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A probabilistic version of the Frequent Hypercyclicity Criterion

by

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Abstract. For a bounded operator T on a separable infinite-dimensional Banach space X, we give a "random" criterion not involving ergodic theory which implies that T is frequently hypercyclic: there exists a vector x such that for every non-empty open subset U of X, the set of integers n such that $T^n x$ belongs to U, has positive lower density. This gives a connection between two different methods for obtaining the frequent hypercyclicity of operators.

1. Introduction. Let X be an infinite-dimensional separable Banach space, and $T \in \mathcal{B}(X)$ a bounded operator on X. In this note we will be concerned with some properties of the linear dynamical system (X, T). A much-studied notion in linear dynamics is hypercyclicity: T is said to be *hypercyclic* if there exists a vector x (a *hypercyclic vector* for T) such that

$$\mathcal{O}rb(x, T) = \{T^n x \; ; \; n \geq 0\}$$

is dense in X. The set of hypercyclic vectors for T is denoted by $\mathrm{HC}(T)$. It is easy to see that T is hypercyclic if and only if it is topologically transitive, i.e. for every pair (U, V) of non-empty open subsets of X, there exists an integer n such that $T^n(U) \cap V \neq \emptyset$. The set $\mathrm{HC}(T)$ is then a residual subset of X. We refer the reader to the two surveys [9] and [10] for more on hypercyclicity and universality properties.

A stronger notion was introduced in [1], that of frequent hypercyclicity:

Definition 1.1. An operator T on X is said to be *frequently hypercyclic* when there exists a vector x such that for every non-empty open subset U of X, the set of integers n such that $T^n x$ belongs to U has positive lower density. In this case, x is called a *frequently hypercyclic vector* for T.

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Recall that the *lower density* of a subset A of $\mathbb{N}$ is

$$\underline{\text{dens}}(A) = \liminf_{N\to\infty} \frac{1}{N} \#\{n \le N \ ; \ n \in A\}.$$

This notion of frequent hypercyclicity deeply differs from the classical hypercyclicity since it does not feature a "global" property of the open sets (topological transitivity), but can be studied only on the orbit of a vector. In particular, no Baire category argument appears in this setting, in contrast to the classical case, and the set $\mathrm{FHC}(T)$ of frequently hypercyclic vectors for T is usually not a residual subset of X (see [1] and [6]).

Frequent hypercyclicity is studied in [1] and [3], using two different kinds of arguments: one of these consists in replacing the Baire category theorem by a measure-theoretic argument, and building a probability measure m on the space X with respect to which T defines an ergodic measure-preserving transformation of X. In this case $\mathrm{FHC}(T)$ is a set of m-measure 1. The other one, on which we will focus now, is called in [1] the Frequent Hypercyclicity Criterion. It is patterned after the well known Hypercyclicity Criterion, which gives a sufficient condition for an operator to be hypercyclic (see for instance [8], [4]). Despite its somewhat involved aspect, it is usually quite easy to apply. The Frequent Hypercyclicity Criterion of [1] was improved by Bonilla and Grosse-Erdmann in [6], and we state here their version in the Banach space setting:

THEOREM 1.2. *Suppose that there exist a dense sequence $(x_l)_{l\ge1}$ of vectors of X and a map S defined on X such that*

(1) *for every $l \ge 1$, the series $\sum_{k\ge1} T^k x_l$ is unconditionally convergent,*
(2) *for every $l \ge 1$, the series $\sum_{k\ge1} S^k x_l$ is unconditionally convergent,*
(3) *$TS = I$.*

Then T is frequently hypercyclic.

Recall that a series $\sum y_k$ of vectors of a (real or complex) separable Banach space X is *unconditionally convergent* in X if $\sum \theta_k y_k$ is convergent for every choice of signs $\theta_k = \pm1$.

The study of frequent hypercyclicity which was carried out in [3] and which repeatedly involved Gaussian random sums led naturally to the following question ([3]): can the assumptions of unconditional convergence be replaced by assumptions of almost unconditional convergence? In other words, let $(\varepsilon_k)_{k\ge1}$ be a sequence of independent random Bernoulli variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$: $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$. Can we merely suppose in the criterion above that the random series $\sum_{k\ge1} \varepsilon_k(\omega) T^k x_l$ and $\sum_{k\ge1} \varepsilon_k(\omega) S^k x_l$ converge almost everywhere? The purpose of this note is to provide an affirmative answer to this question when X has finite cotype, and Section 3 below is devoted to the proof of this result. Section 4 is devoted to examples. We show in particular how to retrieve the frequent hypercyclicity of many of the operators involved in [1] and [3] without referring to ergodic theory.

2. Main result. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(g_k)_{k\geq 0}$ be a sequence of independent real-valued standard Gaussian random variables. Our main result can be stated as follows:

THEOREM 2.1. *Let X be an infinite-dimensional real or complex Banach space, and $T \in \mathcal{B}(X)$ an operator such that there exist a dense sequence $(x_l)_{l\geq 1}$ of vectors of X and a map S defined on X such that*

(1) *for every $l \geq 1$, the series $\sum_{k\geq 1} g_k(\omega)T^k x_l$ converges almost everywhere,*
(2) *for every $l \geq 1$, the series $\sum_{k\geq 1} g_k(\omega)S^k x_l$ converges almost everywhere,*
(3) *$TS = I$.*

Then T is frequently hypercyclic.

Note that an operator satisfying the assumptions of Theorem 2.1 is already hypercyclic (and even mixing, i.e. for every pair (U, V) of non-empty open subsets of X, there exists an integer N such that $T^n(U) \cap V \neq \emptyset$) for every $n \geq N$, since it satisfies the Hypercyclicity Criterion: this follows for instance from the fact that the convergence almost everywhere of a random series of the form $\sum_n g_n(\omega)z_n$, $z_n \in X$, implies the convergence almost everywhere of the Bernoulli series $\sum_n \varepsilon_n(\omega)z_n$, and so $\|z_n\|$ tends to 0 as n tends to infinity.

Another remark concerns the case where the underlying space X is complex. It is much more convenient in this case to consider, instead of the real-valued independent Gaussian variables g_k, a sequence of independent standard *complex-valued* Gaussian variables $\widetilde{g}_k = \frac{1}{\sqrt{2}}(g_k + ig'_k)$, where (g_k) and (g'_k) are two mutually independent sequences of real-valued Gaussian variables. It is not difficult to see that the convergence almost everywhere of a series $\sum g_k(\omega)y_k$, where (y_k) is a sequence of vectors of X, is equivalent to the convergence almost everywhere of $\sum \widetilde{g}_k(\omega)y_k$. Indeed, since $\|y_k\|$ goes to zero as k goes to infinity, the convergence almost everywhere of $\sum \widetilde{g}_k(\omega)y_k$ implies the convergence almost everywhere of $\sum h_k(\omega)z_k$, where $h_{2j} = g_j$, $h_{2j+1} = g'_j$, and $z_{2j} = z_{2j+1} = y_j$, Now (h_k) is a sequence of independent symmetric random variables (with the same law) and by the contraction principle (see for instance [13, p. 121]) the series $\sum h_{2k}(\omega)z_{2k} = \sum g_k(\omega)y_k$ converges almost everywhere.

Before passing to the proof of Theorem 2.1, we briefly recall some facts on the geometry of Banach spaces and random sums of vector-valued independent variables. We refer the reader to [7], [12] or [13] for more details. Our main interests lie in random sums $S_n(\omega) = \sum_{k=1}^{n} \chi_k(\omega) x_k$, where the x_k's are elements of a Banach space X and $(\chi_k)_{k\geq 0}$ is a symmetric sequence of independent variables on $(\Omega, \mathcal{F}, \mathbb{P})$. In particular the χ_k's can be either independent Bernoulli variables ε_k or independent standard Gaussian variables g_k. An important result concerning these random series is the equivalence between the following statements:

(a) $\sum_k \chi_k(\omega) x_k$ converges almost surely,
(b) $\sum_k \chi_k(\omega) x_k$ converges in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $1 \leq p < +\infty$,
(c) $\sum_k \chi_k(\omega) x_k$ converges in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for every $1 \leq p < +\infty$.

Bernoulli random sums are involved in the definition of the geometric property of *cotype*: X is of cotype q ($q \geq 2$) if there exists a positive constant C such that for every $N \geq 0$ and any vectors $x_0, \ldots, x_N$ of X,

$$\Big(\sum_{n=0}^{N} \|x_n\|^q\Big)^{1/q} \leq C \int_{\Omega} \Big\|\sum_{n=0}^{N} \varepsilon_n(\omega) x_n\Big\| \, d\mathbb{P}(\omega). \tag{1}$$

Thanks to the Kahane inequalities, the quantity on the right-hand side can be replaced by

$$C_p \Big(\int_{\Omega} \Big\|\sum_{n=0}^{N} \varepsilon_n(\omega) x_n\Big\|^p \, d\mathbb{P}(\omega)\Big)^{1/p}$$

for every $p \geq 1$, C_p being a constant depending only on p. For instance if μ is a measure on some measure space $(\widetilde{\Omega}, \widetilde{\mathcal{B}})$, then $L^r(\mu)$ has cotype r for $r \geq 2$ and cotype 2 for $1 \leq r \leq 2$.

For a Banach space X, being of cotype q is equivalent to being of *Gaussian cotype* q, i.e. (1) holds true with Bernoulli random sums replaced by Gaussian random sums ([14]). In spaces of non-trivial cotype (i.e. $q < +\infty$), the convergence almost everywhere of a series $\sum \varepsilon_n(\omega) x_n$ is equivalent to the convergence almost everywhere of the corresponding Gaussian series $\sum g_n(\omega) x_n$ (cf. [14]). This immediately yields the following corollary, which gives a positive answer to Question 6.6 of [3] in the case where X has non-trivial cotype:

COROLLARY 2.2. *Let X be a space with non-trivial cotype, and let $T \in \mathcal{B}(X)$ be an operator such that there exist a dense sequence $(x_l)_{l\geq 1}$ of vectors of X and a map S defined on X such that*

(1) *for every $l \geq 1$, the series $\sum_{k\geq 1} \varepsilon_k(\omega) T^k x_l$ converges almost everywhere,*
(2) *for every $l \geq 1$, the series $\sum_{k\geq 1} \varepsilon_k(\omega) S^k x_l$ converges almost everywhere,*
(3) *$TS = I$.*

Then T is frequently hypercyclic.

We finally recall some terminology concerning probability measures on Banach spaces, especially Gaussian measures: if m is a probability measure on $(X, \mathcal{B}, m)$, then m is Gaussian if for every $x^* \in X^*$, x^* as a function from X into $\mathbb{R}$ or $\mathbb{C}$ has Gaussian distribution. This measure is *non-degenerate* if $m(U) > 0$ for every non-empty open subset of X. If T is a bounded operator on X, the probability measure m is *T-invariant* if $m(T^{-1}(A)) = m(A)$ for every $A \in \mathcal{B}$.

3. Proof of the main result. The first tool for the proof of Theorem 2.1 is the following simple proposition, which allows us to derive frequent hypercyclicity from a mixed assumption of measure theory and hypercyclicity:

PROPOSITION 3.1. *Let T be a bounded operator on X, and* $\mathrm{HC}(T)$ *the set of its hypercyclic vectors. Suppose that there exists a probability measure m on X such that $m(U) > 0$ for every non-empty open subset U of X, m is T-invariant, and* $m(\mathrm{HC}(T)) = 1$. *Then T is frequently hypercyclic and* $m(\mathrm{FHC}(T)) = 1$.

Proof. Let U be any non-empty open subset of X. Since m is T-invariant, Birkhoff's theorem implies that for m-almost every x in X,

$$\mathrm{dens}\{n \geq 0 \,;\, T^n x \in U\} = \mathbb{E}(1_U \,|\, \mathcal{I})(x),$$

where 1_U is the characteristic function of the set U and $\mathcal{I}$ the σ-algebra of T-invariant subsets of $(X, \mathcal{B}, m)$. What follows is quite classical, but we recall it here for completeness. By definition of the conditional expectation,

$$\int_A \mathbb{E}(1_U \,|\, \mathcal{I})(x)\, dm(x) = m(A \cap U)$$

for every set $A \in \mathcal{I}$. Applying this with $A = \{x \in X \,;\, \mathbb{E}(1_U \,|\, \mathcal{I})(x) = 0\}$, which is T-invariant, we get $m(A \cap U) = 0$, i.e. $\mathbb{E}(1_U \,|\, \mathcal{I})(x)$ is positive (non-zero) almost everywhere on U. Moreover, since $\mathbb{E}(1_U \,|\, \mathcal{I})$ is a T-invariant function, it is positive almost everywhere on the set $\bigcup_{n \geq 0} T^{-n}(U)$. Now our assumption on the hypercyclic vectors comes into play: since

$$\mathrm{HC}(T) \subseteq \bigcup_{n \geq 0} T^{-n}(U)$$

and $\mathrm{HC}(T)$ has measure 1, $\bigcup_{n \geq 0} T^{-n}(U)$ has measure 1 too, and $\mathbb{E}(1_U|\mathcal{I})(x)$ is positive almost everywhere. Taking a countable basis $(U_p)_{p \geq 1}$ of open sets in X, it is clear that m-almost every x is frequently hypercyclic for T. ∎

The second step of the proof of Theorem 2.1 is the construction of a suitable supercyclic vector for T: recall that x is *supercyclic* for T if the scaled orbit $\{\lambda T^n x \,;\, n \geq 0,\, \lambda \in \mathbb{R}/\mathbb{C}\}$ is dense in X.

LEMMA 3.2. *Under the assumptions of Theorem* 2.1, T *admits a supercyclic vector* x *such that the two series*

$$\sum_{n\geq 0} g_n(\omega)T^n x \quad \text{and} \quad \sum_{n\geq 0} g_n(\omega)S^n x$$

converge almost everywhere.

It will be convenient for the rest of the proof to write $T^{-n}x$ instead of $S^n x$ for $n \geq 0$, so as to be able to write $\sum_{n\in\mathbb{Z}} g_n(\omega)T^n x$, where $(g_n)_{n\in\mathbb{Z}}$ is a double-sided sequence of independent standard Gaussian variables, instead of $\sum_{n<0} g_n(\omega)S^n x + \sum_{n\geq 0} g_n(\omega)T^n x$. But this does not mean in any way that T is invertible.

Proof. By Lévy's inequalities, the quantities

$$M_l = \sup_{N,M\geq 0} \int_\Omega \Big\| \sum_{n=-M}^{N} g_n(\omega)T^n x_l \Big\| \, d\mathbb{P}(\omega)$$

are finite for every $l \geq 1$. Fix a sequence $(a_l)_{l\geq 1}$ of non-zero complex numbers such that the series $\sum_{l\geq 1} |a_l| M_l$ is convergent. We know already that $\|T^n x_l\|$ and $\|T^{-n}x_l\|$ tend to zero as n tends to infinity. Using this, it is easy to construct an increasing sequence $(n_k)_{k\geq 1}$ of integers such that the vectors

$$y_r = \sum_{l=1}^{r} a_l T^{-n_l} x_l$$

have the following properties:

$$\|y_r - y_{r-1}\| \leq \frac{1}{2^{r-1}} \quad \text{for every } r \geq 2, \tag{2}$$

$$\Big\| \frac{1}{a_l} T^{n_l} y_r - x_l \Big\| \leq \frac{1}{2^l} \quad \text{for every } l \leq r. \tag{3}$$

Then $x = \lim_{r\to\infty} y_r = \sum_{l=1}^{\infty} a_l T^{-n_l} x_l$ satisfies

$$\Big\| \frac{1}{a_r} T^{n_r} x - x_r \Big\| \leq \frac{1}{2^r}$$

for every $r \geq 1$, and x is a supercyclic vector for T. It remains to prove that the series $\sum_{n\geq 0} g_n(\omega)T^n x$ and $\sum_{n>0} g_{-n}(\omega)T^{-n}x$ converge almost everywhere: if $q \geq p > 0$, then

$$\begin{aligned}\int_\Omega \Big\| \sum_{n=p}^{q} g_n(\omega)T^n x \Big\| \, d\mathbb{P}(\omega) &\leq \sum_{l=1}^{l_0} |a_l| \int_\Omega \Big\| \sum_{n=p}^{q} g_n(\omega) T^{n-n_l} x \Big\| \, d\mathbb{P}(\omega) \\ &\quad + \sum_{l=l_0+1}^{\infty} |a_l| \int_\Omega \Big\| \sum_{n=p}^{q} g_n(\omega) T^{n-n_l} x \Big\| \, d\mathbb{P}(\omega).\end{aligned}$$

By the definition of M_l, the second term in the right-hand bound is less than

$$\sum_{l=l_0+1}^{\infty} |a_l| M_l < \frac{\varepsilon}{2}$$

if l_0 is large enough, so there exists an n_0 such that for $q \geq p \geq n_0$,

$$\int_{\Omega} \Big\| \sum_{n=p}^{q} g_n(\omega) T^n x \Big\| \, d\mathbb{P}(\omega) < \varepsilon.$$

The convergence of the first random series clearly follows, and idem for the second one. ∎

The following proposition allows us to conclude the proof of Theorem 2.1:

PROPOSITION 3.3. *Suppose that $T \in \mathcal{B}(X)$ has a supercyclic vector x such that the two series*

$$\sum_{n \geq 0} g_n(\omega) T^n x \quad \textit{and} \quad \sum_{n \geq 0} g_n(\omega) S^n x$$

converge almost everywhere. Then T admits a non-degenerate invariant Gaussian measure such that $m(\mathrm{HC}(T)) = 1$.

Proof. If X is a real space, consider the function $\phi(\omega) = \sum_{n\in\mathbb{Z}} g_n(\omega) T^n x$, and if X is complex the function $\phi(\omega) = \sum_{n\in\mathbb{Z}} \widetilde{g}_n(\omega) T^n x$, where $(\widetilde{g}_n)$ is a sequence of independent standard complex Gaussian variables. For convenience, we will drop the tilde in the complex case and write $\phi(\omega) = \sum_{n\in\mathbb{Z}} g_n(\omega) T^n x$, but it is to be remembered that the Gaussian variables are real if X is real and complex if X is complex. This function ϕ is defined almost everywhere on Ω, which makes it possible to consider the measure $m = \phi(\mathbb{P})$ on $(X, \mathcal{B})$:

$$m(A) = \mathbb{P}(\{\omega \in \Omega \; ; \; \phi(\omega) \in A\})$$

for every $A \in \mathcal{B}$. Then m is clearly T-invariant, Gaussian, and its support is the closed linear span of the vectors $T^n x$, $n \in \mathbb{Z}$, which is the whole space X, so m is non-degenerate.

Let $\varepsilon > 0$ and $a \in \mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and consider

$$\Omega_{\varepsilon,a} = \{\omega \in \Omega \; ; \text{ there exists a } k \geq 1 \text{ such that } \|T^k \phi(\omega) - ax\| < \varepsilon\}.$$

It suffices to show that $\mathbb{P}(\Omega_{\varepsilon,a}) = 1$ for every $\varepsilon > 0$ and $a \in \mathbb{K}$. Indeed, if this is the case, then $\Omega_{\varepsilon,a}$ is contained in the set

$$\{\omega \in \Omega \; ; \text{ there exists a } k \geq 1 \text{ such that } \|T^k \phi(\omega) - aT^r x\| < \varepsilon \|T^r\|\}$$

for every $r \geq 1$, so that each one of these sets is of probability 1. If $(\varepsilon_p)_{p\geq 1}$ decreases to zero and $(a_q)_{q\geq 1}$ is a dense sequence of elements of $\mathbb{K}$, then $\widetilde{\Omega} = \bigcap_{p,q\geq 1} \Omega_{\varepsilon_p, a_q}$ is a set of probability 1. Hence $A = \phi^{-1}(\widetilde{\Omega})$ is a set of

m-measure 1, and using the fact that x is supercyclic, it is easy to see that A consists of hypercyclic vectors for T. The conclusion follows.

Now for $r \geq 1$, fix a positive integer N_r such that

$$\int_{\Omega} \Big\| \sum_{|n|>N_r} g_n(\omega) T^n x \Big\| \, d\mathbb{P}(\omega) < \frac{1}{4^r},$$

and let $\delta_r > 0$ be such that $(2N_r+1)\delta_r < 2^{-r}$. We denote by $D^{(r)}_{-N_r}, \dots, D^{(r)}_{N_r}$ the following open disks of the complex plane (or of the real line if we are working in $\mathbb{R}$):

$$\begin{aligned} D^{(r)}_0 &= \{z \in \mathbb{C} \,;\, |z-a| \cdot \|x\| < \delta_r\}, \\ D^{(r)}_n &= \{z \in \mathbb{C} \,;\, |z| \cdot \|T^n x\| < \delta_r\} \quad \text{ if } 0 < |n| \leq N_r. \end{aligned}$$

For every $\omega \in \Omega$, denote by $k_r(\omega)$ the smallest positive integer such that

$$(g_{-N_r-k_r(\omega)}, \dots, g_{-k_r(\omega)}, \dots g_{N_r-k_r(\omega)}) \in D^{(r)}_{N_r} \times \dots \times D^{(r)}_0 \times \dots \times D^{(r)}_{N_r}$$

if such an integer exists, and $k_r(\omega) = +\infty$ if not. Clearly k_r is finite almost everywhere. Let $\Theta = \{\omega \in \Omega \,;\, k_r(\omega) < +\infty\}$. For $n \in \mathbb{Z}$, we define the random variables $X^{(r)}_n$ on Θ by $X^{(r)}_n(\omega) = g_{n-k_r(\omega)}(\omega)$.

Fact 3.4. *For $|n| > N_r$, the random variables $X^{(r)}_n$ are independent and identically distributed, their common law being that of g_0 (or g_n).*

This fact follows easily from the independence of the variables g_n. It can also be seen as a (very simple) instance of the strong Markov property. We have, for $\omega \in \Theta$,

$$T^{k_r(\omega)} \phi(\omega) = \sum_{n \in \mathbb{Z}} g_n(\omega)\, T^{n+k_r(\omega)} x = \sum_{n \in \mathbb{Z}} X^{(r)}_n(\omega)\, T^n x,$$

so that

$$\Big\| T^{k_r(\omega)} \phi(\omega) - \sum_{|n| \leq N_r} X^{(r)}_n(\omega) T^n x \Big\| = \Big\| \sum_{|n|>N_r} X^{(r)}_n(\omega) T^n x \Big\|.$$

By Fact 3.4,

$$\int_{\Omega} \Big\| T^{k_r(\omega)} \phi(\omega) - \sum_{|n| \leq N_r} X^{(r)}_n(\omega) T^n x \Big\| \, d\mathbb{P}(\omega) = \int_{\Omega} \Big\| \sum_{|n|>N_r} g_n(\omega)\, T^n x \Big\| \, d\mathbb{P}(\omega) < \frac{1}{4^r}.$$

If

$$A_r = \Big\{ \omega \in \Theta \,;\, \Big\| T^{k_r(\omega)} \phi(\omega) - \sum_{|n| \leq N_r} X^{(r)}_n(\omega) T^n x \Big\| \leq \frac{1}{2^r} \Big\},$$

it follows that $\mathbb{P}(A_r) \geq 1 - 2^{-r}$. Now for every $\omega \in \Theta$,

$$\begin{aligned}\Big\| \sum_{|n|\leq N_r} X_n^{(r)}(\omega) T^n x - ax \Big\| &= \Big\| \sum_{|n|\leq N_r} g_{n-k_r(\omega)}(\omega) T^n x - ax \Big\| \\ &\leq |g_{-k_r(\omega)} - a| \cdot \|x\| + \sum_{0<|n|\leq N_r} |g_{n-k_r(\omega)}(\omega)| \cdot \|T^n x\| \\ &< (2N_r + 1)\delta_r < \frac{1}{2^r}.\end{aligned}$$

Hence if ω is in A_r, then

$$\|T^{k_r(\omega)}\phi(\omega) - ax\| < \frac{1}{2^{r-1}},$$

and if r is large enough, then A_r is contained in $\Omega_{\varepsilon,a}$. It follows that $\Omega_{\varepsilon,a}$ is a set of probability one, and this finishes the proof. ∎

Combining Propositions 3.1 and 3.3 and Lemma 3.2 proves Theorem 2.1.

4. Applications. The random Frequent Hypercyclicity Criterion of Theorem 2.1 applies especially well to operators which have a perfectly spanning set of eigenvectors associated to unimodular eigenvalues with respect to the normalized Lebesgue length measure on the unit circle:

DEFINITION 4.1 ([1]). We say that T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues with respect to the normalized Lebesgue length measure on the unit circle if for every measurable subset A of the unit circle $\mathbb{T}$ of Lebesgue measure equal to 1,

$$\overline{\mathrm{sp}}[\ker(T - \lambda)\ ;\ \lambda \in A] = X.$$

Let $(E_j)_{j\geq 1}$ be a sequence of σ-measurable eigenvector fields (i.e. σ-measurable X-valued functions defined on $\mathbb{T}$ such that $TE_j(\lambda) = \lambda E_j(\lambda)$ for every $\lambda \in \mathbb{T}$), with $\|E_j\|_{\infty,\mathbb{T}} \leq 1$ such that for every $\lambda \in \mathbb{T}$, $\ker(T - \lambda) = \overline{\mathrm{sp}}[E_j(\lambda)\ ;\ j \geq 1]$ (for the existence of such eigenvector fields, see [2]). Using the notation of [1] and [3], we denote again by K_j the operator from $L^2(\mathbb{T})$ into X defined by

$$K_j f = \int_0^{2\pi} f(e^{i\theta}) E_j(e^{i\theta})\, \frac{d\theta}{2\pi} \quad \text{for every } f \in L^2(\mathbb{T}),$$

and by V the unitary operator of multiplication by λ on $L^2(\mathbb{T})$. The equality $TK_j = K_j V$ implies that for every $j \geq 1$ and $n \geq 0$,

$$T^n(K_j f) = \int_0^{2\pi} e^{in\theta} f(e^{i\theta}) E_j(e^{i\theta})\, \frac{d\theta}{2\pi}.$$

In many cases, the series $\sum_{n\geq 0} g_n(\omega)T^n(K_jf)$ is convergent for every $j\geq 1$ and every smooth (for instance $\mathcal{C}^\infty$) function f on $\mathbb{T}$. This happens for instance in the following situations:

- if X has type 2,
- if the E_j's are α-Hölderian for some $\alpha > 1/2$,
- if $X = L^p(\mu)$ for some p less than 2, and the E_j's are α-Hölderian for some $\alpha > 1/2 - 1/p'$, where p' is the conjugate exponent of p.

For the proof of these statements, see [3], and take into account the fact that the regularity of the E_j's passes over to all the fE_j's, where f is a $\mathcal{C}^\infty$ function.

Let now $(f_r)_{r\geq 1}$ be a sequence of $\mathcal{C}^\infty$ functions which is dense in $L^2(\mathbb{T})$, and D be the countable set consisting of finite linear combinations of the vectors K_jf_r, $j,r\geq 1$, with coefficients in $\mathbb{Q}+i\mathbb{Q}$. Order this set D as a sequence $(x_l)_{l\geq 1}$: for each $l\geq 1$, the series $\sum_{n\geq 0} g_n(\omega)T^n x_l$ is convergent almost everywhere. The map S is defined on the vectors K_jf as being $S(K_jf) = K_j(V^{-1}f)$, so that

$$S^n(K_jf) = \int_0^{2\pi} e^{-in\theta} f(e^{i\theta})E_j(e^{i\theta})\,\frac{d\theta}{2\pi}$$

and the series $\sum_{n\geq 0} g_n(\omega)S^n x_l$ converges almost everywhere as well in the situations which were mentioned above. Moreover, the fact that the eigenvector fields E_j are perfectly spanning with respect to the length measure implies that D is dense. So all the conditions of Theorem 2.1 are met, and T is frequently hypercyclic.

This criterion can also be applied to operators which do not have any unimodular eigenvector, unlike the Frequent Hypercyclicity Criterion of Theorem 1.2: if T satisfies the unconditional convergence assumptions of this last theorem, then T is necessarily chaotic (see [6]), so the unimodular eigenvectors span a dense subspace of X (see [5]). If T is the "Kalisch-type" operator on $C_0(\mathbb{T})$ of Example 4.2 of [3] (so named because it is a modification of an example of Kalisch [11]), then all the series $\sum_{n\geq 0} g_n(\omega)T^nKf$ and $\sum_{n\geq 0} g_n(\omega)S^nKf$ for $f\in\mathcal{C}^\infty(\mathbb{T})$ are convergent almost everywhere, and Theorem 2.1 applies, while Theorem 1.2 does not.

Thus Theorem 2.1 establishes a connection between the two methods for frequent hypercyclicity of [1], and shows that in the examples considered above, we do not need to prove that the operators are ergodic with respect to a certain invariant Gaussian measure in order to show that they are frequently hypercyclic. It is true that the proofs do not become fundamentally simpler: the main difficulty, namely to prove that the series $\sum_{n\geq 0} g_n(\omega)T^nKf$ and $\sum_{n\geq 0} g_n(\omega)S^nKf$ for $f\in\mathcal{C}^\infty(\mathbb{T})$ are convergent almost everywhere, remains unchanged. But it is to be hoped that this different method can shed some light on some open questions in frequent hypercyclicity theory: we recall here one of these questions, mentioned already in [3]:

QUESTION 4.2. *If T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues, is T frequently hypercyclic?*

The work of [3] seems to suggest that the answer to this question could be affirmative, but without T necessarily admitting a non-degenerate invariant Gaussian measure with respect to which it would be ergodic. Hence the possible interest of criteria for frequent hypercyclicity not involving ergodicity.

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