STUDIA MATHEMATICA 180 (3) (2007)

On generalized a-Browder's theorem

by

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Abstract. We characterize the bounded linear operators T satisfying generalized a-Browder's theorem, or generalized a-Weyl's theorem, by means of localized SVEP, as well as by means of the quasi-nilpotent part $H_0(\lambda I - T)$ as λ belongs to certain sets of $\mathbb{C}$. In the last part we give a general framework in which generalized a-Weyl's theorem follows for several classes of operators.

1. Preliminaries. Let $L(X)$ denote the space of bounded linear operators on an infinite-dimensional complex Banach space X. For $T \in L(X)$, denote by $\alpha(T)$ the dimension of the kernel $\ker T$, and by $\beta(T)$ the codimension of the range $T(X)$. The operator $T \in L(X)$ is called *upper semi-Fredholm* if $\alpha(T) < \infty$ and $T(X)$ is closed, and *lower semi-Fredholm* if $\beta(T) < \infty$. If T is either upper or lower semi-Fredholm then it is said to be *semi-Fredholm*; finally, T is a *Fredholm operator* if it is both upper and lower semi-Fredholm. If $T \in L(X)$ is semi-Fredholm, then its *index* is defined by $\operatorname{ind} T := \alpha(T) - \beta(T)$.

For every $T \in L(X)$ and a nonnegative integer n we shall denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from $T^n(X)$ into itself (we set $T_{[0]} = T$). Following Berkani ([6], [11] and [8]), $T \in L(X)$ is said to be *semi B-Fredholm* (resp., *B-Fredholm, upper semi B-Fredholm*, *lower semi B-Fredholm*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm (resp., Fredholm, upper semi-Fredholm, lower semi-Fredholm) operator. Note that in this case $T_{[m]}$ is semi-Fredholm for all $m \geq n$ ([11]). This enables one to define the index of a semi B-Fredholm operator as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$. The class of all upper semi B-Fredholm operators

2000 *Mathematics Subject Classification*: Primary 47A10, 47A11; Secondary 47A53, 47A55.

Key words and phrases: SVEP, Fredholm theory, generalized Weyl's theorem, generalized Browder's theorem.

Part of this work was prepared while the second author was a guest of the Department of Mathematics and the Department of Mathematics and Applications of the University of Palermo. He would like to express his gratitude to the University and to his coauthor for their hospitality.

will be denoted by $\mathrm{USBF}(X)$. A bounded operator is said to be *B-Weyl* if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a *Weyl operator*, i.e., $T_{[n]}$ is Fredholm with index 0.

This paper deals with two other classical quantities associated with an operator T. The *ascent* of T is defined as the smallest nonnegative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such an integer does not exist we put $p(T) = \infty$. Analogously, the *descent* of T is defined as the smallest nonnegative integer $q := q(T)$ such that $T^q(X) = T^{q+1}(X)$, and if such an integer does not exist we put $q(T) = \infty$. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ (see [1, Theorem 3.3]). Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent. In this case λ is an eigenvalue and an isolated point of the spectrum (see [21, Prop. 50.2]). The concept of Drazin invertibility [16] has been introduced in a more abstract setting than operator theory. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is *Drazin invertible* (with a finite index) precisely when $p(T) = q(T) < \infty$, and this is equivalent to saying that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent (see [25, Corollary 2.2] and [23, Prop. A]). The *Drazin spectrum* is defined as

$$\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

The concept of Drazin invertibility for bounded operators may be extended as follows.

Definition 1.1 ([10]). An operator $T \in L(X)$ is said to be *left Drazin invertible* if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. The *left Drazin spectrum* is then defined as

$$\sigma_{\mathrm{ld}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\}.$$

Recall that a bounded operator is said to be *bounded below* if it is injective and has closed range. Denote by $\sigma_{\mathrm{a}}(T)$ the classical *approximate point spectrum* of T,

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

Definition 1.2 ([10]). A point $\lambda \in \sigma_{\mathrm{a}}(T)$ is said to be a *left pole* if $\lambda I - T$ is left Drazin invertible.

The single-valued extension property was introduced by Dunford [17], [18] and has an important role in local spectral theory and Fredholm theory (see the recent monographs by Laursen and Neumann [24] and Aiena [1]).

Definition 1.3. An operator $T \in L(X)$ is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc U centered at λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every point of the resolvent set $\varrho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, the identity theorem for analytic functions entails that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has SVEP at every isolated point of the spectrum. Note that the SVEP is inherited by restrictions to closed invariant subspaces, i.e. if T has SVEP at λ_0 and M is a closed T-invariant subspace of X then $T|M$ has SVEP at λ_0. Moreover, if $T \in L(X)$ has SVEP at λ_0 and if $S \in L(Y)$ is similar to T, then S also has SVEP at λ_0.

We have

$$p(\lambda I - T) < \infty \;\Rightarrow\; T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \;\Rightarrow\; T^* \text{ has SVEP at } \lambda \tag{2}$$

(see [1, Theorem 3.8]). Furthermore, from the definition of localized SVEP it easily seen that

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \;\Rightarrow\; T \text{ has SVEP at } \lambda. \tag{3}$$

The *quasi-nilpotent part* of T is defined as the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

From the definition we see that $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$, so $\mathcal{N}^\infty(T) \subseteq H_0(T)$, where $\mathcal{N}^\infty(T) := \bigcup_{n=1}^\infty \ker(T^n)$ denotes the *hyper-kernel* of T. Moreover, T is quasi-nilpotent if and only if $H_0(T) = X$ (see [1, Theorem 1.68]), while if T is invertible then $H_0(T) = \{0\}$. Note that generally $H_0(T)$ is not closed and

$$H_0(\lambda I - T) \text{ closed} \;\Rightarrow\; T \text{ has SVEP at } \lambda. \tag{4}$$

Remark 1.4. All the implications (1)–(4) above become equivalences if we assume that $\lambda I - T$ is semi-Fredholm (see [1, Chapter 3, §2]).

The subspace $H_0(T)$ admits the following local spectral characterization. For an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, let $\mathcal{X}_T(F)$ denote the *glocal spectral subspace* consisting of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ such that $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. By [1, Theorem 2.20], it follows that $H_0(T) = \mathcal{X}_T(\{0\})$.

Definition 1.5. Let $T \in L(X)$ and $d \in \mathbb{N}$. Following Grabiner [19], T is said to have *uniform descent for* $n \geq d$ if $T(X) + \ker T^n = T(X) + \ker T^d$ for all $n \geq d$. If, in addition, $T(X) + \ker T^d$ is closed then T is said to have *topological uniform descent for* $n \geq d$.

Note that if either of the quantities $\alpha(T)$, $\beta(T)$, $p(T)$, $q(T)$ is finite then T has uniform descent. Define

$$\Delta(T) := \{n \in \mathbb{N} : T^n(X) \cap \ker T \subseteq T^m(X) \cap \ker T \text{ for all } m \geq n\}.$$

The *degree of stable iteration* is defined as $\operatorname{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\operatorname{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 1.6. An operator $T \in L(X)$ is said to be *quasi-Fredholm of degree d* if there exists $d \in \mathbb{N}$ such that:

(a) $\operatorname{dis}(T) = d$,
(b) $T^n(X)$ is a closed subspace of X for each $n \geq d$,
(c) $T(X) + \ker T^d$ is a closed subspace of X.

Let $\mathrm{QF}(d)$ denote the set of all quasi-Fredholm operators of degree d. If $T \in \mathrm{QF}(d)$ then T has topological uniform ascent for $n \geq d$ (see [19, Theorem 3.2]).

The following result is a particular case of Lemma 12 of [26].

Lemma 1.7. *If $T \in L(X)$ and $p = p(T) < \infty$ then the following statements are equivalent*:

(i) *there exists $n \geq p + 1$ such that $T^n(X)$ is closed*;

(ii) *$T^n(X)$ is closed for all $n \geq p$.*

Proof. Set $c'_i(T) := \dim(\ker T^{i+1}/\ker T^i)$. It is clear that $p = p(T) < \infty$ entails that $c'_i(T) = 0$ for all $i \geq p$, so $k_i(T) := c'_i(T) - c'_{i+1}(T) = 0$ for all $i \geq p$. The equivalence is then a consequence of Lemma 12 of [26]. ∎

Let $\mathrm{USF}^-(X)$ denote the class of all upper semi-Fredholm operators such that $\operatorname{ind} T \leq 0$, and $\mathrm{USBF}^-(X)$ the class of all upper semi B-Fredholm operators such that $\operatorname{ind} T \leq 0$.

The concepts of left Drazin invertibility and localized SVEP are related as follows:

Theorem 1.8. *For $T \in L(X)$ the following statements are equivalent*:

(i) *$T \in \mathrm{USBF}^-(X)$ and T has SVEP at* 0;
(ii) *$T \in \mathrm{QF}(d)$ for some d and T has SVEP at* 0;
(iii) *there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below*;
(iv) *$T \in \mathrm{USBF}^-(X)$ and $p(T) < \infty$*;
(v) *T is left Drazin invertible.*

Proof. (i)⇒(ii). By Proposition 2.5 of [11] every semi B-Fredholm operator is quasi-Fredholm.

(ii)⇒(iii). By Proposition 3.2 of [7] if $T \in \mathrm{QF}(d)$ then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is semi-regular. By assumption T has

SVEP at 0, so $T_{[n]}$ has SVEP at 0. From Theorem 2.49 of [1] it then follows that $T_{[n]}$ is bounded below.

(iii)$\Rightarrow$(iv). Suppose that $x \in \ker T^{n+1}$. Clearly, $T(T^n x) = 0$ so $T^n x \in \ker T$. As $T^n x \in T^n(X)$ it follows that $T^n x \in \ker T \cap T^n(X) = \ker T_{[n]} = \{0\}$, thus $x \in \ker T^n$. Therefore, $\ker T^{n+1} = \ker T^n$, so T has finite ascent. Since any operator bounded below is upper semi-Fredholm with index less than or equal to 0, it follows that $T \in \mathrm{USBF}^-(X)$.

(iv)$\Rightarrow$(i). Follows from implication (1).

(iii)$\Rightarrow$(v). Suppose that there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below. The same argument of the proof of (iii)$\Rightarrow$(iv) shows that $p := p(T) \le n$. The range of $T_{[n]}$ is the closed subspace $T^{n+1}(X)$, with $p+1 \le n+1$. Therefore $T^{p+1}(X)$ is closed, thus T is left Drazin invertible.

(v)$\Rightarrow$(iii). Suppose that T is left Drazin invertible. Then $p = p(T) < \infty$ and $T^{p+1}(X)$ is closed. From Lemma 1.7 it follows that $T^p(X)$ is closed. By [1, Lemma 3.2] we have $\ker T \cap T^p(X) = \ker T_{[p]} = \{0\}$, so $T_{[p]}$ is injective. The range of $T_{[p]}$ is closed, since it coincides with $T^{p+1}(X)$, hence $T_{[p]}$ is bounded below, so condition (iii) is satisfied. ∎

2. Generalized a-Browder's theorem. We shall denote by $\operatorname{acc} K$ and $\operatorname{iso} K$ the set of accumulation points and the set of isolated points of $K \subseteq \mathbb{C}$, respectively.

Define
$$\sigma_{\mathrm{usbf}^-}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathrm{USBF}^-(X)\}.$$

THEOREM 2.1. *If $T \in L(X)$ then*
$$\sigma_{\mathrm{usbf}^-}(T) \subseteq \sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{a}}(T). \tag{5}$$
More precisely,
$$\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{usbf}^-}(T) \cup \operatorname{acc} \sigma_{\mathrm{a}}(T). \tag{6}$$

Proof. The inclusion $\sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{a}}(T)$ is obvious: if $\lambda \notin \sigma_{\mathrm{a}}(T)$ then $p(\lambda I - T) = 0$ and $(\lambda I - T)(X)$ is closed, so $\lambda \notin \sigma_{\mathrm{ld}}(T)$. The inclusion $\sigma_{\mathrm{usbf}^-}(T) \subseteq \sigma_{\mathrm{ld}}(T)$ is clear from Theorem 1.8.

From the inclusions (5) in order to show that $\sigma_{\mathrm{usbf}^-}(T) \cup \operatorname{acc} \sigma_{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{ld}}(T)$ we only need to prove that $\operatorname{acc} \sigma_{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{ld}}(T)$. For this, let $\lambda_0 \notin \sigma_{\mathrm{ld}}(T)$. By Theorem 1.8 we know that $\lambda_0 I - T$ is quasi-Fredholm and hence has topological uniform descent. Since $p(\lambda_0 I - T) < \infty$ it then follows, from Corollary 4.8 of [19] that $\lambda I - T$ is bounded below in a punctured disc centered at λ_0, so $\lambda \notin \operatorname{acc} \sigma_{\mathrm{a}}(T)$.

To show the opposite inclusion, let $\lambda_0 \notin \sigma_{\mathrm{usbf}^-}(T) \cup \operatorname{acc} \sigma_{\mathrm{a}}(T)$. Since $\lambda_0 \notin \operatorname{acc} \sigma_{\mathrm{a}}(T)$, from the implication (3) we know that T has SVEP at λ_0. Moreover, from $\lambda_0 \notin \sigma_{\mathrm{usbf}^-}(T)$ we see that $\lambda_0 I - T \in \mathrm{USBF}^-(X)$ so, again by Theorem 1.8, $\lambda_0 I - T$ is left Drazin invertible and hence $\lambda_0 \notin \sigma_{\mathrm{ld}}(T)$. ∎

Denote by $\Pi^{\mathrm{a}}(T)$ the set of left poles of T. Clearly, $\Pi^{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$. Note that

$$\Pi^{\mathrm{a}}(T) \subseteq \operatorname{iso} \sigma_{\mathrm{a}}(T) \quad \text{for all } T \in L(X). \tag{7}$$

In fact, if $\lambda_0 \in \Pi^{\mathrm{a}}(T)$ then $\lambda_0 I - T$ is left Drazin invertible and hence, by Theorem 1.8, $\lambda_0 I - T \in \mathrm{QF}(d)$. This implies that $\lambda_0 I - T$ has topological uniform descent, and since $p(\lambda_0 I - T) < \infty$, it follows from Corollary 4.8 of [19] that $\lambda I - T$ is bounded below in a punctured disc centered at λ_0.

Define $\Delta^{\mathrm{a}}(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbf}^-}(T)$.

LEMMA 2.2. *If $T \in L(X)$ then*

$$\Delta^{\mathrm{a}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathrm{USBF}^-(X),\, 0 < \alpha(\lambda I - T)\}. \tag{8}$$

Furthermore, $\Pi^{\mathrm{a}}(T) \subseteq \Delta^{\mathrm{a}}(T)$.

Proof. The inclusion

$$\{\lambda \in \mathbb{C} : \lambda I - T \in \mathrm{USBF}^-(X),\, 0 < \alpha(\lambda I - T)\} \subseteq \Delta^{\mathrm{a}}(T)$$

is obvious. To show the opposite inclusion, suppose that $\lambda \in \Delta^{\mathrm{a}}(T)$. There is no harm if we assume $\lambda = 0$. Then $T \in \mathrm{USBF}^-(X)$ and $0 \in \sigma_{\mathrm{a}}(T)$. Both conditions entail that $\alpha(T) > 0$ (if $\alpha(T) = 0$, and hence $p(T) = 0$, then by Lemma 1.7, we would have $T(X)$ closed, thus $0 \notin \sigma_{\mathrm{a}}(T)$, a contradiction). Therefore the equality (8) holds.

To show the inclusion $\Pi^{\mathrm{a}}(T) \subseteq \Delta^{\mathrm{a}}(T)$, assume that $\lambda \in \Pi^{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$. Since $\lambda I - T$ is a left pole we have $\lambda \in \sigma_{\mathrm{a}}(T)$, and by Theorem 1.8, $\lambda I - T \in \mathrm{USBF}^-(X)$. Therefore $\Pi^{\mathrm{a}}(T) \subseteq \Delta^{\mathrm{a}}(T)$. ∎

DEFINITION 2.3. An operator $T \in L(X)$ is said to satisfy *generalized a-Browder's theorem* if $\Delta^{\mathrm{a}}(T) = \Pi^{\mathrm{a}}(T)$, or equivalently $\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{ld}}(T)$.

From Theorem 2.1 we readily see that

$$T \text{ satisfies generalized } a\text{-Browder's theorem} \iff \operatorname{acc} \sigma_{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{usbf}^-}(T).$$

In the following result we give a local spectral characterization of the operators satisfying generalized a-Browder's theorem.

THEOREM 2.4. *An operator $T \in L(X)$ satisfies generalized a-Browder's theorem if and only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$.*

Proof. If T satisfies generalized a-Browder's theorem then $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{usbf}^-}(T)$ and hence T has SVEP at every $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$, since $p(\lambda I - T) < \infty$.

Conversely, assume that T has SVEP at every $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$. If $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$ then, by Theorem 1.8, $\lambda \notin \sigma_{\mathrm{ld}}(T)$. Hence, $\sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{usbf}^-}(T)$, and since the opposite inclusion holds for all operators, we have $\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{ld}}(T)$, and thus T satisfies generalized a-Browder's theorem. ∎

COROLLARY 2.5. *If $T \in L(X)$ has SVEP then T satisfies generalized a-Browder's theorem.*

Let

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}.$$

An operator $T \in L(X)$ is said to satisfy *generalized Browder's theorem* if $\sigma(T) \setminus \sigma_{\mathrm{bw}}(T)$ coincides with the set of poles of T, or equivalently $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$. Let $\mathrm{UB}(X)$ denote the class of all *upper Browder operators* consisting of all upper semi-Fredholm operators $T \in L(X)$ such that $p(T) < \infty$, and denote by $\mathrm{UW}(X)$ the class of all *upper Weyl operators* consisting of all upper semi-Fredholm operators $T \in L(X)$ such that $\operatorname{ind} T \leq 0$. Set

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathrm{UB}(X)\},$$
$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathrm{UW}(X)\},$$

An operator T is said to satisfy *a-Browder's theorem* if $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T)$.

The next result is an immediate consequence of the fact that each of the Browder type theorems introduced before corresponds to the SVEP at the points of certain sets:

COROLLARY 2.6. *If $T \in L(X)$ satisfies generalized a-Browder's theorem then both generalized Browder's theorem and a-Browder's theorem hold for T.*

Proof. Generalized Browder's theorem for T is equivalent to the SVEP of T at the points $\lambda \notin \sigma_{\mathrm{bw}}(T)$ [5], and obviously, $\sigma_{\mathrm{usbf}^-}(T) \subseteq \sigma_{\mathrm{bw}}(T)$. By Theorem 2.4 generalized a-Browder's theorem for T implies generalized Browder's theorem for T. In a similar way, a-Browder's theorem for T is equivalent to the SVEP of T at the points $\lambda \notin \sigma_{\mathrm{uw}}(T)$, and obviously, $\sigma_{\mathrm{usbf}^-}(T) \subseteq \sigma_{\mathrm{uw}}(T)$. Therefore, by Theorem 2.4, generalized a-Browder's theorem for T implies a-Browder's theorem for T ∎

Denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions defined on an open neighborhood of $\sigma(T)$, and for $f \in \mathcal{H}(\sigma(T))$ let $f(T)$ be defined by means of the classical functional calculus. It is known that the spectral mapping theorem holds for $\sigma_{\mathrm{ld}}(T)$ whenever f is not constant on any component of its domain [26]. From this we easily obtain the following result:

COROLLARY 2.7. *Suppose that T has SVEP and that $f \in \mathcal{H}(\sigma(T))$. Then generalized a-Browder's theorem holds for $f(T)$. Moreover, if f is not constant on any component of its domain, then $f(\sigma_{\mathrm{usbf}^-}(T)) = \sigma_{\mathrm{usbf}^-}(f(T))$.*

Proof. If T has SVEP then $f(T)$ has SVEP (see Theorem 2.40 of [1]), so generalized a-Browder's theorem holds for $f(T)$. Moreover, if f is not constant on any component of its domain, we have

$$f(\sigma_{\mathrm{usbf}^-}(T)) = f(\sigma_{\mathrm{ld}}(T)) = \sigma_{\mathrm{ld}}(f(T)) = \sigma_{\mathrm{usbf}^-}(f(T)). \ ∎$$

Given $n \in \mathbb{N}$ we shall denote by $\widehat{T}_n : X/\ker T^n \to X/\ker T^n$ the canonical quotient map defined by $\widehat{T}_n \widehat{x} := \widehat{Tx}$ for each $\widehat{x} \in X/\ker T^n$, where $x \in \widehat{x}$.

Lemma 2.8. *If $T \in L(X)$, $T^n(X)$ is closed and $T_{[n]}$ is upper semi-Fredholm then $\widehat{T}_n$ is upper semi-Fredholm with* $\operatorname{ind}\widehat{T}_n = \operatorname{ind} T_{[n]}$. *Moreover, if T has SVEP at 0 then so does $\widehat{T}_n$.*

Proof. The operator $[T^n] : X/\ker T^n \to T^n(X)$ defined by

$$[T^n]\widehat{x} = T^n x, \qquad \text{where } x \in \widehat{x},$$

is a bijection and it is easy to check that $[T^n]\widehat{T}_n = T_{[n]}[T^n]$, from which the first assertion follows. If T has SVEP at 0 then the restriction $T_{[n]}$ has SVEP at 0 and the SVEP of $\widehat{T}_n$ at 0 follows from Theorem 2.15 of [1]. ∎

Theorem 2.9. *If $T \in L(X)$ and T^* has SVEP then generalized a-Browder's theorem holds for T. Moreover,*

$$\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T). \tag{9}$$

Proof. Suppose that $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$. Without loss of generality, we may assume that $\lambda = 0$. Then, by Lemma 2.8, there exists $n \geq 0$ such that $\widehat{T}_n$ is upper semi-Fredholm with $\operatorname{ind}\widehat{T}_n \leq 0$. Since $(\widehat{T}_n)^* = T^*|(\ker T^n)^\perp$, $(\ker T^n)^\perp$ being the annihilator of $\ker T^n$, it follows that $(\widehat{T}_n)^*$ has SVEP and thus (see Remark 1.4) $q(\widehat{T}_n) < \infty$. Since $\widehat{T}_n$ is semi-Fredholm it follows from [1, Theorem 3.4] that $\operatorname{ind}\widehat{T}_n \geq 0$. Therefore, $\operatorname{ind}\widehat{T}_n = 0$ and, again by Theorem 3.4 of [1], we may conclude that $p(\widehat{T}_n) < \infty$. Thus 0 is a pole of the resolvent of $\widehat{T}_n$, in particular 0 is isolated in $\sigma(\widehat{T}_n)$.

We claim that 0 is an isolated point of $\sigma(T)$. Indeed, if $\mathbb{D}(0,\varepsilon)$ is an open ball centered at 0 such that $\mathbb{D}(0,\varepsilon) \setminus \{0\} \subseteq \varrho(\widehat{T}_n)$, then $\mathbb{D}(0,\varepsilon) \setminus \{0\} \subseteq \varrho(T|\ker T^n)$, since $T|\ker T^n$ is nilpotent. Set $\widehat{X} := X/\ker T^n$. If $x \in X$ and $0 < |\lambda| < \varepsilon$, then there exists $y \in X$ such that $(\lambda I - \widehat{T}_n)\widehat{y} = \widehat{x}$, and hence $x - (\lambda I - T)y \in \ker T^n$. Thus there exists $z \in \ker T^n$ so that

$$x - (\lambda I - T)y = (\lambda I - T)z,$$

from which it follows that $x \in (\lambda I - T)(X)$. From $\ker(\lambda I - \widehat{T}_n) = \{0\}$ and $\ker(\lambda I - T|\ker T^n) = \{0\}$ we easily see that $\ker(\lambda I - T) = \{0\}$, so $\mathbb{D}(0,\varepsilon) \setminus \{0\} \subseteq \varrho(T)$, and hence 0 is an isolated point of $\sigma(T)$, as claimed. Therefore, T has SVEP at 0, and this implies by Theorem 1.8 that $0 \notin \sigma_{\mathrm{ld}}(T)$ so that $\sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{usbf}^-}(T)$. The opposite inclusion is always true, so $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{usbf}^-}(T)$ and hence T satisfies generalized a-Browder's theorem.

To show the equalities (9) observe first that $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{bw}}(T)$, since by Corollary 2.6, T satisfies generalized Browder's theorem. Hence it suffices to prove that $\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{d}}(T)$. The inclusion $\sigma_{\mathrm{usbf}^-}(T) \subseteq \sigma_{\mathrm{bw}}(T)$ is true for every operator. To show the opposite inclusion, suppose that $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$. Then $\lambda I - T_{[n]}$ is upper semi-Fredholm for some $n \in \mathbb{N}$, and proceeding as in the first part of the proof we see that $\operatorname{ind}(\lambda I - \widehat{T}_n) = 0$. By Lemma 2.8 it then follows that $\operatorname{ind}(\lambda I - T_{[n]}) = 0$, thus $\lambda \notin \sigma_{\mathrm{bw}}(T)$. Hence, the proof of (9) is complete. ∎

COROLLARY 2.10. *If $T \in L(X)$ and T^* has SVEP then generalized a-Browder's theorem holds for $f(T)$ for each $f \in \mathcal{H}(\sigma(T))$. Moreover, the spectral mapping theorem holds for $\sigma_{\mathrm{usbf}^-}(T)$.*

Proof. $(f(T))^* = f(T^*)$ has SVEP by Theorem 2.40 of [1], so, by Theorem 2.9, generalized a-Browder's theorem holds for $f(T)$. Since the spectral mapping theorem holds for $\sigma_{\mathrm{d}}(T)$ (see [12]), it follows that

$$f(\sigma_{\mathrm{usbf}^-}(T)) = f(\sigma_{\mathrm{d}}(T)) = \sigma_{\mathrm{d}}(f(T)) = \sigma_{\mathrm{usbf}^-}(f(T)),$$

so also the last assertion is proved. ∎

The following theorem gives a precise spectral picture of operators satisfying generalized a-Browder's theorem.

THEOREM 2.11. *For $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies a-generalized Browder's theorem;*
(ii) *every $\lambda \in \Delta^{\mathrm{a}}(T)$ is an isolated point of $\sigma_{\mathrm{a}}(T)$;*
(iii) *$\Delta^{\mathrm{a}}(T) \subseteq \partial\sigma_{\mathrm{a}}(T)$, where $\partial\sigma_{\mathrm{a}}(T)$ is the topological boundary of $\sigma_{\mathrm{a}}(T)$;*
(iv) $\operatorname{int} \Delta^{\mathrm{a}}(T) = \emptyset$;
(v) $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{usbf}^-}(T) \cup \partial\sigma_{\mathrm{a}}(T)$.
(vi) $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{usbf}^-}(T) \cup \operatorname{iso}\sigma_{\mathrm{a}}(T)$

Proof. (i)⇒(ii). Clear, since by (7) we have $\Delta^{\mathrm{a}}(T) = \Pi^{\mathrm{a}}(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$.

(ii)⇒(iii). Obvious.

(iii)⇒(iv). Clear, since $\operatorname{int}\partial\sigma_{\mathrm{a}}(T) = \emptyset$.

(iv)⇒(v). Suppose that $\operatorname{int}\Delta^{\mathrm{a}}(T) = \emptyset$. Let $\lambda_0 \in \Delta^{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbf}^-}(T)$ and suppose that $\lambda_0 \notin \partial\sigma_{\mathrm{a}}(T)$. Then there exists an open disc $\mathbb{D}$ centered at λ_0 contained in $\sigma_{\mathrm{a}}(T)$. Since $\lambda_0 I - T$ is upper semi B-Fredholm there exists a punctured open disc $\mathbb{D}_1$ contained in $\mathbb{D}$ such that $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in \mathbb{D}_1$ (see [11]). Moreover, $0 < \alpha(\lambda I - T)$ for all $\lambda \in \mathbb{D}_1$. In fact, if $0 = \alpha(\lambda I - T)$, then $(\lambda I - T)(X)$ being closed, we would have $\lambda \notin \sigma_{\mathrm{a}}(T)$, a contradiction. By Lemma 2.2 then λ_0 belongs to $\operatorname{int}\Delta^{\mathrm{a}}(T)$, and this contradicts $\operatorname{int}\Delta^{\mathrm{a}}(T) = \emptyset$. This shows that $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{usbf}^-}(T) \cup \partial\sigma_{\mathrm{a}}(T)$, as desired.

(v)⇒(vi). Let $\lambda_0 \in \partial\sigma_{\mathrm{a}}(T)$ and $\lambda_0 \notin \sigma_{\mathrm{usbf}^-}(T)$. Let $\mathbb{D}$ be an open disc centered at λ_0 and suppose that $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$. If $\mu \in \mathbb{D}$ and $\mu \notin \sigma_{\mathrm{a}}(T)$ then T has SVEP at μ, so $f \equiv 0$ in an open disc $\mathbb{U} \subseteq \mathbb{D}$ centered at μ. The identity theorem for analytic functions entails that $f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$, so T has SVEP at λ_0. Since $\lambda_0 I - T$ is upper semi B-Fredholm it follows from Theorem 1.8 that $\lambda_0 I - T$ is left Drazin invertible, and hence $\lambda_0 \in \Pi^{\mathrm{a}}(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$. Therefore, $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{usbf}^-}(T) \cup \operatorname{iso}\sigma_{\mathrm{a}}(T)$.

(vi)⇒(i). We show that $\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{ld}}(T)$. Let $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$. If $\lambda \notin \sigma_{\mathrm{a}}(T)$ then, by Theorem 2.1, $\lambda \notin \sigma_{\mathrm{ld}}(T)$. Suppose that $\lambda \in \sigma_{\mathrm{a}}(T)$. Then $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbf}^-}(T)$ and hence $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$. This implies that T has SVEP at λ. Since $\lambda I - T$ is upper semi B-Fredholm it follows by Theorem 1.8 that $\lambda I - T$ is left Drazin invertible, so $\lambda \notin \sigma_{\mathrm{ld}}(T)$. This proves the inclusion $\sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{usbf}^-}(T)$. The opposite inclusion is satisfied by every operator, so $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{usbf}^-}(T)$ and hence T satisfies a-generalized Browder's theorem. ∎

Corollary 2.12. *If $T \in L(X)$ has SVEP and* $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \emptyset$ *then* $\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{a}}(T)$.

Proof. T satisfies generalized a-Browder's theorem and hence by Theorem 2.11, $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{usbf}^-}(T) \cup \operatorname{iso}\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{usbf}^-}(T)$. ∎

The result of Corollary 2.12 applies in particular to non-quasi-nilpotent shift operators on $\ell^p(\mathbb{N})$ (see §5 of Chapter 2 in [1]).

We shall denote by $\mathcal{H}(\Omega, Y)$ the Fréchet space of all analytic functions from the open set $\Omega \subseteq \mathbb{C}$ to the Banach space Y.

Theorem 2.13. *If $T \in L(X)$ is upper semi B-Fredholm and has SVEP at* 0 *then there exists $\nu \in \mathbb{N}$ such that $H_0(T) = \ker T^\nu$.*

Proof. By Lemma 2.8 there exists $n \in \mathbb{N}$ such that $\widehat{T}_n$ is upper semi-Fredholm and $\widehat{T}_n$ has SVEP at 0. By Theorem 3.16 of [1], $\widehat{T}_n$ has finite ascent p and $H_0(\widehat{T}_n) = \ker(\widehat{T}_n)^p$. Suppose now that $x \in H_0(T)$. We show that $\widehat{x} \in H_0(\widehat{T}_n)$. Indeed, since $H_0(T) = \mathcal{X}_T(\{0\})$ there exists $g \in \mathcal{H}(\mathbb{C} \setminus \{0\}, X)$ such that

$$x = (\mu I - T)g(\mu) \quad \text{for all } \mu \in \mathbb{C} \setminus \{0\}.$$

If $\phi : X \to \widehat{X} = X/\ker T^n$ denotes the canonical quotient map then $\widehat{g} := \phi \circ g \in \mathcal{H}(\mathbb{C} \setminus \{0\}, \widehat{X})$ and

$$\widehat{x} = (\mu I - \widehat{T}_n)\widehat{g(\mu)} = (\mu I - \widehat{T}_n)\widehat{g}(\mu) \quad \text{for all } \mu \in \mathbb{C} \setminus \{0\}.$$

Thus $\widehat{x} \in \widehat{\mathcal{X}}_{\widehat{T}_n}(\{0\}) = H_0(\widehat{T}_n) = \ker(\widehat{T}_n)^p$, i.e.,

$$\widehat{T}_n^p \widehat{x} = \widehat{T^p x} = \widehat{0},$$

and so $T^p x \in \ker T^n$. This shows that $H_0(T) \subseteq \ker T^{p+n}$. The opposite inclusion holds for every operator, so $H_0(T) = \ker T^\nu$, where $\nu := p + n$. ∎

Theorem 2.14. *For $T \in L(X)$ the following statements are equivalent*:

(i) *T satisfies generalized a-Browder's theorem*;

(ii) *for each $\lambda \in \Delta^{\mathrm{a}}(T)$ there exists $\nu := \nu(\lambda) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^\nu; \tag{10}$$

(iii) *$H_0(\lambda I - T)$ is closed for all $\lambda \in \Delta^{\mathrm{a}}(T)$.*

Proof. (i)$\Rightarrow$(ii). Assume that T satisfies generalized a-Browder's theorem, and let $\lambda_0 \in \Delta^{\mathrm{a}}(T)$. We may assume that $\lambda_0 = 0$. Then $T \in \mathrm{USBF}^-(X)$ and by Theorem 2.11 we have $0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$, thus T has SVEP at 0. By Theorem 2.13 it then follows that $H_0(T) = \ker T^{\nu}$ for some $\nu \in \mathbb{N}$.

(ii)$\Rightarrow$(iii). Clear.

(iii)$\Rightarrow$(i). Suppose that $H_0(\lambda I - T)$ is closed for all $\lambda \in \Delta^{\mathrm{a}}(T)$. Let $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$. If $\lambda \notin \sigma_{\mathrm{a}}(T)$ then T has SVEP at λ, by (3). If $\lambda \in \sigma_{\mathrm{a}}(T)$ then $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbf}^-}(T) = \Delta^{\mathrm{a}}(T)$, so by (4), T has SVEP at λ. Therefore, by Theorem 2.4, T satisfies generalized a-Browder's theorem. ∎

We shall need the following punctured disc theorem which is a particular case of a result proved in [11, Corollary 3.2] for operators having topological uniform descent for $n \geq d$.

THEOREM 2.15. *If $T \in L(X)$ is upper semi B-Fredholm then there exists an open disc $\mathbb{D}(0, \varepsilon)$ such that $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ and* $\operatorname{ind}(\lambda I - T) = \operatorname{ind} T$ *for all $\lambda \in \mathbb{D}(0, \varepsilon)$. Moreover, if $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$ then*

$$\alpha(\lambda I - T) = \dim(\ker T \cap T^d(X)) \quad \textit{for some } d \in \mathbb{N},$$

so that $\alpha(\lambda I - T)$ is constant as λ varies in $\mathbb{D}(0, \varepsilon) \setminus \{0\}$ and

$$\alpha(\lambda I - T) \leq \alpha(T) \quad \textit{for all } \lambda \in \mathbb{D}(0, \varepsilon).$$

Let M, N be two closed linear subspaces of X and define

$$\delta(M, N) := \sup\{\operatorname{dist}(u, N) : u \in M,\ \|u\| = 1\},$$

in the case $M \neq \{0\}$, otherwise set $\delta(\{0\}, N) = 0$ for any subspace N. According to [22, §2, Chapter IV], the *gap* between M and N is defined by

$$\widehat{\delta}(M, N) := \max\{\delta(M, N), \delta(N, M)\}.$$

The function $\widehat{\delta}$ is a metric on the set of all closed linear subspaces of X and the convergence $M_n \to M$ is obviously defined by $\widehat{\delta}(M_n, M) \to 0$ as $n \to \infty$.

THEOREM 2.16. *For $T \in L(X)$ the following statements are equivalent*:

(i) *T satisfies generalized a-Browder's theorem*;

(ii) *The mapping $\lambda \mapsto \ker(\lambda I - T)$ is discontinuous at every $\lambda \in \Delta^{\mathrm{a}}(T)$ in the gap metric.*

Proof. (i)$\Rightarrow$(ii). By Theorem 2.11 if T satisfies generalized Browder's theorem then $\Delta^{\mathrm{a}}(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$. If $\lambda_0 \in \Delta^{\mathrm{a}}(T)$ then, by Lemma 2.2, $\alpha(\lambda_0 I - T) > 0$, and since $\Delta^{\mathrm{a}}(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$ there exists a punctured open disc $\mathbb{D}(\lambda_0)$ centered at λ_0 such that $\alpha(\lambda I - T) = 0$ for all $\lambda \in \mathbb{D}(\lambda_0)$. Hence $\lambda \mapsto \ker(\lambda I - T)$ is discontinuous at λ_0 in the gap metric.

(ii)$\Rightarrow$(i). We show that $\Delta^{\mathrm{a}}(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$, so Theorem 2.11 applies. Let $\lambda_0 \in \Delta^{\mathrm{a}}(T)$. Then $\lambda_0 I - T$ is upper semi B-Fredholm with $\operatorname{ind}(\lambda_0 I - T) \leq 0$.

By Theorem 2.15 there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that $\lambda I - T$ is upper semi B-Fredholm for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, $\alpha(\lambda I - T)$ is constant as λ ranges over $\mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$,

$$\operatorname{ind}(\lambda I - T) = \operatorname{ind}(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon),$$

and

$$0 \le \alpha(\lambda I - T) \le \alpha(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon).$$

Since $\lambda \mapsto \ker(\lambda I - T)$ is discontinuous at every $\lambda \in \Delta^{\mathrm{a}}(T)$,

$$0 \le \alpha(\lambda I - T) < \alpha(\lambda_0 I - T) \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}.$$

We claim that

$$\alpha(\lambda I - T) = 0 \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}. \tag{11}$$

To see this, suppose that there exists $\lambda_1 \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ with $\alpha(\lambda_1 I - T) > 0$. Clearly, $\lambda_1 \in \Delta^{\mathrm{a}}(T)$, so arguing as for λ_0 we obtain a $\lambda_2 \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0, \lambda_1\}$ such that

$$0 < \alpha(\lambda_2 I - T) < \alpha(\lambda_1 I - T),$$

and this is impossible since $\alpha(\lambda I - T)$ is constant for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Therefore (11) is satisfied and since $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, the range $(\lambda I - T)(X)$ is closed for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, thus $\lambda_0 \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$, as desired. ■

Define

$$E(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\},$$
$$E^{\mathrm{a}}(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T)\}.$$

Clearly, $E(T) \subseteq E^{\mathrm{a}}(T)$ for every $T \in L(X)$. Moreover, from the inclusion (7) and Lemma 2.2 we have

$$\Pi^{\mathrm{a}}(T) \subseteq E^{\mathrm{a}}(T) \quad \text{for all } T \in L(X). \tag{12}$$

Set $\Delta(T) := \sigma(T) \setminus \sigma_{\mathrm{bw}}(T)$. By Theorem 1.5 of [5] we have

$$\Delta(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is B-Weyl and } 0 < \alpha(\lambda I - T)\}.$$

Obviously, $\Delta(T) \subseteq \Delta^{\mathrm{a}}(T)$ for every $T \in L(X)$.

Definition 2.17. An operator $T \in L(X)$ is said to satisfy *generalized Weyl's theorem* if $\Delta(T) = E(T)$, and to satisfy *generalized a-Weyl's theorem* if $\Delta^{\mathrm{a}}(T) = E^{\mathrm{a}}(T)$.

Define

$$\Delta_1^{\mathrm{a}}(T) := \Delta^{\mathrm{a}}(T) \cup E^{\mathrm{a}}(T).$$

Theorem 2.18. *For $T \in L(X)$ the following statements are equivalent*:

(i) *T satisfies generalized a-Weyl's theorem*;
(ii) *T satisfies generalized a-Browder's theorem and $E^{\mathrm{a}}(T) = \Pi^{\mathrm{a}}(T)$*;

(iii) *for every* $\lambda \in \varDelta_1^a(T)$ *there exists* $p := p(\lambda) \in \mathbb{N}$ *such that* $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ *and* $(\lambda I - T)^n(X)$ *is closed for all* $n \geq p$.

Proof. The equivalence (i)⇔(ii) has been proved in [10, Corollary 3.2].

(ii)⇒(iii). If T satisfies generalized a-Browder's theorem then $\varDelta^{\mathrm{a}}(T) = \varPi^{\mathrm{a}}(T)$ and from the assumption $E^{\mathrm{a}}(T) = \varPi^{\mathrm{a}}(T)$ we have

$$\varDelta_1^{\mathrm{a}}(T) = \varDelta^{\mathrm{a}}(T) = \varPi^{\mathrm{a}}(T).$$

Let $\lambda \in \varDelta_1^{\mathrm{a}}(T)$. Then from Theorem 2.14 there exists $m \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^m$. Clearly, $p := p(\lambda I - T)$ is finite, and $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Since $\lambda \in \varPi^{\mathrm{a}}(T)$, $\lambda I - T$ is left Drazin invertible, thus $(\lambda I - T)^{p+1}(X)$ is closed, and hence, by Lemma 1.7, $(\lambda I - T)^n(X)$ is closed for all $n \geq p$.

(iii)⇒(ii). Since $\varDelta^{\mathrm{a}}(T) \subseteq \varDelta_1^{\mathrm{a}}(T)$ Theorem 2.14 entails that T satisfies generalized a-Browder's theorem. To show that $E^{\mathrm{a}}(T) = \varPi^{\mathrm{a}}(T)$ it suffices by (12) to prove that $E^{\mathrm{a}}(T) \subseteq \varPi^{\mathrm{a}}(T)$. Suppose that $\lambda \in E^{\mathrm{a}}(T)$. Then there exists $\nu \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^\nu$ and this implies that $\lambda I - T$ has ascent $p = p(\lambda I - T) \leq \nu$. Thus it follows from our assumption that $(\lambda I - T)^{p+1}(X)$ is closed, i.e. $\lambda \in \varPi^{\mathrm{a}}(T)$. ∎

Generalized Weyl's theorem may also be described by the quasi-nilpotent part $H_0(\lambda I - T)$ as λ ranges over a suitable subset of $\mathbb{C}$. In fact, if we define $\varDelta_1(T) := \varDelta(T) \cup E(T)$, in [5] it is shown that generalized Weyl's theorem holds for T if and only if for every $\lambda \in \varDelta_1(T)$ there exists $p := p(\lambda) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Since

$$\varDelta_1(T) = \varDelta(T) \cup E(T) \subseteq \varDelta^{\mathrm{a}}(T) \cup E^{\mathrm{a}}(T) = \varDelta_1^{\mathrm{a}}(T)$$

we easily deduce from Theorem 2.18 the following implication, already proved in [10, Theorem 1.37] by using different arguments:

generalized a-Weyl's theorem for T $\Rightarrow$ generalized Weyl's theorem for T.

In an important situation this implication may be reversed:

Theorem 2.19. *If* T^* *has SVEP then generalized a-Weyl's theorem holds for* T *if and only if generalized Weyl's theorem holds for* T.

Proof. Suppose that T satisfies generalized Weyl's theorem. Since T^* has SVEP we have $\sigma(T) = \sigma_{\mathrm{a}}(T)$ (see [1, Corollary 2.45]), and hence $E(T) = E^{\mathrm{a}}(T)$. By Theorem 2.9 we also have $\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{bw}}(T)$, so that

$$\varDelta^{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbf}^-}(T) = \sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = \varDelta(T) = E(T) = E^{\mathrm{a}}(T),$$

and hence generalized a-Weyl's theorem holds for T. ∎

An operator $T \in L(X)$ is said to be *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent operator $(\lambda I - T)^{-1}$, or equivalently, if $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ for every $\lambda \in \operatorname{iso}\sigma(T)$.

Corollary 2.20. *If T^* has SVEP and T is polaroid then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. By Theorem 1.20 of [5], $f(T)$ satisfies generalized Weyl's theorem. By [1, Theorem 2.40], $f(T^*) = f(T)^*$ has SVEP. ∎

In [27] Oudghiri studied the class $H(p)$ of operators on Banach spaces for which there exists $p := p(\lambda) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{13}$$

Clearly, if $T \in H(p)$ then T has SVEP by (4). The class $H(p)$ is rather large, for instance it contains the generalized scalar operators and subscalar operators, and hence the p-hyponormal, log-hyponormal, M-hyponormal operators on Hilbert spaces (see [27]).

The condition (13) (with $p := p(\lambda) = 1$ for every $\lambda \in \mathbb{C}$) is also satisfied by multipliers of commutative semisimple Banach algebras, in particular by convolution operators on group algebras $L_1(G)$ (see [3]). Also totally paranormal operators on Banach spaces, and $*$-paranormal operators on Hilbert spaces satisfy the condition $H(1)$ (see [1, p. 116] and [20]).

The condition (13) is not satisfied, in general, by algebraically paranormal operators on Hilbert spaces (see [4, Example 2.3]), but every algebraically paranormal operator on Hilbert space satisfies SVEP ([4]). Note that if $T \in H(p)$ or $T \in L(H)$ is algebraically paranormal then T is polaroid (see [2, Lemmas 3.3 and 4.3]).

Remark 2.21. For an operator defined on a Hilbert space H, instead of the dual T^* of $T \in L(H)$ it is more appropriate to consider the Hilbert adjoint T' of $T \in L(H)$. However, we have [2]

$$T' \text{ has SVEP } \Leftrightarrow T^* \text{ has SVEP},$$

so that the result of Corollary 2.20 holds if we suppose that T' has SVEP.

Theorem 2.22. *If $T \in L(X)$ and $T^* \in H(p)$, then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$. Analogously, if $T' \in L(H)$, H a Hilbert space, is algebraically paranormal then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. Suppose first that $T^* \in H(p)$. Then, by Lemma 3.3 of [2], for every $\lambda \in \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T^*)$, we have $p := p(\lambda I^* - T^*) = q(\lambda I^* - T^*) < \infty$. Therefore [1, Theorem 3.74] implies that $X^* = \ker(\lambda I^* - T^*)^p \oplus (\lambda I^* - T^*)^p(X^*)$ and that $(\lambda I^* - T^*)^p(X^*)$ is closed. By the classical closed range theorem, it follows that $(\lambda I - T)^p(X)$ is also closed and

$$X = {}^\perp\ker(\lambda I^* - T^*)^p \oplus {}^\perp(\lambda I^* - T^*)^p(X^*) = (\lambda I - T)^p(X) \oplus \ker(\lambda I - T)^p,$$

where $^\perp M$, denotes the pre-annihilator of $M \subset X^*$. By Theorem 3.6 of [1] we then conclude that $p(\lambda I - T) = q(\lambda I - T) < \infty$. Hence T is polaroid, so Corollary 2.20 applies.

Suppose that $T' \in L(H)$ is algebraically paranormal and $\lambda \in \operatorname{iso}\sigma(T)$. Since $\sigma(T) = \overline{\sigma(T')}$, $\overline{\lambda}$ is isolated in $\sigma(T')$ and hence a pole of the resolvent of T', i.e. $p := p(\overline{\lambda} I - T') = q(\overline{\lambda} I - T') < \infty$. We have $H = \ker(\overline{\lambda} I - T')^p \oplus (\overline{\lambda} I - T')^p(H)$ and since $(\overline{\lambda} I - T')^p(H)$ is closed it then follows that $(\lambda I - T)^p(H)$ is closed. We also have

$$H = (\ker(\overline{\lambda} I - T')^p)^\perp \oplus ((\overline{\lambda} I - T')^p(H))^\perp = (\lambda I - T)^p(H) \oplus \ker(\lambda I - T)^p,$$

where $M^\perp$ denotes the orthogonal complement of M in the Hilbert space sense. Again by Theorem 3.6 of [1] we conclude that $p(\lambda I - T) = q(\lambda I - T) < \infty$, i.e. λ is a pole of the resolvent of T. Therefore, T is polaroid, so Corollary 2.20 applies also in this case. ∎

Theorem 2.22 generalizes results from [2], [14], [15], [20] (where a-Weyl's theorem was proved for $f(T)$), and subsumes results from [13, Theorem 3.3], [28, Theorem 3.2], [9] (where generalized a-Weyl's theorem was proved, separately, for each class).

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Received June 5, 2006
Revised version March 9, 2007 (5935)