independent of X and such that $Y_\lambda X \stackrel{d}{=} X - \lambda X'$. One can easily see that Y_λ is also finitely supported. We have $b + \lambda a \in V_\lambda$ so that there exist $c, d \neq 0$ such that $\mathbf{P}\{Y_\lambda = c\} > 0$, $d \in V$ and $cd = b + \lambda a$. Also for any $d' \in V$ we have $cd' \in V_\lambda$, so that $|d'| > |d|$ would imply $|cd'| > b + \lambda a$, contrary to the fact that $b + \lambda a$ has the maximal absolute value among all elements of V_λ. Hence d must have maximal absolute value among all elements of V and therefore $d = b$ so that $c = 1 + \lambda a/b$.

We deduce that
$$-\frac{a}{b}\,(b + \lambda a) = c \cdot (-a) \in V_\lambda$$
and therefore there exist $v, w \in V$ such that $-(a/b)(b + \lambda a) = v - \lambda w$, and consequently $\lambda(w - a^2/b) = v + a$. Assume that $a^2/b \notin V$. Then the last equation may be satisfied for finitely many values of the parameter λ only (because v and w can be chosen from a finite set only). It was proved for all $\lambda \in (0,1)$, however. Hence $a^2/b \in V$. Therefore
$$\frac{a^2}{b^2}\,(b + \lambda a) = c \cdot a^2/b \in V_\lambda$$
and again, there exist $v, w \in V$ such that $(a^2/b^2)(b + \lambda a) = v - \lambda w$ so that $\lambda(w + a^3/b^2) = v - a^2/b$. As before we infer that $-a^3/b^2 \in V$. By iterating this reasoning we prove that $(-1)^{k+1} a^{k+1}/b^k \in V$ for every $k \in \mathbb{N}$. Since $0 < a/b < 1$ this implies that V contains an infinite subset, contradicting our assumptions. The case $a > b$ is excluded in a similar way. Hence $a = b$.

Now, let $-\alpha$ be the greatest negative element and β the least positive element of V. Consider $X - \lambda X'$ for $0 < \lambda < \min(\alpha, \beta)/a$. Clearly, the least positive element of V_λ is $\beta - \lambda a$, whereas $-(\alpha - \lambda a)$ is the greatest negative one. Assume without loss of generality that $\beta \le \alpha$ so that $\beta - \lambda a$ has the least absolute value among all elements of V_λ (otherwise consider $-X$ instead of X). Again, we choose Y_λ and parameters $c, d \neq 0$ such that $\mathbf{P}\{Y_\lambda = c\} > 0$, $d \in V$ and $cd = \beta - \lambda a$. We obtain $d \in \{-\alpha, \beta\}$ by a similar reasoning—no element can be both at the same side of zero as d and closer to zero than d because multiplying by c we would get a positive element of V_λ less than $\beta - \lambda a$. Hence $c \in \{(\beta - \lambda a)/\beta, -(\beta - \lambda a)/\alpha\}$. However, $ca \in V_\lambda$ so that there exist $v, w \in V$ such that $ca = v - \lambda w$, which means that $\lambda(w - a^2/\beta) = v - a$ or $\lambda(w + a^2/\alpha) = v + a\beta/\alpha$. Since we proved this alternative for infinitely many λ's and we know that v and w can have only finitely many values we infer that $a^2/\beta \in V$ (if $d = \beta$) or $-a^2/\alpha \in V$ (if $d = -\alpha$).

We have proved that $V \subset [-a, a]$, so that $\beta = a$ if $d = \beta$, or $\alpha = a$ if $d = -\alpha$. Anyway, $|d| = a$ so that $|c| = \beta/a - \lambda$. Since $\{-a, a\} \subset V$ we have $\{-ca, ca\} \subset V_\lambda$ and therefore also $-(\beta - \lambda a) \in V_\lambda$. We have assumed though that $\beta - \lambda a$ has the least absolute value among all elements of V_λ, so in particular $-(\alpha - \lambda a) \le -(\beta - \lambda a)$. Since $-(\alpha - \lambda a)$ is the greatest negative element of V_λ we also have $-(\alpha - \lambda a) \ge -(\beta - \lambda a)$. Hence $\alpha = \beta$.

We have proved earlier that $\alpha = a$ or $\beta = a$, so finally $\alpha = \beta = a$ and the support of μ is $\{-a, a\}$. Theorem 4 implies that μ is symmetric. ∎

LEMMA 11. *Let X be a real random variable with distribution $\mu \ne \delta_0$ and let X' be its independent copy. Assume that μ is weakly stable*, *so that for any $\lambda \in \mathbb{R}$ there exists a real random variable Y_λ independent of X such that $X - \lambda X' \stackrel{d}{=} Y_\lambda X$. If X is symmetric*, *assume additionally that $Y_\lambda \ge 0$ a.s. Then the map*

$$\lambda \mapsto \mathcal{L}(Y_\lambda)$$

is well defined and continuous on $\mathbb{R}$.

Proof. The existence and uniqueness of distribution of Y_λ follows from Theorems 2 and 3. We only need to prove that $\lambda_n \to \lambda$ implies that $Y_{\lambda_n} \stackrel{d}{\to} Y_\lambda$ as $n \to \infty$. Suppose not. Then we can find $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that for any k the law of $Y_{\lambda_{n_k}}$ is ε-separated from the law of Y_λ in Lévy's metric. Since

$$Y_{\lambda_{n_k}} X \stackrel{d}{=} X - \lambda_{n_k} X' \stackrel{d}{\to} X - \lambda X' \stackrel{d}{=} Y_\lambda X,$$

by Lemma 2 we can choose a subsequence $\{n_{k_l}\} \subset \{n_k\}$ such that $Y_{\lambda_{n_{k_l}}} \stackrel{d}{\to} Z$ for some real random variable Z as $l \to \infty$. Hence $\mathcal{L}(Z) \ne \mathcal{L}(Y_\lambda)$. Moreover $Z \ge 0$ a.s. if X is symmetric because then all Y_{λ_n}'s are nonnegative a.s.

On the other hand, $Z'X \stackrel{d}{=} Y_\lambda X$, where Z' is a copy of Z independent of X since the map $\lambda \mapsto \mathcal{L}(X - \lambda X')$ is continuous. Therefore $Y_\lambda \stackrel{d}{=} Z' \stackrel{d}{=} Z$, by Theorems 2 and 3 (or by Remark 1). The contradiction obtained ends the proof. ∎

REMARK 6. Let $\alpha \in [1, 2]$. Note that if X is a random variable with a weakly stable distribution μ and $\mathbf{E}|X|^p < \infty$ for all $p \in (0, \alpha)$ then

$$1 + |\lambda| \ge \begin{cases} |Y_\lambda| \text{ a.s.} & \text{if } \alpha = 2, \\ \|Y_\lambda\|_\alpha & \text{if } \alpha < 2. \end{cases}$$

Indeed, by Theorem 5 there exists Θ independent of X such that $X\Theta$ is strictly α-stable. If $\alpha < 2$ then $\mathbf{E}|X\Theta|^\beta < \infty$ for every $\beta < \alpha$, thus $\mathbf{E}|X|^\beta < \infty$ for every $\beta < \alpha$. If $\alpha = 2$ then $X\Theta$ is Gaussian so $\mathbf{E}|X|^\beta < \infty$

for every $\beta > 0$. Now it is enough to notice that for $\beta \ge 1$ we have

$$\|Y_\lambda\|_\beta \|X\|_\beta = \|Y_\lambda X\|_\beta = \|X - \lambda X'\|_\beta \le \|X\|_\beta + |\lambda|\, \|X'\|_\beta = (1+|\lambda|)\|X\|_\beta.$$

The case $\beta = \alpha$ can be obtained by observing that $\|Y_\lambda\|_\alpha = \lim_{\beta\to\alpha^-} \|Y_\lambda\|_\beta$. If $\alpha = 2$ the inequality holds for all $\beta \ge 1$, which implies that $\|Y_\lambda\|_\infty \le 1 + |\lambda|$.

Lemma 12. *Let X be a real random variable with distribution μ. If μ is weakly stable and supported on a countable set then there exists $a \in \mathbb{R}$ such that $\mu = \delta_a$ or there exists $a \neq 0$ such that $\mu = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$.*

Proof. Assume that the support of μ is an infinite countable set. For $\lambda \in (0,1)$ we have $X - \lambda X' \stackrel{d}{=} Y_\lambda X$, where X' and Y_λ are defined as in the preceding lemma (so that if X is symmetric then $Y_\lambda \ge 0$ a.s.). By Lemma 11, $Y_\lambda \xrightarrow{d} Y_0 = 1$ as $\lambda \to 0$. Let

$$\mu = \sum_{n=1}^{\infty} p_n \delta_{x_n},$$

where x_n's are nonzero (by Lemma 8) and pairwise different, and $(p_n)_{n=1}^{\infty}$ is a nonincreasing sequence of positive numbers. Let

$$M = \left\{ \frac{x_i - x_j}{x_k - x_l} : k \neq l \right\}.$$

Clearly, M is a countable set. We see that for $\lambda \notin M$ the equality $x_k - \lambda x_i = x_l - \lambda x_j$ implies $i = j$ and $k = l$. Finally, let $N \in \mathbb{N}$ be such that $\sum_{n>N} p_n \le p_1^2/2$. Then for $\lambda \in (0,1) \setminus M$ we have

$$\begin{aligned} p_1^2 &= \mathbf{P}\{X = x_1, X' = x_1\} = \mathbf{P}\{X - \lambda X' = x_1 - \lambda x_1\} \\ &= \mathbf{P}\{Y_\lambda X = x_1(1-\lambda)\} = \sum_{n=1}^{\infty} \mathbf{P}\left\{ Y_\lambda = \frac{x_1}{x_n}(1-\lambda) \right\} \cdot p_n \\ &\le \mathbf{P}\{Y_\lambda = 1-\lambda\} \cdot p_1 + \frac{p_1^2}{2} + \sum_{n=2}^{N} \mathbf{P}\left\{ \frac{Y_\lambda}{1-\lambda} = \frac{x_1}{x_n} \right\} \cdot p_n, \end{aligned}$$

and the summands for $2 \le n \le N$ tend to zero as $\lambda \to 0$ (since $\frac{Y_\lambda}{1-\lambda} \xrightarrow{d} 1$) so that

$$\liminf_{\lambda\to 0,\, \lambda\in(0,1)\setminus M} \mathbf{P}\{Y_\lambda = 1-\lambda\} \ge p_1/2.$$

On the other hand, for $\lambda \in (0,1) \setminus M$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} p_k^2 &= \mathbf{P}\{X = x_k,\, X' = x_k\} = \mathbf{P}\{X - \lambda X' = x_k(1-\lambda)\} \\ &= \mathbf{P}\{Y_\lambda X = x_k(1-\lambda)\} \ge \mathbf{P}\{Y_\lambda = 1-\lambda\} \cdot p_k, \end{aligned}$$

so that

$$\limsup_{\lambda\to 0,\, \lambda\in(0,1)\setminus M} \mathbf{P}\{y_\lambda = 1-\lambda\} \le p_k.$$

Hence $p_k \ge p_1/2$ for any $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} p_k = \infty$, which is clearly not possible. This proves that μ has finite support and the assertion follows from Lemma 10. ∎

THEOREM 6. *Let μ be a weakly stable probability measure on a separable Banach space $\mathbb{E}$. Then either there exists $a \in \mathbb{E}$ such that $\mu = \delta_a$, or there exists $a \in \mathbb{E}\setminus\{0\}$ such that $\mu = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$, or $\mu(\{a\}) = 0$ for all $a \in \mathbb{E}$.*

Proof. Assume first that $\mathbb{E} = \mathbb{R}$. One can express μ as $p\mu_1 + (1-p)\mu_2$, where $p \in [0,1]$, μ_1 is a discrete probability measure and $\mu_2(\{x\}) = 0$ for any $x \in \mathbb{R}$. The case $p = 0$ is trivial, so assume that $p > 0$. Lemma 9 implies that μ_1 is weakly stable, and therefore by Lemma 12, $\mu_1 = \delta_a$ for some $a \in \mathbb{R}$ or $\mu_1 = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$ for some $a \ne 0$.

CASE 1: $\mu_1 = \delta_a$. If $a = 0$ then by Lemma 8 we have $p = 1$ and the proof is finished. If $a \ne 0$, note that for $\lambda \in (0,1)$ the random variable $X - \lambda X' \stackrel{d}{=} Y_\lambda X$ has exactly one atom with mass p^2 at $(1-\lambda)a$. Hence Y_λ has an atom with mass p at $1-\lambda$. Since $Y_\lambda \xrightarrow{d} Y_1$ as $\lambda \to 1$ we have $\mathbf{P}\{Y_1 = 0\} \ge p$, and therefore $\mathbf{P}\{X - X' = 0\} = \mathbf{P}\{Y_1 X = 0\} \ge p$. On the other hand, $\mathbf{P}\{X - X' = 0\} = \mathbf{P}\{X = X'\} = p^2$ because X has only one atom, at a. Hence $p^2 \ge p$ so that $p = 1$.

CASE 2: $\mu_1 = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$ for some $a \ne 0$. For $\lambda \in (0,1)$ the random variable $X - \lambda X' \stackrel{d}{=} Y_\lambda X$ has exactly four atoms, with mass $p^2/4$ each, at $\pm(1-\lambda)a$ and $\pm(1+\lambda)a$. Hence Y_λ has atoms with total mass $p/2$ at $\pm(1-\lambda)$ (and atoms with total mass $p/2$ at $\pm(1+\lambda)$). Since $Y_\lambda \xrightarrow{d} Y_1$ as $\lambda \to 1$ we have $\mathbf{P}\{Y_1 = 0\} \ge p/2$, and therefore $\mathbf{P}\{X - X' = 0\} = \mathbf{P}\{Y_1 X = 0\} \ge p/2$. On the other hand, $\mathbf{P}\{X - X' = 0\} = \mathbf{P}\{X = X' = a\} + \mathbf{P}\{X = X' = -a\}$ $= p^2/2$ so that $p^2/2 \ge p/2$ and $p = 1$.

Let now $\mathbb{E}$ be an arbitrary separable Banach space. By making use of the above result for real random variables $\xi(X)$, where $\xi \in \mathbb{E}^*$, we can easily finish the proof. ∎

REMARK 7. Let X be a given random variable. It may happen that for some $a, b \in \mathbb{R}$ there exists a random variable $Q_{a,b}$ independent of X such that $aX + bX' \stackrel{d}{=} XQ_{a,b}$, and for some other a, b such a random variable does not exist.

Consider, for example, X with exponential distribution and characteristic function $(1-it)^{-1}$. It follows from Theorem 4 that the class $\mathcal{M}((1-it)^{-1})$ is not closed under convolutions. However, if $a, b \in \mathbb{R}$ are such that $ab < 0$

then the characteristic function of $aX + bX'$ can be written as

$$\mathbf{E}\exp\{i(aX+bX')t\} = \frac{1}{1-iat}\,\frac{1}{1-ibt} = \int \frac{1}{1-ist}\,\lambda(ds),$$

where $\lambda(\{a\}) = p = 1 - \lambda(\{b\})$, with $p = a/(a-b)$. This means that $aX + bX' \stackrel{d}{=} XQ_{a,b}$, where $Q_{a,b}$ is independent of X and has distribution λ.

Assume that for some $a, b > 0$ there exists $Q \geq 0$ such that $aX + bX' \stackrel{d}{=} XQ$. Then the density g of $aX + bX'$ can be written as

$$g(x) = \int_0^\infty e^{-x/s}s^{-1}\,\mathcal{L}(Q)(ds).$$

On the other hand, we have

$$g(x) = \int_0^\infty e^{-x/s}s^{-1}\,\lambda(ds).$$

The uniqueness of the Laplace transform for signed σ-finite measures implies that $\mathcal{L}(Q) = \lambda$, which is impossible since $\mathcal{L}(Q)$ is a probability measure while λ is a signed measure only. Similar arguments can be used for $a, b < 0$. Finally, if $ab > 0$ then $aX + bX'$ cannot have the same distribution as XQ for any random variable Q independent of X.

References

[1] S. Cambanis, R. Keener and G. Simons, *On α-symmetric distributions*, J. Multivariate Anal. 13 (1983), 213–233.

[2] J. Kucharczak and K. Urbanik, *Quasi-stable functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), 263–268.

[3] —, —, *Transformations preserving weak stability*, Bull. Polish Acad. Sci. Math. 34 (1986), 475–486.

[4] L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, Van Nostrand, Toronto, 1953.

[5] G. Mazurkiewicz, *When a scale mixture is equivalent with rescaling*, J. Math. Sci. (New York) 111 (2002), 3851–3853.

[6] J. K. Misiewicz, *Positive definite functions on l^∞*, Statist. Probab. Lett. 8 (1989), 255–260.

[7] —, *Elliptically contoured measures on R^∞*, Bull. Polish Acad. Sci. Math. 30 (1982), 283–290.

[8] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian Random Processes. Stochastic Models with Infinite Variance*, Chapman & Hall, 1994.

[9] I. J. Schoenberg, *Metric spaces and completely monotonic functions*, Ann. of Math. 38 (1938), 811–841.

[10] K. Urbanik, *Remarks on $\mathcal{B}$-stable probability distributions*, Bull. Polish Acad. Sci. Math. 24 (1976), 783–787.

[11] V. Vol'kovich, *On symmetric stochastic convolutions*, J. Theoret. Probab. 5 (1992), 417–430.

[12] V. Vol'kovich, *Infinitely divisible distributions in algebras with stochastic convolution*, J. Soviet Math. 40 (1988), 459–467.

[13] —, *Multidimensional $\mathcal{B}$-stable distributions and realizations of generalized convolutions*, in: Stability Problems of Stochastic Models (Moscow, 1984), VNIISI, Moscow, 1984, 40–54 (in Russian).

Department of Mathematics, Informatics and Econometry
University of Zielona Góra
Podgórna 50
65-246 Zielona Góra, Poland
E-mail: j.misiewicz@wmie.uz.zgora.pl

Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: koles@mimuw.edu.pl

Institute of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: urbanik@math.uni.wroc.pl

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