

Growth and smooth spectral synthesis in the Fourier algebras of Lie groups

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Abstract. Let G be a Lie group and $A(G)$ the Fourier algebra of G . We describe sufficient conditions for complex-valued functions to operate on elements $u \in A(G)$ of certain differentiability classes in terms of the dimension of the group G . Furthermore, generalizing a result of Kirsch and Müller [Ark. Mat. 18 (1980), 145–155] we prove that closed subsets E of a smooth m -dimensional submanifold of a Lie group G having a certain cone property are sets of smooth spectral synthesis. For such sets we give an estimate of the degree of nilpotency of the quotient algebra $I_A(E)/J_A(E)$, where $I_A(E)$ and $J_A(E)$ are the largest and the smallest closed ideals in $A(G)$ with hull E .

1. Introduction. In this paper we study two questions concerning the Fourier algebra $A(G)$ of a (noncommutative) Lie group G . The first one deals with the functional calculus in $A(G)$ and the other with problems of spectral synthesis in that algebra.

Functional calculus is one of the basic tools in the theory of Banach algebras and in its applications. In particular, it plays a fundamental role in some parts of harmonic analysis, where one of the important algebras is the Fourier algebra $A(G)$ for a locally compact group G . One says that a complex-valued function φ defined on \mathbb{C} *operates on* $u \in A(G)$ if the composition $\varphi \circ u : G \rightarrow \mathbb{C}$ belongs to $A(G)$. It is known that any analytic function defined on a neighbourhood of the image of $u \in A(G)$ operates on u . Furthermore H. Helson, J.-P. Kahane, Y. Katznelson and W. Rudin [HKKR] for the case of G abelian, and then C. Dunkl [D] and D. Rider [Ri] for infinite compact groups G , showed that exactly the real-analytic functions $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ operate on each element of $A(G)$. Some necessary conditions on the functions φ operating on certain classes of elements of $A(G)$ for abelian G can be found for example in [Ka]. In order to determine the class $[u]$ of functions which operate on a given element u of $A(G)$, it is essential to have control over the growth of the norms $\|e^{itu}\|_{A(G)}$ for $t \in \mathbb{R}$ and real functions u

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in $A(G)$. In this paper we study this question for Lie groups. First we give an estimate of the growth in t of the norms $\|e^{itu}\|_{A(G)}$ for a connected compact Lie group and real-valued functions u of a certain differentiability class in terms of the dimension of G . Our method is a development of the method given in [Ka] for the one-dimensional torus. For a general Lie group G , we use a Laplacian and its heat kernel to derive for smooth functions u on G a similar estimate for $\|e^{itu}\|_{A(G)}$. This result is slightly rougher than in the compact case. These estimates allow us to find necessary conditions for a function φ to operate on smooth functions $u \in A(G)$.

Section 4 deals with questions of spectral synthesis in $A(G)$ for Lie groups G . It is well known that for any closed subset $E \subset G$ there exists a largest closed ideal $I_A(E)$ and a smallest closed ideal $J_A(E)$ with hull E . We say that E is a *set of spectral synthesis* if $I_A(E) = J_A(E)$. If E is not spectral, it may nevertheless happen that the algebra $I_A(E)/J_A(E)$ is nilpotent (see [W], where such sets E are studied). We say that E is a *set of smooth spectral synthesis* if $\overline{I_A(E) \cap \mathcal{D}(G)} = I_A(E)$, where $\mathcal{D}(G)$ is the set of test functions on G . Generalizing a method in [KM] we show that closed subsets E of a smooth m -dimensional submanifold of a Lie group G with a certain cone property are sets of smooth synthesis. Then using Herz's arguments [H] we obtain an estimate of the nilpotency degree of the algebra $I_A(E)/J_A(E)$.

In the last section we give some applications to linear operator equations and to Varopoulos algebras.

2. Preliminaries and notations. Let G be a locally compact σ -compact separable group with left Haar measure $m = dg$. Let $L^p(G)$, $p = 1, 2$, denote the space of p -integrable functions with norm $\|\cdot\|_p$ and let $C(G)$ denote the algebra of continuous complex-valued functions on G . The convolution algebra $L^1(G)$ is an involutive algebra with involution defined by $f^*(s) = \Delta^{-1}(s)f(s^{-1})$, where Δ is the modular function of the group. Let $\lambda : G \rightarrow B(L^2(G))$ be the left regular representation given by $\lambda(s)f(g) = f(s^{-1}g)$. We denote by $\text{VN}(G)$ the von Neumann algebra of G , that is,

$$\text{VN}(G) = \overline{\text{span}}^{\text{WOT}} \{ \lambda(g) : g \in G \} \subset B(L^2(G)).$$

The *Fourier algebra* is the family of functions $s \mapsto (\lambda(s)\xi, \eta) = \bar{\eta} * \check{\xi}$, $\xi, \eta \in L^2(G)$, $\check{\xi}(s) = \xi(s^{-1})$, $s \in G$, as defined by Eymard in [E]. The Banach space $A(G)$ can be identified with the predual $\text{VN}(G)_*$ via $\langle (\lambda(s)\xi, \eta), T \rangle = (T\xi, \eta)$, and thus is a normed algebra with the norm denoted by $\|\cdot\|_A$. It is known that for $u \in A(G)$ there exist $\xi, \eta \in L^2(G)$ such that $u = \xi * \bar{\eta}$ and $\|u\|_A = \|\xi\|_2 \cdot \|\eta\|_2$. The algebra $A(G)$ is a semisimple regular Banach algebras with spectrum G (see [E]).

3. Growth in the Fourier algebra of a Lie group and functional calculus in $A(G)$

3.1. The compact Lie case. The following description of the dual space of a connected compact Lie group G has been taken from [Wa]. Let \mathfrak{g} be the Lie algebra of G . Then $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1$ with \mathfrak{z} the centre of \mathfrak{g} and $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ a compact Lie algebra. Let $\langle \cdot, \cdot \rangle$ be an inner product on \mathfrak{g} satisfying (1) $\langle \mathfrak{g}_1, \mathfrak{z} \rangle = (0)$ and (2) $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_1 \times \mathfrak{g}_1} = -B_{\mathfrak{g}_1}$ (here $B_{\mathfrak{k}}$ denotes the Killing form of a Lie algebra \mathfrak{k}). Let $U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} . Let X_1, \dots, X_n be an orthonormal basis of \mathfrak{g} such that $\{X_1, \dots, X_r\}$ is a basis of \mathfrak{z} . Set $\Omega = \sum_i X_i^2 \in U(\mathfrak{g})$. Then Ω is independent of the choice of the orthonormal basis of \mathfrak{g} and Ω is central in $U(\mathfrak{g})$.

Let \mathfrak{t} be a maximal abelian subalgebra of \mathfrak{g}_1 and let $T = \exp(\mathfrak{t})$. Let also $\lambda_1, \dots, \lambda_r$ be complex-valued linear forms on \mathfrak{z} defined by $\lambda_j(X_i) = 2\pi(-1)^{1/2}\delta_{i,j}$. Let P be a Weyl chamber of T . Let A_1, \dots, A_l be defined by $2A_i(H_{\alpha_j})/\alpha_j(H_{\alpha_j}) = \delta_{i,j}$, where $\alpha_1, \dots, \alpha_l$ are the simple roots relative to P , and the H_{α_j} the corresponding vectors in \mathfrak{t} . To every γ in the dual space \widehat{G} of G corresponds a unique element $A_\gamma = \sum_i n_i \lambda_i + \sum_j m_j A_j$ with the n_i integers and the m_j nonnegative integers. Set $\|\gamma\| = \max_{i,j} \{|n_i|, m_j\}$.

We know from [Wa, Lemma 5.6.4] (with the notations of that lemma) that for every $\gamma \in \widehat{G}$ and $\pi_\gamma \in \gamma$,

$$-\pi_\gamma(\Omega) = (\langle A_\gamma + \varrho, A_\gamma + \varrho \rangle - \langle \varrho, \varrho \rangle) \mathbb{I}_{\mathcal{H}_\gamma} =: c(\gamma) \mathbb{I}_{\mathcal{H}_\gamma},$$

where ϱ is half the sum of the positive roots of G related to the Weyl chamber of T . Then by [Wa, Lemma 5.6.6], there are positive constants c_1, c_2 such that

$$(3.1) \quad c_1 \|\gamma\|^2 \leq c(\gamma) \leq c_2 \|\gamma\|^2$$

and by [Wa, Lemma 5.6.7], the series

$$(3.2) \quad \sum_{\gamma \in \widehat{G}} d(\gamma)^2 (1 + \|\gamma\|^2)^{-s}$$

converges if $s > \dim(G)/2$. Here $d(\gamma)$ denotes the dimension of the Hilbert space \mathcal{H}_γ of γ .

Also for a real number a , let $[a]$ be the integer part of a and let $d(G)$ denote the dimension of the group G .

THEOREM 3.1. *Let G be a connected compact Lie group and let $u = \bar{u}$ be a self-adjoint element of $A(G)$, which is differentiable of class $C^{[d(G)/2]+1}$. Then there exists a positive constant $C = C(u)$ such that*

$$(3.3) \quad \|e^{itu}\|_{A(G)} \leq C(1 + |t|)^{d(G)/2}, \quad t \in \mathbb{R}.$$

Proof. By (3.1) and (3.2), we know that

$$(3.4) \quad \sum_{\gamma \in \widehat{G}} \frac{d(\gamma)^2}{(1 + c(\gamma))^s} < \infty, \quad \forall s > d(G)/2.$$

Take $N \in \mathbb{N}^*$ and let $\widehat{G}_N = \{\gamma \in \widehat{G} : \|\gamma\| \leq N\}$. By the Plancherel theorem, for all real $m > d(G)/4$ there exists an L^2 -function E_m on G such that

$$\gamma(E_m) = \frac{1}{(1 + c(\gamma))^m} \mathbb{I}_{\mathcal{H}_\gamma}, \quad \forall \gamma \in \widehat{G}.$$

Then

$$(3.5) \quad E_m * (1 - \Omega)^m g = g$$

for every k -times differentiable function g on G , where $k = 2m$ if $2m$ is an integer and $k = [2m] + 1$ otherwise. Denote also by F_N the element in $L^2(G)$ for which

$$\gamma(F_N) = \begin{cases} \mathbb{I}_{\mathcal{H}_\gamma}, & \forall \gamma \in \widehat{G}_N, \\ 0, & \text{otherwise,} \end{cases}$$

and by $E_{m,N}$ the element in $L^2(G)$ for which

$$\gamma(E_{m,N}) = \begin{cases} \gamma(E_m), & \forall \gamma \notin \widehat{G}_N, \\ 0, & \text{otherwise.} \end{cases}$$

Write now g_t for e^{itu} , $t \in \mathbb{R}$. Since $u \in C^{[d(G)/2]+1}$, we also have $g_t \in C^{[d(G)/2]+1}$. We decompose g_t into

$$g_t = a_{t,N} + b_{t,N},$$

where $a_{t,N}$ and $b_{t,N}$ are defined by

$$\gamma(a_{t,N}) = \begin{cases} \gamma(g_t), & \forall \gamma \in \widehat{G}_N, \\ 0, & \text{otherwise,} \end{cases} \quad \gamma(b_{t,N}) = \begin{cases} 0, & \forall \gamma \in \widehat{G}_N, \\ \gamma(g_t), & \text{otherwise.} \end{cases}$$

Then $a_{t,N}$ is a C^∞ -vector and so $b_{t,N} = g_t - a_{t,N}$ is of class $C^{[d(G)/2]+1}$. By the definition of F_N , we have

$$F_N * a_{t,N} = a_{t,N}.$$

Hence

$$\|a_{t,N}\|_{A(G)} \leq \|F_N\|_2 \|a_{t,N}\|_2 \leq \|F_N\|_2 \|g_t\|_2 \leq \|F_N\|_2 \|g_t\|_{L^\infty(G)}.$$

Now, by [Wa, proof of 5.6.7], if we set $n = d(G)$ for simplicity, we have

$$\|F_N\|_2^2 = \sum_{\|\gamma\| \leq N} d_\gamma^2 \leq c_3 \sum_{j=0}^N j^{n-l-r} (2r+l)(2j+1)^{r+l-1} \leq c_4 \sum_{j=0}^N j^{n-1} \leq c_5 N^n.$$

Hence, since u is a continuous real-valued function, we have $g_t = e^{itu} \in L^\infty(G)$, $t \in \mathbb{R}$, and $\|g_t\|_\infty = 1$, and we see that

$$\|a_{t,N}\|_{A(G)} \leq CN^{d(G)/2}$$

for a certain constant $C > 0$. Now for the norm of the element $b_{t,N}$ we get, using (3.5) for $g = b_{t,N}$ and $m = \frac{1}{2}([d(G)/2] + 1)$ (which is easily checked to be strictly larger than $d(G)/4$),

$$E_m * (1 - \Omega)^m b_{t,N} = E_{m,N} * (1 - \Omega)^m g_t$$

and so

$$\|b_{t,N}\|_{A(G)} = \|E_{m,N} * (1 - \Omega)^m g_t\|_{A(G)} \leq \|E_{m,N}\|_2 \|(1 - \Omega)^m g_t\|_2.$$

If m is an integer then, since Ω is a differential operator of order 2, we have $(1 - \Omega)^m g_t = (1 + t^{2m})e^{itu}w_{t,m}$ for some $w_{t,m} \in L^2(G)$ such that $\|w_{t,m}\|_2 \leq C_1$ for all $t \in \mathbb{R}$ (for some constant $C_1 > 0$). If m is a half integer $k + 1/2$ then we use the inequality

$$(3.6) \quad \|(1 - \Omega)^{1/2}\eta\|_2 \leq \left(\|\eta\|_2^2 + \sum_j \|X_j\eta\|_2^2 \right)^{1/2}$$

for any differentiable η . Indeed, if η is twice differentiable, then

$$\|(1 - \Omega)^{1/2}\eta\|_2^2 = \langle (1 - \Omega)\eta, \eta \rangle = \|\eta\|_2^2 - \sum_j \langle X_j^2\eta, \eta \rangle = \|\eta\|_2^2 + \sum_j \|X_j\eta\|_2^2.$$

We can now approximate a once differentiable η with respect to the Sobolev norm $\|\eta\|_S := (\|\eta\|_2^2 + \sum_j \|X_j\eta\|_2^2)^{1/2}$ by twice differentiable functions to obtain our inequality (3.6). Therefore, for $\eta_t := (1 - \Omega)^k g_t$,

$$\begin{aligned} \|(1 - \Omega)^{k+1/2}g_t\|_2^2 &= \|(1 - \Omega)^{1/2}\eta_t\|_2^2 \\ &\leq \|\eta_t\|_2^2 + \sum_j \|X_j\eta_t\|_2^2 \\ &= \|(1 - \Omega)^k g_t\|_2^2 + \sum_j \|X_j(1 - \Omega)^k g_t\|_2^2 \\ &= \|(1 + t^{2k})e^{itu}v_{t,k}\|_2^2 + \|(1 + t^{2k+1})e^{itu}w_{t,k}\|_2^2 \end{aligned}$$

for some $w_{t,k}, v_{t,k} \in L^2(G)$ with L^2 -norms uniformly bounded in t . Hence

$$\|(1 - \Omega)^m g_t\|_2 \leq C(1 + |t|^{2m}), \quad t \in \mathbb{R},$$

for some new constant $C > 0$ and for $m = k + 1/2$.

Therefore we get the following estimate of the $A(G)$ -norm of $b_{t,N}$: for $t \in \mathbb{R}$,

$$\begin{aligned} \|b_{t,N}\|_{A(G)} &\leq C_1 \left(\sum_{\|\gamma\|>N} \frac{d(\gamma)^2}{(1 + c(\gamma))^{2m}} \right)^{1/2} (1 + |t|)^{2m} \\ &\leq C_2 \left(\sum_{j>N} j^{n-1} \frac{1}{(1 + j)^{4m}} \right)^{1/2} (1 + |t|)^{2m} \leq C_3 \frac{1}{N^{2m-n/2}} (1 + |t|)^{2m}. \end{aligned}$$

Hence, if we let N be the smallest integer $\geq |t|$ we obtain

$$\begin{aligned} \|e^{itu}\|_{A(G)} &\leq \|a_{t,N}\|_{A(G)} + \|b_{t,N}\|_{A(G)} \\ &\leq \frac{C}{2} (1 + |t|)^{d(G)/2} + \frac{C}{2} (1 + |t|)^{d(G)/2} \end{aligned}$$

for a new constant $C > 0$. ■

REMARK 3.2. In the case of the one-dimensional torus $G = \mathbb{T}$ it is shown in [Ka] that if $u = \bar{u} \neq \text{const}$ is differentiable of class C^2 , the estimate (3.3) is sharp, i.e. then $\|e^{inu}\|_{A(\mathbb{T})} \approx \sqrt{n}$. We could also ask for which class of functions $u = \bar{u} \in A(G)$ on a compact Lie group G we can have $\|e^{itu}\|_{A(\mathbb{T})} \approx t^{d(G)/2}$.

3.2. The case of a general nondiscrete Lie group. We now consider a Lie group G of positive dimension and we let G_0 be its connected component. Choose a basis $\{X_1, \dots, X_n\}$ of the Lie algebra \mathfrak{g} of G and take again a Laplacian $\Omega = \sum_{i=1}^n X_i^2$ on G . Let furthermore $(h_t)_{t>0}$ be the heat kernel on G_0 associated to Ω (see [VSC] for the definition).

Define the function E_m ($m \in \mathbb{N}$, $m > d(G)/4$) on G_0 by the formula

$$(3.7) \quad E_m = \frac{1}{m!} \int_0^\infty t^{m-1} e^{-t} h_t dt.$$

The function E_m is clearly in $L^1(G_0) \subset L^1(G)$ and $\int_G E_m(g) dg = 1$, since $\|h_t\|_1 = \int_G E_m(g) dg = 1$ for all $t > 0$.

Furthermore, denoting by λ the left regular representation of G on $L^2(G)$, we find that $\lambda(E_m)$ is the inverse of the operator $(1 - \Omega)^m$. In fact, let

$$-\Omega = \int_0^\infty s dP_s$$

be the spectral resolution of $-\Omega$. Then

$$(1 - \Omega)^m = \int_0^\infty (1 + s)^m dP_s, \quad \lambda(h_t) = e^{t\Omega} = \int_0^\infty e^{-ts} dP_s$$

by the definition of the heat kernel and

$$\begin{aligned} \lambda(E_m) &= \frac{1}{m!} \int_0^\infty t^{m-1} e^{-t} \int_0^\infty e^{-ts} dP_s dt \\ &= \frac{1}{m!} \int_0^\infty \int_0^\infty t^{m-1} e^{-(1+s)t} dt dP_s \\ &= \frac{1}{m!} \int_0^\infty \frac{1}{(1+s)^m} m! dP_s = (1 - \Omega)^{-m}. \end{aligned}$$

This means in particular that

$$(3.8) \quad E_m * (1 - \Omega)^m f = f$$

for any $f \in \mathcal{D}(G)$, the space of compactly supported C^∞ -functions on G .

For the estimation of the $A(G)$ norm, we must show that the functions \check{E}_m defined by $\check{E}_m(g) := E_m(g^{-1})$, $g \in G$, are in $L^2(G)$ for $m > d(G)/4$.

LEMMA 3.3. *There exists a Laplacian Ω on G such that the functions \check{E}_m are in $L^2(G)$ for $m > d(G)/4$.*

Proof. If G_0 is not unimodular, let G_1 be the kernel of the modular function Δ in G_0 . Since $\Delta(\exp(tU)) = e^{-t\delta(U)}$, $t \in \mathbb{R}$, $U \in \mathfrak{g}$, where $\delta : \mathfrak{g} \rightarrow \mathbb{R}$ denotes the trace of $\text{ad}(U)$, it follows that G_1 is of codimension one in G and that G is the topological product of \mathbb{R} and G_1 , since now the function δ is not 0. Let $\mathfrak{g}_1 = \ker(\delta)$ be the Lie algebra of G_1 . We choose a basis vector $X = X_1$ in $\mathfrak{g} \setminus \mathfrak{g}_1$ and vectors X_2, \dots, X_n in \mathfrak{g}_1 . We can also assume that

$$(3.9) \quad \delta(X) = -1/2, \quad \text{i.e.} \quad \Delta(\exp(sX)) = e^{s/2}, \quad s \in \mathbb{R}.$$

It is easy to see then that

$$(3.10) \quad \Delta(g) \leq e^{d\varrho(g)}, \quad g \in G,$$

where $\varrho(g)$ denotes the Carnot–Carathéodory distance of g to the origin and where d is some positive constant (see [VSC]).

Since the linear operators $\lambda(h_t)$, $t > 0$, are self-adjoint and the functions h_t are real-valued, it follows that

$$(3.11) \quad \check{h}_t(g) = h_t(g^{-1}) = \Delta(g)h_t(g)$$

for every $g \in G$. Hence by (3.7), $\check{E}_m = \Delta E_m$. By [VSC, V.4.3 and IX.1.3] and (3.10), it follows that for some constant $C > 0$,

$$(3.12) \quad \Delta(g)h_t(g) \leq C \frac{1}{t^{d(G)/2}}, \quad 0 < t \leq \varepsilon, \quad g \in G,$$

for some constant $\varepsilon > 0$.

Define for $t > 0$ the function p_t on \mathbb{R} by

$$p_t(s) := \int_{G_1} h_t(\exp(sX)g_1) dg_1, \quad s \in \mathbb{R}.$$

The functions p_t are then of class C^∞ and they satisfy the equations

$$\begin{aligned} X_j p_t(s) &= \int_{G_1} X_j h_t(\exp(sX)g_1) dg_1 \\ &= \int_{G_1} \frac{d}{du} h_t(\exp(-uX_j) \exp(sX)g_1)_{u=0} dg_1 \end{aligned}$$

$$\begin{aligned}
 &= \int_{G_1} \frac{d}{du} h_t(\exp(sX) \exp(-u \operatorname{Ad}(\exp(-sX))X_j)g_1)_{u=0} dg_1 \\
 &= \int_{G_1} \frac{d}{du} h_t(\exp(sX)g_1)_{u=0} dg_1 = 0
 \end{aligned}$$

for $j = 2, \dots, n$, and therefore

$$\begin{aligned}
 (3.13) \quad \left(\frac{\partial}{\partial s}\right)^2 p_t(s) &= \int_{G_1} X^2 h_t(\exp(sX)g_1) dg_1 = \int_{G_1} \Omega h_t(\exp(sX)g_1) dg_1 \\
 &= \int_{G_1} \frac{\partial}{\partial t} h_t(\exp(sX)g_1) dg_1 = \frac{\partial}{\partial t} p_t(s).
 \end{aligned}$$

Hence p_t is the heat kernel on \mathbb{R} and so $p_t(s) = (c/\sqrt{t})e^{-s^2/4t}$, $s \in \mathbb{R}$ (for some $c > 0$). This shows that

$$\begin{aligned}
 (3.14) \quad \|\Delta h_t\|_1 &= \int_G \Delta(g)h_t(g) dg \\
 &= \int_{\mathbb{R}} \Delta(\exp(sX)) \int_{G_1} h_t(\exp(sX)g_1) dg_1 ds \\
 &= \int_{\mathbb{R}} e^{s^2/2} p_t(s) ds \quad (\text{by 3.9}) \\
 &= \int_{\mathbb{R}} e^{s^2/2} c \frac{1}{\sqrt{t}} e^{-s^2/4t} ds = c' e^{t/4}, \quad t > 0,
 \end{aligned}$$

for some new $c' > 0$.

From the inequalities (3.12) and (3.14) we deduce that for a certain constant $C > 0$,

$$(3.15) \quad \|\Delta h_t\|_2 \leq \sqrt{\|\Delta h_t\|_\infty} \sqrt{\|\Delta h_t\|_1} \leq C \frac{1}{t^{d(G)/4}}, \quad t \in]0, \varepsilon],$$

and if $t > \varepsilon$, then $\Delta h_t = \Delta h_{t-\varepsilon/2} * \Delta h_{\varepsilon/2}$ and so

$$(3.16) \quad \|\Delta h_t\|_2 = \|\Delta h_{t-\varepsilon/2} * \Delta h_{\varepsilon/2}\|_2 \leq \|\Delta h_{t-\varepsilon/2}\|_1 \|\Delta h_{\varepsilon/2}\|_2 \leq D e^{t/4}.$$

Hence, for $m > d(G)/4$,

$$\begin{aligned}
 \|\check{E}_m\|_2 = \|\Delta E_m\|_2 &= \frac{1}{m!} \left\| \int_{\mathbb{R}} t^m e^{-t} \Delta h_t dt \right\|_2 \\
 &\leq \frac{1}{m!} \int_0^\varepsilon t^m \frac{1}{t^{d(G)/4}} dt + \frac{1}{m!} \int_\varepsilon^\infty t^m D e^{-3t/4} dt < \infty.
 \end{aligned}$$

The unimodular case is similar, but easier. ■

We can now prove the following theorem.

THEOREM 3.4. *Let $u = \bar{u}$ be a self-adjoint element of $A(G) \cap L^2(G)$ which is differentiable of class $C^{2([d(G)/4]+1)}$ with bounded partial derivatives contained in $L^2(G)$. Then there exists a positive constant $C = C(u)$ such that*

$$(3.17) \quad \|e^{itu} - 1\|_{A(G)} \leq C(1 + |t|)^{2[d(G)/4]+2}, \quad t \in \mathbb{R}.$$

Proof. Let $m = [d(G)/4] + 1$. As in the proof of Theorem 3.1, we see that

$$(3.18) \quad \begin{aligned} e^{itu} &= E_m * (1 - \Omega)^m(e^{itu}) \\ &= E_m * (e^{iut} + (1 + |t|^{2m})e^{iut}w_{t,m}) \\ &= E_m * \left(1 + itu \left(\frac{e^{itu} - 1}{itu} \right) + (1 + |t|^{2m})e^{iut}w_{t,m} \right) \\ &= E_m * 1 + tE_m * \left(iu \left(\frac{e^{itu} - 1}{itu} \right) \right) + (1 + |t|^{2m})E_m * (e^{iut}w_{t,m}) \\ &= 1 + (1 + |t|^{2m})E_m * v_{t,m} \end{aligned}$$

for some $w_{t,m}, v_{t,m} \in L^2(G)$ whose $L^2(G)$ norms are bounded by some constant $C > 0$.

Hence using (3.18), we obtain

$$\|e^{itu} - 1\|_{A(G)} \leq (1 + |t|)^{2m} \|\check{E}_m\|_2 \|v_{t,m}\|_2.$$

Thus we get

$$\|e^{itu} - 1\|_{A(G)} \leq C(1 + |t|)^{2([d(G)/4]+1)}, \quad t \in \mathbb{R},$$

for some constant $C > 0$. ■

REMARK 3.5. Let G be a discrete group. Then the Hilbert space $l^2(g)$ is contained in $A(G)$ and for every $u \in l^2(G)$ we have $\|u\|_{A(G)} = \|\delta * u\|_{A(G)} \leq \|\delta\|_2 \|u\|_2 = \|u\|_2$ (here δ denotes the identity of $l^1(G)$). Hence if $u = \bar{u}$ is an element of $l^2(G) \subset A(G)$ then

$$\|e^{itu} - 1\|_{A(G)} \leq C(1 + |t|), \quad t \in \mathbb{R},$$

for some constant $C > 0$. Indeed,

$$\begin{aligned} \|e^{itu} - 1\|_{A(G)} &= \left\| itu \left(\frac{e^{itu} - 1}{itu} \right) \right\|_{A(G)} \\ &\leq (1 + |t|) \left\| \frac{e^{itu} - 1}{itu} \right\|_{\infty} \|u\|_2, \quad t \in \mathbb{R}. \end{aligned}$$

COROLLARY 3.6.

- (a) *Let G be a connected compact Lie group. Let $u = \bar{u}$ be a self-adjoint element of $A(G)$ which is differentiable of class $C^{([d(G)/2]+1)}$. Then*

every function φ of class C^m with $m > [d(G)/2] + 1$, vanishing at 0, operates on u .

- (b) Let G be a Lie group of positive dimension. Let $u = \bar{u}$ be a self-adjoint element of $A(G) \cap L^2(G)$ which is differentiable of class $C^{2([d(G)/4]+1)}$ with bounded partial derivatives contained in $L^2(G)$. Then every function φ of class C^m with $m > 2[d(G)/4] + 3$, vanishing at 0, operates on u .

Proof. (a) Let $u = \bar{u} \in A(G)$ as in Theorem 3.1. Let $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ be of class C^m with $m > d(G)/2 + 1$ such that $\varphi(0) = 0$. Let φ_0 be any C^m function on \mathbb{R} with compact support such that $\varphi_0 = \varphi$ on the interval $[-\|u\|_{A(G)} - 1, \|u\|_{A(G)} + 1]$. Then the Fourier transform $\widehat{\varphi}_0(t) = \int_{\mathbb{R}} \varphi_0(x)e^{-2i\pi tx} dx, t \in \mathbb{R}$, of φ_0 satisfies the inequality

$$(3.19) \quad |\widehat{\varphi}_0(t)| \leq \frac{C_{\varphi_0}}{(1 + |t|)^m}, \quad t \in \mathbb{R},$$

where C_{φ_0} is some constant depending on φ_0 . Therefore by (3.3) the integral

$$v := \int_{\mathbb{R}} \widehat{\varphi}_0(t)e^{i2\pi tu} dt$$

converges in $A(G)$, hence also in $C(G)$, and so for every $s \in G$, by the Fourier inversion formula we have

$$v(s) = \int_{\mathbb{R}} \widehat{\varphi}_0(t)e^{2i\pi tu(s)} dt = \varphi_0(u(s)) = \varphi(u(s)).$$

Hence $\varphi \circ u = v \in A(G)$. The proof of (b) is similar. ■

4. Smooth synthesis. Let A be a semisimple, regular, commutative Banach algebra with X_A as spectrum; for any $a \in A$ we denote by $\widehat{a} \in C_0(X_A)$ its Gelfand transform. Let also $E \subset X_A$ be a closed subset. We then define

$$I_A(E) = \{a \in A : \widehat{a}^{-1}(0) \text{ contains } E\},$$

$$J_A^0(E) = \{a \in A : \widehat{a}^{-1}(0) \text{ contains a nbhd of } E\}, \quad J_A(E) = \overline{J_A^0(E)}.$$

It is known that $I_A(E)$ and $J_A(E)$ are the largest and the smallest closed ideals with E as hull, i.e., if I is a closed ideal such that $\{x \in X_A : f(x) = 0 \text{ for all } f \in I\} = E$ then

$$J_A(E) \subset I \subset I_A(E).$$

We say that E is a *set of spectral synthesis* for A if $J_A(E) = I_A(E)$, and of *weak synthesis* if $I_A(E)^d = J_A(E)$ for some integer d (see [W]).

Let A^* be the dual of A . For $a \in A$ we set $\text{supp}(a) = \overline{\{x \in X_A : \widehat{a}(x) \neq 0\}}$ and $\text{null}(a) = \{x \in X_A : \widehat{a}(x) = 0\}$. For $\tau \in A^*$ and $a \in A$ define $a\tau$ in A^*

by $a\tau(b) = \tau(ab)$ and define the *support* of τ by

$$\text{supp}(\tau) = \{x \in X_A : a\tau \neq 0 \text{ whenever } \widehat{a}(x) \neq 0\}.$$

It is known that $\text{supp}(\tau)$ consists of all $x \in X_A$ such that for any neighbourhood U of x there exists $a \in A$ for which $\text{supp}(a) \subset U$ and $\tau(a) \neq 0$. Then, for every closed set $E \subset X_A$,

$$J_A(E)^\perp = \{\tau \in A^* : \text{supp}(\tau) \subset E\}$$

and E is spectral for A if and only if $\tau(a) = 0$ for any $a \in A$ and $\tau \in A^*$ such that $\text{supp}(\tau) \subset E \subset \text{null}(a)$.

In what follows we write $I_A(E)$ for $I_{A(G)}(E)$ and $J_A(E)$ for $J_{A(G)}(E)$.

Let G be a Lie group and let $\mathcal{D}(G)$ be the space of all compactly supported C^∞ -functions on G . For a closed subset E of G , we denote by $J_{\mathcal{D}}(E)$ the space of all elements of $\mathcal{D}(G)$ which vanish on E . Let

$$B(E) := \{\lambda(\mu) : \mu \text{ a bounded measure on } G \text{ supported in } E\}.$$

Here $\lambda(\mu)f = \mu * f$ denotes the convolution of the function f with the measure μ .

The closure of $B(E)$ is the annihilator of $I_A(E)$ in $\text{VN}(G)$, where the closure is taken in the weak* topology $\sigma(\text{VN}(G), A(G))$.

DEFINITION 4.1. The closed subset E of G is said to be of *smooth synthesis* if $\overline{J_{\mathcal{D}}(E)} = I_A(E)$.

Let us remark that in the papers of Müller [M] and Guo [G] the word “weak” is used instead of “smooth”.

REMARK 4.2. Let G be a Lie group and f an element in $\mathcal{D}(G)$. Then by (3.8), for any integer $m > d(G)/4$ we have $f = E_m * \check{g}$, where $g = ((1 - \Omega)^m f)^\sim \in \mathcal{D}(G)$. Here $\check{u}(x) := u(x^{-1})$ for any function u on G and $x \in G$. Hence, for $t \in G$, denoting by $\varrho(t)$ the right translation by t , we have

$$\begin{aligned} \varrho(t)f(x) &= f(xt) = \int_G E_m(u)g(t^{-1}x^{-1}u) du \\ &= \int_G E_m(u)\lambda(t)g(x^{-1}u) du = E_m * (\lambda(t)g)^\sim(x). \end{aligned}$$

This shows that the mapping $t \mapsto \varrho(t)f$ from G to the Banach space $A(G)$ is smooth.

THEOREM 4.3. Let G be a Lie group of dimension n . Let M be a smooth submanifold of dimension $m < n$ and let E be a compact subset of M . Then $\overline{J_{\mathcal{D}}(E)}^{[m/2]+1} = J_A(E)$.

Proof. It suffices to show that $J_{\mathcal{D}}(E)^{[m/2]+1} \subset J_A(E)$. Let $f \in J_{\mathcal{D}}(E)$. We will use first a procedure given in [H] to estimate the distance between

$f^{[m/2]+1}$ and $J_A(E)$. For $0 < \varepsilon < \|f\|_\infty$ let

$$\begin{aligned} W_\varepsilon &= \{x \in G : \|\varrho(x)f - f\|_{A(G)} < \varepsilon\} \\ &\subset \Omega_\varepsilon = \{x \in G : \|\varrho(x)f - f\|_\infty < \varepsilon\}. \end{aligned}$$

Since f is a C^∞ -function with compact support, the mapping $g \mapsto \varrho(g)f \in A(G)$ is C^∞ by Remark 4.2. Therefore there exist a constant $K > 0$ and an open neighbourhood W of 0 in the Lie algebra \mathfrak{g} of G such that

$$\|\varrho(\exp(X))f - f\|_{A(G)} \leq K\|X\|$$

for every $X \in W$ and some fixed norm $\|\cdot\|$ on \mathfrak{g} . For $\varepsilon > 0$, let $V_\varepsilon = \exp(B_\varepsilon)$, where B_ε denotes the ball of radius $\varepsilon/2K$ of centre 0 in \mathfrak{g} . For ε small enough we have $B_\varepsilon \subset W$. There exist constants $C_1 > C_2 > 0$ such that for every $\varepsilon > 0$,

$$C_1\varepsilon^n > |V_\varepsilon| > C_2\varepsilon^n$$

and $V_\varepsilon \subset W_\varepsilon \subset \Omega_\varepsilon$. In particular, for every $x = x_0v \in EV_\varepsilon$, $x_0 \in E$, $v \in V_\varepsilon$, we have $f(x) = 0$ since $f \in J_{\mathcal{D}}(E)$, and therefore

$$\begin{aligned} |f(x)| &= |f(x_0v)| \leq |f(x_0v) - f(x_0)| + |f(x_0)| \\ &= |(\varrho(v)f - f)(x_0)| \leq \|(\varrho(v)f - f)\|_\infty < \varepsilon. \end{aligned}$$

Hence

$$(4.20) \quad |f^{[m/2]+1}(x)| \leq \varepsilon^{[m/2]+1}.$$

Take $\nu = f^{[m/2]+1}$ on EV_ε and $\nu = 0$ elsewhere and let $u \in L^2(G)$, $u \geq 0$, be such that $\int_G u(x) dx = 1$ and $\text{supp}(u) \subset V_\varepsilon$. Consider now the function

$$\varphi(s) = (f^{[m/2]+1} - \nu) * \check{u}(s) = \int_G (f^{[m/2]+1} - \nu)(st)u(t) dt.$$

Clearly $\varphi \in A(G)$, and $\varphi(s) = 0$ if $s \cdot \text{supp}(u) \subset EV_\varepsilon$. Since $E \subset \{s : s \cdot \text{supp}(u) \subset EV_\varepsilon\}$ and the set $\{s : s \cdot \text{supp}(u) \subset EV_\varepsilon\}$ is open, $\text{supp}(\varphi)$ is disjoint from E and therefore $\varphi \in J_A(E)$. We have

$$f^{[m/2]+1} - \varphi = (f^{[m/2]+1} - f^{[m/2]+1} * \check{u}) + \nu * \check{u}.$$

As $\text{supp}(u) \subset V_\varepsilon \subset W_\varepsilon$, and $\|f^{[m/2]+1} - \varrho(x)f^{[m/2]+1}\|_{A(G)} \leq K\varepsilon$ for all $x \in W_\varepsilon$ and some constant $K = K(m) > 0$ which is independent of ε , it follows that

$$\begin{aligned} \|f^{[m/2]+1} - f^{[m/2]+1} * \check{u}\|_{A(G)} &= \left\| \int_G (f^{[m/2]+1} - \varrho(x)f^{[m/2]+1})u(x) dx \right\|_{A(G)} \\ &\leq \int_G \|f^{[m/2]+1} - \varrho(x)f^{[m/2]+1}\|_{A(G)} u(x) dx \leq K\varepsilon. \end{aligned}$$

We also have $\|\nu * \check{u}\|_{A(G)} \leq \|\nu\|_2 \cdot \|u\|_2$. As the greatest lower bound for $\|u\|_2$

is $|V_\varepsilon|^{-1/2}$, we obtain

$$\begin{aligned} \text{dist}(f^{[m/2]+1}, J_A(E)) &\leq \|f^{[m/2]+1} - \varphi\|_{A(G)} \\ &\leq K\varepsilon + |V_\varepsilon|^{-1/2} \left(\int_{EV_\varepsilon} |f^{[m/2]+1}(x)|^2 dx \right)^{1/2} \\ &\leq K\varepsilon + |V_\varepsilon|^{-1/2} \sup_{x \in EV_\varepsilon} |f^{[m/2]+1}(x)| |EV_\varepsilon|^{1/2} \\ &\leq K\varepsilon + \frac{\varepsilon^{-n/2}}{C_2^{1/2}} \varepsilon^{[m/2]+1} |EV_\varepsilon|^{1/2}. \end{aligned}$$

Let us estimate $|EV_\varepsilon|$. Since E is compact, we can assume that EV_ε is covered by a finite number of charts and hence is contained in one chart, $\psi : U \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-m}$, such that $\psi(M \cap U) \subset \mathbb{R}^m \times \{0\}$. Furthermore, there exists a constant $C > 0$ and a bounded set B in \mathbb{R}^m , which do not depend on ε , and a rectangle R_ε of measure $C\varepsilon^{n-m}$ in \mathbb{R}^{n-m} such that $\psi((M \cap U)V_\varepsilon) \subset B \times R_\varepsilon$. Since there exists a continuous positive function $F : U \rightarrow \mathbb{R}$ such that

$$\int_U \varphi(g) dg = \int_{\psi(U)} F(x) \varphi(\psi^{-1}(x)) dx, \quad \varphi \in C_c(U),$$

we get, for every small ε ,

$$|EV_\varepsilon| = \int_{\psi(EV_\varepsilon)} F(x) dx \leq \int_{B \times R_\varepsilon} F(x) dx \leq C' \varepsilon^{n-m}.$$

Hence, for $\varepsilon > 0$ small enough,

$$\begin{aligned} \text{dist}(f^{[m/2]+1}, J_A(E)) &\leq K\varepsilon + C'' \varepsilon^{-n/2} \varepsilon^{[m/2]+1} \varepsilon^{(n-m)/2} \\ &= K\varepsilon + C'' \varepsilon^{[m/2]+1-m/2} \end{aligned}$$

for a new constant C'' which does not depend on ε . Thus $f^{[m/2]+1} \in J_A(E)$. It now follows from standard arguments that $J_{\mathcal{D}}(E)^{[m/2]+1} \subset J_A(E)$. ■

COROLLARY 4.4. *Let E be a compact subset of a smooth m -dimensional submanifold of the Lie group G . If E is a set of smooth synthesis, then E is of weak synthesis with $I_A(E)^{[m/2]+1} = J_A(E)$.*

DEFINITION 4.5.

1. Let G be a group and let $\text{Aut}(G)$ be the group of automorphisms of G . Let $t \in G$ and $a \in \text{Aut}(G)$. We call the mapping $G \rightarrow G, s \mapsto a(ts)$, an *affine transformation* of G .
2. Let G be a Lie group. We say that a group A is a *group of affine transformations* of G if A is a Lie group which acts smoothly by affine transformations on G . Smoothly here means that the mapping $A \times G \rightarrow G, (a, x) \mapsto a(x)$, is smooth.

DEFINITION 4.6. Let G be a Lie group and let M be a smooth m -dimensional submanifold of G . We say that a subset E of M has the *cone property* if

1. E is closed in G ,
2. for every $x \in E$ there exists an open neighbourhood U_x of x in G and a C^∞ mapping ψ_x from an open subset $W_x \subset \mathbb{R}^m$ containing 0 into a Lie group of affine transformations A_x on G such that $\psi_x(0) = \mathbb{I}_G$ and there exists an open subset W_x^0 in W_x such that
 - (a) 0 is in the closure of W_x^0 ,
 - (b) for every $y \in U_x \cap E$, $\psi_x(W_x^0)y$ is contained in E and open in M and the mapping $W_x^0 \rightarrow \psi_x(W_x^0)y$, $t \mapsto \psi_x(t)y$, is a diffeomorphism.

REMARK 4.7. Let B be a Lie group acting continuously by continuous automorphisms on our Lie group G . Then the group B also acts continuously by continuous automorphisms on the Fourier algebra $A(G)$, since for any $b \in B$, the representation $\lambda \circ b$ is equivalent to λ .

THEOREM 4.8. *Let G be a Lie group such that $A(G)$ has an approximate identity, let M be a smooth m -dimensional submanifold of G and let $E \subset M$ be a subset with the cone property. Then E is a set of smooth synthesis.*

Proof. Let T be an element of $J_{\mathcal{D}}(E)^\perp$. We must prove that T annihilates $I_A(E)$. Since $A(G)$ has an approximate identity, we can assume that T has compact support. In order to prove that $\langle T, I_A(E) \rangle = \{0\}$, it suffices to show that T can be approximated in the weak* topology by elements $T_\nu = \lambda(m_\nu)$ of $B(E)$.

We shall show first that for every $x \in E$, there exists an open neighbourhood U_x^0 of x contained in U_x such that each $T \in J_{\mathcal{D}}(E)^\perp$ which is supported in $E \cap U_x^0$ is the weak* limit of a sequence $\{T_\nu \in B(E)\}_\nu$. Fix x . To shorten the notation, we shall omit the index x for neighbourhoods and mappings.

We can assume that $U = U_x$ is the domain of a chart $S : U \rightarrow \mathbb{R}^n$ for which

$$S(M \cap U) = B \times \{0_{\mathbb{R}^{n-m}}\}$$

and B is an open subset of \mathbb{R}^m . By shrinking $W^0 = W_x^0$ and U if necessary, we can also assume that there exists an open relatively compact subset O in B such that $S(x) \in O \times \{0_{\mathbb{R}^{n-m}}\}$ and $\psi(t)(S^{-1}(O \times \{0_{\mathbb{R}^{n-m}}\})) \subset M \cap U$ for all $t \in W_x^0$. For $t \in W^0 = W_x^0$ let

$$\phi(t) : O \rightarrow B, \quad (\phi(t)(c), 0) := S(\psi(t)(S^{-1}((c, 0))))), \quad t \in W_x^0, c \in O.$$

Now assume that T is supported in $U \cap E$ and consider T as a distribution on G . We transform it into a distribution \tilde{T} on $\mathbb{R}^m \times \mathbb{R}^{n-m}$ via the diffeomorphism S . Since the support of T is contained in $U \cap M$, the support of \tilde{T} is contained in $B \times \{0_{\mathbb{R}^{n-m}}\}$ and so by the classical formula of L. Schwartz, there exist a finite collection of distributions d^β on B ($\beta \in \mathbb{N}^{n-m}$) such that for any $f \in \mathcal{D}(G)$,

$$\langle T, f \rangle = \langle \tilde{T}, f \circ S^{-1} \rangle = \sum_{\beta} \langle d^\beta, \partial_2^\beta (f \circ S^{-1})(\cdot, 0) \rangle,$$

where the partial derivative ∂_2^β acts on the variable in \mathbb{R}^{n-m} . It follows from the properties of T that \tilde{T} vanishes on every function a in $\mathcal{D}(\mathbb{R}^n)$ which is zero on $\mathbb{R}^m \times \{0_{\mathbb{R}^{n-m}}\}$. Therefore, by standard arguments (see for instance [KM]) $d^\beta = 0$ for any $\beta \neq 0$ and so

$$\begin{aligned} (4.21) \quad \langle T, f \rangle &= \langle \tilde{T}, f \circ S^{-1} \rangle = \langle d^0, (f \circ S^{-1})(\cdot, 0) \rangle \\ &= \sum_{\alpha} \int_B \varphi_{\alpha}(r) \partial^\alpha (f \circ S^{-1})(r, 0) dr, \end{aligned}$$

where the φ_{α} are continuous functions and the partial derivatives ∂^α act only on the variable $r \in \mathbb{R}^m$.

Choose a decreasing sequence $(W_{\nu})_{\nu}$ of open neighbourhoods of 0 in W and let $W_{0,\nu} = W^0 \cap W_{\nu}$ for all ν . Since 0 is contained in the closure of W^0 we know that $W_{0,\nu} \neq \emptyset$ for ν large enough. Let $y \in O$ and let $V_{y,\nu} := \{\phi(t)y : t \in W_{0,\nu}\}$. Then $V_{y,\nu}$ is an open subset of B and the mappings $\Phi_{y,\nu} : W_{0,\nu} \rightarrow V_{y,\nu}, t \mapsto \phi(t)y$, are diffeomorphisms by the condition on ψ . We can assume that there exists an open subset $B_0 \subset O$ such that $S(x) \in B_0 \times \{0_{\mathbb{R}^{n-m}}\}$, for every ν a small open set $W_{1,\nu}$ in $W_{0,\nu}$, and an open neighbourhood C_0^ν of 0 in \mathbb{R}^m , such that the open subset $O_{y,\nu} := \{\phi(t)(y) + z : t \in W_{1,\nu}, y \in B_0, z \in C_0^\nu\}$ of B is contained in $V_{y,\nu}$ for every $\nu \in \mathbb{N}$. For $z \in C_0^\nu$ and $y \in B_0$ we denote the open subset $\{\phi(t)(y) + z : t \in W_{1,\nu}\}$ of $O_{y,\nu}$ by $O_{y,\nu}^z$. Let $\varrho_{y,\nu}^z$ be the restriction to $O_{y,\nu}^z$ of the inverse of the mapping $W_{1,\nu} \rightarrow O_{y,\nu}^z, t \mapsto \phi(t)y + z$.

Let now $\beta_{\nu} \in C_c^\infty(W_{1,\nu})$ such that $\beta_{\nu}(t) \geq 0$ for all $t \in W_{1,\nu}$ and $\int_{W_{1,\nu}} \beta_{\nu}(t) dt = 1$. Hence for every $f \in \mathcal{D}(B)$, $y \in B_0$ and $z \in C_0^\nu$ we have, by the change of variable formula,

$$(4.22) \quad \int_{W_{\nu}^0} \beta_{\nu}(t) f(\phi(t)(y) + z) dt = \int_{O_{y,\nu}} \beta_{\nu}(\varrho_{y,\nu}^z(r)) \text{Jac}(\varrho_{y,\nu}^z)(r) f(r) dr.$$

Hence, differentiating this equation in z and using again the change of variable formula for the mapping $t \mapsto \phi(t)(y)$, for every multi-index $\alpha \in \mathbb{N}^m$,

we see that

$$\begin{aligned}
 (4.23) \quad \int_{W_\nu^0} \beta_\nu(t) \partial^\alpha f(\phi(t)y) dt &= \int_{O_{y,\nu}} p_{\alpha,\nu,y}(r) f(r) dr \\
 &= \int_{W_\nu^0} q_{\alpha,\nu}(t,y) f(\phi(t)y) dt
 \end{aligned}$$

for some function $p_{\alpha,\nu,y} \in \mathcal{D}(O_{y,\nu})$ which is C^∞ in y , and some function $q_{\alpha,\nu}(\cdot, y)$ in $\mathcal{D}(W_{0,\nu})$ which is also C^∞ in y .

Since $\psi(t)$ is affine for every $t \in W_0$ and the mapping $(t, s) \mapsto \psi(t)s \in G$ is continuous (even C^∞), it follows that for every $u \in A(G)$ the mapping $W_x \ni t \mapsto \psi(t)u$, where $\psi(t)u(x) := u(\psi(t)x)$, $x \in G$, is also continuous, and so u_ν defined by

$$u_\nu := \beta_\nu \cdot u := \int_{W_{1,\nu}} \beta_\nu(t) \psi(t)u dt,$$

i.e.

$$u_\nu(g) := \int_{W_{1,\nu}} \beta_\nu(t) u(\psi(t)g) dt, \quad g \in G,$$

is an element of $A(G)$. Furthermore, since the support of β_ν is contained in $W_{1,\nu}$ and since $\psi(0) = \mathbb{I}$, we have $\lim_{\nu \rightarrow \infty} u_\nu = u$. This tells us also that the new elements $T_\nu \in \text{VN}(G)$, $\nu \in \mathbb{N}$, defined by

$$\langle T_\nu, u \rangle := \langle T, u_\nu \rangle = \langle T, \beta_\nu \cdot u \rangle$$

converge in the weak* topology to T .

Let now $U_x^0 := S^{-1}(B_0 \times \mathbb{R}^{n-m} \cap S(U_x))$. Then by (4.21) and (4.23), for $f \in \mathcal{D}(U_x^0)$, we have

$$\begin{aligned}
 \langle T_\nu, f \rangle &= \langle T, f_\nu \rangle = \sum_\alpha \int_B \varphi_\alpha(y) \partial^\alpha (f_\nu \circ S^{-1})(y, 0) dy \\
 &= \sum_\alpha \int_B \int_{W_{1,\nu}} \varphi_\alpha(y) \beta_\nu(t) \partial^\alpha (f \circ S^{-1})(\phi(t)y, 0) dt dy \\
 &= \sum_\alpha \int_B \int_{W_{1,\nu}} \varphi_\alpha(y) q_{\alpha,\nu}(t,y) (f \circ S^{-1})(\phi(t)y, 0) dt dy.
 \end{aligned}$$

This shows that T_ν is in fact a measure for every ν . Furthermore if f is zero on E , then for any $y \in S(U_x^0 \cap E)$, we have

$$0 = f(\psi(t)y) = f \circ S^{-1}(\phi(t)(S(y)), 0) \quad \text{for all } t \in W_{1,\nu},$$

since $\psi(t)y \in E$ for all $y \in U_x^0$ and $t \in W_{1,\nu}$. Hence $\langle T_\nu, f \rangle = 0$ for all ν . This implies that $\langle T, u \rangle = 0$ for every $u \in A(G)$ which vanishes on E .

Let now $T \in J_{\mathcal{D}}(E)^\perp$ with compact support. Since $\text{supp } T \subset E$, there exists a finite set L in E such that $\text{supp } T \subset \bigcup_{x \in L} U_x^0$. For every $x \in L$, let $\varphi_x \in \mathcal{D}(G) \subset A(G)$ be such that $\text{supp}(\varphi_x) \subset U_x^0$ and $\sum_{x \in L} \varphi_x = 1$ on a

compact neighbourhood C of $\text{supp } T$. Then $\varphi_x T$ is in $J_{\mathcal{D}}(E)^\perp$ and has its support in $U_x^0 \cap E$ for every $x \in L$ and $T = \sum_{x \in L} \varphi_x T$. Therefore, by what we have seen above,

$$\varphi_x T = w^* \lim \lambda(m_\nu^x) \quad \text{with } m_\nu^x \in B(E)$$

and so $T = w^* \lim_\nu \lambda(\sum_{x \in L} m_\nu^x)$. ■

The following corollary generalizes the result of Kirsch–Müller in [KM].

COROLLARY 4.9. *Let G be a Lie group and let B be a group of affine transformations of G . Let $\omega \subset G$ be a closed m -dimensional B -orbit in G . Then ω is a set of weak synthesis with $I_A(\omega)^{[m/2]+1} = J_A(\omega)$.*

Proof. Let $z \in \omega$ and let B_0 be the stabilizer of z in B . Then $\omega \simeq B/B_0$ via the diffeomorphism $B/B_0 \rightarrow \omega, tB_0 \mapsto t(z) =: tB_0 \cdot z$. We choose neighbourhoods $M \subset L$ in B/B_0 of \mathbb{I}_G such that L is the domain of a chart $S : L \rightarrow S(L) \subset \mathbb{R}^m$ (for some $m \in \mathbb{N}$), $\mathfrak{b} := S(M)$ is a closed euclidean ball with centre 0, and $S(\mathbb{I} \bmod B_0) = 0$. We can also suppose that there exists a smooth section $\sigma : L \rightarrow B$ with $\sigma(M)M \subset L$. Furthermore, since the mapping $t \mapsto \sigma(S^{-1}(t)) \bmod B_0$ is regular in a neighbourhood of 0, we can assume that \mathfrak{b} is small enough so that the mappings $\mathfrak{b} \ni t \mapsto \sigma(S^{-1}(t))S^{-1}(u) \in B/B_0$ are regular and hence are diffeomorphisms onto their images for every $u \in \mathfrak{b}$. In the chart $S : L \rightarrow \mathbb{R}^m$, we can write

$$(4.24) \quad S(\sigma(S^{-1}(v)) \cdot S^{-1}(u)) = u + R(u)v + v_0(u), \quad v, u \in \mathfrak{b},$$

where $R(u)$ is an invertible linear mapping of \mathbb{R}^m which depends smoothly on u and where $v_0(u)$ is a vector in \mathbb{R}^m which also varies smoothly with u and is of length $o(\|v\|)\|v\|$, where $\| \cdot \|$ denotes the euclidean norm on \mathbb{R}^m .

LEMMA 4.10. *The closed subset $E_z = \sigma(S^{-1}(\mathfrak{b}))(z)$ of ω has the cone property.*

Proof. Fix a point $x = \sigma(S^{-1}(u))(z) \in E_z$ ($u \in \mathfrak{b}$). Using (4.24) we choose a small open cone C' in \mathbb{R}^m containing 0 in its closure, such that the radius r of \mathfrak{b} satisfies

$$r - \|u' + c'\| \geq K\|c'\|$$

for all u' in an open neighbourhood \mathfrak{k} of u in \mathfrak{b} and for all $c' \in C'$ (for some constant $K > 0$), and $C := R(u)^{-1}(C') \subset \mathfrak{b}$. For $c = R(u)^{-1}c' \in C$, $u' \in \mathfrak{k}$, we then have

$$\begin{aligned} r - \|S(\sigma(S^{-1}(c)) \cdot S^{-1}(u'))\| &= r - \|u' + R(u')c + c_0(u')\| \\ &\geq r - \|u' + c'\| - \|R(u')R(u)^{-1}(c') - c'\| - \|c_0(u')\| \\ &\geq K\|c'\| - \|R(u')R(u)^{-1}(c') - c'\| - \|c_0(u')\| > 0 \end{aligned}$$

for \mathfrak{k} small enough. This implies by (4.24) that

$$(4.25) \quad S(\sigma(S^{-1}(c)) \cdot S^{-1}(u')) \in \mathfrak{b}$$

for all $u' \in \mathfrak{k}$ and $c \in C$, if we choose \mathfrak{k} small enough. Let

$$V_x =: \sigma(S^{-1}(\mathfrak{k})) \cdot z = \sigma(S^{-1}(\mathfrak{k}))(z).$$

Then V_x is an open subset of E_z containing x . Let U_x be any open subset in G such that $U_x \cap E_z = V_x$. Define also

$$W_x := \mathfrak{b}, \quad W_x^0 := C \quad \text{and} \quad \psi_x(c) := \sigma(S^{-1}(c)) \in B, \quad c \in W_x.$$

Then for every $y = \sigma(S^{-1}(u'))(z) \in V_x$, $u' \in \mathfrak{k}$, by (4.25) we have

$$\psi_x(c)(y) = \sigma(S^{-1}(c))S^{-1}(u') \cdot z \in \sigma(S^{-1}(\beta))z \in E_z$$

for any $c \in C$. This shows that E_z has the cone property. ■

Since ω is the union of the sets E_z , $z \in A$, and since being of smooth synthesis is a local property, ω is a set of smooth synthesis and so by Corollary 4.4 it is of weak synthesis with $I_A(\omega)^{[m/2]+1} = J_A(\omega)$, which completes the proof of Corollary 4.9. ■

5. Some applications to the Varopoulos algebra $V(G)$ and linear operator equations. Let $C(G)$ denote the algebra of continuous complex-valued functions on a compact group G and let

$$V(G) = C(G) \hat{\otimes} C(G),$$

the projective tensor product of $C(G)$. $V(G)$ is the Banach algebra of complex-valued continuous functions w on $G \times G$ which admit representations

$$w(s, t) = \sum_{i=1}^{\infty} \varphi_i(s)\psi_i(t)$$

($w = \sum_{i=1}^{\infty} \varphi_i \otimes \psi_i$) such that

$$\sum_{i=1}^{\infty} \|\varphi_i\|_{\infty} \|\psi_i\|_{\infty} < \infty$$

with the norm defined by

$$\|w\|_V = \inf \left\{ \sum_{i=1}^{\infty} \|\varphi_i\|_{\infty} \|\psi_i\|_{\infty} : w = \sum_{i=1}^{\infty} \varphi_i \otimes \psi_i \right\}.$$

$V(G)$ is a semisimple regular commutative Banach algebra with spectrum $G \times G$. In [V] Varopoulos proved that if G is a compact Abelian group then the question of spectral synthesis for the Fourier algebra $A(G)$ is closely related to the one for the algebra $V(G)$. This result was extended in [ST] to arbitrary compact groups G .

For a closed subset $E \subset G$, put $E^* = \{(t, s) : st^{-1} \in E\}$. Repeating mainly the arguments in [ST, Theorem 4.6] one also obtains the following

PROPOSITION 5.1. *Let G be a compact group, and $E \subset G$ a closed subset. Assume that $I_A(E)^m = J_A(E)$ for some $m \in \mathbb{N}^*$. Then $I_V(E^*)^m = J_V(E^*)$.*

In [ShT, LT] it was shown that spectral synthesis for $A(G)$ can be applied to the study of linear operator equations. Below we will see that the results of Section 4 are also applicable to equations of this type.

Consider the space $V^\infty(G)$ of all (marginal equivalence classes ([A, ShT]) of) functions $w(s, t)$ that can be written in the form

$$w(s, t) = \sum_{n=1}^{\infty} f_n(s)g_n(t)$$

with $f_n \in L^\infty(G, m)$, $g_n \in L^\infty(G, m)$ and

$$\sum_{n=1}^{\infty} |f_n(s)|^2 \leq C, \quad s \in G, \quad \sum_{n=1}^{\infty} |g_n(t)|^2 \leq C, \quad t \in G.$$

In tensor notations $V^\infty(G) = L^\infty(G, m) \hat{\otimes}^{w^*h} L^\infty(G, m)$, the weak* Haagerup tensor product ([BSm]). For $w \in V^\infty(G)$ one defines $\text{null } w$ as the largest set (up to a marginally null set, see [A, ShT]) where w is zero.

For $w \in V^\infty(G)$, we denote by Δ_w the operator $T \mapsto w \cdot T = \sum_{i=1}^{\infty} M_{g_i} T M_{f_i}$ on $B(L^2(G))$ (the sum converges in the strong operator topology). For a linear operator Δ on $B(L^2(G))$ the *ascent*, $\text{asc } \Delta$, of Δ is the least positive integer such that $\ker(\Delta^m) = \ker(\Delta^{m+1})$.

PROPOSITION 5.2. *Let G be a locally compact group. If $I_A(E)^m = J_A(E)$ for a closed subset $E \subset G$ and $m \in \mathbb{N}^*$ then $\text{asc } \Delta_w \leq m$ for any $w \in V^\infty(G)$ such that $\text{null } w = E^*$.*

Proof. By [ShT, Proposition 5.1], if $w^k \cdot T = 0$ for $k \in \mathbb{N}^*$ then $\text{supp}(T) \subset \text{null } w = E^*$ (see [ShT] for the definition of the support of an operator). On the other hand, under the assumptions of the proposition one can prove, using arguments similar to those in [LT], that $w^m \cdot T = 0$ for any T supported in E^* . Thus we have the inclusions

$$\{T : \text{supp}(T) \subset E^*\} \subset \ker(\Delta_w^m) \subset \ker(\Delta_w^{m+1}) \subset \{T : \text{supp}(T) \subset E^*\},$$

giving the statement. ■

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