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Operator Segal algebras in Fourier algebras

by

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Abstract. Let G be a locally compact group, $\mathrm{A}(G)$ its Fourier algebra and $\mathrm{L}^1(G)$ the space of Haar integrable functions on G. We study the Segal algebra $\mathrm{S}^1\mathrm{A}(G) = \mathrm{A}(G) \cap \mathrm{L}^1(G)$ in $\mathrm{A}(G)$. It admits an operator space structure which makes it a completely contractive Banach algebra. We compute the dual space of $\mathrm{S}^1\mathrm{A}(G)$. We use it to show that the restriction operator $u \mapsto u|_H : \mathrm{S}^1\mathrm{A}(G) \to \mathrm{A}(H)$, for some non-open closed subgroups H, is a surjective complete quotient map. We also show that if N is a non-compact closed subgroup, then the averaging operator $\tau_N : \mathrm{S}^1\mathrm{A}(G) \to \mathrm{L}^1(G/N)$, $\tau_N u(sN) = \int_N u(sn)\,dn$, is a surjective complete quotient map. This puts an operator space perspective on the philosophy that $\mathrm{S}^1\mathrm{A}(G)$ is "locally $\mathrm{A}(G)$ while globally L^1". Also, using the operator space structure we can show that $\mathrm{S}^1\mathrm{A}(G)$ is operator amenable exactly when when G is compact; and we can show that it is always operator weakly amenable. To obtain the latter fact, we use E. Samei's theory of hyper-Tauberian Banach algebras.

1. Operator Segal algebras

1.1. *Notation.* For any Banach space $\mathcal{X}$ we let $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of bounded linear operators from $\mathcal{X}$ to itself, and $\mathrm{b}_1(\mathcal{X})$ the set of all vectors of norm not exceeding 1.

For details on classical harmonic analysis, we use [15, 23]. We will always let G denote a locally compact group with a fixed left invariant Haar measure m. For $1 \leq p \leq \infty$, $\mathrm{L}^p(G)$ is the usual L^p-space with respect to m. If f, g are Borel measurable functions and $s \in G$, then for almost every t in G we denote by

$$s * f(t) = f(s^{-1}t), \qquad f * g(t) = \int_G f(s) s * g(t)\,dt \quad \text{and} \quad \check{f}(t) = f(t^{-1})$$

the left group action, convolution (when the integrand makes sense) and inversion. We note that $\mathrm{L}^1(G)$ is a Banach algebra with respect to convolution.

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We let $\mathrm{A}(G)$ and $\mathrm{B}(G)$ denote the *Fourier* and *Fourier–Stieltjes algebras* of G, which are Banach algebras of continuous functions on G and were introduced in [7]. We recall, from that article, that $\mathrm{A}(G)$ consists exactly of functions on G of the form $u(s) = \langle \lambda(s)f \mid g\rangle = \overline{g} * \check{f}(s)$, where $\lambda : G \to \mathcal{B}(\mathrm{L}^2(G))$ is the left regular representation given by $\lambda(s)f = s * f$. The dual of $\mathrm{A}(G)$ is the von Neumann algebra $\mathrm{VN}(G)$, which is generated by $\lambda(G)$ in $\mathcal{B}(\mathrm{L}^2(G))$.

Our standard references for operator spaces are [6, 22]. An operator space is a complex Banach space $\mathcal{V}$ equipped with an *operator space structure*: for each space of matrices $\mathrm{M}_n(\mathcal{V})$ with entries in $\mathcal{V}$, $n \in \mathbb{N}$, we have a norm $\|\cdot\|_{\mathrm{M}_n(\mathcal{V})}$, and the norms satisfy Ruan's axioms in addition to that $\|\cdot\|_{\mathrm{M}_1(\mathcal{V})}$ is the norm on $\mathcal{V} = \mathrm{M}_1(\mathcal{V})$. A map T from $\mathcal{V}$ to another operator space $\mathcal{W}$ is said to be *completely bounded* if the family of linear operators $[v_{ij}] \mapsto [Tv_{ij}] : \mathrm{M}_n(\mathcal{V}) \to \mathrm{M}_n(\mathcal{W})$ is uniformly bounded over n. If $\mathcal{A}$ is an algebra and an operator space for which $\mathcal{V}$ is a left module over $\mathcal{A}$, we say $\mathcal{V}$ is a *completely bounded $\mathcal{A}$-module* if there is $C > 0$ so that for each $[a_{ij}]$ in $\mathrm{M}_n(\mathcal{A})$ and each $[v_{kl}]$ in $\mathrm{M}_m(\mathcal{V})$ we have

$$\|[a_{ij}v_{kl}]\|_{\mathrm{M}_{nm}(\mathcal{V})} \leq C\|[a_{ij}]\|_{\mathrm{M}_n(\mathcal{A})}\|[v_{kl}]\|_{\mathrm{M}_m(\mathcal{V})}.$$

We say $\mathcal{V}$ is a *completely contractive $\mathcal{A}$-module* if we can set $C = 1$. This is the same as asserting that the module multiplication extends to a map on the *operator projective tensor product* $\mathcal{A} \widehat{\otimes} \mathcal{V} \to \mathcal{V}$ and is bounded at all matrix levels by C. We say $\mathcal{A}$ is a *completely bounded* (*contractive*) *Banach algebra* if it itself is a completely bounded (contractive) $\mathcal{A}$-module. We note that any C*-algebra admits a canonical operator space structure. The algebras $\mathrm{L}^1(G)$ and $\mathrm{A}(G)$ will always have the *standard predual* structures (see [3]), in their respective roles as the preduals of $\mathrm{L}^\infty(G)$ and $\mathrm{VN}(G)$. With these operator space structures, these are completely contractive Banach algebras.

1.2. *Abstract operator Segal algebras.* Let $\mathcal{A}$ be a completely contractive Banach algebra. A (*contractive*) [*left*] *operator Segal algebra* in $\mathcal{A}$ is a dense [left] ideal $\mathrm{S}\mathcal{A}$ equipped with an operator space structure $\{\|\cdot\|_{\mathrm{M}_n(\mathrm{S}\mathcal{A})} : \mathrm{M}_n(\mathrm{S}\mathcal{A}) \to \mathbb{R}^{\geq 0}\}$ under which

(OSA1) $(\mathrm{S}\mathcal{A}, \|\cdot\|_{\mathrm{S}\mathcal{A}})$ is a Banach space,
(OSA2) the identity map $\mathrm{S}\mathcal{A} \hookrightarrow \mathcal{A}$ is completely bounded (contractive),
(OSA3) $\mathrm{S}\mathcal{A}$ is a completely bounded (contractive) $\mathcal{A}$-bimodule.

Note that (OSA2) and (OSA3) imply that $\mathrm{S}\mathcal{A}$ is a completely bounded Banach algebra. Moreover $\mathrm{S}\mathcal{A}$ is a completely contractive Banach algebra if the associated maps and module actions are completely contractive.

Let us see that operator Segal algebras are reasonably common.

(i) For any Banach space $\mathcal{X}$, let $\max \mathcal{X}$ denote the maximal operator space whose underlying Banach space is $\mathcal{X}$. If $\mathcal{A}$ is any Banach algebra with abstract Segal subalgebra $\mathrm{S}\mathcal{A}$, then $\max \mathrm{S}\mathcal{A}$ is an operator Segal algebra in $\max \mathcal{A}$.

(ii) Let $\mathcal{H}$ be a Hilbert space, $\mathcal{K}(\mathcal{H})$ the C*-algebra of compact operators on $\mathcal{H}$, and for $1 \leq p < \infty$, $\mathcal{S}^p(\mathcal{H})$ be the Schatten p-class operators. Since we have $\mathcal{S}^1(\mathcal{H}) \cong \mathcal{K}(\mathcal{H})^*$ under the dual pairing $(s,k) \mapsto \mathrm{trace}(sk)$, we see that $\mathcal{S}^1(\mathcal{H})$ is a completely contractive $\mathcal{K}(\mathcal{H})$-module in its dual operator space structure, and hence is an operator Segal algebra. For $1 < p < \infty$, we assign to $\mathcal{S}^p(\mathcal{H})$ the interpolated operator space structure $\mathcal{S}^p(\mathcal{H}) = (\mathcal{K}(\mathcal{H}), \mathcal{S}^1(\mathcal{H}))_{1/p}$. See [22] for more on this. By the functorial properties of operator interpolation we can verify that (OSA2) and (OSA3) obtain for $\mathcal{S}^p(\mathcal{H})$, and, in fact, $\mathcal{S}^p(\mathcal{H})$ is a contractive Segal algebra in $\mathcal{K}(\mathcal{H})$.

(iii) Similarly to (ii) above, we can see that if I is any set and $c_0(I)$ denotes the C*-algebra of functions on I vanishing at infinity, then $\ell^1(I) \cong c_0(I)^*$ is a contractive operator Segal algebra in $c_0(I)$. With the operator interpolation structure $\ell^p(I) = (c_0(I), \ell^1(I))_{1/p}$ $(1 \leq p \leq \infty)$ we see that $\ell^p(I)$ is a contractive operator Segal algebra in $c_0(I)$.

1.3. *The 1-Segal Fourier algebra.* We define the 1-*Segal Fourier algebra* to be the space

$$\mathrm{S}^1\mathrm{A}(G) = \mathrm{A}(G) \cap \mathrm{L}^1(G).$$

For u in $\mathrm{S}^1\mathrm{A}(G)$ we let

$$\|u\|_{\mathrm{S}^1\mathrm{A}} = \|u\|_{\mathrm{A}} + \|u\|_{\mathrm{L}^1}.$$

In [12] this space is denoted $\mathrm{LA}(G)$. It is shown in Lemma 1.1 of that article that it is complete; and in Proposition 2.5 that it is a Segal algebra in $\mathrm{A}(G)$. We will reserve the notation $\mathrm{LA}(G)$ for $\mathrm{S}^1\mathrm{A}(G)$, when it is treated as a Segal algebra in $\mathrm{L}^1(G)$, and call it the *Lebesgue–Fourier algebra.* If G is abelian, with dual group $\widehat{G}$, then there is an isometric algebra isomorphism $\mathrm{S}^1\mathrm{A}(G) \cong \mathrm{LA}(\widehat{G})$.

Let us assign to $\mathrm{S}^1\mathrm{A}(G)$ a natural operator space structure. The norm on $\mathrm{S}^1\mathrm{A}(G)$ was gained via the embedding $\mathrm{S}^1\mathrm{A}(G) \hookrightarrow \mathrm{A}(G) \oplus_1 \mathrm{L}^1(G) : u \mapsto (u,u)$. The space $\mathrm{A}(G) \oplus_1 \mathrm{L}^1(G)$ is the predual of the von Neumann algebra $\mathrm{VN}(G) \oplus_\infty \mathrm{L}^\infty(G)$ and as such inherits a natural operator space structure [3, 6]. Thus we identify the matrix space $\mathrm{M}_n(\mathrm{S}^1\mathrm{A}(G))$ as a subspace of $\mathrm{A}(G) \oplus_1 \mathrm{L}^1(G) \cong \mathcal{CB}^\sigma(\mathrm{VN}(G) \oplus_\infty \mathrm{L}^\infty(G), \mathrm{M}_n)$. Hence if $[u_{ij}] \in \mathrm{M}_n(\mathrm{S}^1\mathrm{A}(G))$ we obtain

$$\|[u_{ij}]\|_{\mathrm{M}_n(\mathrm{S}^1\mathrm{A})} = \sup\{\|[\langle T_{pq}, u_{ij}\rangle + \langle \varphi_{pq}, u_{ij}\rangle]\|_{\mathrm{M}_{nr}}\}$$

where the supremum is taken over $[T_{pq}]$ in $\mathrm{b}_1(\mathrm{M}_r(\mathrm{VN}(G)))$, $[\varphi_{pq}]$ in $\mathrm{b}_1(\mathrm{M}_r(\mathrm{L}^\infty(G)))$ and r in $\mathbb{N}$, and where $\langle T_{pq}, u_{ij}\rangle$ indicates the $\mathrm{VN}(G)$-$\mathrm{A}(G)$ dual pairing and $\langle \varphi_{pq}, u_{ij}\rangle = \int_G \varphi_{pq}(s) u_{ij}(s)\, ds$ is the $\mathrm{L}^\infty(G)$-$\mathrm{L}^1(G)$ dual pairing, for each quadruple index p,q,i,j. It follows immediately that $\|[u_{ij}]\|_{\mathrm{M}_n(\mathrm{S}^1\mathrm{A})} \geq \|[u_{ij}]\|_{\mathrm{M}_n(\mathrm{A})}$, so (OSA2) is satisfied.

It remains to check the axiom (OSA3). Suppose $[v_{kl}] \in \mathrm{b}_1(\mathrm{M}_m(\mathrm{A}(G)))$. Then $\|[v_{kl}]\|_{\mathrm{M}_m(\mathrm{L}^\infty)} \leq \|[v_{kl}]\|_{\mathrm{M}_m(\mathrm{A})} \leq 1$, since the injection $\mathrm{A}(G) \hookrightarrow \mathrm{L}^\infty(G)$ is a contraction, hence a complete contraction, so $\|[\varphi_{pq} v_{kl}]\|_{\mathrm{M}_{rm}(\mathrm{L}^\infty)} \leq 1$ for each $[\varphi_{pq}]$ in $\mathrm{b}_1(\mathrm{M}_r(\mathrm{L}^\infty(G)))$. Hence

$$\begin{aligned}\|[v_{kl}u_{ij}]\|_{\mathrm{M}_{mn}(\mathrm{S}^1\mathrm{A})} &= \sup\{\|[\langle T_{pq}, v_{kl}u_{ij}\rangle + \langle \varphi_{pq}, v_{kl}u_{ij}\rangle]\|_{\mathrm{M}_{nr}}\} \\ &= \sup\{\|[\langle T_{pq}v_{kl}, u_{ij}\rangle + \langle \varphi_{pq}v_{kl}, u_{ij}\rangle]\|_{\mathrm{M}_{nr}}\} \\ &\leq \sup\{\|[\langle T_{pq}, u_{ij}\rangle + \langle \varphi_{pq}, u_{ij}\rangle]\|_{\mathrm{M}_{nr}}\} = \|[u_{ij}]\|_{\mathrm{M}_n(\mathrm{S}^1\mathrm{A})}\end{aligned}$$

where the suprema are taken over $[T_{pq}]$ in $\mathrm{b}_1(\mathrm{M}_r(\mathrm{VN}(G)))$, $[\varphi_{pq}]$ in $\mathrm{b}_1(\mathrm{M}_r(\mathrm{L}^\infty(G)))$ and varying r in $\mathbb{N}$. Thus if $[v_{kl}] \in \mathrm{b}_1(\mathrm{M}_m(\mathrm{A}(G)))$ and $[u_{ij}] \in \mathrm{M}_n(\mathrm{S}^1\mathrm{A}(G))$ then

$$\|[v_{kl}u_{ij}]\|_{\mathrm{M}_{mn}(\mathrm{S}^1\mathrm{A})} \leq \|[v_{kl}]\|_{\mathrm{M}_m(\mathrm{A})}\|[u_{ij}]\|_{\mathrm{M}_n(\mathrm{S}^1\mathrm{A})}.$$

Collecting these facts together we obtain

Proposition 1.1. $\mathrm{S}^1\mathrm{A}(G)$ *is a contractive operator Segal algebra in* $\mathrm{A}(G)$.

The convolution algebra $\mathrm{LA}(G)$ was shown in [12] to be a left Segal algebra in $\mathrm{L}^1(G)$. It is immediate, under the operator space structure developed above, that the injection $\mathrm{LA}(G) \hookrightarrow \mathrm{L}^1(G)$ is a complete contraction and, since $\mathrm{L}^1(G)$ has the maximal operator space structure, that $\mathrm{LA}(G)$ is a completely contractive $\mathrm{L}^1(G)$-module. Thus $\mathrm{LA}(G)$ is an operator Segal algebra as well.

1.4. *Other Segal algebras.* If $1 < p < \infty$ we define the *Segal p-Fourier algebra* by

$$\mathrm{S}^p\mathrm{A}(G) = \mathrm{A}(G) \cap \mathrm{L}^p(G).$$

It is standard to show that if we embed $\mathrm{S}^p\mathrm{A}(G) \hookrightarrow \mathrm{A}(G) \oplus_1 \mathrm{L}^p(G)$, then we obtain a Segal algebra in $\mathrm{A}(G)$. If G is abelian with dual group $\widehat{G}$, then $\mathrm{S}^p\mathrm{A}(G)$ is the Segal algebra in $\mathrm{L}^1(\widehat{G})$ given by $\{f \in \mathrm{L}^1(\widehat{G}) : \widehat{f} \in \mathrm{L}^p(G)\}$; see [23]. If we admit on $\mathrm{L}^p(G)$ an operator space structure for which it is a completely contractive $\mathrm{L}^\infty(G)$-module—we might refer to this as an "L^∞-homogeneous operator space structure"—then we may assign an operator space structure to $\mathrm{S}^p\mathrm{A}(G)$ analogously to that which we assigned to $\mathrm{S}^1\mathrm{A}(G)$ above. The result is still a contractive operator Segal algebra. If $p = 2$, there are many candidate operator space structures for $\mathrm{L}^2(G)$, including row, column and the "operator Hilbert space" $\mathrm{OL}^2(G)$. For any p, the interpolated structures $\mathrm{L}^p(G) = (\min \mathrm{L}^\infty(G), \max \mathrm{L}^1(G))_{1/p}$ suffice. See [22] for information on interpolation and $\mathrm{OL}^2(G)$. By [20, Theo. 3.5] or [19] there are certain row and column operator space structures $\mathrm{ROW}(\mathrm{L}^p(G))$ and $\mathrm{COL}(\mathrm{L}^p(G))$ which are also L^∞-homogeneous operator space structures.

For $0 < q < \infty$ the Figà-Talamanca–Herz algebra $\mathrm{A}_q(G)$ has been shown in [20] to admit an operator space structure under which it is a completely bounded Banach algebra. If $1 < p < \infty$ we can define the *Segal* p, q*-Figà-Talamanca–Herz algebra* by

$$\mathrm{S}^p\mathrm{A}_q(G) = \mathrm{A}_q(G) \cap \mathrm{L}^p(G).$$

If on $\mathrm{L}^p(G)$ we admit an L^∞-homogeneous operator space structure, then just as above we obtain an operator Segal algebra in $\mathrm{A}_q(G)$. Note that $\mathrm{S}^p\mathrm{A}_q(G)$ may not be a completely *contractive* Banach algebra as $\mathrm{A}_q(G)$ is not known to be.

2. Dual spaces. In this section we develop the dual of $\mathrm{S}^1\mathrm{A}(G)$ and apply this to determining some restriction and averaging theorems.

We begin with a useful lemma. We denote the group action of right translation by

$$t \cdot f(s) = f(st)$$

for t, s in G, where f is any function on G. If $\mathrm{S}^1(G)$ is a left Segal algebra in $\mathrm{L}^1(G)$, then we say that $\mathrm{S}^1(G)$ has *continuous right translations* if for any u in $\mathrm{S}^1(G)$, $t \cdot u \in \mathrm{S}^1(G)$, and $t \mapsto t \cdot u : G \to \mathrm{S}^1(G)$ is continuous. For example, if $u \in \mathrm{LA}(G)$ we have

$$\|t \cdot u - u\|_{\mathrm{LA}} = \|t \cdot u - u\|_{\mathrm{L}^1} + \|t \cdot u - u\|_{\mathrm{A}} \xrightarrow{t \to e} 0.$$

However, the right action of G on $\mathrm{LA}(G)$ is isometric (bounded) if and only if G is unimodular—in which case we say $\mathrm{LA}(G)$ is a *symmetric* Segal algebra in $\mathrm{L}^1(G)$. Indeed, $\|t \cdot u\|_{\mathrm{LA}} = \Delta(t)\|u\|_{\mathrm{L}^1} + \|u\|_{\mathrm{A}}$, where Δ is the Haar modular function.

Lemma 2.1. *Let $\mathcal{U}$ denote a neigbourhood basis of relatively compact symmetric neighbourhoods of the identity e in G, which is a directed set via reverse inclusion. For each U in $\mathcal{U}$ let $e_U = (1/m(u))1_U$ (normalised indicator function). If $\mathrm{S}^1(G)$ is any Segal algebra in $\mathrm{L}^1(G)$ with continuous right translations, then, for any u in $\mathrm{S}^1(G)$, $u * e_U \in \mathrm{S}^1(G)$ for each U and $\lim_{U\in\mathcal{U}} \|u * e_U - u\|_{\mathrm{S}^1} = 0$.*

Proof. If $u \in \mathrm{S}^1(G)$ then for each U in $\mathcal{U}$ and almost every s in G we have

$$u * e_U(s) = \int_G u(t)e_U(t^{-1}s)\,dt = \int_G u(st)e_U(t)\,dt = \frac{1}{m(U)}\int_U f(st)\,dt$$

where we used symmetry of U to obtain $\check{e}_U = e_U$. Since right translation is continuous on G, and U is relatively compact, we may regard

$$u * e_U = \frac{1}{m(U)}\int_U t \cdot u\,dt$$

as a Bochner integral, converging in $\mathrm{S}^1(G)$. We then obtain

$$\begin{aligned}\|u * e_U - u\|_{\mathrm{S}^1} &= \Big\| \frac{1}{m(U)} \int_U (t\cdot u - u)\, dt \Big\| \\ &\le \frac{1}{m(U)} \int_U \|t\cdot u - u\|_{\mathrm{S}^1} dt \le \sup_{t\in U} \|t\cdot u - u\|_{\mathrm{S}^1} \xrightarrow{U\in\mathcal{U}} 0.\end{aligned}$$

We note that $\{e_U\}_{U\in\mathcal{U}}$ is a well-known symmetric bounded approximate identity in $\mathrm{L}^1(G)$. ∎

2.1. L^∞*-convolvers.* Since the metrical structure on $\mathrm{S}^1\mathrm{A}(G)$ is determined by an embedding $\mathrm{S}^1\mathrm{A}(G) \hookrightarrow \mathrm{A}(G) \oplus_1 \mathrm{L}^1(G)$, the dual $\mathrm{S}^1\mathrm{A}(G)^*$ is a quotient of $\mathrm{VN}(G) \oplus_\infty \mathrm{L}^\infty(G)$ by the annihilator $\mathrm{S}^1\mathrm{A}(G)^\perp$. The L^∞-convolvers allow us to describe this annihilator.

PROPOSITION 2.2. *Let $\varphi \in \mathrm{L}^\infty(G)$. Then the following statements are equivalent*:

(i) $\sup\{\|\varphi * f\|_{\mathrm{L}^2} : f \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee$ *and* $\|f\|_{\mathrm{L}^2} \le 1\} < \infty$,
(ii) $\sup\{|\int_G \varphi(s)u(s)\, ds| : u \in \mathrm{S}^1\mathrm{A}(G)$ *and* $\|u\|_{\mathrm{A}} \le 1\} < \infty$.

In this case the operator $f \mapsto \varphi * f : \mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee \to \mathrm{L}^2(G)$ *extends uniquely to a bounded linear operator $\Lambda(\varphi)$ on $\mathrm{L}^2(G)$. Furthermore $\Lambda(\varphi) \in \mathrm{VN}(G)$, and the quantities in* (i) *and* (ii) *are each equal to* $\|\Lambda(\varphi)\|_{\mathrm{VN}}$.

In (i), $\mathrm{L}^1(G)^\vee = \{\check{f} : f \in \mathrm{L}^1(G)\}$; which is $\mathrm{L}^1(G)$ itself exactly when G is unimodular. By [15, 20.16], if $\varphi \in \mathrm{L}^\infty(G)$ and $f \in \mathrm{L}^1(G)^\vee$, then $\varphi * f$ makes sense; however, in general, it is not square integrable, even if we further assume that f is. For example, if G is non-compact then $1 * f = (\int_G \check{f}\, dm)\, 1 \notin \mathrm{L}^2(G)$ if $\int_G \check{f}\, dm \neq 0$.

We call a φ which satisfies the conditions of the proposition an L^∞-*convolver*, and $\Lambda(\varphi)$ its *convolution operator*. We define

$$\mathrm{Conv}^\infty(G) = \{\varphi \in \mathrm{L}^\infty(G) : \varphi \text{ is a convolver}\}.$$

Proof of Proposition 2.2. (i)⇒(ii). Since $\mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee$ defines a dense subspace of $\mathrm{L}^2(G)$, the operator $f \mapsto \varphi * f : \mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee \to \mathrm{L}^2(G)$ is a continuous linear operator and hence extends uniquely to a bounded linear operator $\Lambda(\varphi)$ in $\mathcal{B}(\mathrm{L}^2(G))$.

If $\varrho : G \to \mathcal{B}(\mathrm{L}^2(G))$ is the right regular representation given by $\varrho(t)f = \Delta(t)^{-1/2} t\cdot f$ for f in $\mathrm{L}^2(G)$ and t in G, then it is easy to check that $(\Lambda(\varphi)\varrho(t) - \varrho(t)\Lambda(\varphi))f = 0$ for each f in $\mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee$, and hence for each f in $\mathrm{L}^2(G)$. This implies that $\Lambda(\varphi) \in \mathrm{VN}(G)$.

If $f \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee$ and $g \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G)$, then the associated coefficient function satisfies

$$\langle \lambda(\cdot) f \mid g\rangle = \overline{g} * \check{f} \in \mathrm{S}^1\mathrm{A}(G). \tag{2.1}$$

Moreover, in the VN(G)-A(G) dual pairing we obtain

$$\langle \Lambda(\varphi), \overline{g} * \check{f}\rangle = \langle \varphi * f \mid g\rangle = \int_G \Big(\int_G \varphi(s) f(s^{-1}t)\, ds\Big)\overline{g(t)}\, dt \tag{2.2}$$
$$= \int_G \varphi(s)\Big(\int_G f(s^{-1}t)\overline{g(t)}\, dt\Big)\, ds = \int_G \varphi(s)\overline{g} * \check{f}(s)\, ds$$

where the use of Fubini's Theorem is justified by our choices for f and g. Now if $u \in \mathrm{S}^1\mathrm{A}(G)$ then $\overline{u} \in \mathrm{L}^\infty(G) \cap \mathrm{L}^1(G) \subseteq \mathrm{L}^2(G)$, so $\overline{u} \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G)$. Moreover, if $(e_U)_{U\in\mathcal{U}}$ is the bounded approximate identity from Lemma 2.1, then each e_U is in $\mathrm{L}^2(G)\cap\mathrm{L}^1(G)^\vee$ with $\check{e}_U = e_U$. Also, $\lim_{U\in\mathcal{U}} \|u * e_U - u\|_{\mathrm{S}^1\mathrm{A}} = 0$. We then have $\langle \lambda(\cdot)e_U \mid \overline{u}\rangle = u * e_U$ and

$$\langle \Lambda(\varphi), u\rangle = \lim_{U\in\mathcal{U}} \langle \Lambda(\varphi), u * e_U\rangle \tag{2.3}$$
$$= \lim_{U\in\mathcal{U}} \int_G \varphi(s) u * e_U(s)\, ds = \int_G \varphi(s) u(s)\, ds.$$

Hence

$$\sup\Big\{\Big|\int_G \varphi(s)u(s)\, ds\Big| : u \in \mathrm{S}^1\mathrm{A}(G) \text{ and } \|u\|_{\mathrm{A}} \le 1\Big\}$$
$$\le \sup\{|\langle \Lambda(\varphi), u\rangle| : u \in \mathrm{b}_1(\mathrm{A}(G))\} = \|\Lambda(\varphi)\|_{\mathrm{VN}} < \infty.$$

(ii)⇒(i). Since $\mathrm{L}^2(G) \cap \mathrm{L}^1(G)$ is dense in $\mathrm{L}^2(G)$ we have, for each f in $\mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee$,

$$\|\varphi * f\|_{\mathrm{L}^2} = \sup\{|\langle \varphi * f \mid g\rangle| : g \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G) \text{ and } \|g\|_{\mathrm{L}^2} \le 1\}.$$

Restating (2.2) and (2.1) we deduce for $f \in \mathrm{L}^2(G)\cap\mathrm{L}^1(G)^\vee$ and $g \in \mathrm{L}^2(G)\cap \mathrm{L}^1(G)$ that

$$\langle \varphi * f \mid g\rangle = \int_G \varphi(s)\langle \lambda(s)f \mid g\rangle\, ds \quad \text{and} \quad \langle \lambda(\cdot)f \mid g\rangle \in \mathrm{S}^1\mathrm{A}(G).$$

Thus

$$\sup\{\|\varphi * f\|_{\mathrm{L}^2} : f \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee \text{ and } \|f\|_{\mathrm{L}^2} \le 1\}$$
$$= \sup\left\{\Big|\int_G \varphi(s)\langle \lambda(s)f \mid g\rangle\, ds\Big| : \begin{array}{l} f \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G)^\vee, \\ g \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G) \\ \text{and } \|f\|_{\mathrm{L}^2}\|g\|_{\mathrm{L}^2} \le 1 \end{array}\right\}$$
$$\le \sup\Big\{\Big|\int_G \varphi(s)u(s)\, ds\Big| : u \in \mathrm{S}^1\mathrm{A}(G) \text{ and } \|u\|_{\mathrm{A}} \le 1\Big\} < \infty.$$

By density of $\mathrm{L}^2(G)\cap\mathrm{L}^1(G)^\vee$ in $\mathrm{L}^2(G)$, the left hand side quantity above is $\|\Lambda(\varphi)\|$. ∎

Unfortunately, we cannot determine if formula (2.3) obtains for any u in $\mathrm{A}(G)$, since if $u \notin \mathrm{S}^1\mathrm{A}(G)$, then the limit of the integrals in that equation cannot be expected to hold.

We note that if G is discrete, then it is well-known that every element of $\mathrm{VN}(G)$ is an ℓ^2-convolver, and in particular an ℓ^∞-convolver, i.e. $\mathrm{VN}(G) = \Lambda(\mathrm{Conv}^\infty(G))$. This corresponds to the fact that $\mathrm{S}^1\mathrm{A}(G) = \mathrm{A}(G) \cap \ell^1(G) = \ell^1(G)$.

If G is compact, then $\mathrm{L}^\infty(G) \subseteq \mathrm{L}^1(G)$, and hence $\mathrm{L}^\infty(G) = \mathrm{Conv}^\infty(G)$. This corresponds to the fact that $\mathrm{S}^1\mathrm{A}(G) = \mathrm{A}(G)$ in this case. Here we see that $\Lambda(\mathrm{Conv}^\infty(G))$ is a dense $*$-subalgebra of the (reduced) group C^*-algebra $\mathrm{C}^*_\mathrm{r}(G)$.

There are natural questions concerning the extent of $\mathrm{Conv}^\infty(G)$. We note that in both cases above, $\mathrm{Conv}^\infty(G) \subseteq \mathrm{L}^2(G)$. Moreover, $\mathrm{L}^\infty(G) \cap \mathrm{L}^1(G)$, which is necessarily always contained in $\mathrm{Conv}^\infty(G)$, is always a subset of $\mathrm{L}^2(G)$. Thus we ask: *for which G is* $\mathrm{Conv}^\infty(G)$ *contained in* $\mathrm{L}^2(G)$*?* It is also natural to wonder about the norm closure $\overline{\Lambda(\mathrm{Conv}^\infty(G))}$. *When is* $\overline{\Lambda(\mathrm{Conv}^\infty(G))}$ *a $*$-subalgebra of* $\mathrm{VN}(G)$*? How big is* $\overline{\Lambda(\mathrm{Conv}^\infty(G))}$*?* We note that $\overline{\Lambda(\mathrm{Conv}^\infty(G))}$ always contains the reduced C^*-algebra $\mathrm{C}^*_r(G)$ of G.

2.2. *The dual of the 1-Segal Fourier algebra*

Theorem 2.3. *There is an isometric isomorphism*

$$\mathrm{S}^1\mathrm{A}(G)^* \cong \mathrm{VN}(G) \oplus_\infty \mathrm{L}^\infty(G)/\{(\Lambda(\varphi), -\varphi) : \varphi \in \mathrm{Conv}^\infty(G)\}.$$

In particular, $\mathrm{Conv}^\infty(G)$ *is linearly isomorphic to a norm-closed subspace of* $\mathrm{VN}(G) \oplus_\infty \mathrm{L}^\infty(G)$.

Proof. We have an isometry $\mathrm{S}^1\mathrm{A}(G) \hookrightarrow \mathrm{A}(G) \oplus_1 \mathrm{L}^1(G)$. Hence by the Hahn–Banach Theorem every continuous linear functional may be realised as one from $\mathrm{VN}(G) \oplus_\infty \mathrm{L}^\infty(G) \cong (\mathrm{A}(G) \oplus_1 \mathrm{L}^1(G))^*$ via the form

$$\langle (T, \varphi), u\rangle = \langle T, u\rangle + \int_G \varphi u \, dm$$

for (T, φ) in $\mathrm{VN}(G) \oplus_\infty \mathrm{L}^\infty(G)$ and u in $\mathrm{S}^1\mathrm{A}(G)$, where $\langle T, u\rangle$ denotes the $\mathrm{VN}(G)$-$\mathrm{A}(G)$ dual pairing. It follows that the annihilator of $\mathrm{S}^1\mathrm{A}(G)$ will consist exactly of those pairs (T, φ) for which $\langle T, u\rangle = -\int_G \varphi u \, dm$ for each u in $\mathrm{S}^1\mathrm{A}(G)$. But then

$$\sup\Big\{\Big|\int_G \varphi u \, dm\Big| : u \in \mathrm{S}^1\mathrm{A}(G) \text{ and } \|u\|_\mathrm{A} \le 1\Big\} \le \sup\{|\langle T, u\rangle| : u \in \mathrm{b}_1(\mathrm{A}(G))\} = \|T\|_{\mathrm{VN}} < \infty.$$

Hence $\varphi \in \mathrm{Conv}^\infty(G)$. Moreover, since $\mathrm{S}^1\mathrm{A}(G)$ is dense in $\mathrm{A}(G)$, it follows that $T = -\Lambda(\varphi)$. ∎

We have found no obvious direct computation which allows us to obtain the following corollary, in general. We note that, by [4, Theo. 2.1], $\mathrm{S}^1\mathrm{A}(G)$, being a Segal algebra in $\mathrm{A}(G)$, has spectrum G. If $\mathrm{A}(G)$ were assumed to have an approximate unit (maybe unbounded!), then the corollary would be a consequence of [4, Theo. 1.1]. We let $\mathrm{A_c}(G)$ denote the subalgebra of compactly supported elements of $\mathrm{A}(G)$ and recall that $\mathrm{A}(G)$ is *Tauberian*, as $\mathrm{A_c}(G)$ is dense in $\mathrm{A}(G)$.

COROLLARY 2.4. $\mathrm{S^1A}(G)$ *is an essential* $\mathrm{A}(G)$*-module*, *i.e.*, $\mathrm{A}(G)\cdot\mathrm{S^1A}(G)$ *is a dense subspace of* $\mathrm{S^1A}(G)$. *In particular* $\mathrm{S^1A}(G)$ *is a Tauberian Banach algebra.*

Proof. Let $(T, -\varphi)$ in $\mathrm{VN}(G) \oplus_\infty \mathrm{L}^\infty(G)$ be such that

$$\langle T, uv\rangle - \int_G \varphi uv\, dm = 0$$

for every u in $\mathrm{A}(G)$ and v in $\mathrm{S^1A}(G)$. Then we see from the theorem above that $Tu = \Lambda(\varphi u)$ for every u in $\mathrm{A}(G)$. It is clear that $\mathrm{A_c}(G) \subset \mathrm{S^1A}(G)$. Thus, for any $v \in \mathrm{A_c}(G)$ we see that

$$\langle T, uv\rangle = \langle Tu, v\rangle = \int_G \varphi uv\, dm = \langle \Lambda(\varphi), uv\rangle$$

for all u in $\mathrm{A}(G)$, from which it follows that $\langle T, w\rangle = \langle \Lambda(\varphi), w\rangle$ for any $w \in \mathrm{A_c}(G)$. Since $\mathrm{A_c}(G)$ is dense in $\mathrm{A}(G)$, $T = \Lambda(\varphi)$. It follows, again for the theorem above, that $(T, -\varphi)$ is the zero functional on $\mathrm{S^1A}(G)$. The Hahn–Banach Theorem then implies that $\mathrm{A}(G)\cdot\mathrm{S^1A}(G)$ is dense in $\mathrm{S^1A}(G)$.

We observe that $\mathrm{A_c}(G) = \mathrm{A_c}(G)\cdot\mathrm{S^1A}(G)$ consists of the compactly supported elements in $\mathrm{S^1A}(G)$ and is dense in there too. ∎

2.3. *The dual of the 2-Segal Fourier algebra.* We note that the computations in this section work analogously for $\mathrm{S^2A}(G)$. Let us briefly indicate how. Since translations are continuous on $\mathrm{S^2A}(G)$ we can deduce that $\lim_{U\in\mathcal{U}} \|u * e_U - u\|_{\mathrm{S^2A}} = 0$ just as in Lemma 2.1. Then we find for a fixed h in $\mathrm{L}^2(G)$ that the following two quantities:

(i) $\sup\{\|h * f\|_{\mathrm{L}^2} : f \in \mathrm{L}^2(G) \cap \mathrm{L}^2(G)^\vee \text{ and } \|f\|_{\mathrm{L}^2} \le 1\}$,
(ii) $\sup\{|\int_G hu\, dm| : u \in \mathrm{S^2A}(G) \text{ and } \|u\|_{\mathrm{A}} \le 1\}$

are equal, in particular both are finite if one of them is. We call such a function h an L^2-*convolver* and denote the set of them by $\mathrm{Conv}^2(G)$. If $h \in \mathrm{Conv}^2(G)$ then it defines a bounded operator $\Lambda_2(h)$ on $\mathrm{L}^2(G)$ which is an element of $\mathrm{VN}(G)$. We thus obtain the linear isometric identification

$$\mathrm{S^2A}(G)^* \cong \mathrm{VN}(G) \oplus_\infty \mathrm{L}^2(G)/\{(\Lambda_2(h), -h) : h \in \mathrm{Conv}^2(G)\}. \tag{2.4}$$

We note that if G is an abelian group with dual group $\widehat{G}$ and $U : \mathrm{L}^2(G) \to \mathrm{L}^2(\widehat{G})$ is the Plancherel unitary, then $U(\mathrm{Conv}^2(G)) = \mathrm{L}^\infty(\widehat{G}) \cap \mathrm{L}^2(\widehat{G})$. *Is there an analogous description for* $U\Lambda(\mathrm{Conv}^\infty(G))U^*$*?*

3. Restriction and averaging operations. Let us first note that if H is an open subgroup of G, then we have restrictions $\mathrm{A}(G)|_H = \mathrm{A}(H)$ and

$\mathrm{L}^1(G)|_H = \mathrm{L}^1(H)$. Both restriction operations are quotient maps, the latter so provided the Haar measure on H is the restricted Haar measure from G. Hence we obtain the restriction

$$\mathrm{S}^1\mathrm{A}(G)|_H = \mathrm{S}^1\mathrm{A}(H).$$

We shall see that for a non-open closed subgroup, this result can be much different.

3.1. *A distance formula.* If H is any closed subgroup of G, the restriction map $u \mapsto u|_H : \mathrm{A}(G) \to \mathrm{A}(H)$ is also a surjective quotient map by [14, 28]. The kernel of this map is the ideal $\mathrm{I}(H) = \{u \in \mathrm{A}(G) : u|_H = 0\}$. Dual to this is the fact that the subalgebra $\mathrm{VN}_H(G)$, which is the weak operator closure of the span of $\{\lambda(s) : s \in H\}$ in $\mathrm{VN}(G)$, is the annihilator of $\mathrm{I}(H)$ in G and hence $*$-isomorphic to $\mathrm{A}(H)^* \cong \mathrm{VN}(H)$.

If H is a closed subgroup of G, we say H admits a *bounded approximate indicator in* G (after [1]) if there is a bounded net (u_α) from $\mathrm{B}(G)$ such that

(i) $\lim_\alpha u_\alpha|_H v = v$ for each v in $\mathrm{A}(H)$,
(ii) $\lim_\alpha u_\alpha u = 0$ for each u in $\mathrm{I}(H)$.

By [1, Theo. 3.7] we can suppose that $\|u_\alpha\|_{\mathrm{B}} \leq 1$ for each α. We will have a bounded approximate indicator for H in G provided:

- H is neutral in G, i.e. there is a neighbourhood basis $\mathcal{V}$ of e for which $VH = HV$ for each V in $\mathcal{V}$ (see [18, Prop. 2.2])—this is always true if G is a small invariant neighbourhood group; or
- G is amenable (see [9, Theo. 1.3] and [1, Prop. 4.1]).

Since there is a completely contractive injection $\mathrm{S}^1\mathrm{A}(G) \hookrightarrow \mathrm{A}(G)$ with dense range, there is a completely contractive injection $\mathrm{VN}(G) \hookrightarrow \mathrm{S}^1\mathrm{A}(G)^*$. Using Theorem 2.3 we can obtain a lower bound on the norms of the range of this map: if $T \in \mathrm{VN}(G)$ then

$$\begin{aligned}\|T\|_{\mathrm{S}^1\mathrm{A}^*} &= \inf\{\max\{\|T + \Lambda(\varphi)\|_{\mathrm{VN}}, \|\varphi\|_{\mathrm{L}^\infty}\} : \varphi \in \mathrm{Conv}^\infty(G)\} \\ &\geq \inf\{\|T + \Lambda(\varphi)\|_{\mathrm{VN}} : \varphi \in \mathrm{Conv}^\infty(G)\} = \mathrm{dist}_{\mathrm{VN}}(T, \Lambda(\mathrm{Conv}^\infty(G))).\end{aligned}$$

We obtain a similar bound for matricial norms: if $[T_{ij}] \in \mathrm{M}_n(\mathrm{VN}(G))$, then

$$\|[T_{ij}]\|_{\mathrm{M}_n(\mathrm{S}^1\mathrm{A}^*)} \geq \mathrm{dist}_{\mathrm{M}_n(\mathrm{VN})}([T_{ij}], \mathrm{M}_n(\Lambda(\mathrm{Conv}^\infty(G)))). \tag{3.1}$$

Under certain assumptions, the quantity on the right is as big as it can be.

THEOREM 3.1. *If H is a non-open closed subgroup of G which admits a contractive approximate indicator (u_α), then for any $[T_{ij}]$ in $\mathrm{M}_n(\mathrm{VN}_H(G))$,*

$$\mathrm{dist}_{\mathrm{M}_n(\mathrm{VN})}([T_{ij}], \mathrm{M}_n(\Lambda(\mathrm{Conv}^\infty(G)))) = \|[T_{ij}]\|_{\mathrm{M}_n(\mathrm{VN})}.$$

Proof. Let us first suppose that $n = 1$. Given any $\varepsilon > 0$ find $u \in \mathrm{b}_1(\mathrm{A}(G))$ for which $\mathrm{supp}(u)$ is compact and

$$|\langle T, u\rangle| > \|T\|_{\mathrm{VN}} - \varepsilon.$$

Now let $\varphi \in \mathrm{Conv}^\infty(G)$. We then have, for any α,

$$\langle T + \Lambda(\varphi), uu_\alpha\rangle = \langle T, uu_\alpha\rangle + \int_G \varphi uu_\alpha \, dm. \tag{3.2}$$

If we can find α_0 for which

$$|\langle T, u\rangle - \langle T, uu_{\alpha_0}\rangle| < \varepsilon \quad \text{and} \quad \Big|\int_G \varphi uu_{\alpha_0}\, dm\Big| < \varepsilon \tag{3.3}$$

then it follows that

$$|\langle T + \Lambda(\varphi), uu_{\alpha_0}\rangle| > \|T\|_{\mathrm{VN}} - 3\varepsilon. \tag{3.4}$$

But then $\mathrm{dist}_{\mathrm{VN}}(T, \Lambda(\mathrm{Conv}^\infty(G))) \geq \|T\|_{\mathrm{VN}}$, whence equality must hold.

Let us verify the first inequality from (3.3). Since $T \in \mathrm{VN}_H(G)$ there is T_H in $\mathrm{VN}(H)$ such that $\langle T, v\rangle = \langle T_H, v|_H\rangle$ for each v in $\mathrm{A}(G)$. Hence by condition (i) in the definition of (u_α) we have

$$\langle T, uu_\alpha\rangle = \langle T_H, (uu_\alpha)|_H\rangle \xrightarrow{\alpha} \langle T_H, u|_H\rangle = \langle T, u\rangle$$

and we can find α_1 for which the desired inequality is satisfied for any $\alpha_0 \geq \alpha_1$. Now let us verify the second inequality from (3.3). Since H is not open it is locally null [15, 20.17], so $m(S \cap H) = 0$ where $S = \mathrm{supp}(u)$. Let C be any compact neighbourhood of $S \cap H$ with

$$m(C) < \frac{\varepsilon}{2\|\varphi\|_{\mathrm{L}^\infty} + 1}.$$

Let U be a symmetric neighbourhood of e for which $(S \cap H)U^2 \subseteq C$. Then if for s in G we let

$$v(s) = \frac{1}{m(U)}\langle \lambda(s)1_U \mid 1_{(S\cap H)U}\rangle = \frac{m(sU \cap (S \cap H)U)}{m(U)}$$

we obtain

$$\|v\|_{\mathrm{L}^\infty} = 1, \qquad v|_{S\cap H} = 1, \qquad \mathrm{supp}(v) \subseteq (S \cap H)U^2 \subseteq C.$$

Now we have

$$\int_G \varphi uu_\alpha \, dm = \int_G \varphi uu_\alpha v \, dm + \int_G \varphi uu_\alpha (1 - v)\, dm,$$

where $\|uu_\alpha v\|_{\mathrm{L}^\infty} \leq 1$, so

$$\Big|\int_G \varphi uu_\alpha v\, dm\Big| \leq \int_C |\varphi|\, dm \leq \|\varphi\|_{\mathrm{L}^\infty} m(C) < \frac{\varepsilon}{2},$$

while

$$\|u_\alpha u(1 - v)\|_{\mathrm{L}^\infty} \leq \|u_\alpha u(1 - v)\|_{\mathrm{A}} \xrightarrow{\alpha} 0$$

as $u(1 - v) \in \mathrm{I}(H)$, so

$$\Big|\int_G \varphi uu_\alpha(1 - v)\, dm\Big| \leq \int_S \|\varphi\|_{\mathrm{L}^\infty}\|uu_\alpha(1 - v)\|_{\mathrm{L}^\infty}\, dm \xrightarrow{\alpha} 0.$$

Hence we can find α_2 so that the desired inequality holds for $\alpha_0 \geq \alpha_2$. To obtain (3.3), we now choose $\alpha_0 \geq \alpha_1, \alpha_2$.

We can now obtain the theorem for $n > 1$. First, given $\varepsilon > 0$ we can choose m and $[u_{kl}]$ in $\mathrm{b}_1(\mathrm{M}_m(\mathrm{A}(G)))$ for which each $\mathrm{supp}(u_{ij})$ is compact and

$$\|\langle T_{ij}, u_{kl}\rangle\|_{\mathrm{M}_{mn}} > \|[T_{ij}]\|_{\mathrm{M}_n(\mathrm{VN})} - \varepsilon.$$

Then, as in (3.2), if $[\varphi_{ij}] \in \mathrm{M}_n(\mathrm{Conv}^\infty(G))$ we have

$$[\langle T_{ij} + \Lambda(\varphi_{ij}), u_{kl}u_\alpha\rangle] = [\langle T_{ij}, u_{kl}u_\alpha\rangle] + \Big[\int\limits_G \varphi_{ij}u_{kl}u_\alpha\,dm\Big].$$

As above we can arrange to find an α_0 for which

$$|\langle T_{ij}, u_{kl}u_{\alpha_0}\rangle - \langle T_{ij}, u_{kl}\rangle| < \frac{\varepsilon}{n^2} \quad \text{and} \quad \Big|\int\limits_G \varphi_{ij}u_{kl}u_{\alpha_0}\,dm\Big| < \frac{\varepsilon}{n^2},$$

and then the matricial analogue of (3.4) is satisfied. ■

3.2. *A restriction theorem.* We conjecture that the following theorem holds for general closed non-open subgroups of G.

THEOREM 3.2. *If H is a closed non-open subgroup of G which admits a bounded approximate indicator in G, then the restriction map*

$$u \mapsto u|_H : \mathrm{S}^1\mathrm{A}(G) \to \mathrm{A}(H)$$

is a surjective complete quotient map.

Proof. The adjoint of the restriction map is the composition of the injective $*$-homomorphism $J : \mathrm{VN}(H) \to \mathrm{VN}_H(G) \subset \mathrm{VN}(G)$ with the completely contractive injection $\mathrm{VN}(G) \hookrightarrow \mathrm{S}^1\mathrm{A}(G)^*$. It follows from (3.1) and Theorem 3.1 that for $[T_{ij}]$ in $\mathrm{M}_n(\mathrm{VN}(H))$, we have

$$\begin{aligned}\|[JT_{ij}]\|_{\mathrm{M}_n(\mathrm{S}^1\mathrm{A}^*)} &\geq \mathrm{dist}_{\mathrm{M}_n(\mathrm{VN})}([JT_{ij}], \mathrm{M}_n(\Lambda(\mathrm{Conv}^\infty(G))))\\ &= \|[JT_{ij}]\|_{\mathrm{M}_n(\mathrm{VN})} = \|[T_{ij}]\|_{\mathrm{M}_n(\mathrm{VN}_H)}.\end{aligned}$$

Hence $J : \mathrm{VN}_H(G) \to \mathrm{S}^1\mathrm{A}(G)^*$ is a complete isometry, so the restriction map must be a surjective complete quotient map. ■

We recall that for the operator projective tensor product $\widehat{\otimes}$ we have the formulas

$$\mathrm{L}^1(G) \mathbin{\widehat{\otimes}} \mathrm{L}^1(G) \cong \mathrm{L}^1(G{\times}G), \qquad \mathrm{A}(G) \mathbin{\widehat{\otimes}} \mathrm{A}(G) \cong \mathrm{A}(G{\times}G).$$

It is thus a little surprising that the 1-Segal Fourier algebra $\mathrm{S}^1\mathrm{A}(G)$ admits no such formula.

COROLLARY 3.3. *If G is a non-compact, non-discrete group which is either amenable or admits small invariant neighbourhoods, then*

$$\mathrm{S}^1\mathrm{A}(G) \mathbin{\widehat{\otimes}} \mathrm{S}^1\mathrm{A}(G) \ncong \mathrm{S}^1\mathrm{A}(G{\times}G).$$

Proof. If the two spaces were isomorphic, then the multiplication map $\mathrm{S}^1\mathrm{A}(G)\,\widehat{\otimes}\,\mathrm{S}^1\mathrm{A}(G) \to \mathrm{S}^1\mathrm{A}(G) \subset \mathrm{A}(G)$ would be isomorphic to the restriction map $w \mapsto w|_D : \mathrm{S}^1\mathrm{A}(G\times G) \to \mathrm{A}(D) \cong \mathrm{A}(G)$, where $D = \{(s,s) : s \in G\}$ is the diagonal subgroup. Since our assumptions on G allow D to admit a bounded approximate indicator in $G\times G$ by [1, Theo. 2.4], the restriction map is surjective. But this contradicts the fact that $\mathrm{S}^1\mathrm{A}(G) \subsetneq \mathrm{A}(G)$; see [12, Prop. 2.6]. ∎

3.3. *An averaging theorem.* We note an analogous result to Theorem 3.2 which applies to $\mathrm{LA}(G)$, which we recall is $\mathrm{L}^1(G) \cap \mathrm{A}(G)$, the Segal algebra of $\mathrm{L}^1(G)$.

Let N be a normal subgroup of G, admitting left Haar integral $\int_N \cdots dn$ (normalised if N is compact). It is well-known that the N-averaging operator $\tau_N : \mathrm{L}^1(G) \to \mathrm{L}^1(G/N)$ given by

$$\tau_N f(tN) = \int_N f(tn)\, dn$$

for almost every tN in G/N is a contractive surjective algebra homomorphism. See [23, III.4], for example.

Let us note that τ_N is a quotient map, but offer an alternative proof to the standard one. We let

$$\mathrm{L}^\infty(G\!:\!N) = \left\{ \varphi \in \mathrm{L}^\infty(G) : \begin{array}{l} \text{for locally almost every } t \text{ in } G, \\ \varphi(tn) = \varphi(t) \text{ for every } n \text{ in } N \end{array} \right\},$$

which is clearly a closed subspace of $\mathrm{L}^\infty(G)$. We note that if $\varphi \in \mathrm{L}^\infty(G\!:\!N)$, then for locally almost every t in G and every n in N we have $\varphi(nt) = \varphi(t\cdot t^{-1}nt) = \varphi(t)$. If $q : G \to G/N$ is the quotient map then $\varphi \mapsto \varphi \circ q$ is a linear isometry from $\mathrm{L}^\infty(G/N)$ onto $\mathrm{L}^\infty(G\!:\!N)$. Now let us compute the adjoint of τ_N: if $\varphi \in \mathrm{L}^\infty(G/N)$ and $f \in \mathrm{L}^1(G)$, then

$$\begin{aligned} \langle \tau_N^*\varphi, f\rangle = \langle \varphi, \tau_N f\rangle &= \int_{G/N} \varphi(tN) \int_N f(tn)\, dn\, dtN \\ &= \int_{G/N}\int_N \varphi \circ q(tn) f(tn)\, dn\, dtN = \int_G \varphi \circ q(t) f(t)\, dt = \langle \varphi \circ q, f\rangle \end{aligned}$$

by Weil's integral formula, so $\tau_N^*\varphi = \varphi \circ q$. Thus $\tau_N^* : \mathrm{L}^\infty(G/N) \to \mathrm{L}^\infty(G)$ is an isometry, so τ_N is a quotient map.

If N is a compact group, then we may let $\mathrm{L}^1(G\!:\!N)$ be defined similarly to $\mathrm{L}^\infty(G\!:\!N)$, and we note it is a subalgebra of $\mathrm{L}^1(G)$ which is isometrically algebraically isomorphic to $\mathrm{L}^1(G/N)$. We may consider τ_N to have range $\mathrm{L}^1(G : N)$. It is well-known that $\tau_N\mathrm{A}(G) = \mathrm{A}(G : N) = \{u \in \mathrm{A}(G) : u \text{ is constant on cosets of } N\} \cong \mathrm{A}(G/N)$, and that $\tau_N u = u$ for u in $\mathrm{A}(G)$.

Hence it follows that

$$\tau_N(\mathrm{LA}(G)) = \mathrm{LA}(G\!:\!N) \cong \mathrm{LA}(G/N)$$

where $\mathrm{LA}(G\!:\!N) = \{u \in \mathrm{LA}(G) : u$ is constant on cosets of $N\}$. This result does not hold if N is not compact. In the case that G is abelian, the following is due to Krogstad, whose unpublished result is announced in [8]. Our proof is for general locally compact G.

THEOREM 3.4. *If N is a non-compact closed normal subgroup of G, then the N-averaging operator $\tau_N : \mathrm{LA}(G) \to \mathrm{L}^1(G/N)$ is a surjective complete quotient map.*

As with Theorem 3.2, the proof relies on a distance formula.

LEMMA 3.5. *If N is a non-compact closed normal subgroup of G, and $[\varphi_{ij}] \in \mathrm{M}_n(\mathrm{L}^\infty(G\!:\!N))$, then*

$$\mathrm{dist}_{\mathrm{M}_n(\mathrm{L}^\infty)}([\varphi_{ij}], \mathrm{M}_n(\mathrm{Conv}^\infty(G))) = \|[\varphi_{ij}]\|_{\mathrm{M}_n(\mathrm{L}^\infty)}.$$

Proof. Let us begin with the "scalar" case. Let $\varphi \in \mathrm{L}^\infty(G\!:\!N)$ and $\varepsilon > 0$. Let $f \in (\mathrm{L}^1(G) \cap \mathrm{L}^2(G))^\vee$ be so that $\|f\|_{\mathrm{L}^1} = 1$ and $|\langle \varphi, f\rangle| > \|\varphi\|_{\mathrm{L}^\infty} - \varepsilon$. Then for any $\psi \in \mathrm{Conv}^\infty(G)$ and any $n \in N$ we have

$$\begin{aligned}\langle \varphi + \psi, n * f\rangle &= \langle n^{-1} * \varphi, f\rangle + \langle \psi, n * f\rangle = \langle \varphi, f\rangle + \int_G \psi(t)\check{f}(t^{-1}n)\,dt\\ &= \langle \varphi, f\rangle + \psi * \check{f}(n).\end{aligned}$$

By [15, 20.16] we find that $h = \psi * \check{f}$ is left uniformly continuous. Since $\psi \in \mathrm{Conv}^\infty(G)$, $h \in \mathrm{L}^2(G)$. It then follows that h is a continuous function vanishing at ∞. Indeed, if not, then there is a $\delta > 0$ and a net (t_α) in G such that $\lim_\alpha t_\alpha = \infty$ and for which $|h(t_\alpha)| > \delta$ for each α. Then, by uniform continuity, there is a compact neighbourhood U of the identity in G such that $|h(t_\alpha s)| > \delta/2$ for $s \in U$. By dropping to a subnet, we may assume that $t_\alpha U \cap t_\beta U = \emptyset$ if $\alpha \neq \beta$. But then, selecting any finite collection F of indices we find

$$\int_G |h|^2\,dm \ge \sum_{\alpha\in F} \int_{t_\alpha U} |h|^2\,dm \ge \frac{|F|\delta^2}{4},$$

which, since $|F|$ can be chosen arbitrarily large, contradicts that $h \in \mathrm{L}^2(G)$.

Thus, since N is non-compact, we may find n in N for which

$$|\langle \varphi + \psi, n * f\rangle| > \|\varphi\|_{\mathrm{L}^\infty} - \varepsilon.$$

Hence $\mathrm{dist}_{\mathrm{L}^\infty}(\varphi, \mathrm{Conv}^\infty(G)) = \|\varphi\|_{\mathrm{L}^\infty}$.

The general matricial case can be deduced from the scalar case, exactly as in the proof of Theorem 3.1. ∎

Proof of Theorem 3.4. The adjoint of $\tau_N : \mathrm{LA}(G) \to \mathrm{L}^1(G/N)$ is the composition of the injective $*$-homomorphism $\varphi \mapsto \varphi \circ q : \mathrm{L}^\infty(G/N) \to$

$\mathrm{L}^\infty(G:N) \subset \mathrm{L}^\infty(G)$ with the contractive inclusion $\mathrm{L}^\infty(G) \hookrightarrow \mathrm{LA}(G)^*$. We recall that $\mathrm{LA}(G)^* = \mathrm{S}^1\mathrm{A}(G)^*$, which is described in Theorem 2.3. Now if $[\varphi_{ij}] \in \mathrm{M}_n(\mathrm{L}^\infty(G/N))$, then we obtain, just as in (3.4), and using the lemma above,

$$\begin{aligned}\|[\varphi_{ij}]\|_{\mathrm{M}_n(\mathrm{L}^\infty)} &\geq \|[\tau_N^*\varphi_{ij}]\|_{\mathrm{M}_n(\mathrm{LA}^*)}\\ &\geq \mathrm{dist}_{\mathrm{M}_n(\mathrm{L}^\infty)}([\varphi_{ij}\circ q], \mathrm{M}_n(\mathrm{Conv}^\infty(G)))\\ &= \|[\varphi_{ij}\circ q]\|_{\mathrm{M}_n(\mathrm{L}^\infty)} = \|[\varphi_{ij}]\|_{\mathrm{M}_n(\mathrm{L}^\infty)}.\end{aligned}$$

Hence $\tau_N^* : \mathrm{L}^\infty(G/N) \to \mathrm{LA}(G)^*$ is a complete isometry, which implies that $\tau_N : \mathrm{LA}(G) \to \mathrm{L}^1(G/N)$ is a surjective complete quotient map. ■

It was shown in [13] that if G is a unimodular group—so $\mathrm{LA}(G)$ is a symmetric Segal algebra—then $\mathrm{LA}(G)$ *is Arens regular if and only if* G *is compact.* Let us briefly note that this result holds without assuming *a priori* that G is unimodular. We refer the reader to the survey article [5] for details on, and functorial properties of, Arens regularity.

COROLLARY 3.6. *If* G *is not unimodular*, *then* $\mathrm{LA}(G)$ *is not Arens regular.*

Proof. Let us note that if G admits a continuous homomorphism $\delta : G \to \mathbb{R}$ for which $K = \ker\delta$ is compact, then G is an extension of K by an abelian group. Since abelian groups are small invariant neighbourhood groups, by [21, 12.1.31] we deduce that G is an invariant neighbourhood group. Then it is well-known (see [21, 21.1.9], for example) that G is unimodular.

Hence, if G is not unimodular, then $N = \ker\Delta$ is necessarily a non-compact closed subgroup of G, with G/N isomorphic to an infinite subgroup of $\mathbb{R}$. Hence $\mathrm{L}^1(G/N)$ is not Arens regular and $\tau_N : \mathrm{LA}(G) \to \mathrm{L}^1(G/N)$ is a quotient homomorphism. ■

4. Cohomological properties. In this section we discuss amenability and operator amenability of the 1-Segal Fourier algebra $\mathrm{S}^1\mathrm{A}(G)$. We have opted to keep our presentation simple by focusing on $\mathrm{S}^1\mathrm{A}(G)$. With suitable modifications to the proofs, we believe that all of our results hold for $\mathrm{S}^p\mathrm{A}_q(G)$, which was defined and given an operator space structure in Section 1.4, where $1 \leq p < \infty$, $1 < q < \infty$.

4.1. *Amenability.* Ruan's result, that $\mathrm{A}(G)$ is *operator amenable* if and only if G is amenable [24], is one of the most important results which justifies treating $\mathrm{A}(G)$ as an operator space. We recall that a completely contractive Banach algebra $\mathcal{A}$ is *operator amenable* if every completely bounded derivation $D : \mathcal{A} \to \mathcal{V}^*$, where $\mathcal{V}^*$ is the operator dual space to a completely bounded $\mathcal{A}$-bimodule, is inner. We note that there exist compact groups, for example $G = \mathrm{SO}(3)$, such that $\mathrm{A}(G)$ is not amenable [17].

PROPOSITION 4.1. $\mathrm{S}^1\mathrm{A}(G) = \mathrm{A}(G)$ *completely isomorphically if and only if G is compact.*

Proof. By [12, Prop. 2.6], $\mathrm{S}^1\mathrm{A}(G) = \mathrm{A}(G)$ if and only if G is compact. Thus it remains to show that if G is compact then the identity map $j : \mathrm{A}(G) \to \mathrm{S}^1\mathrm{A}(G)$ is completely bounded. For $[u_{ij}]$ in $\mathrm{M}_n(\mathrm{A}(G))$ we have

$$\|[u_{ij}]\|_{\mathrm{M}_n(\mathrm{S}^1\mathrm{A})} = \|[u_{ij}1]\|_{\mathrm{M}_n(\mathrm{S}^1\mathrm{A})} \leq \|[u_{ij}]\|_{\mathrm{M}_n(\mathrm{A})}\|1\|_{\mathrm{S}^1\mathrm{A}} = 2\|[u_{ij}]\|_{\mathrm{M}_n(\mathrm{A})}.$$

Thus j is completely bounded with $\|j\|_{cb} \leq 2$. In fact, since $2 = \|j(1)\|$ we have $2 \leq \|j\| \leq \|j\|_{cb}$ as well. Since $j^{-1} : \mathrm{S}^1\mathrm{A}(G) \to \mathrm{A}(G)$ is completely contractive we obtain $\mathrm{S}^1\mathrm{A}(G) = \mathrm{A}(G)$ completely isomorphically. ∎

Thus we obtain the main result of this section.

THEOREM 4.2. $\mathrm{S}^1\mathrm{A}(G)$ *is operator amenable if and only if G is compact.*

Proof. (⇒) $\mathrm{S}^1\mathrm{A}(G)$ is operator amenable only if it has a bounded approximate identity [24, 16]. This happens if and only if G is compact by [12, Prop. 2.6].

(⇐) By Proposition 4.1 above, any completely bounded $\mathrm{S}^1\mathrm{A}(G)$-module $\mathcal{V}$ is a completely bounded $\mathrm{A}(G)$-module. Moreover, every completely bounded derivation $D : \mathrm{S}^1\mathrm{A}(G) \to \mathcal{V}^*$ induces a completely bounded derivation $D \circ j : \mathrm{A}(G) \to \mathcal{V}^*$. By [24], $D \circ j$ is inner, whence so too is D. ∎

The main result of [10] states that $\mathrm{A}(G)$ is amenable if and only if G admits an abelian subgroup of finite index. Combining this with [12, Cor. 2.7] we obtain the following.

COROLLARY 4.3. $\mathrm{S}^1\mathrm{A}(G)$ *is amenable if and only if G is compact and admits an abelian subgroup of finite index.*

4.2. *Weak amenability.* The theory of hyper-Tauberian Banach algebras, developed by Samei [26], extends very easily to Segal algebras. We are grateful to E. Samei for pointing out an error we made, applying his work, in an earlier draft of this article.

Let us first recall some basic definitions. Let $\mathcal{A}$ be a semisimple abelian (completely contractive) Banach algebra with Gelfand spectrum X. Hence we identify $\mathcal{A}$ as a subspace of $\mathcal{C}_0(X)$. We define for a in $\mathcal{A}$ its support by $\mathrm{supp}(a) = \overline{\{x \in X : a(x) \neq 0\}}$. We say that $\mathcal{A}$ is *Tauberian* if the subalgebra of compactly supported elements, $\mathcal{A}_{\mathrm{c}}$, is dense in $\mathcal{A}$. If $\mathcal{V}$ is a symmetric Banach $\mathcal{A}$-module and $v \in \mathcal{V}$, we define the *support* of v *over* $\mathcal{A}$ by

$$\mathrm{supp}_{\mathcal{A}}(v) = \{x \in X : a(x) = 0 \text{ whenever } a \cdot v = 0\}.$$

If $\mathcal{V} = \mathcal{A}^*$, with the usual dual module action $a \cdot f(b) = f(ba)$, this agrees with the usual notion of support of a linear functional. If $\mathcal{V} = \mathcal{A}$, then $\mathrm{supp}_{\mathcal{A}}(a) = \mathrm{supp}(a)$ for any a in $\mathcal{A}$. If $\mathcal{V}$ and $\mathcal{W}$ are (completely bounded)

Banach $\mathcal{A}$-modules, a linear operator $T : \mathcal{V} \to \mathcal{W}$ is called $\mathcal{A}$-*local* if

$$\mathrm{supp}_{\mathcal{A}}(Tv) \subseteq \mathrm{supp}_{\mathcal{A}}(v) \quad \text{for any } v \text{ in } \mathcal{V}.$$

We say that $\mathcal{A}$ is (*operator*) *hyper-Tauberian* if every (completely) bounded local operator $T : \mathcal{A} \to \mathcal{A}^*$ is an $\mathcal{A}$-module map.

Now suppose that the semisimple abelian (completely contractive) Banach algebra $\mathcal{A}$ has an abstract (operator) Segal algebra $\mathrm{S}\mathcal{A}$. By [4, Theo. 2.1], $\mathrm{S}\mathcal{A}$ is also semisimple with Gelfand spectrum X.

Theorem 4.4. *If $\mathcal{A}$ is* (*operator*) *hyper-Tauberian and* $\mathrm{S}\mathcal{A}$ *is an essential $\mathcal{A}$-module*, *i.e.*, $\mathcal{A} \cdot \mathrm{S}\mathcal{A}$ *is dense in* $\mathrm{S}\mathcal{A}$, *then* $\mathrm{S}\mathcal{A}$ *is* (*operator*) *hyper-Tauberian.*

Note that the converse follows from [26, Theo. 4.6] and (OSA2).

Proof. It is immediate that for v in $\mathrm{S}\mathcal{A}$,

$$\mathrm{supp}_{\mathcal{A}}(v) = \mathrm{supp}(v) = \mathrm{supp}_{\mathrm{S}\mathcal{A}}(v). \tag{4.1}$$

Also, since $\mathrm{S}\mathcal{A}$ is a subalgebra of $\mathcal{A}$, we see that if $f \in \mathrm{S}\mathcal{A}^*$ then

$$\mathrm{supp}_{\mathcal{A}}(f) \subseteq \mathrm{supp}_{\mathrm{S}\mathcal{A}}(f). \tag{4.2}$$

Now suppose $T : \mathrm{S}\mathcal{A} \to \mathrm{S}\mathcal{A}^*$ is a (completely) bounded $\mathrm{S}\mathcal{A}$-local operator. Then we have for u in $\mathrm{S}\mathcal{A}$, combining (4.2) and (4.1),

$$\mathrm{supp}_{\mathcal{A}}(Tu) \subseteq \mathrm{supp}_{\mathrm{S}\mathcal{A}}(Tu) \subseteq \mathrm{supp}(u) = \mathrm{supp}_{\mathcal{A}}(u).$$

Hence T is also an $\mathcal{A}$-local operator. Hence by [26, Prop. 2.3], T is an $\mathcal{A}$-module map. Thus it is an $\mathrm{S}\mathcal{A}$-module map. ∎

One of the main motivations for studying hyper-Tauberian algebras is the result [26, Theo. 3.2]: *if $\mathcal{A}$ is* (*operator*) *hyper-Tauberian*, *then it is* (*operator*) *weakly amenable.* We recall that $\mathcal{A}$ is (*operator*) *weakly amenable* if every (completely) bounded derivation $D : \mathcal{A} \to \mathcal{A}^*$ is inner [2]. We also recall that the subalgebra $\mathrm{A_c}(G)$ of the Fourier algebra of compactly supported elements lies within $\mathrm{S}^1\mathrm{A}(G)$, and is exactly the subalgebra of compactly supported elements there.

Corollary 4.5.

(i) $\mathrm{S}^1\mathrm{A}(G)$ *is always operator weakly amenable.*
(ii) $\mathrm{S}^1\mathrm{A}(G)$ *is weakly amenable if the connected component G_e of G is abelian.*

Proof. In [26, Theo. 7.4] it is shown that $\mathrm{A}(G)$ is always operator hyper-Tauberian. Thus (i) follows from [26, Theo. 3.2], whose statement was mentioned above, the preceding theorem, and Corollary 2.4. In [26, Theo. 6.5] it is shown that $\mathrm{A}(G)$ is hyper-Tauberian if G_e is abelian. Hence (ii) follows, similarly to (i) above. ∎

We note that weak amenability of $\mathrm{A}(G)$ is discussed in [17] and [10], and operator weak amenability in [11], [27] and [25].

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