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Common zero sets of equivalent singular inner functions II

by

KEIJI IZUCHI (Niigata)

Abstract. We study connected components of a common zero set of equivalent singular inner functions in the maximal ideal space of the Banach algebra of bounded analytic functions on the open unit disk. To study topological properties of zero sets of inner functions, we give a new type of factorization theorem for inner functions.

1. Introduction. Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disk D with the supremum norm. We denote by $M(H^\infty)$ the maximal ideal space of H^∞, the space of non-zero multiplicative linear functionals on H^∞ with the weak*-topology. We identify a function in H^∞ with its Gelfand transform. We may think of D as an open subset of $M(H^\infty)$. By the well known corona theorem due to Carleson [2], D is dense in $M(H^\infty)$, so a function f in H^∞ defined on D can be extended continuously and uniquely onto $M(H^\infty)$. Also we identify $f \in H^\infty$ with its radial limit $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ for almost all points $e^{i\theta} \in \partial D$, so we may think of H^∞ as a closed subalgebra of $L^\infty(\partial D)$. We write

$$Z(f) = \{x \in M(H^\infty) : f(x) = 0\}$$

for the zero set of f in $M(H^\infty)$. Also for $r > 0$, we write

$$\{|f| < r\} = \{x \in M(H^\infty) : |f(x)| < r\}.$$

For a subset E of $M(H^\infty)$, we denote by $\overline{E}$ the closure of E in $M(H^\infty)$. A function $\varphi \in H^\infty$ is called *inner* if $|\varphi^*(e^{i\theta})| = 1$ a.e. on ∂D (see [11]).

We denote by C the space of continuous functions on the unit circle ∂D. Sarason's theorem tells us that $H^\infty + C$ is a closed subalgebra of $L^\infty(\partial D)$ and $M(H^\infty + C) = M(H^\infty) \setminus D$ (see [18]).

Let M_s^+ be the set of bounded positive singular Borel measures on ∂D with respect to the Lebesgue measure on ∂D. We use familiar measure-

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theoretic notations; for $\mu, \nu \in M_{\mathrm{s}}^{+}$, $\mu \ll \nu$ (absolutely continuous), $\mu \perp \nu$ (mutually singular), $\mu \sim \nu$ (equivalent, i.e., $\mu \ll \nu$ and $\nu \ll \mu$), $\mu \wedge \lambda$ (the lower bound), and $\delta_{e^{i\theta}}$ (the unit point mass at $e^{i\theta} \in \partial D$). We denote by $\operatorname{supp}(\mu)$ the closed support set of μ. For each $\mu \in M_{\mathrm{s}}^{+}$, we have the singular inner function ψ_μ defined by

$$\psi_\mu(z) = \exp\biggl(-\int_{\partial D} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(e^{i\theta})\biggr), \qquad z \in D.$$

Division in H^∞ is well understood. Axler, Gorkin, Guillory, Mortini, Sarason, and the author [1, 5, 7, 9, 10, 17] studied division in $H^\infty + C$. Division involving Blaschke products has been well-studied. In this paper, we are motivated by the study of division involving singular inner functions. The following problem of Guillory and Sarason remains unsolved: Are there singular inner functions ψ_μ and ψ_ν with $\mu \perp \nu$ which are codivisible in $H^\infty + C$, that is, $\psi_\mu/\psi_\nu \in H^\infty + C$ and $\psi_\nu/\psi_\mu \in H^\infty + C$? This is an interesting problem that will reveal information about the structure of $M(H^\infty)$ and singular inner functions. The author believes that the answer is negative, but in order to make progress on this question it is necessary to understand the boundary behavior of singular inner functions. The author has studied this problem from this point of view in [15] and [16]. The goal of this paper is to better understand the zero sets of singular inner functions.

We denote by

$$E(\mu) = \{\nu \in M_{\mathrm{s}}^{+} : \nu \sim \mu\}$$

the set of measures equivalent to μ. The singular inner functions ψ_μ and ψ_ν are called *equivalent* if $\nu \in E(\mu)$. So, for each $\mu \in M_{\mathrm{s}}^{+}$, we have a family of equivalent singular inner functions $\{\psi_\nu : \nu \in E(\mu)\}$. In [16], the author considered the common zero set of $\{\psi_\nu : \nu \in E(\mu)\}$,

$$\mathcal{Z}(\mu) = \bigcap_{\nu \in E(\mu)} Z(\psi_\nu), \tag{1.1}$$

and proved that $\mathcal{Z}(\mu) \neq \emptyset$ and $\mathcal{Z}(\mu) \cap \mathcal{Z}(\nu) = \emptyset$ if $\mu \perp \nu$. To understand the boundary behavior of ψ_ν, $\nu \in E(\mu)$, it is important to know various properties of $\mathcal{Z}(\mu)$ in $M(H^\infty)$.

In this paper, we study the topological properties of $\mathcal{Z}(\mu)$. In [16], we have defined a closed subset $\Phi_\mu(x)$ of $\mathcal{Z}(\mu)$ for every point x in $M(L^\infty(\mu))$, the maximal ideal space of $L^\infty(\mu)$. In Section 3, we prove that $\Phi_\mu(x)$ is a connected set and $\{\Phi_\mu(x) : x \in M(L^\infty(\mu))\}$ is the family of connected components of $\mathcal{Z}(\mu)$. This answers a problem posed in [16, p. 253]. To show these facts, we need a new factorization theorem. In Section 2, we show that for an inner function φ, if U and V are non-empty open and closed subsets of $Z(\varphi)$ with $Z(\varphi) = U \cup V$ and $U \cap V = \emptyset$, then there is a factorization

$\varphi = \varphi_1\varphi_2$ such that $Z(\varphi_1) = U$ and $Z(\varphi_2) = V$ for some inner functions φ_1 and φ_2. This is interesting enough in its own right, and it can be applied to study the topological properties of $Z(\varphi)$ for inner functions φ.

2. Factorization of inner functions. There are several factorization theorems for Blaschke products (see [8, 12, 14]). The following factorization theorem for inner functions is of a new type.

THEOREM 2.1. *Let φ be an inner function. Suppose that U_1 and U_2 are non-empty open and closed subsets of $Z(\varphi)$ satisfying $Z(\varphi) = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Then there exists a factorization $\varphi = \varphi_1\varphi_2$, where φ_1 and φ_2 are inner with $Z(\varphi_j) = U_j$ for $j = 1, 2$.*

Proof. Let $\widetilde{U}_1$ and $\widetilde{U}_2$ be open subsets of $M(H^\infty)$ such that $U_j \subset \widetilde{U}_j$ and $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$. There exists r, $0 < r < 1$, satisfying $\{|\varphi| < r\} \subset \widetilde{U}_1 \cup \widetilde{U}_2$. Let

$$W_j = \{|\varphi| < r\} \cap \widetilde{U}_j \cap D.$$

By the corona theorem, $W_j \neq \emptyset$ and

$$|\varphi| = r \quad \text{on } \partial W_j \cap D \tag{2.1}$$

for $j = 1, 2$. By Frostman's theorem [4, p. 79], there is a sequence $\{\alpha_n\}_n$ of complex numbers with $0 < |\alpha_n| \leq r/3$ such that $\alpha_n \to 0$ and

$$B_n(z) := \frac{\varphi(z) - \alpha_n}{1 - \overline{\alpha}_n\varphi(z)} \tag{2.2}$$

is a Blaschke product for every n. The zeros in D of B_n are contained in $W_1 \cup W_2$. Let $B_{n,1}$ and $B_{n,2}$ be the Blaschke products with zeros of B_n in W_1 and W_2, respectively. Note that $B_n = B_{n,1}B_{n,2}$. By (2.1) and (2.2),

$$|B_n| \geq \frac{r - |\alpha_n|}{1 + |\alpha_n|r} \geq \frac{r}{2} \quad \text{on } (\partial W_1 \cap D) \cup (\partial W_2 \cap D).$$

Since $|B_{n,j}| \geq |B_n|$, we have

$$|B_{n,1}| \geq r/2 \quad \text{on } \partial W_2 \cap D \tag{2.3}$$

and

$$|B_{n,2}| \geq r/2 \quad \text{on } \partial W_1 \cap D. \tag{2.4}$$

We shall prove that

$$|B_{n,1}| \geq r/2 \quad \text{on } W_2 \quad \text{and} \quad |B_{n,2}| \geq r/2 \quad \text{on } W_1. \tag{2.5}$$

We only prove the first statement. Indeed, assume that $|B_{n,1}(z_n)| < r/2$ for some $z_n \in W_2$. Write $B_{n,1,k}$ for the kth partial product of $B_{n,1}$. Since $B_{n,1,k} \to B_{n,1}$ uniformly on each compact subset of D as $k \to \infty$, there exists a positive integer k satisfying $|B_{n,1,k}(z_n)| < r/2$. Since $B_{n,1,k}$ is a finite Blaschke product, $B_{n,1,k}$ is a continuous function on the closed unit disk and $|B_{n,1,k}| = 1$ on ∂D. Since $B_{n,1,k}$ has no zeros in W_2, there exists

$w_n \in \partial W_2 \cap D$ such that $|B_{n,1,k}(w_n)| < r/2$. Therefore $|B_{n,1}(w_n)| < r/2$. This contradicts (2.3). Thus we get (2.5).

Since $\{B_{n,j}\}_n$ is a normal family, there are subsequences $\{B_{n_i,j}\}_i$ of $\{B_{n,j}\}_n$, $j = 1, 2$, such that $B_{n_i,1} \to \varphi_1$ and $B_{n_i,2} \to \varphi_2$ uniformly on each compact subset of D as $i \to \infty$. Then $\varphi_j \in H^\infty$ and $\|\varphi_j\|_\infty \le 1$. By (2.2), $\|B_n - \varphi\|_\infty \to 0$ as $n \to \infty$. Since $B_{n_i} = B_{n_i,1}B_{n_i,2}$, we get $\varphi = \varphi_1\varphi_2$. One easily sees that both φ_1 and φ_2 are inner functions. By (2.5), $|\varphi_1| \ge r/2$ on W_2 and $|\varphi_2| \ge r/2$ on W_1. By the corona theorem, $U_j \subset \overline{W}_j$, so that $|\varphi_1| \ge r/2$ on U_2 and $|\varphi_2| \ge r/2$ on U_1. Since

$$Z(\varphi_1) \cup Z(\varphi_2) = Z(\varphi) = U_1 \cup U_2,$$

we get $Z(\varphi_j) = U_j$ for $j = 1, 2$. ∎

Corollary 2.2. *Let φ be an inner function. Suppose that U_1 and U_2 are non-empty open and closed subsets of $Z(\varphi) \setminus D$ satisfying $Z(\varphi) \setminus D = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Then there exists a factorization $\varphi = \varphi_1\varphi_2$, where φ_1 and φ_2 are inner with $Z(\varphi_j) \setminus D = U_j$ for $j = 1, 2$.*

Proof. Let $\widetilde{U}_1$ and $\widetilde{U}_2$ be open subsets of $M(H^\infty)$ such that $U_j \subset \widetilde{U}_j$ and $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$. If φ has a Blaschke factor, discarding some finite Blaschke factor from φ, we may assume that $Z(\varphi) \subset \widetilde{U}_1 \cup \widetilde{U}_2$. By Theorem 2.1, there is a factorization $\varphi = \varphi_1\varphi_2$ such that $Z(\varphi_j) = Z(\varphi) \cap \widetilde{U}_j$ for $j = 1, 2$. Hence $Z(\varphi_j) \setminus D = U_j$ for $j = 1, 2$. ∎

It is well known that $Z(\psi_{\delta_{e^{i\theta}}})$ is a connected set. The next two corollaries show that, in general, if an inner function is discontinuous at more than one point of the unit circle, then the zero set is disconnected.

Corollary 2.3. *Let $\mu \in M_{\mathrm{s}}^+$. Then $Z(\psi_\mu)$ is a connected set if and only if* $\mathrm{supp}(\mu)$ *is a one-point set.*

Proof. Suppose that $\mu = a\delta_{e^{i\theta}}$ with $a > 0$. By Theorem 2.1, one easily sees that $Z(\psi_\mu) = Z(\psi_{\delta_{e^{i\theta}}})$ is a connected set.

Next, suppose that $\mathrm{supp}(\mu)$ is not a one-point set. Since $|\psi_\mu^*| = 1$ a.e. on ∂D, there exist two points $e^{i\theta_1}, e^{i\theta_2} \in \partial D$ with $0 < \theta_1 < \theta_2 < 2\pi$ such that

$$\lim_{r\to 1} |\psi_\mu(re^{i\theta_k})| = 1 \tag{2.6}$$

and $\mu(J_k) \ne 0$ for $k = 1, 2$, where $J_1 = \{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ and $J_2 = \partial D \setminus \overline{J}_1$. By (2.6), we have $\mu(\{e^{i\theta_1}, e^{i\theta_2}\}) = 0$. Write $\mu_k = \mu|_{J_k}$ for $k = 1, 2$. Then $\mu = \mu_1 + \mu_2$, so $\psi_\mu = \psi_{\mu_1}\psi_{\mu_2}$. Since $\mu_k \ne 0$, $Z(\psi_{\mu_k}) \ne 0$ for $k = 1, 2$. Since $Z(\psi_\mu) = Z(\psi_{\mu_1}) \cup Z(\psi_{\mu_2})$, to prove that $Z(\psi_\mu)$ is not connected it is sufficient to prove that $Z(\psi_{\mu_1}) \cap Z(\psi_{\mu_2}) = \emptyset$. To prove this, let

$$D_k = \{z \in D : z = re^{i\theta},\ r > 0,\ e^{i\theta} \in J_k\}, \qquad k = 1, 2.$$

Since $\mu_2(J_1) = 0$, $|\psi^*_{\mu_2}(e^{i\theta})| = 1$ for all $e^{i\theta} \in J_1$. For $z \in D_1$, we have

$$\begin{aligned}|\psi_{\mu_2}(z)| &= \exp\Big(-\int_{\partial D} P_z(e^{i\theta})\, d\mu_2(e^{i\theta})\Big)\\ &\geq \min_{k=1,2} \exp\Big(-\int_{\partial D} P_{|z|e^{i\theta_k}}(e^{i\theta})\, d\mu_2(e^{i\theta})\Big)\\ &= \min_{k=1,2} |\psi_{\mu_2}(|z|e^{i\theta_k})| \geq \min_{k=1,2} |\psi_{\mu}(|z|e^{i\theta_k})|\\ &\to 1 \quad \text{as } |z| \to 1 \text{ by (2.6)},\end{aligned}$$

where

$$P_z(e^{i\theta}) = \frac{1-|z|^2}{|e^{i\theta}-z|^2}, \qquad z \in D,$$

is the Poisson kernel for a point $z \in D$. Hence $|\psi_{\mu_2}| > 0$ on $\overline{D}_1$, so

$$Z(\psi_{\mu_2}) \subset M(H^\infty) \setminus \overline{D}_1.$$

Similarly, we have

$$Z(\psi_{\mu_1}) \subset M(H^\infty) \setminus \overline{D}_2.$$

Since $M(H^\infty) = \overline{D}_1 \cup \overline{D}_2$ by the corona theorem, we get $Z(\psi_{\mu_1}) \cap Z(\psi_{\mu_2}) = \emptyset$. Thus $Z(\psi_\mu)$ is disconnected. ∎

For a discontinuous inner function φ, we denote by $\mathrm{Sing}(\varphi)$ the set of $e^{i\theta} \in \partial D$ at which $\varphi(z)$ does not have a continuous extension. As in the proof of Corollary 2.3, we can prove the following which is of interest when φ has a Blaschke factor.

COROLLARY 2.4. *Let φ be a discontinuous inner function such that* $\mathrm{Sing}(\varphi)$ *has more than one point. Then $Z(\varphi) \setminus D$ is disconnected.*

Proof. By our assumption, there are two points $e^{i\theta_1}, e^{i\theta_2} \in \partial D$ with $0 < \theta_1 < \theta_2 < 2\pi$ such that

$$\lim_{r\to 1} |\varphi(re^{i\theta_k})| = 1 \tag{2.7}$$

and $\mathrm{Sing}(\varphi) \cap J_k \neq \emptyset$ for $k = 1, 2$, where $J_1 = \{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ and $J_2 = \partial D \setminus \overline{J}_1$. By (2.7), there is a number R with $0 < R < 1$ such that

$$\min_{k=1,2} \min_{R\leq r<1} |\varphi(re^{i\theta_k})| > \delta > 0 \tag{2.8}$$

for some $\delta > 0$. Moreover we may assume that

$$\min_{\theta_1\leq\theta\leq\theta_2} |\varphi(Re^{i\theta})| > \delta > 0. \tag{2.9}$$

Let

$$D_1 = \{z \in D : z = re^{i\theta},\ R < r < 1,\ e^{i\theta} \in J_1\}$$

and $D_2 = D \setminus D_1$. Write $\varphi(z) = B(z)\psi_\mu(z)$, where B and ψ_μ are a Blaschke factor and a singular inner factor of $\varphi(z)$, respectively. By (2.7), we have

$\mu(\{e^{i\theta_1}, e^{i\theta_2}\}) = 0$. Write $\mu_k = \mu|_{J_k}$ for $k = 1, 2$. Then $\mu = \mu_1 + \mu_2$. Let B_1 and B_2 be Blaschke factors of B with zeros in D_1 and D_2, respectively. Then $B = B_1 B_2$. By the proof of Corollary 2.3, (2.8), and (2.9), we have $|\psi_{\mu_2}(z)| \geq \delta$ for every $z \in D_1$. Also by the proof of Theorem 2.1, (2.8), and (2.9), $|B_2(z)| \geq \delta$ for every $z \in D_1$. Hence $|(B_2\psi_{\mu_2})(z)| \geq \delta^2 > 0$ for every $z \in D_1$. Therefore we get $|B_2\psi_{\mu_2}| \geq \delta^2$ on $\overline{D}_1$. Similarly, $|B_1\psi_{\mu_1}| \geq \delta^2$ on $\overline{D}_2$. Therefore we get

$$Z(B_2\psi_{\mu_2}) \subset M(H^\infty) \setminus \overline{D}_1 \quad \text{and} \quad Z(B_1\psi_{\mu_1}) \subset M(H^\infty) \setminus \overline{D}_2.$$

Since $\overline{D}_1 \cup \overline{D}_2 = M(H^\infty)$, $Z(B_1\psi_{\mu_1}) \cap Z(B_2\psi_{\mu_2}) = \emptyset$. As $\varphi = B_1\psi_{\mu_1}B_2\psi_{\mu_2}$,

$$Z(\varphi) \setminus D = (Z(B_1\psi_{\mu_1}) \setminus D) \cup (Z(B_2\psi_{\mu_2}) \setminus D).$$

Since $\operatorname{Sing}(\varphi) \cap J_k \neq \emptyset$, $Z(B_k\psi_{\mu_k}) \setminus D \neq \emptyset$ for $k = 1, 2$. Thus $Z(\varphi) \setminus D$ is disconnected. ∎

3. Connected components of common zero sets. For $\mu \in M_s^+$, we denote by $M(L^\infty(\mu))$ the maximal ideal space of the Banach algebra $L^\infty(\mu)$. It is a totally disconnected compact Hausdorff space. For $f \in L^\infty(\mu)$, we denote by $\widehat{f}$ the Gelfand transform of f. For a measurable subset S of $\operatorname{supp}(\mu)$, we have $\widehat{\chi}_S^2 = \widehat{\chi}_S$ on $M(L^\infty(\mu))$, where χ_S is the characteristic function for S, so there exists an open and closed subset $\widehat{S}$ of $M(L^\infty(\mu))$ with $\widehat{\chi}_S = \chi_{\widehat{S}}$. The family $\{\chi_{\widehat{S}}\}_S$ coincides with the set of idempotents in $C(M(L^\infty(\mu)))$, the space of continuous functions on $M(L^\infty(\mu))$. We have $\widehat{S^c} = (\widehat{S})^c$, where $S^c = \operatorname{supp}(\mu) \setminus S$ and $(\widehat{S})^c = M(L^\infty(\mu)) \setminus \widehat{S}$. See [3, pp. 17–18].

For each point $x \in M(L^\infty(\mu))$, write

$$\Phi_\mu(x) = \bigcap_{\{S\,:\,x\in\widehat{S}\}} \mathcal{Z}(\mu|_S). \tag{3.1}$$

In [16, Theorem 6.3], the author proved the following.

Theorem A.

(i) $\emptyset \neq \Phi_\mu(x) \subset \mathcal{Z}(\mu)$ *for every* $x \in M(L^\infty(\mu))$.
(ii) $\Phi_\mu(x) \cap \Phi_\mu(y) = \emptyset$ *if* $x, y \in M(L^\infty(\mu))$ *and* $x \neq y$.
(iii) $\mathcal{Z}(\mu) = \bigcup_{x\in M(L^\infty(\mu))} \Phi_\mu(x)$.

By this theorem, the family

$$\{\Phi_\mu(x) : x \in M(L^\infty(\mu))\}$$

is like an atomic decomposition of $\mathcal{Z}(\mu)$, and we may consider $\mathcal{Z}(\mu)$ as a shadow of the measure μ on ∂D in the maximal ideal space of H^∞.

The following is proved in [16, Theorem 2.2].

LEMMA 3.1. *Let* $\mu, \nu \in M_s^+$. *If* $\mu \perp \nu$, *then* $\mathcal{Z}(\mu) \cap \mathcal{Z}(\nu) = \emptyset$.

The following is proved in [16, Lemma 3.3].

LEMMA 3.2. *Let* $\mu, \nu \in M_s^+$.

(i) *If* $\mu \ll \nu$, *then* $\mathcal{Z}(\mu) \subset \mathcal{Z}(\nu)$.
(ii) *If* $\mu \wedge \nu \neq 0$, *then* $\mathcal{Z}(\mu \wedge \nu) = \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)$.
(iii) $\mathcal{Z}(\mu + \nu) = \mathcal{Z}(\mu) \cup \mathcal{Z}(\nu)$.

We now state our main theorem, which answers a problem posed in [16, Problem 6.4]. To prove it, we use Theorem 2.1.

THEOREM 3.3. *Let* $\mu \in M_s^+$. *Then*:

(i) *For each* $x \in M(L^\infty(\mu))$, $\Phi_\mu(x)$ *is a connected set.*
(ii) $\{\Phi_\mu(x) : x \in M(L^\infty(\mu))\}$ *is the family of connected components of* $\mathcal{Z}(\mu)$.

Proof. (i) Suppose that there exist non-empty open and closed subsets U_1 and U_2 of $\Phi_\mu(x)$ such that

$$\Phi_\mu(x) = U_1 \cup U_2 \quad \text{and} \quad U_1 \cap U_2 = \emptyset.$$

Then there are open subsets $\widetilde{U}_1$ and $\widetilde{U}_2$ of $M(H^\infty)$ such that $U_j \subset \widetilde{U}_j$ for $j = 1, 2$ and $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$. Since $M(H^\infty) \setminus (\widetilde{U}_1 \cup \widetilde{U}_2)$ is compact, by the definition of $\Phi_\mu(x)$, there are measurable subsets $S_1, \ldots, S_n$ of $\operatorname{supp}(\mu)$ such that

$$x \in \bigcap_{j=1}^n \widehat{S}_j = \Big(\bigcap_{j=1}^n S_j\Big)^\wedge \quad \text{and} \quad \bigcap_{j=1}^n \mathcal{Z}(\mu|_{S_j}) \subset \widetilde{U}_1 \cup \widetilde{U}_2.$$

Write

$$S_0 = \bigcap_{j=1}^n S_j.$$

By (3.1) and Lemma 3.2(ii),

$$\Phi_\mu(x) \subset \mathcal{Z}(\mu|_{S_0}) = \bigcap_{j=1}^n \mathcal{Z}(\mu|_{S_j}) \subset \widetilde{U}_1 \cup \widetilde{U}_2.$$

By the definition of $\mathcal{Z}(\mu|_{S_j})$, there exist $\nu_1, \ldots, \nu_k$ in M_s^+ such that $\nu_j \sim \mu|_{S_j}$ and

$$\Phi_\mu(x) \subset \mathcal{Z}(\mu|_{S_0}) \subset \bigcap_{j=1}^n Z(\psi_{\nu_j}) \subset \widetilde{U}_1 \cup \widetilde{U}_2.$$

Let

$$\nu = \bigwedge_{j=1}^k \nu_j.$$

Then $\nu \sim \mu|_{S_0}$ and

$$\Phi_\mu(x) \subset \mathcal{Z}(\mu|_{S_0}) \subset Z(\psi_\nu) \subset \bigcap_{j=1}^{n} Z(\psi_{\nu_j}) \subset \widetilde{U}_1 \cup \widetilde{U}_2.$$

Since $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$, by Theorem 2.1 we have a decomposition $\nu = \sigma_1 + \sigma_2$ such that

$$Z(\psi_{\sigma_1}) = Z(\psi_\nu) \cap \widetilde{U}_1 \quad \text{and} \quad Z(\psi_{\sigma_2}) = Z(\psi_\nu) \cap \widetilde{U}_2.$$

Note that $\sigma_1 \perp \sigma_2$. Then there exist measurable subsets R_1 and R_2 of S_0 such that $S_0 = R_1 \cup R_2$, $R_1 \cap R_2 = \emptyset$, $\sigma_1 = \nu|_{R_1}$, and $\sigma_2 = \nu|_{R_2}$. Therefore $\sigma_1 \sim \mu|_{R_1}$ and $\sigma_2 \sim \mu|_{R_2}$. Since $x \in \widehat{S}_0 = \widehat{R}_1 \cup \widehat{R}_2$ and $\widehat{R}_1 \cap \widehat{R}_2 = \emptyset$, either $x \in \widehat{R}_1$ or $x \in \widehat{R}_2$. If $x \in \widehat{R}_1$, then

$$\Phi_\mu(x) \subset \mathcal{Z}(\mu|_{R_1}) = \mathcal{Z}(\sigma_1) \subset Z(\psi_{\sigma_1}) = Z(\psi_\nu) \cap \widetilde{U}_1.$$

But

$$\emptyset \neq U_2 = \Phi_\mu(x) \cap U_2 \subset Z(\psi_\nu) \cap \widetilde{U}_1 \cap U_2 = \emptyset.$$

This is the desired contradiction. Thus we get (i).

(ii) Let S be a measurable subset of $\operatorname{supp}(\mu)$ with $0 < \mu(S) < \mu(\partial D)$. Since $\mu|_S \perp \mu|_{S^c}$, by Lemma 3.1 we have $\mathcal{Z}(\mu|_S) \cap \mathcal{Z}(\mu|_{S^c}) = \emptyset$. By Lemma 3.2(iii),

$$\mathcal{Z}(\mu) = \mathcal{Z}(\mu|_S) \cup \mathcal{Z}(\mu|_{S^c}).$$

Hence $\mathcal{Z}(\mu|_S)$ is a non-empty open and closed subset of $\mathcal{Z}(\mu)$. Therefore by (3.1), $\Phi_\mu(x)$ is a union of connected components of $\mathcal{Z}(\mu)$. By (i), it is a single connected component. ∎

Remark. If we think of each set $\Phi_\mu(x)$ as a one-point set $\widetilde{\Phi}_\mu(x)$ and if we equip $\widetilde{\mathcal{Z}}(\mu) := \{\widetilde{\Phi}_\mu(x) : x \in M(L^\infty(\mu))\}$ with the quotient topology of $\mathcal{Z}(\mu)$, then

$$\widetilde{\Phi}_\mu : M(L^\infty(\mu)) \ni x \mapsto \widetilde{\Phi}_\mu(x) \in \widetilde{\mathcal{Z}}(\mu)$$

is a homeomorphism.

We give some additional properties of $\Phi_\mu(x)$.

Proposition 3.4. *Let $\mu, \nu \in M_{\mathrm{s}}^+$, $x \in M(L^\infty(\mu))$, and $y \in M(L^\infty(\nu))$. Then either $\Phi_\mu(x) = \Phi_\nu(y)$ or $\Phi_\mu(x) \cap \Phi_\nu(y) = \emptyset$.*

Proof. Suppose that $\Phi_\mu(x) \cap \Phi_\nu(y) \neq \emptyset$. We write $\mu + \nu = \mu_1 + \mu_2 + \mu_3$, where $\{\mu_1, \mu_2, \mu_3\}$ is a set of mutually singular measures with $\mu \sim \mu_1 + \mu_2$ and $\nu \sim \mu_2 + \mu_3$. We may write

$$\begin{aligned} M(L^\infty(\mu)) &= M(L^\infty(\mu_1)) \uplus M(L^\infty(\mu_2)), \\ M(L^\infty(\nu)) &= M(L^\infty(\mu_2)) \uplus M(L^\infty(\mu_3)), \\ M(L^\infty(\mu+\nu)) &= M(L^\infty(\mu_1)) \uplus M(L^\infty(\mu_2)) \uplus M(L^\infty(\mu_3)), \end{aligned}$$

where $\uplus$ denotes disjoint union. Now, we have $x, y \in M(L^\infty(\mu+\nu))$, $\Phi_\mu(x) = \Phi_{\mu+\nu}(x)$, and $\Phi_\nu(y) = \Phi_{\mu+\nu}(y)$. Hence by Theorem A, we get the assertion. ∎

Let

$$QC = (H^\infty + C) \cap \overline{H^\infty + C} \subset L^\infty(\partial D),$$

where $\overline{H^\infty + C}$ is the set of complex conjugates of functions in $H^\infty + C$. For each point $\zeta \in M(H^\infty) \setminus D$, let

$$Q(\zeta) = \{\xi \in M(H^\infty) \setminus D : f(\xi) = f(\zeta) \text{ for every } f \in QC\}.$$

The set $Q(\zeta)$ is called the *QC-level set* containing ζ (see [18, 19]). Generally, for a subset E of $M(H^\infty)$, $\bigcup_{\zeta \in E} Q(\zeta)$ is fairly bigger than E. Here we have the following.

PROPOSITION 3.5. *If $\mu \in M_s^+$ and $x, y \in M(L^\infty(\mu))$ with $x \neq y$, then*

$$\overline{\bigcup_{\zeta \in \Phi_\mu(x)} Q(\zeta)} \cap \overline{\bigcup_{\zeta \in \Phi_\mu(y)} Q(\zeta)} = \emptyset.$$

Proof. By Theorem A, $\Phi_\mu(x) \cap \Phi_\mu(y) = \emptyset$. There exist measurable subsets S_1 and S_2 of $\operatorname{supp}(\mu)$ such that $x \in \widehat{S}_1$, $y \in \widehat{S}_2$, and $S_1 \cap S_2 = \emptyset$. By (3.1),

$$\Phi_\mu(x) \subset \mathcal{Z}(\mu|_{S_1}) \quad \text{and} \quad \Phi_\mu(y) \subset \mathcal{Z}(\mu|_{S_2}).$$

Since $\mu|_{S_1} \perp \mu|_{S_2}$, by [16, Theorem 2.1] there exist $\nu_1, \nu_2 \in M_s^+$ such that $\nu_1 \sim \mu|_{S_1}$, $\nu_2 \sim \mu|_{S_2}$, and

$$\{\zeta \in M(H^\infty) \setminus D : |\psi_{\nu_1}(\zeta)| < 1\} \cap \{\zeta \in M(H^\infty) \setminus D : |\psi_{\nu_2}(\zeta)| < 1\} = \emptyset.$$

We have

$$\begin{aligned} \Phi_\mu(x) &\subset \mathcal{Z}(\mu|_{S_1}) = \mathcal{Z}(\nu_1) \subset \{\zeta \in M(H^\infty) \setminus D : |\psi_{\nu_1}(\zeta)| < 1\}, \\ \Phi_\mu(y) &\subset \mathcal{Z}(\mu|_{S_2}) = \mathcal{Z}(\nu_2) \subset \{\zeta \in M(H^\infty) \setminus D : |\psi_{\nu_2}(\zeta)| < 1\}. \end{aligned}$$

Therefore by [13, Corollary 3], we get the assertion. ∎

We may think of $M(L^\infty)$ as a closed subset of $M(H^\infty)$. For each $\zeta \in M(H^\infty)$, it is known that there exists a unique probability measure μ_ζ on $M(L^\infty)$ such that

$$f(\zeta) = \int_{M(L^\infty)} f \, d\mu_\zeta$$

for every $f \in H^\infty$ (see [4]). Note that $\operatorname{supp} \mu_\zeta \subset Q(\zeta)$ (see [18]). There are many studies of the representing measures μ_ζ; here we just mention [6].

COROLLARY 3.6. *Let $\mu \in M_s^+$ and $x, y \in M(L^\infty(\mu))$ with $x \neq y$. Then $\operatorname{supp} \mu_\zeta \cap \operatorname{supp} \mu_\xi = \emptyset$ for every $\zeta \in \Phi_\mu(x)$ and $\xi \in \Phi_\mu(y)$.*

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Department of Mathematics
Niigata University
Niigata 950-2181, Japan
E-mail: izuchi@math.sc.niigata-u.ac.jp

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