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A double commutant theorem for purely large C^*-subalgebras of real rank zero corona algebras

by

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Abstract. Let $\mathcal{A}$ be a unital separable simple nuclear C^*-algebra such that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has real rank zero. Suppose that $\mathcal{C}$ is a separable simple liftable and purely large unital C^*-subalgebra of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$. Then the relative double commutant of $\mathcal{C}$ in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$ is equal to $\mathcal{C}$.

1. Introduction. A basic result in the theory of von Neumann algebras is von Neumann's double commutant theorem, which says that if $\mathcal{A}_0$ is a unital C^*-subalgebra of $\mathbb{B}(\mathcal{H})$, then the double commutant of $\mathcal{A}_0$ is equal to the weak operator closure of $\mathcal{A}_0$ [11]. (We note that in our terminology, a *unital C^*-subalgebra* of $\mathbb{B}(\mathcal{H})$ contains the unit of $\mathbb{B}(\mathcal{H})$. Hence, such an algebra acts nondegenerately on $\mathcal{H}$.)

In [13], [14] (see also [1]), Voiculescu proved an interesting C^*-algebraic version of von Neumann's result for the case of the Calkin algebra. Specifically, he showed that if $\mathcal{A}_0$ is a separable unital C^*-subalgebra of $\mathbb{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, then the relative double commutant of $\mathcal{A}_0$ in $\mathbb{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is equal to $\mathcal{A}_0$ itself.

Attempts have been made to generalize Voiculescu's theorem to more general corona algebras than the Calkin algebra. Generalizations to the case of hereditary C^*-subalgebras (which need not be separable) of a corona algebra have been. Specifically, in [6], Kucerovsky showed that if $\mathcal{B}$ is a stable separable C^*-algebra with a "purely large" type property (more precisely, for every positive element $c\in\mathcal{M}(\mathcal{B})-\mathcal{B}$, the hereditary C^*-subalgebra $\overline{c\mathcal{B}c}$ contains a full stable hereditary C^*-subalgebra of $\mathcal{B}$) then for every nonunital, hereditary, σ-unital C^*-subalgebra $\mathcal{C}$ of the corona algebra $\mathcal{M}(\mathcal{B})/\mathcal{B}$, the relative double commutant of $\mathcal{C}$ in $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is equal to the unitization of $\mathcal{C}$. In [5], Elliott and Kucerovsky showed that if $\mathcal{B}$ is a σ-unital simple

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stable C^*-algebra, and if $\mathcal{C}$ is a singly generated hereditary C^*-subalgebra of $\mathcal{M}(\mathcal{B})/\mathcal{B}$, then the relative double commutant of $\mathcal{C}$ in $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is equal to the unitization of $\mathcal{C}$.

In this paper, we also extensively use the theory of absorbing extensions as in [6], [4] and [5], but we approach the problem in a different manner and do not require the initial algebra to be a hereditary C^*-subalgebra of the corona algebra. However, we do require that the initial algebra be a *purely large* C^*-subalgebra.

For a C^*-algebra $\mathcal{B}$, let $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be the natural quotient map.

Definition 1.1. Let $\mathcal{A}$ be a unital separable simple C^*-algebra.

(1) Let $\mathcal{D}$ be a separable simple unital C^*-subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. Then $\mathcal{D}$ is said to be *purely large* if for every nonzero positive element $a \in \mathcal{D}$, the hereditary C^*-subalgebra $\overline{a(\mathcal{A} \otimes \mathcal{K})a}$ contains a full stable hereditary C^*-subalgebra of $\mathcal{A} \otimes \mathcal{K}$.
(2) Let $\mathcal{C}$ be a unital separable simple C^*-algebra, and let $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be a unital $*$-homomorphism (which is necessarily injective). Then ϕ is said to be *purely large* if $\phi(\mathcal{C})$ is a purely large C^*-subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.
(3) Let $\mathcal{C}$ be a separable simple unital C^*-subalgebra of the quotient $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K})$. Let $i : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K})$ be the natural inclusion map. Then $\mathcal{C}$ is said to be *liftable and purely large* if there exists a purely large unital $*$-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that $i = \pi \circ \phi$.

(We note that, in the literature, the notion of *purely large* is defined without the condition of simplicity: see, for example, [4]. However, adding this condition makes the definition and the paper in general less complicated.)

Our main result is the following:

Theorem 1.2. *Suppose that $\mathcal{A}$ is a unital separable simple nuclear C^*-algebra such that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has real rank zero. Suppose that $\mathcal{C}$ is a simple separable liftable and purely large unital C^*-subalgebra of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$. Then the relative double commutant of $\mathcal{C}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K})$ is equal to $\mathcal{C}$.*

As a corollary, we get the following result:

Theorem 1.3. *Let $\mathcal{A}$ be a unital separable simple nuclear C^*-algebra with $K_1(\mathcal{A}) = 0$ such that either*

(1) *$\mathcal{A}$ has real rank zero, stable rank one and weak unperforation, or*
(2) *$\mathcal{A}$ is purely infinite.*

Let $\mathcal{C}$ be a simple separable unital C^*-subalgebra of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$, and $i : \mathcal{C} \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$ the natural inclusion map. Suppose that there exists a unital $*$-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that $i = \pi \circ \phi$. Then the relative double commutant of $\mathcal{C}$ in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$ is equal to $\mathcal{C}$ itself.

In this paper, we will use the following notation: Suppose that $\mathcal{A}$ is a unital separable simple C^*-algebra and suppose that $\mathcal{C}$ is a C^*-subalgebra of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$. Then $\mathcal{C}'$ will denote the relative commutant of $\mathcal{C}$ in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$. In other words, $\mathcal{C}' := \{d \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K}) : dc = cd, \forall c \in \mathcal{C}\}$. Thus, $\mathcal{C}''$ will be the relative commutant of $\mathcal{C}'$ in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$; i.e., $\mathcal{C}''$ is the relative double commutant of $\mathcal{C}$ in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$.

2. Main theorem

LEMMA 2.1. *Let $\mathcal{A}$ be a unital separable C^*-algebra. Then there is no sequence $\{a_n\}_{n=1}^{\infty}$ of norm one elements in $\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K}$ such that for all $a \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})\otimes 1_{\mathcal{M}(\mathcal{K})}$,*

$$\|aa_n - a_n a\| \to 0 \quad \text{as } n \to \infty.$$

Proof. Firstly, let $\{e_{i,j}\}_{1\leq i,j<\infty}$ be a system of matrix units for $\mathcal{K}$. Hence, $\{1_{\mathcal{A}} \otimes e_{i,j}\}_{1\leq i,j<\infty}$ is a system of matrix units for $1_{\mathcal{A}}\otimes\mathcal{K}$. Since there will be no confusion, we will identify $e_{i,j}$ with $1_{\mathcal{A}}\otimes e_{i,j}$ for all i,j. For all $n \geq 1$, let $e_n := \sum_{l=1}^{n} e_{l,l}$. Hence, $\{f_n = \bigoplus^n e_n\}_{n=1}^{\infty}$ is an approximate identity for $\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K}$.

Suppose, to the contrary, that $\{a_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K}$ such that $\|a_n\| = 1$ for all $n \geq 1$ and $\|a_n a - a a_n\| \to 0$ as $n \to \infty$ for all $a \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})\otimes 1_{\mathcal{M}(\mathcal{K})}$. We may assume that each a_n is positive, and that $\{r_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that $a_n \in \mathcal{A}\otimes\mathcal{K}\otimes\mathbb{M}_{r_n}$ for every n.

CLAIM 1. *For every $n \geq 1$, there exist integers m, m' with $m, m' \geq n$ such that*

$$\|a_{m'} - f_m a_{m'} f_m\| \geq 1/3,$$

Suppose, to the contrary, that $n \geq 1$ is such that for all $m, m' \geq n$,

$$\|a_{m'} - f_m a_{m'} f_m\| \leq 1/3.$$

Then, for all $m' \geq n$,

$$\|a_{m'} - f_n a_{m'} f_n\| \leq 1/3.$$

In other words, for all $m' \geq n$,

$$(*) \qquad \|f_n a_{m'}(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})} - f_n) + (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})} - f_n)a_{m'} f_n + (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})} - f_n)a_{m'}(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})} - f_n)\| \leq 1/3.$$

Therefore, since each a_k has norm one, we must have, for all $m' \geq n$,

$$(**) \qquad \|f_n a_{m'} f_n\| \geq 2/3.$$

Let v' be a partial isometry in $\mathcal{A} \otimes \mathcal{K}$ with range projection e_n and initial projection contained in $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - e_n$. Let v be the partial isometry in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})}$ given by $v := v' \otimes 1_{\mathcal{M}(\mathcal{K})}$ (so v has range projection $e_n \otimes 1_{\mathcal{M}(\mathcal{K})}$ and initial projection contained in $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})} - e_n \otimes 1_{\mathcal{M}(\mathcal{K})}$). Then we deduce from $(*)$ that for all $m' \geq n$,

$$(***) \qquad \|va_{m'}\| = \|v(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})} - e_n \otimes 1_{\mathcal{M}(\mathcal{K})})a_{m'}\| \leq 1/3.$$

On the other hand, by $(**)$, for all $m' \geq n$,

$$\|a_{m'}v\| = \|a_{m'}(e_n \otimes 1_{\mathcal{M}(\mathcal{K})})\| \geq 2/3.$$

From this and $(***)$, we have $\|a_{m'}v - va_{m'}\| \geq 1/3$ for all $m' \geq n$. This contradicts our assumption that $\{a_m\}_{m=1}^{\infty}$ asymptotically commutes with every element of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})}$. This ends the proof of Claim 1.

We will use Claim 1 to derive a contradiction and thus prove the nonexistence of a sequence $\{a_n\}_{n=1}^{\infty}$ (of positive norm one elements of $\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K}$) which asymptotically commutes with every element of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})}$.

We now construct a partial isometry $b \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})}$. We do this by constructing two sequences $\{b_k\}_{k=1}^{\infty}$, $\{v_k\}_{k=1}^{\infty}$ of partial isometries in $\mathcal{A}\otimes\mathcal{K}\otimes 1_{\mathcal{M}(\mathcal{K})}$ such that $b_{k+1} = b_k + v_{k+1}$ for all k, and $b_k \to b$ in the strict topology in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\otimes 1_{\mathcal{M}(\mathcal{K})} \cong \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ as $k \to \infty$. In the process, we also construct four subsequences $\{l_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$, $\{n_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ of positive integers. The construction will be by induction on k (i.e., in the kth step, we construct v_k, b_k, l_k, m_k, n_k and s_k).

Basis step $k = 1$. By Claim 1, let l_1 and m_1 be positive integers such that

$$\|a_{l_1} - f_{m_1} a_{l_1} f_{m_1}\| \geq 1/3.$$

Choose an integer $n_1 \geq m_1$ such that the following hold:

$$(+) \qquad \begin{aligned} &(1)\ \|a_{l_1} - f_{n_1} a_{l_1} f_{n_1}\| < 1/100, \\ &(2)\ \|(f_{n_1} - f_{m_1})a_{l_1}\| = \|a_{l_1}(f_{n_1} - f_{m_1})\| \geq 1/7, \\ &(3)\ \|(f_{n_1} - f_{m_1})a_{l_1}(f_{n_1} - f_{m_1})\| \geq 1/49. \end{aligned}$$

Now let $s_1 \geq n_1$ be a positive integer and $b_1' \in \mathcal{A}\otimes\mathcal{K}$ be a partial isometry such that b_1' has initial projection $e_{n_1} - e_{m_1}$ and range projection contained in $e_{s_1} - e_{n_1}$. Take $v_1 = b_1 := b_1' \otimes 1_{\mathcal{M}(\mathcal{K})}$.

Induction step: Suppose that b_k, v_k, l_k, m_k, n_k and s_k have been constructed for $k \leq K$. We now construct the corresponding quantities for $k = K+1$. Firstly, by Claim 1, choose positive integers l_{K+1}, m_{K+1} with $m_{K+1}, l_{K+1} \geq 1 + s_K$ such that

$$\|a_{l_{K+1}} - f_{m_{K+1}} a_{l_{K+1}} f_{m_{K+1}}\| \geq 1/3,$$

Next choose an integer $n_{K+1} \geq m_{K+1}$ such that the following hold:

$$(++)\qquad \begin{aligned} &(1)\ \|a_{l_k} - f_{n_{K+1}} a_{l_k} f_{n_{K+1}}\| < 1/(100)^{K+1} \text{ for all } k \leq K+1,\\ &(2)\ \|(f_{n_{K+1}} - f_{m_{K+1}}) a_{l_{K+1}}\| = \|a_{l_{K+1}}(f_{n_{K+1}} - f_{m_{K+1}})\| \geq 1/7,\\ &(3)\ \|(f_{n_{K+1}} - f_{m_{K+1}}) a_{l_{K+1}} (f_{n_{K+1}} - f_{m_{K+1}})\| \geq 1/49. \end{aligned}$$

Now let $s_{K+1} \geq n_{K+1}$ be a positive integer and $v'_{K+1} \in \mathcal{A} \otimes \mathcal{K}$ a partial isometry with initial projection $e_{n_{K+1}} - e_{m_{K+1}}$ and range projection contained in $e_{s_{K+1}} - e_{n_{K+1}}$. Let $v_{K+1} := v'_{K+1} \otimes 1_{\mathcal{M}(\mathcal{K})}$ and $b_{K+1} := b_K + v_{K+1}$. Note that b_K and v_{K+1} are orthogonal (i.e., have orthogonal initial projections and orthogonal range projections). Hence, as b_K and v_{K+1} are partial isometries, b_{K+1} is a partial isometry. This completes the inductive construction.

We have thus constructed a sequence $\{b_k\}_{k=1}^{\infty}$. By construction, $\{b_k\}_{k=1}^{\infty}$ converges in the strict topology to an element $b \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.

Claim 2. *For all $k \geq 1$, $\|ba_{l_k} - a_{l_k} b\| \geq 1/100$.*

To prove Claim 2, it suffices to prove that for all $k \geq 1$ and $k' \geq k$,

$$\text{(V)}\qquad \|b_{k'} a_{l_k} - a_{l_k} b_{k'}\| \geq 1/100.$$

To prove (V), fix $k \geq 1$ and $k' \geq k$. Let t be the projection in $\mathcal{A} \otimes \mathcal{K} \otimes 1_{\mathcal{M}(\mathcal{K})}$ given by $t := (e_{s_k} - e_{m_k}) \otimes 1_{\mathcal{M}(\mathcal{K})}$. Then

$$\text{(VV)}\qquad \begin{aligned} \|b_{k'} a_{l_k} - a_{l_k} b_{k'}\| &\geq \|t(b_{k'} a_{l_k} - a_{l_k} b_{k'})t\| = \|v_k a_{l_k} t - t a_{l_k} v_k\|\\ &\geq \|v_k a_{l_k} t\| - \|t a_{l_k} v_k\|. \end{aligned}$$

By the definition of v_k and (++)(3), we have $\|v_k a_{l_k} t\| \geq 1/49$. But by the definition of v_k and (++)(1), we have $\|t a_{l_k} v_k\| < 1/100^k$. From this and (VV), we see that

$$\|b_{k'} a_{l_k} - a_{l_k} b_{k'}\| \geq 1/49 - 1/100 \geq 1/100.$$

Since k and $k' \geq k$ are arbitrary, we have proven statement (V) and hence Claim 2.

Claim 2 implies that $\{a_n\}_{n=1}^{\infty}$ does not asymptotically commute with every element of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, which contradicts our assumption at the beginning of the proof. This proves Lemma 2.1. ∎

We note that the above lemma implies the same statement, but with $\mathcal{A} \otimes \mathcal{K}$ replacing $\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K}$ and with $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ replacing $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})}$. However, the proof of our main result involves reducing to the case of the Calkin algebra $\mathbb{B}(\mathcal{H})/\mathcal{K}$ and the stronger statement of the above lemma is required.

For a unital C^*-algebra $\mathcal{A}$, we let $\pi : \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K})$ denote the natural quotient map. Also, for a C^*-algebra $\mathcal{D}$ and for subsets

$S \subseteq \mathcal{D}$ and $T \subseteq \mathcal{D}$, we define $\mathrm{dist}(S,T) := \inf\{\|s-t\| : s \in S,\, t \in T\}$. For $a \in \mathcal{D}$, we set $\mathrm{dist}(a,T) := \mathrm{dist}(\{a\},T)$.

LEMMA 2.2. *Let $\mathcal{A}$ be a unital simple separable C^*-algebra such that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has real rank zero. Suppose that $c \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ is such that $\pi(c)$ commutes with every element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$. Then $c \in \mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K}$.*

Proof. Case 1: Suppose that c is positive. Since $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has real rank zero, it follows by [15] that there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of pairwise orthogonal projections of $\mathcal{A}\otimes\mathcal{K}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive real numbers such that the following statements hold:

(1) $\sum_{n=1}^{\infty} p_n$ converges in the strict topology in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.
(2) $\sum_{n=1}^{\infty} \lambda_n p_n$ converges in the strict topology in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.
(3) $b := c - \sum_{n=1}^{\infty} \lambda_n p_n$ is an element of $\mathcal{A}\otimes\mathcal{K}$.

Suppose, to the contrary, that $c \notin \mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K}$. Let $r > 0$ be such that $\mathrm{dist}(\pi(c), \pi(\mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})})) > r$. Then $\mathrm{dist}(c, \mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K}) > r > 0$. Choose an $\varepsilon > 0$ such that $r > 100\varepsilon$. It follows, then, that for all n, there exist integers $n', n'' \geq n$ such that $|\lambda_{n'} - \lambda_{n''}| > r - \varepsilon$.

So let $\{N_n\}_{n=1}^{\infty}$ and $\{M_n\}_{n=1}^{\infty}$ be two subsequences of positive integers such that for all n,

$$n \leq M_n < N_n < M_{n+1} \quad \text{and} \quad |\lambda_{M_n} - \lambda_{N_n}| \geq r - \varepsilon.$$

Since $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has real rank zero, $\mathcal{A}\otimes\mathcal{K}$ has real rank zero. Hence, for all $n \geq 1$, choose nonzero projections $r_n, s_n \in \mathcal{A}\otimes\mathcal{K}$ such that $r_n \leq p_{M_n}$, $s_n \leq p_{N_n}$ and r_n is Murray–von Neumann equivalent to q_n in $\mathcal{A}\otimes\mathcal{K}$.

For each $n \geq 1$, let $w_n \in \mathcal{A}\otimes\mathcal{K}$ be a partial isometry with initial projection r_n and range projection s_n. Let $v_n := w_n + (w_n)^*$. Let $v \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be the partial isometry given by $v := \sum_{n=1}^{\infty} v_n$ where the sum converges in the strict topology in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. One can check that $\|\pi(v)\pi(c) - \pi(c)\pi(v)\| \geq r - 2\varepsilon > 0$. Hence, $\pi(v)$ does not commute with $\pi(c)$, which contradicts our hypothesis on c.

Case 2: Suppose now that c is an arbitrary element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Then using [15] and the polar decomposition of c, we can represent c as $c = \sum_{n=1}^{\infty} \lambda_n x_n + b'$ where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers, $\{x_n\}_{n=1}^{\infty}$ is a sequence of partial isometries with pairwise orthogonal initial projections and pairwise orthogonal range projections, $b' \in \mathcal{A}\otimes\mathcal{K}$ and the sum converges in the strict topology in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. The proof is a technical modification of the proof of Case 1. ∎

Let $\mathcal{A}$ be a unital C^*-algebra. Let $\{e_{i,j}\}_{1\leq i,j<\infty}$ be a system of matrix units for $\mathcal{K}$. Since no confusion will occur, for each i, j we will use $e_{i,j}$ to denote both the element in $\mathcal{K}$ and $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \otimes e_{i,j} \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$. For each $c \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ and any i, j, we let $c_{i,j}$ denote $e_{i,i}ce_{j,j}$.

LEMMA 2.3. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra such that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ has real rank zero. Suppose that $c \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ is such that $\pi(c)$ commutes with $\pi(a\otimes 1_{\mathcal{M}(\mathcal{K})})$ for all $a \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Then $c_{i,j} \in \mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes K)} \otimes e_{i,j} + \mathcal{A}\otimes\mathcal{K}\otimes e_{i,j}$ for $1 \le i, j < \infty$.*

Proof. Fix i, j with $1 \le i, j < \infty$. Note that $c_{i,j} \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K}) \otimes e_{i,j}$, and also π is a $*$-homomorphism. Let $d_{i,j} := e_{1,i}c_{i,j}e_{j,1} \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K}) \otimes e_{1,1}$. Hence, for all $a \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$,

$$\begin{aligned}
&\pi((a\otimes e_{1,1})d_{i,j})\\
&\qquad = \pi((a\otimes e_{1,1})e_{1,i}ce_{j,1}) = \pi((a\otimes 1_{\mathcal{M}(\mathcal{K})})e_{1,i}ce_{j,1})\\
&\qquad = \pi(e_{1,i}(a\otimes 1_{\mathcal{M}(\mathcal{K})})ce_{j,1}) = \pi(e_{1,i})\pi(a\otimes 1_{\mathcal{M}(\mathcal{K})})\pi(c)\pi(e_{j,1})\\
&\qquad = \pi(e_{1,i})\pi(c)\pi(a\otimes 1_{\mathcal{M}(\mathcal{K})})\pi(e_{j,1}) = \pi(e_{1,i})\pi(c(a\otimes 1_{\mathcal{M}(\mathcal{K})}))\pi(e_{j,1})\\
&\qquad = \pi(e_{1,i}c(a\otimes 1_{\mathcal{M}(\mathcal{K})})e_{j,1}) = \pi(e_{1,i}ce_{j,1}(a\otimes e_{1,1})) = \pi(d_{i,j}(a\otimes e_{1,1})).
\end{aligned}$$

(Here, we are using $e_{s,t}$ to mean both an element of $\mathcal{K}$ and $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \otimes e_{s,t}$, for all s, t.)

Hence, by Lemma 2.2, $d_{i,j} \in \mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \otimes e_{1,1} + \mathcal{A}\otimes\mathcal{K}\otimes e_{1,1}$. So, $c_{i,j} = e_{i,1}d_{i,j}e_{1,j} \in \mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \otimes e_{i,j} + \mathcal{A}\otimes\mathcal{K}\otimes e_{i,j}$ as required. ■

LEMMA 2.4. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra such that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ has real rank zero. Suppose that $c \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ is such that $\pi(c)$ commutes with every element of $\pi(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\otimes 1_{\mathcal{M}(\mathcal{K})})$, so (by Lemma 2.3)*

$$c_{i,j} = \alpha_{i,j}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \otimes e_{i,j} + f_{i,j}\otimes e_{i,j}$$

for all i, j, where $\alpha_{i,j} \in \mathbb{C}$ and $f_{i,j} \in \mathcal{A}\otimes\mathcal{K}$. Then

$$g := \sum_{1\le i,j<\infty} \alpha_{i,j}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \otimes e_{i,j} \in 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \otimes \mathbb{B}(\mathcal{H}).$$

(*In particular, the infinite sum, viewed as being the limit of the net of all sums over finitely many terms, converges in the strict topology on $\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$.*)

Proof. Let $M = \|c\| > 0$. It suffices to prove that for all $N \ge 1$, $\|\sum_{1\le i,j\le N} \alpha_{i,j}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} \otimes e_{i,j}\| \le 2M$.

Let $\varepsilon > 0$ be given. Decreasing $\varepsilon > 0$ if necessary, we may assume that $M > 100\varepsilon$. Since the $f_{i,j}$s are all elements of $\mathcal{A}\otimes\mathcal{K}$, choose a nonzero projection $p \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that for $1 \le i, j \le N$,

$(*)$ $\qquad pf_{i,j}$ and $f_{i,j}p$ have norm strictly less than $\varepsilon/(2N^2)$.

Now let $P\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ be the projection given by $P:=\sum_{1\leq i\leq N} p\otimes e_{i,i}$. Since $\|c\|\leq M$, we have $\|PcP\|\leq M$. Hence,

$$\Big\|\sum_{1\leq i,j\leq N}\alpha_{i,j}p\otimes e_{i,j}+(pf_{i,j}p)\otimes e_{i,j}\Big\|\leq M.$$

By $(*)$,

$$\Big\|\sum_{1\leq i,j\leq N}(pf_{i,j}p)\otimes e_{i,j}\Big\|\leq\varepsilon/2.$$

Hence,

$$\Big\|\sum_{1\leq i,j\leq N}\alpha_{i,j}p\otimes e_{i,j}\Big\|\leq M+\varepsilon\leq 2M.$$

From this, it follows that

$$\Big\|\sum_{1\leq i,j\leq N}\alpha_{i,j}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}\otimes e_{i,j}\Big\|\leq 2M$$

as required. ■

Lemma 2.5. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra such that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ has real rank zero. Let $c\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ be such that $\pi(c)$ commutes with every element of $\pi(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\otimes 1_{\mathcal{M}(\mathcal{K})})$, so by Lemma 2.3*

$$c_{i,j}=\alpha_{i,j}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}\otimes e_{i,j}+f_{i,j}\otimes e_{i,j}$$

for all i,j, where $\alpha_{i,j}\in\mathbb{C}$ and $f_{i,j}\in\mathcal{A}\otimes\mathcal{K}$. Then $\sum_{1\leq i,j<\infty}f_{i,j}\otimes e_{i,j}\in\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K}$. (In particular, the infinite sum converges in the norm topology, as a limit over the net of finite sums.)

Proof. By Lemma 2.4, $g:=\sum_{1\leq i,j<\infty}\alpha_{i,j}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}\otimes e_{i,j}$ is an element of $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}\otimes\mathbb{B}(\mathcal{H})$. Hence,

$$f:=c-g=\sum_{1\leq i,j<\infty}f_{i,j}\otimes e_{i,j}$$

is an element of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$ and has norm less than or equal to $\|c\|+\|g\|$. (Here, as in Lemma 2.4, we view the sums as being the limits of (nets of) finite sums in the strict topology on $\mathcal{M}(\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K})$.)

Moreover, since $\pi(c)$ and $\pi(g)$ both commute with every element of $\pi(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\otimes 1_{\mathcal{M}(\mathcal{K})})$,

$(*)$ $\quad\pi(f)=\pi(c)-\pi(g)$

commutes with every element of $\pi(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\otimes 1_{\mathcal{M}(\mathcal{K})})$.

Suppose, to the contrary, that $f\in\mathcal{A}\otimes\mathcal{K}\otimes\mathcal{K}$. Then there exists an $r>0$ such that for every positive integer $N\geq 1$,

$$\Big\|f-\sum_{1\leq i,j\leq N}f_{i,j}\otimes e_{i,j}\Big\|>r.$$

Hence, we can choose a subsequence $\{N_n\}_{n=1}^{\infty}$ of positive integers such that for all $n \geq 1$, $N_n + 1 \leq N_{n+1}$ and $f_n := \sum_{N_n+1 \leq \max\{i,j\} \leq N_{n+1}} f_{i,j} \otimes e_{i,j}$ has norm greater than r. But since $\pi(f)$ commutes with every element of $\pi(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})})$ (see $(*)$), for all $a \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ we have

$$\|(a \otimes 1_{\mathcal{M}(\mathcal{K})})f_n - f_n(a \otimes 1_{\mathcal{M}(\mathcal{K})})\| \to 0$$

as $n \to \infty$. This contradicts Lemma 2.1. ∎

LEMMA 2.6. *Let $\mathcal{A}$ be a unital simple separable C^*-algebra such that $\mathcal{M}(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})$ has real rank zero. Then*

$$\pi(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})})' \subseteq \pi(1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} \otimes \mathbb{B}(\mathcal{H})).$$

Proof. This follows from Lemmas 2.4 and 2.5. ∎

We note that the above lemma would not be true if we replaced $\pi(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})})'$ by $\pi(\mathcal{A} \otimes 1_{\mathcal{M}(\mathcal{K} \otimes \mathcal{K})})'$. A counterexample can be found where $\mathcal{A}$ is a unital simple separable infinite-dimensional *AF*-algebra.

THEOREM 2.7. *Let $\mathcal{A}$ be a unital separable simple nuclear C^*-algebra such that $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has real rank zero. Suppose that $\mathcal{C}$ is a simple liftable and purely large unital C^*-subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K})$. Then $\mathcal{C}'' = \mathcal{C}$.*

Proof. Note that $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K}$ and $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \cong \mathcal{M}(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})$. So we may assume that we are working in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})$.

Let $i : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})$ be the natural inclusion map. Since $\mathcal{C}$ is a liftable and purely large C^*-subalgebra, there exists a unital $*$-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})$ such that $\phi(\mathcal{C})$ is a purely large C^*-subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})$ and $i = \pi \circ \phi$.

Let $\psi' : \mathcal{C} \to \mathbb{B}(\mathcal{H})$ be any unital $*$-homomorphism (which is automatically faithful since $\mathcal{C}$ is simple). Let $\psi : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})$ be the unital $*$-homomorphism given by $\psi := 1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} \otimes \psi'$. Then by [2, Theorem 15.12.4] and [4], ψ also has the purely large property. Hence, as $\mathcal{A}$ is nuclear, it follows, by [4], that there is a unitary $u \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{K})$ such that $\pi(u)c\pi(u)^* = \pi(u)\pi \circ \phi(c)\pi(u)^* = \pi \circ \psi(c)$ for all $c \in \mathcal{C}$. Therefore, $\pi(u)\mathcal{C}\pi(u)^* = \pi \circ \psi(\mathcal{C})$. Hence, $\pi(u)\mathcal{C}'\pi(u)^* = \pi \circ \psi(\mathcal{C})'$ and $\pi(u)\mathcal{C}''\pi(u)^* = \pi \circ \psi(\mathcal{C})''$. Thus, to show that $\mathcal{C}'' = \mathcal{C}$, it suffices to prove that $\pi \circ \psi(\mathcal{C})'' = \pi \circ \psi(\mathcal{C})$.

Since $\pi \circ \psi(\mathcal{C}) \subseteq \pi(1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} \otimes \mathbb{B}(\mathcal{H}))$, we have $\pi(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})}) \subseteq \pi \circ \psi(\mathcal{C})'$. Hence, by Lemma 2.6,

$$\pi \circ \psi(\mathcal{C})'' \subseteq \pi(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})})' \subseteq \pi(1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} \otimes \mathbb{B}(\mathcal{H})).$$

Consequently, by Voiculescu's theorem ([13], [14] and [1]), we have $\pi \circ \psi(\mathcal{C})'' = \pi \circ \psi(\mathcal{C})$ as required. ∎

THEOREM 2.8. *Suppose that $\mathcal{A}$ is a unital simple separable nuclear C^*-algebra with $K_1(\mathcal{A}) = 0$ such that either*

(1) *$\mathcal{A}$ has real rank zero, stable rank one and weak unperforation, or*
(2) *$\mathcal{A}$ is purely infinite.*

Suppose that $\mathcal{C} \subseteq \mathcal{M}(\mathcal{A} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K})$ is a simple separable unital C^*-subalgebra such that there exists a unital $*$-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ with $\pi \circ \phi = i$, where $i : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K})$ is the natural inclusion map. Then $\mathcal{C}'' = \mathcal{C}$.

Proof. By [8], [9] and [16], the real rank of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is zero. By [7], every simple unital separable C^*-subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is purely large. Hence, the result follows from Theorem 2.7. ∎

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