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Marcinkiewicz integrals on product spaces

by

H. Al-Qassem (Irbid), A. Al-Salman (Irbid),
L. C. Cheng (Bryn Mawr, PA) and Y. Pan (Pittsburgh, PA)

Abstract. We prove the L^p boundedness of the Marcinkiewicz integral operators μ_Ω on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$ under the condition that $\Omega \in L(\log L)^{k/2}(\mathbb{S}^{n_1-1} \times \cdots \times \mathbb{S}^{n_k-1})$. The exponent $k/2$ is the best possible. This answers an open question posed in [7].

1. Introduction. Marcinkiewicz integrals have been studied by many authors, dating back to the investigations of such operators by Zygmund on the circle and by Stein on $\mathbb{R}^n$.

We shall be primarily concerned with Marcinkiewicz integrals on the product space $\mathbb{R}^n \times \mathbb{R}^m$, since the more general setting of $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$ can be handled similarly (see Section 4).

For $n, m \geq 2$, $x \in \mathbb{R}^n \backslash \{0\}$, $y \in \mathbb{R}^m \backslash \{0\}$, we let $x' = x/|x|$ and $y' = y/|y|$. Let $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ be a function satisfying the following cancellation conditions:

$$\begin{cases} \int_{\mathbb{S}^{n-1}} \Omega(x', \cdot)\, d\sigma(x') = 0, \\ \int_{\mathbb{S}^{m-1}} \Omega(\cdot, y')\, d\sigma(y') = 0. \end{cases} \tag{1.1}$$

Then the Marcinkiewicz integral operator μ_Ω is given by

$$\mu_\Omega(f)(x,y) = \Big(\iint_{\mathbb{R}^2_+} |F_{t,s}(x,y)|^2 \frac{dt\, ds}{(ts)^3} \Big)^{1/2}, \tag{1.2}$$

where

$$F_{t,s}(x,y) = \iint_{\{|\xi| \leq t,\, |\eta| \leq s\}} \frac{\Omega(\xi', \eta')}{|\xi|^{n-1} |\eta|^{m-1}} f(x-\xi, y-\eta)\, d\xi\, d\eta. \tag{1.3}$$

It has been known for a while that the L^p boundedness of μ_Ω holds for $1 < p < \infty$ under the condition $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ (see Ding [6] and Chen *et al.* [3]). On the other hand, by adapting an argument of Walsh ([18]) to the product space setting, it can be shown that, for every $\varepsilon > 0$, the L^2 boundedness of μ_Ω fails to hold for some Ω in $L(\log L)^{1-\varepsilon}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. In

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this sense the condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, if sufficient, would be the best possible.

For the special case $p = 2$, Choi ([5]) verified that μ_Ω is indeed bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ for all $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Ding subsequently conjectured in [7] that the L^p boundedness of μ_Ω holds under the condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for $1 < p < \infty$.

As a more recent progress in this investigation, Chen, Fan and Yang obtained the following:

THEOREM 1 ([4]). *Suppose that* $p \in (1, \infty)$, $r = \min\{p, p'\}$, *and*

$$\Omega \in L(\log L)^{2/r}(\log\log L)^{8(1-2/r')}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}).$$

Then μ_Ω *is bounded on* $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Since the condition in Theorem 1 becomes $\Omega \in L(\log L)$ when $p = 2$, it recovers Choi's L^2 result. But, for $p \neq 2$, it still falls short of what is conjectured by Ding.

The main purpose of this paper is to establish the following:

THEOREM 2. *If* $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *and* $p \in (1, \infty)$, *then* μ_Ω *is bounded on* $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one at each occurrence.

2. Main lemma. Given a two-parameter family $\nu = \{\nu_{t,s} : t, s \in \mathbb{R}\}$ of measures on $\mathbb{R}^n \times \mathbb{R}^m$, we define the maximal operator ν^* by

$$\nu^*(f) = \sup_{t,s\in\mathbb{R}} \big||\nu_{t,s}| * f\big| \tag{2.1}$$

and the corresponding square function by

$$G_\nu(f)(x, y) = \Big(\int_{\mathbb{R}\times\mathbb{R}} |\nu_{t,s} * f(x, y)|^2 \, dt \, ds\Big)^{1/2}. \tag{2.2}$$

Also, we write $t^{\pm\alpha} = \min\{t^\alpha, t^{-\alpha}\}$ and use $\|\nu_{t,s}\|$ to denote the total variation of $\nu_{t,s}$.

The following is our main lemma:

LEMMA 2.1. *Let* $a, b \geq 2$, $\alpha, \beta, q > 1$ *and* $A > 0$. *Suppose that the family* $\{\nu_{t,s} : t, s \in \mathbb{R}\}$ *of measures satisfies the following*:

(i) $\|\nu_{t,s}\| \leq A$ *for* $t, s \in \mathbb{R}$;
(ii) $|\widehat{\nu}_{t,s}(\xi, \eta)| \leq A|a^t\xi|^{\pm\alpha/\ln a}|b^s\eta|^{\pm\beta/\ln b}$ *for* $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$ *and* $t, s \in \mathbb{R}$;
(iii) $\|\nu^*(f)\|_q \leq A\|f\|_q$ *for* $f \in L^q(\mathbb{R}^n \times \mathbb{R}^m)$.

Then, for every p satisfying $|1/p - 1/2| < 1/(2q)$, there exists a positive constant C_p which is independent of a and b such that

$$\|G_\nu(f)\|_p \le C_p \|f\|_p \tag{2.3}$$

for $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Two propositions are needed for the proof of Lemma 2.1.

PROPOSITION 2.2. *Suppose that* (i) *and* (iii) *in Lemma* 2.1 *are satisfied and $|1/p_0 - 1/2| = 1/(2q)$. Let $F(x,y,t,s)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^2$ and $F_{t,s}(x,y) = F(x,y,t,s)$ for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(t,s) \in \mathbb{R}^2$. Then*

$$\Big\|\Big(\int_{\mathbb{R}^2} |\nu_{t,s} * F_{t,s}|^2 \, dt\, ds\Big)^{1/2}\Big\|_{L^{p_0}(\mathbb{R}^n\times\mathbb{R}^m)} \le \sqrt{A}\Big\|\Big(\int_{\mathbb{R}^2} |F_{t,s}|^2 \, dt\, ds\Big)^{1/2}\Big\|_{L^{p_0}(\mathbb{R}^n\times\mathbb{R}^m)}.$$

The above proposition can be proved by using the proof of Lemma 14 in [8], after some minor modifications.

For $\lambda > 2$, let $\phi^{(\lambda)} : \mathbb{R} \to [0,1]$ be a C^∞ function supported in $[4/(5\lambda), (5\lambda)/4]$ such that

$$\int_0^\infty \frac{\phi^{(\lambda)}(t)}{t}\, dt = 2 \ln \lambda. \tag{2.4}$$

For $a, b > 2$, let $\Psi \in C^\infty(\mathbb{R}^n)$ and $\Gamma \in C^\infty(\mathbb{R}^m)$ be given by

$$\widehat{\Psi}(\xi) = \phi^{(a)}(|\xi|^2), \qquad \widehat{\Gamma}(\eta) = \phi^{(b)}(|\eta|^2)$$

for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$. For $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $t, s > 0$, set

$$\Psi_t(x) = t^{-n}\Psi(x/t), \qquad \Gamma_s(y) = s^{-m}\Gamma(y/s)$$

and

$$\Phi_{t,s}(x,y) = \Psi_t(x) \cdot \Gamma_s(y).$$

Define the square function operator S_Φ on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$(S_\Phi f)(x,y) = \Big(\int_{\mathbb{R}\times\mathbb{R}} |(\Phi_{a^t,b^s} * f)(x,y)|^2 \, dt\, ds\Big)^{1/2}. \tag{2.5}$$

PROPOSITION 2.3. *For every $p \in (1,\infty)$, there exists a positive constant C_p independent of a and b such that*

$$\|S_\Phi f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le C_p \|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$$

for $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Proposition 2.3 can be established by using an argument of Fefferman and Stein in [12] (pp. 123–124) which is rooted in the theory of vector-valued singular integrals ([16, p. 46]). A careful tracking of the constant at each step shows its independence from the parameters a and b, which is a key feature of Proposition 2.3. Details of the proof are omitted.

Proof of Lemma 2.1. It suffices to prove (2.3) for all Schwartz functions f on $\mathbb{R}^n \times \mathbb{R}^m$. For $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, it follows from (2.4) that

$$f = \int_{\mathbb{R}\times\mathbb{R}} (\Phi_{a^t,b^s} * f)\, dt\, ds. \tag{2.6}$$

By (2.6) and Minkowski's inequality,

$$\begin{aligned} G_\nu(f)(x,y) &= \Big(\int_{\mathbb{R}\times\mathbb{R}} \Big| \int_{\mathbb{R}\times\mathbb{R}} \Phi_{a^{t+u},b^{s+v}} * \nu_{t,s} * f(x,y)\, du\, dv \Big|^2 dt\, ds \Big)^{1/2} \\ &\le \int_{\mathbb{R}\times\mathbb{R}} (H_{u,v} f)(x,y)\, du\, dv, \end{aligned} \tag{2.7}$$

where

$$(H_{u,v} f)(x,y) = \Big(\int_{\mathbb{R}\times\mathbb{R}} |\Phi_{a^{t+u},b^{s+v}} * \nu_{t,s} * f(x,y)|^2\, dt\, ds \Big)^{1/2}.$$

First we shall obtain the following L^2 estimate:

$$\|H_{u,v}\|_{2,2} \le \frac{A}{2\sqrt{\alpha\beta}}\, e^{(\alpha+\beta)} e^{-\alpha|u|} e^{-\beta|v|}. \tag{2.8}$$

We shall present the proof of (2.8) for the case $u, v \ge 0$ only. The remaining cases can be handled similarly. Let

$$E_{u,v,\xi,\eta} = \Big\{ (t,s) \in \mathbb{R}\times\mathbb{R} : \frac{4}{5a} \le a^{2(t+u)}|\xi|^2 \le \frac{5a}{4},\ \frac{4}{5b} \le b^{2(s+v)}|\eta|^2 \le \frac{5b}{4} \Big\}.$$

By Plancherel's theorem and assumption (ii), we have

$$\begin{aligned} \|H_{u,v} f\|_2^2 &= \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 \Big(\int_{\mathbb{R}\times\mathbb{R}} \big|\phi^{(a)}(a^{2(t+u)}|\xi|^2)\phi^{(b)}(b^{2(s+v)}|\eta|^2)\big|^2 \\ &\qquad\qquad \times |\widehat{\nu}_{t,s}(\xi,\eta)|^2\, dt\, ds \Big)\, d\xi\, d\eta \\ &\le A^2 \int_{\mathbb{R}^n\times\mathbb{R}^m} |\widehat{f}(\xi,\eta)|^2 \Big(\int_{E_{u,v,\xi,\eta}} |a^t\xi|^{2\alpha/\ln a} |b^s\eta|^{2\beta/\ln b}\, dt\, ds \Big)\, d\xi\, d\eta \\ &\le \frac{A^2}{4\alpha\beta}\, e^{2\alpha-2\alpha|u|} e^{2\beta-2\beta|v|} \|f\|_2^2, \end{aligned}$$

which yields (2.8).

Next we let p_0 satisfy $|1/p_0 - 1/2| = 1/(2q)$. Then by Propositions 2.2 and 2.3, there exists a positive constant C such that

$$\|H_{u,v} f\|_{p_0} \le C\|S_\Phi f\|_{p_0} \le C\|f\|_{p_0}. \tag{2.9}$$

By interpolating between (2.8) and (2.9) we obtain

$$\|H_{u,v}f\|_p \le C(\alpha,\beta,p)e^{-\alpha_p|u|}e^{-\beta_p|v|}\|f\|_p \tag{2.10}$$

for $(u,v)\in\mathbb{R}\times\mathbb{R}$ and p satisfying $|1/p-1/2|<1/(2q)$, where $C(\alpha,\beta,p)$, α_p and β_p are positive constants independent of u,v,a and b. Lemma 2.1 now follows from (2.7) and (2.10).

3. Proof of the main theorem. Assume $\Omega\in L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ and Ω satisfies (1.1). For $k\in\mathbb{N}$, let $E_k=\{(x,y)\in\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}: 2^{k-1}\le |\Omega(x,y)|<2^k\}$. Let $D=\{k\in\mathbb{N}:|E_k|>2^{-4k}\}$, where $|\cdot|$ denotes the product measure on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$. We now define Ω_k by

$$\begin{aligned}\Omega_k(x,y)&=\Omega(x,y)\chi_{E_k}(x,y)-\int_{\mathbb{S}^{n-1}}\Omega(x,y)\chi_{E_k}(x,y)\,d\sigma(x)\\&\quad-\int_{\mathbb{S}^{m-1}}\Omega(x,y)\chi_{E_k}(x,y)\,d\sigma(y)+\int_{E_k}\Omega(x,y)\,d\sigma(x)\,d\sigma(y)\end{aligned}\tag{3.1}$$

for $k\in\mathbb{N}$, and

$$\Omega_0(x,y)=\Omega(x,y)-\sum_{k\in D}\Omega_k(x,y). \tag{3.2}$$

Thus, for $k\in\mathbb{N}\cup\{0\}$, Ω_k satisfies (1.1). Since $\|\Omega_0\|_{L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\le 8$, it follows that μ_{Ω_0} is a bounded operator on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ for all $p\in(1,\infty)$.

For $k\in D$ we let

$$a_k=2^k,\qquad A_k=\frac{16\pi^{(n+m)/2}\|\Omega\chi_{E_k}\|_1}{(\Gamma(n/2)\Gamma(m/2))}.$$

We then define the family of measures $\nu^{(k)}=\{\nu_{k,t,s}:t,s\in\mathbb{R}\}$ on $\mathbb{R}^n\times\mathbb{R}^m$ by

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f\,d\nu_{k,t,s}=\left(\frac{1}{A_k a_k^{t+s}}\right)\int_{\{|x|\le a_k^t,\,|y|\le a_k^s\}}\frac{\Omega_k(x',y')}{|x|^{n-1}|y|^{m-1}}f(x,y)\,dx\,dy. \tag{3.3}$$

Thus

$$\|\nu_{k,t,s}\|\le 1. \tag{3.4}$$

By the cancellation properties of Ω_k, we have

$$\begin{aligned}&|\widehat{\nu}_{k,t,s}(\xi,\eta)|\\&\quad\le\left(\frac{1}{A_k a_k^{t+s}}\right)\int_{\{|x|\le a_k^t,\,|y|\le a_k^s\}}|e^{i\xi\cdot x}-1|\,\frac{|\Omega_k(x',y')|\,dx\,dy}{|x|^{n-1}|y|^{m-1}}\le a_k^t|\xi|.\end{aligned}\tag{3.5}$$

Similarly,

$$|\widehat{\nu}_{k,t,s}(\xi,\eta)|\le a_k^s|\eta|. \tag{3.6}$$

By the proof of Corollary 4.1 in [9],

$$\Big|a_k^{-s}\int\limits_{|y|\le a_k^s} e^{i\eta\cdot y}\,\frac{\Omega_k(x',y')}{|y|^{m-1}}\,dy\Big| \le C(a_k^s|\eta|)^{-1/6}\Big(\int\limits_{\mathbb{S}^{m-1}}|\Omega_k(x',y')|^2\,d\sigma(y')\Big)^{1/2}.$$

Thus, for $k\in D$ and $t,s\in\mathbb{R}$,

$$\begin{aligned}(3.7)\qquad |\widehat{\nu}_{k,t,s}(\xi,\eta)| &\le \left(\frac{1}{A_k a_k^{t+s}}\right)\int\limits_{|x|\le a_k^t}\frac{1}{|x|^{n-1}}\Big|\int\limits_{|y|\le a_k^s} e^{i\eta\cdot y}\,\frac{\Omega_k(x',y')}{|y|^{m-1}}\,dy\Big|\,dx\\ &\le CA_k^{-1}(a_k^s|\eta|)^{-1/6}\|\Omega_k\|_{L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\\ &\le C2^{2k+1}(a_k^s|\eta|)^{-1/6}.\end{aligned}$$

Similarly,

$$(3.8)\qquad |\widehat{\nu}_{k,t,s}(\xi,\eta)| \le C2^{2k+1}(a_k^t|\xi|)^{-1/6}.$$

By (3.4)–(3.8) we obtain

$$(3.9)\qquad |\widehat{\nu}_{k,t,s}(\xi,\eta)| \le C|a_k^t\xi|^{\pm1/(6k)}|a_k^s\eta|^{\pm1/(6k)}.$$

By the boundedness of the strong maximal function on $\mathbb{R}^2$ we see that

$$\|(\nu^{(k)})^*(f)\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m)} \le B_q\|f\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m)}$$

for $1<q\le\infty$, where B_q is independent of k. Applying Lemma 2.1, we get

$$(3.10)\qquad \|G_{\nu^{(k)}}(f)\|_p \le C_p\|f\|_p.$$

Finally, by Minkowski's inequality and (3.10), we have

$$\begin{aligned}(3.11)\qquad \|\mu_\Omega(f)\|_p &\le \|\mu_{\Omega_0}(f)\|_p + \sum_{k\in D}(\ln a_k)A_k\|G_{\nu^{(k)}}(f)\|_p\\ &\le C_p\Big(1+\sum_{k\in D}k\int\limits_{E_k}|\Omega(x,y)|\,d\sigma(x)\,ds(y)\Big)\|f\|_p\\ &\le C_p(1+\|\Omega\|_{L(\log L)})\|f\|_p\end{aligned}$$

for $1<p<\infty$ and $f\in L^p(\mathbb{R}^n\times\mathbb{R}^m)$. This proves Theorem 2.

4. Concluding remarks. Let $k\in\mathbb{N}$, $n_1,\dots,n_k\ge2$ and $\Omega(x_1',\dots,x_k')$ be an integrable function on $\mathbb{S}^{n_1-1}\times\cdots\times\mathbb{S}^{n_k-1}$. Suppose that Ω satisfies the following cancellation condition:

$$(4.1)\qquad \int\limits_{\mathbb{S}^{n_j-1}}\Omega(x_1',\dots,x_k')\,d\sigma(x_j')=0 \quad \text{for } j=1,\dots,k.$$

The corresponding Marcinkiewicz integral operator on $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$ is defined by

$$(4.2)\qquad \mu_\Omega(f)(x_1,\dots,x_k)=\Big(\int\limits_0^\infty\cdots\int\limits_0^\infty|F_{t_1,\dots,t_k}(x_1,\dots,x_k)|^2\,\frac{dt_1\cdots dt_k}{t_1^3\cdots t_k^3}\Big)^{1/2},$$

where

$$(4.3)\qquad F_{t_1,\ldots,t_k}(x_1,\ldots,x_k) = \int\limits_{|y_1|\le t_1} \cdots \int\limits_{|y_k|\le t_k} \frac{\Omega(y_1',\ldots,y_k')}{|y_1|^{n_1-1}\ldots|y_k^{n_k-1}} f(x_1-y_1,\ldots,x_k-y_k)\,dy_1\ldots dy_k.$$

Theorem 2 admits the following generalization:

THEOREM 3. *For Ω, μ_Ω as above, if $\Omega\in L(\log L)^{k/2}(\mathbb{S}^{n_1-1}\times\cdots\times\mathbb{S}^{n_k-1})$ and $p\in(1,\infty)$, then μ_Ω is bounded on $L^p(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k})$. The exponent $k/2$ is the best possible.*

When $k=1$ (i.e. the underlying space is not a product space), the L^p boundedness of μ_Ω under the condition $\Omega\in L(\log L)^{1/2}$ was obtained first for $p=2$ in [18], and then for all $p\in(1,\infty)$ in [1]. Historically, this is the case that had received the most amount of attention. For a sampling of past studies, see [2], [11], [13], [14], [18], [19]. Related results can also be found in [8], [15], and [17].

Theorem 2 takes care of the case $k=2$. The proof of Theorem 2 easily extends to the case $k>2$. We omit the details.

References

[1] A. Al-Salman, H. Al-Qassem, L. Cheng and Y. Pan, *L^p bounds for the function of Marcinkiewicz*, Math. Res. Lett. 9 (2002), 697–700.

[2] A. Benedek, A. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. 48 (1962), 356–365.

[3] J. Chen, D. Fan and Y. Ying, *Rough Marcinkiewicz integrals with $L(\log^+ L)^2$ kernels on product spaces*, Adv. Math. (China) 30 (2001), 179–181.

[4] —, —, —, *The method of rotation and Marcinkiewicz integrals on product domains*, Studia Math. 153 (2002), 41–58.

[5] Y. Choi, *Marcinkiewicz integrals with rough homogeneous kernels of degree zero in product domains*, J. Math. Anal. Appl. 261 (2001), 53–60.

[6] Y. Ding, *L^2-boundedness of Marcinkiewicz integral with rough kernel*, Hokkaido Math. J. 27 (1998), 105–115.

[7] —, *On Marcinkiewicz integral*, in: Proc. Conf. "Singular Integrals and Related Topics III", Osaka, 2001, 28–38.

[8] J. Duoandikoetxea, *Multiple singular integrals and maximal functions along hypersurfaces*, Ann. Inst. Fourier (Grenoble) 36 (1986), 185–206.

[9] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. 84 (1986), 541–561.

[10] D. Fan and Y. Pan, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math. 119 (1997), 799–839.

[11] D. Fan and S. Sato, *Weak type $(1,1)$ estimates for Marcinkiewicz integrals with rough kernels*, Tohoku Math. J. 53 (2001), 265–284.

[12] R. Fefferman and E. Stein, *Singular integrals on product spaces*, Adv. Math. 45 (1982), 117–143.

[13] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–140.
[14] E. Stein, *On the functions of Littlewood–Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc. 88 (1958), 430–466.
[15] —, *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*, Princeton Univ. Press, 1993.
[16] —, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.
[17] E. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. 84 (1978), 1239–1295.
[18] T. Walsh, *On the function of Marcinkiewicz*, Studia Math. 44 (1972), 203–217.
[19] A. Zygmund, *On certain integrals*, Trans. Amer. Math. Soc. 55 (1944), 170–204.

Department of Mathematics
Yarmouk University
Irbid, Jordan
E-mail: hussain@yu.edu.jo
alsalman@yu.edu.jo

Department of Mathematics
Bryn Mawr College
Bryn Mawr, PA 19010, U.S.A.
E-mail: lcheng@brynmawr.edu

Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260, U.S.A.
E-mail: yibiao+@pitt.edu

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