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Optimal Sobolev imbedding spaces

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Abstract. This paper continues our study of Sobolev-type imbedding inequalities involving rearrangement-invariant Banach function norms. In it we characterize when the norms considered are optimal. Explicit expressions are given for the optimal partners corresponding to a given domain or range norm.

1. Introduction. Our aim is to further study those rearrangement-invariant Banach function spaces which are optimal in the Sobolev imbeddings considered in [2] and [4].

We begin by briefly describing the content of [4]. Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Let $\partial^\alpha/\partial x^\alpha := \partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$ be a differential operator of order $|\alpha| := \alpha_1 + \cdots + \alpha_n$, where $\alpha_i \in \mathbb{Z}_+ \cup \{0\}$, $i = 1, \ldots, n$. Denote by $|D^m u|$ the Euclidean length of the vector, $D^m u$, $1 \leq m \leq n-1$, of all derivatives of u of order m or less, whenever such derivatives exist on Ω in the weak sense. In [4] we considered Sobolev imbedding inequalities of the form

$$\sigma(u) \leq C\varrho(|D^m u|), \tag{1.1}$$

in which ϱ and σ are rearrangement-invariant (r.i.) norms (such as those of Lebesgue, Orlicz and Lorentz) and u belongs to the r.i. Sobolev space

$$W^{m,\varrho}(\Omega) := \{u : \Omega \to \mathbb{R} : \varrho(|D^m u|) < \infty\};$$

that is, we investigated when

$$W^{m,\varrho}(\Omega) \hookrightarrow L_\sigma(\Omega) := \{f : \Omega \to \mathbb{R} : \sigma(f) < \infty\}.$$

The focus was on cases in which ϱ and/or σ is optimal, namely $W^{m,\varrho}(\Omega)$ cannot be made larger and/or $L_\sigma(\Omega)$ cannot be made smaller. Expressions were given for the optimal partners of ϱ and σ in (1.1). They involved related r.i. norms, $\bar{\varrho}$ and $\bar{\sigma}$, defined at functions on $I_\Omega := (0, |\Omega|)$. Thus, for σ, the

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optimal ϱ, called ϱ_σ, had

$$\varrho_\sigma(f) := \sup \overline{\sigma}\Big(\int_t^{|\Omega|} h(s)s^{m/n-1}\,ds\Big), \qquad f:\Omega\to\mathbb{R},$$

the supremum being over all h on I_Ω such that

$$|\{t\in\mathbb{R}_+ : |h(t)|>\lambda\}| = |\{x\in\Omega : |f(x)|>\lambda\}|, \qquad \lambda\in\mathbb{R}_+;$$

as usual, $\mathbb{R}_+ := (0,\infty)$. Again, for ϱ, the optimal σ, denoted by σ_ϱ, satisfied

$$\sigma_\varrho'(g) := \bar{\varrho}'\Big(t^{m/n-1}\int_0^t g^*(s)\,ds\Big), \qquad g:\Omega\to\mathbb{R}, \tag{1.2}$$

where σ_ϱ' and $\bar{\varrho}'$ are the Köthe dual norms of σ_ϱ and $\bar{\varrho}$ discussed in Section 2 below and

$$g^*(t) := \inf\{\lambda>0 : \mu_g(\lambda)\le t\}, \qquad t\in I_\Omega,$$

with

$$\mu_g(\lambda) := |\{x\in\Omega : |g(x)|>\lambda\}|, \qquad \lambda\in\mathbb{R}_+,$$

is the decreasing rearrangement of g on I_Ω.

Proposition 5.2 in [4] proved that the formula for ϱ_σ can be dramatically improved if σ is optimal in (1.1). There is also a more explicit formula for σ_ϱ when ϱ is optimal in (1.1). These expressions, together with precise criteria for the optimality of ϱ and σ in (1.1), are the subject of Theorem A below.

To state the theorem we must, first of all, introduce two supremum operators, namely,

$$(S_{n/m}f)(t) := t^{m/n-1}\sup_{0<s\le t} s^{1-m/n}f^*(s)$$

and

$$(T_{n/m}f)(t) := t^{-m/n}\sup_{t\le s<|\Omega|} s^{m/n}f^*(s), \qquad f:I_\Omega\to\mathbb{R},\ t\in I_\Omega.$$

Observe that for $S_{n/m}f$ to be finite one requires

$$\sup_{0<s\le|\Omega|} s^{1-m/n}f^*(s)<\infty,$$

or, as we will write, $f\in L_{n/(n-m),\infty}(I_\Omega)$. Also, one has

$$(S_{n/m}f)^{**}(t)\approx(S_{n/m}f^{**})(t)\approx(S_{n/m}f)(t), \qquad f\in\mathfrak{M}_+(I_\Omega),\ t\in I_\Omega.$$

(We recall the notation $X\approx Y$, which signifies that each of X and Y is dominated by a constant multiple of the other, the constants being independent of all functions involved. More generally, $X\lesssim Y$ means X is no bigger than a constant times Y, with the constant independent of all functions involved.) For any measurable subset E of $\mathbb{R}^n$, we define

$$\mathfrak{M}(E) := \{f:E\to\mathbb{R} : f \text{ is measurable}\}$$

and denote by $\mathfrak{M}_+(E)$ the class of nonnegative functions in $\mathfrak{M}(E)$.

THEOREM A. *Fix $m, n \in \mathbb{Z}_+$, with $n \geq 2$ and $1 \leq m \leq n-1$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then, an r.i. norm ϱ on $\mathfrak{M}_+(\Omega)$, associated to the r.i. norm $\bar{\varrho}$ on $\mathfrak{M}_+(I_\Omega)$, with $L_{\bar{\varrho}}(I_\Omega) \supsetneq L_{n/m,1}(I_\Omega)$, is optimal in* (1.1) *for some r.i. norm σ on $\mathfrak{M}_+(\Omega)$ if and only if*

$$S_{n/m} : L_{\bar{\varrho}'}(I_\Omega) \to L_{\bar{\varrho}'}(I_\Omega). \tag{1.3}$$

In that case,

$$\sigma_\varrho(f) \approx \bar{\varrho}(t^{-m/n}[f^{**}(t) - f^*(t)]) + \int_0^1 f^*(t)\,dt, \qquad f \in \mathfrak{M}_+(\Omega), \tag{1.4}$$

where $f^{**}(t) := t^{-1}\int_0^t f^*(s)\,ds$.

Again, an r.i. norm σ on $\mathfrak{M}_+(\Omega)$, associated to the r.i. norm $\overline{\sigma}$ on $\mathfrak{M}_+(I_\Omega)$, is optimal in (1.1) *for some r.i. norm ϱ on $\mathfrak{M}_+(\Omega)$ if and only if*

$$T_{n/m} : L_{\overline{\sigma}'}(I_\Omega) \to L_{\overline{\sigma}'}(I_\Omega), \tag{1.5}$$

in which case

$$\varrho_\sigma(f) \approx \overline{\sigma}\Big(\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big), \qquad f \in \mathfrak{M}_+(\Omega). \tag{1.6}$$

In practice, one starts with a Sobolev space, $W^{m,\varrho}(\Omega)$, and seeks to find its optimal imbedding space, $L_{\sigma_\varrho}(\Omega)$. One can then go on to form $\varrho_D := \varrho_{\sigma_\varrho}$. It is readily seen that

$$W^{m,\varrho}(\Omega) \hookrightarrow W^{m,\varrho_D}(\Omega) \hookrightarrow L_{\sigma_\varrho}(\Omega)$$

and, indeed, that $W^{m,\varrho_D}(\Omega)$ is the largest Sobolev space that imbeds into $L_{\sigma_\varrho}(\Omega)$. Accordingly, we refer to ϱ_D as the optimal r.i. hull norm for ϱ in (1.1). Our new description of ϱ_D is given in

THEOREM B. *Fix $m, n \in \mathbb{Z}_+$, with $n \geq 2$ and $1 \leq m \leq n-1$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and suppose ϱ is an r.i. norm on $\mathfrak{M}_+(\Omega)$, associated to the r.i. norm $\bar{\varrho}$ on $\mathfrak{M}_+(I_\Omega)$. Then,*

$$\varrho_D(f) \approx \mu'(f^*), \qquad f \in \mathfrak{M}_+(\Omega),$$

where

$$\mu(g) := \bar{\varrho}'(S_{n/m}g^{**}), \qquad g \in \mathfrak{M}_+(I_\Omega).$$

The basic technical result on which the proofs of Theorems A and B depend is

PROPOSITION C. *Fix $m, n \in \mathbb{Z}_+$, with $n \geq 2$ and $1 \leq m \leq n-1$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and suppose ϱ is an r.i. norm on $\mathfrak{M}_+(\Omega)$, associated to the r.i. norm $\bar{\varrho}$ on $\mathfrak{M}_+(I_\Omega)$ satisfying $L_{\bar{\varrho}}(I_\Omega) \supsetneq L_{n/m,1}(I_\Omega)$.*

Then,

$$(1.7)\qquad \sigma_\varrho(f) \approx \sup_{\bar{\varrho}'(S_{n/m}g)\le 1} \int_0^{|\Omega|} t^{-m/n}[f^{**}(t)-f^*(t)]g^*(t)\,dt + \int_0^{|\Omega|} f^*(t)\,dt,$$

where $f \in \mathfrak{M}_+(\Omega)$, $g \in \mathfrak{M}_+(I_\Omega)$.

The structure of the paper is as follows. Section 2 contains background material on r.i. norms and an interpolation-theoretic result involving $S_{n/m}$ and $T_{n/m}$ needed later on. The optimal range, σ_ϱ, corresponding to a given ϱ, is treated in Section 3, which begins with the proof of Proposition C. Theorems A and B are proved in Section 4.

Theorem A is illustrated in the context of Orlicz spaces in the last section, using results from [3]. A property of the so-called level function, f°, of $f \in \mathfrak{M}(I_\Omega)$, necessary to obtain (1.4), is proved in an appendix.

Finally, we mention that, in [5], Proposition C turns out to be crucial to characterizing when the imbedding

$$W^{m,\varrho}(\Omega) \hookrightarrow L_\sigma(\Omega)$$

is compact.

2. Rearrangement-invariant norms. The decreasing rearrangement defined above satisfies [1, Chapter 2, Theorem 2.2]

$$(2.1)\qquad \int_\Omega f(x)g(x)\,dx \le \int_0^{|\Omega|} f^*(t)g^*(t)\,dt, \qquad f,g \in \mathfrak{M}_+(\Omega).$$

The operation of rearrangement is not sublinear, though for the Hardy average of h^*, namely $h^{**}(t) := t^{-1}\int_0^t h^*(s)\,ds$, $t \in I_\Omega$, we have [1, Chapter 2, Proposition 3.3]

$$(2.2)\qquad (f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \qquad f,g \in \mathfrak{M}_+(\Omega),\ t \in I_\Omega.$$

Definition 2.1. A *rearrangement-invariant* (*r.i.*) *Banach function norm* ϱ on $\mathfrak{M}_+(\Omega)$ satisfies the following seven axioms:

- (A_1) $\varrho(f) \ge 0$, with $\varrho(f) = 0$ if and only if $f = 0$ a.e. on Ω;
- (A_2) $\varrho(cf) = c\varrho(f)$, $c \ge 0$;
- (A_3) $\varrho(f+g) \le \varrho(f) + \varrho(g)$;
- (A_4) $f_n \uparrow f$ implies $\varrho(f_n) \uparrow \varrho(f)$;
- (A_5) $\varrho(\chi_E) < \infty$ for measurable $E \subset \Omega$, $|E| < \infty$;
- (A_6) $\int_E f(x)\,dx \le C_E\varrho(f)$, with $E \subset \Omega$, $|E| < \infty$, $C_E > 0$ independent of f;
- (A_7) $\varrho(f) = \varrho(g)$ whenever $\mu_f = \mu_g$.

According to a fundamental result of Luxemburg [1, Chapter 2, Theorem 4.10], to every r.i. norm ϱ on $\mathfrak{M}_+(\Omega)$ there corresponds an r.i. norm, $\bar{\varrho}$, on $\mathfrak{M}_+(I_\Omega)$, such that

$$\varrho(f) = \bar{\varrho}(f^*), \qquad f \in \mathfrak{M}_+(\Omega). \tag{2.3}$$

The basic technique for working with an r.i. norm ϱ involves the *Hardy–Littlewood–Pólya* (*HLP*) *Principle* (see [1, Chapter 2, Proposition 4.6]), which asserts that

$$f^{**}(t) \le g^{**}(t),\ t \in I_\Omega, \quad \text{implies} \quad \varrho(f) \le \varrho(g).$$

It is based on the following result of Hardy: if $f, g, h \in \mathfrak{M}_+(I_\Omega)$, then

$$\int_0^t f(s)\,ds \le \int_0^t g(s)\,ds, \quad t \in I_\Omega, \tag{2.4}$$
$$\Rightarrow \int_0^{|\Omega|} f(t)h^*(t)\,dt \le \int_0^{|\Omega|} g(t)h^*(t)\,dt.$$

The *Köthe dual* of an r.i. norm ϱ on $\mathfrak{M}_+(\Omega)$ is another such norm, ϱ', with

$$\varrho'(g) := \sup_{\varrho(h)\le 1} \int_\Omega g(x)h(x)\,dx, \qquad g, h \in \mathfrak{M}_+(\Omega).$$

It obeys the *Principle of Duality*,

$$\varrho'' := (\varrho')' = \varrho. \tag{2.5}$$

Further, the *Hölder inequality*,

$$\int_\Omega f(x)g(x)\,dx \le \varrho(f)\varrho'(g),$$

holds for all $f, g \in \mathfrak{M}_+(\Omega)$, and this inequality is saturated, in the sense that, given $f \in \mathfrak{M}_+(\Omega)$ and $\varepsilon > 0$, there exists $g_0 \in \mathfrak{M}_+(\Omega)$ such that $\varrho'(g_0) = 1$ and

$$\int_\Omega f(x)g_0(x)\,dx > (1-\varepsilon)\varrho(f).$$

Finally, $\overline{\varrho'} = \bar{\varrho}'$.

A smaller functional dual to the r.i. norm $\bar{\varrho}$ on $\mathfrak{M}(I_\Omega)$ will also be of interest to us, namely the *down dual norm*, $\bar{\varrho}'_d$, defined by

$$\bar{\varrho}'_d(g) := \sup_{\bar{\varrho}(h)\le 1} \int_0^{|\Omega|} g(t)h^*(t)\,dt, \qquad g, h \in \mathfrak{M}_+(I_\Omega).$$

One connection between ϱ' and $\bar{\varrho}'_d$, observed in [2, p. 312], is

$$\varrho'(g) = \bar{\varrho}'_d(g^*), \qquad g \in \mathfrak{M}_+(I_\Omega).$$

Recently, G. Sinnamon [7] proved

$$\bar{\varrho}'_d(g) = \bar{\varrho}'(g^\circ), \qquad g \in \mathfrak{M}_+(I_\Omega), \tag{2.6}$$

in which g°, referred to as the *level function* of g, is the (nonincreasing) derivative of the least concave majorant of $\int_0^t g(s)\,ds$, $t \in I_{\Omega}$. One has

$$\int_0^t g^*(s)\,ds \ge \int_0^t g^{\circ}(s)\,ds \approx t \sup_{t\le s<|\Omega|} s^{-1}\int_0^s g(y)\,dy, \qquad g \in \mathfrak{M}_+(I_{\Omega}). \tag{2.7}$$

The inequality in (2.7) is almost obvious. The equivalence was pointed out to us by G. Sinnamon ([8]); a proof of it, due to A. Gogatishvili, appears in the appendix at the end of this paper.

Corresponding to an r.i. norm ϱ on $\mathfrak{M}_+(\Omega)$ is the set

$$L_{\varrho}(\Omega) := \{f \in \mathfrak{M}(\Omega) : \varrho(|f|) < \infty\},$$

which becomes a Banach space when

$$\|f\|_{L_{\varrho}(\Omega)} := \varrho(|f|), \qquad f \in L_{\varrho}(\Omega);$$

indeed, it is a so-called *rearrangement-invariant Banach function space*, or, for short, an *r.i. space*. A detailed treatment of such spaces appears in [1, Chapters 1 and 2].

The *dilation operator* E_s, $s \in \mathbb{R}_+$, given at $f \in \mathfrak{M}_+(I_{\Omega})$, $t \in I_{\Omega}$, by

$$(E_s f)(t) := \begin{cases} f(t/s), & 0 < t \le |\Omega| s, \\ 0, & |\Omega| s < t < |\Omega|, \end{cases}$$

if $s \in (0,1)$, and by

$$(E_s f)(t) := f(t/s), \qquad 0 < t \le |\Omega|,$$

if $s \in [1,\infty)$, is bounded on any r.i. space $L_{\bar{\varrho}}(I_{\Omega})$ ([1, Chapter 3, Proposition 5.11]).

The *Lorentz norms*, $\varrho_{p,q}$, with $1 < p < \infty$, $1 \le q \le \infty$, are defined by

$$\varrho_{p,q}(f) := \Big(\int_0^{|\Omega|} (f^{**}(t) t^{1/p-1/q})^q\,dt\Big)^{1/q} \quad \text{when } q < \infty, \tag{2.8}$$

and

$$\varrho_{p,\infty}(f) := \sup_{0<t<|\Omega|} t^{1/p} f^{**}(t), \quad f \in \mathfrak{M}_+(\Omega).$$

In view of a well-known inequality of Hardy,

$$\varrho_{p,p}(f) \approx \|f\|_p := \Big(\int_{\Omega} f(x)^p\,dx\Big)^{1/p} = \Big(\int_0^{|\Omega|} f^*(t)^p\,dt\Big)^{1/p}, \qquad f \in \mathfrak{M}_+(\Omega).$$

We denote $L_{\varrho_{p,q}}(\Omega)$ by $L_{p,q}(\Omega)$.

To conclude, we record a special interpolation-theoretic result.

Suppose X_0, X_1 and X are r.i. spaces of functions in $\mathfrak{M}_+(\Omega)$ satisfying

$$X_0 \subset X \subset X_1 \quad \text{or} \quad X_0 \supset X \supset X_1.$$

We say that X is an *interpolation space* between X_0 and X_1, denoted $X \in \mathrm{Int}(X_0, X_1)$, if, for any linear operator T,

$$T : X_0 \to X_0 \text{ and } T : X_1 \to X_1 \quad \text{implies} \quad T : X \to X.$$

For example, if ϱ is any r.i. norm on $\mathfrak{M}_+(\Omega)$, then

$$L_1(\Omega) \supset L_\varrho(\Omega) \supset L_\infty(\Omega) \quad \text{and} \quad L_\varrho(\Omega) \in \mathrm{Int}(L_1(\Omega), L_\infty(\Omega));$$

see [1, Chapter 3, Theorem 2.12].

When X_0 and X_1 are certain Lorentz spaces, there are simple tests for $L_\varrho(\Omega) \in \mathrm{Int}(X_0, X_1)$ involving the supremum operators $S_{n/m}$ and $T_{n/m}$. More specifically, we have

THEOREM 2.2. *Let $m, n \in \mathbb{Z}_+$ with $n \geq 2$ and $1 \leq m \leq n-1$, and suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let ϱ be an r.i. norm on $\mathfrak{M}_+(\Omega)$. Then $L_\varrho(\Omega) \supset L_{n/m,1}(\Omega)$, and*

$$L_\varrho(\Omega) \in \mathrm{Int}(L_1(\Omega), L_{n/m,1}(\Omega)) \tag{2.9}$$

if and only if (1.3) *holds.*

Again, given $L_\varrho(\Omega) \subset L_{n/(n-m),1}(\Omega)$, we have

$$L_\varrho(\Omega) \in \mathrm{Int}(L_{n/(n-m),1}(\Omega), L_\infty(\Omega))$$

if and only if (1.5) *holds.*

The "if" parts are consequences of [4, Corollary 3.7 and Theorem 3.12]. The "only if" parts follow by standard arguments (see, for example, [1, Chapter 4, Section 4]) from the endpoint estimates for $S_{n/m}$ and $T_{n/m}$, in [4, Lemma 3.5], combined with their "quasisubadditivity" properties

$$(S_{n/m}[f+g])(t) \leq (S_{n/m}f)(t/2) + (S_{n/m}g)(t/2)$$

and

$$(T_{n/m}[f+g])(t) \leq (T_{n/m}f)(t/2) + (T_{n/m}g)(t/2), \quad f, g \in \mathfrak{M}_+(I_\Omega),\ t \in I_\Omega,$$

and the boundedness of the dilation operators on every r.i. space.

One readily sees from [4, Theorem A] that

$$\sigma_{\varrho_1} = \varrho_{n/(n-m),1} \quad \text{and} \quad \varrho_{\varrho_\infty} = \varrho_{n/m,1}.$$

Thus, when considering ϱ and σ in (1.1) one may safely assume

$$L_\varrho(\Omega) \supset L_{n/m,1}(\Omega) \quad \text{and} \quad L_\sigma(\Omega) \subset L_{n/(n-m),1}(\Omega).$$

3. The optimal range norm σ_ϱ. In the first part of this section we prove Proposition C. The strategy of the proof is as follows. According to (1.2),

$$\sigma_\varrho'(g) = \bar{\varrho}'\Big(t^{m/n-1}\int_0^t g^*(s)\,ds\Big) =: \lambda(g), \quad g \in \mathfrak{M}_+(\Omega). \tag{3.1}$$

Thus, we must show $\lambda'(f)$ is equivalent to the right side of (1.7).

We begin with two lemmas essential to the proof.

LEMMA 3.1. *Fix $b > 0$ and set $I_b := (0, b)$. Let $\bar{\varrho}$ be an r.i. norm on $\mathfrak{M}_+(I_b)$ such that $L_{\bar{\varrho}'}(I_b) \subsetneq L_{n/(n-m),\infty}(I_b)$. Then,*

$$\mu(f) := \sup_{\bar{\varrho}'(S_{n/m}g)\le 1} \int_0^b f^*(t)\, d \operatorname*{csup}_{0<s\le t} s^{1-m/n} g^*(s) + \int_0^{|\Omega|} f^*(s)\, ds, \tag{3.2}$$

$$f, g \in \mathfrak{M}_+(I_b),$$

is also an r.i. norm on $\mathfrak{M}_+(I_b)$; in (3.2), $\operatorname{csup}_{0<s\le t} s^{1-m/n} g^*(s) =: \alpha(t)$ *denotes the least concave majorant of* $\sup_{0<s\le t} s^{1-m/n} g^*(s) =: \beta(t)$, $t \in I_b$, *and*

$$\int_0^b f^*(t)\, d\alpha(t) := \lim_{\varepsilon\to 0+} \int_\varepsilon^{b-\varepsilon} f^*(t)\, d\alpha(t).$$

Proof. To start, observe that $\beta(t)$ is quasiconcave (so $\beta(t) \le \alpha(t) \le 2\beta(t)$) and that $\bar{\varrho}'(S_{n/m}g) < \infty$ implies $\beta(b-) = (S_{n/m}g)(b-) < \infty$. Thus, $\alpha(t)$ is continuous on I_b (in fact, locally Lipschitz of order 1) and hence

$$\int_\varepsilon^{1-\varepsilon} f^*(t)\, d\alpha(t)$$

is well defined as a (Riemann) Stieltjes integral, for all ε with $0 < \varepsilon < b/2$. Indeed,

$$\int_\varepsilon^{b-\varepsilon} f^*(t)\, d\alpha(t) = \int_\varepsilon^{b-\varepsilon} f^*(t) h(t)\, dt,$$

hence

$$\int_0^b f^*(t)\, d\alpha(t) = \int_0^b f^*(t) h(t)\, dt,$$

where $h(t) := d\alpha(t)/dt$ is nonincreasing.

As for μ being an r.i. norm, only the subadditivity requires comment. But, it readily follows once we observe that, given $f_1, f_2 \in \mathfrak{M}_+(I_b)$, (2.2) and (2.4) ensure

$$\begin{aligned}\int_0^b (f_1+f_2)^*(t)\, d\alpha(t) &= \int_0^b (f_1+f_2)^*(t) h(t)\, dt \le \int_0^b [f_1^*(t) + f_2^*(t)] h(t)\, dt \\ &= \int_0^b f_1^*(t)\, d\alpha(t) + \int_0^b f_2^*(t)\, d\alpha(t). \ \blacksquare\end{aligned}$$

LEMMA 3.2. *Suppose ϱ is an r.i. norm on $\mathfrak{M}_+(\Omega)$, associated to the r.i. norm $\bar{\varrho}$ on $\mathfrak{M}_+(I_\Omega)$, with $L_{\bar{\varrho}}(I_\Omega) \subsetneq L_{n/m,\infty}(I_\Omega)$, and let λ be defined as* *in* (3.1). *Then*, $\lambda' \approx \tau$ *with*

$$\tau(f) := \sup_{\bar{\varrho}'(S_{n/m} g)\le 1} \int_0^{|\Omega|} -t^{1-m/n} g^*(t)\, df^*(t) + \int_0^{|\Omega|} f^*(t)\, dt$$

for $f \in \mathfrak{M}(\Omega)$, $g \in C(I_\Omega)$.

Proof. In view of Corollary 3.7 and Theorem 3.13 of [4], we may assume

$$\lambda(g) \approx \nu\Big(t^{m/n-1} \int_0^t g^*(s)\, ds\Big), \qquad g \in \mathfrak{M}(\Omega), \tag{3.3}$$

where

$$\nu(h) = \bar{\varrho}'(S_{n/m} h^{**}) \approx \bar{\varrho}'(S_{n/m} h), \qquad h \in \mathfrak{M}(I_\Omega), \tag{3.4}$$

and

$$S_{n/m} : L_\nu(I_\Omega) \to L_\nu(I_\Omega). \tag{3.5}$$

We first show that $\tau' \lesssim \lambda$. For any $f, g \in C(I_\Omega)$, with $f^*(0+) < \infty$ and $f^*(|\Omega|-) = 0$, we have

$$\begin{aligned}
\int_0^{|\Omega|} g^*(t) f^*(t)\, dt &\le \int_0^{|\Omega|} g^*(t) \int_t^{|\Omega|} -df^*(s)\, dt = \int_0^{|\Omega|} -\int_0^t g^*(s)\, ds\, df^*(t) \\
&= \int_0^{|\Omega|} -t^{1-m/n} t^{m/n-1} \int_0^t g^*(s)\, ds\, df^*(t) \\
&\le \int_0^{|\Omega|} -t^{1-m/n} \sup_{t\le s<|\Omega|} s^{m/n-1} \int_0^s g^*(y)\, dy\, df^*(t) \\
&\lesssim \lambda(g) \nu\Big(t^{m/n-1} \int_0^t g^*(s)\, ds\Big)^{-1} \int_0^{|\Omega|} -t^{1-m/n} \sup_{t\le s<|\Omega|} s^{m/n-1} \int_0^s g^*(y)\, dy\, df^*(t) \\
&\lesssim \lambda(g) \nu\Big(\sup_{t\le s<|\Omega|} s^{m/n-1} \int_0^s g^*(y)\, dy\Big)^{-1} \\
&\qquad \times \int_0^{|\Omega|} -t^{1-m/n} \sup_{t\le s<|\Omega|} s^{m/n-1} \int_0^s g^*(y)\, dy\, df^*(t) \\
&\lesssim \lambda(g) \sup_{\nu(h)\le 1} \int_0^{|\Omega|} -t^{1-m/n} h^*(t)\, df^*(t) \lesssim \lambda(g)\tau(f),
\end{aligned}$$

in which (3.3), Theorem 3.9 of [4] and (3.4) combined with (3.5) were used to obtain the fourth last, third last and second last inequalities, respectively. Thus, $\tau' \lesssim \lambda$.

To prove $\lambda \lesssim \tau'$ we show the existence of $C > 0$ such that to each $g \in \mathfrak{M}_+(\Omega)$, $\lambda(g) < \infty$, there corresponds $f_0 \in \mathfrak{M}_+(\Omega)$ satisfying $f_0^*(0+) < \infty$, $f_0^*(|\Omega|-) = 0$, $\tau(f_0) \le C$ and

$$\int_0^{|\Omega|} g^*(t) f_0^*(t)\, dt \ge C^{-1}\lambda(g).$$

Now, $\lambda(g) < \infty$ implies the existence of $k_0 \in \mathfrak{M}_+(I_\Omega)$, with $\bar{\varrho}(k_0) \le 1$, such that

$$\int_0^{|\Omega|} k_0(t) t^{m/n-1} \int_0^t g^*(s)\, ds\, dt > \frac{1}{2}\lambda(g).$$

Take f_0 such that

$$f_0^*(t) = \int_t^{|\Omega|} k_0(s) s^{m/n-1}\, ds, \qquad t \in I_\Omega.$$

Then, for $h = h^* \in \mathfrak{M}_+(I_\Omega)$ with $\nu(h) \le 1$,

$$\begin{aligned}\int_0^{|\Omega|} -t^{1-m/n} h^*(t)\, df_0(t) &= \int_0^{|\Omega|} h^*(t) k_0^*(t)\, dt \le \nu(h)\nu'(k_0)\\ &\lesssim \nu(h)\bar{\varrho}(k_0) \quad (\bar{\varrho}' \le \nu \text{ implies } \nu' \le \bar{\varrho})\\ &\le C,\end{aligned}$$

and

$$\begin{aligned}\int_0^{|\Omega|} f_0^*(t)\, dt &= \int_0^{|\Omega|}\int_t^{|\Omega|} k_0(s) s^{m/n-1}\, ds\, dt = \int_0^{|\Omega|} k_0(t) t^{m/n}\, dt\\ &\lesssim \int_0^{|\Omega|} k_0(t)\, dt \lesssim \varrho(k_0) \le C,\end{aligned}$$

so $\tau(f_0) \le C$. Further,

$$\begin{aligned}\int_0^{|\Omega|} g^*(t) f_0^*(t)\, dt &= \int_0^{|\Omega|} g^*(t) \int_t^1 k_0(s) s^{m/n-1}\, ds\, dt\\ &= \int_0^{|\Omega|} k_0(t) t^{m/n-1} \int_0^t g^*(s)\, ds\, dt \ge \frac{1}{2}\lambda(g).\end{aligned}$$

The result will follow by the Principle of Duality once we verify

$$\tau(f) \approx \mu(f), \qquad f \in \mathfrak{M}_+(\Omega),\ f^*(0+) < \infty,\ f^*(|\Omega|-) = 0,$$

where $\mu(f)$ is defined as in (3.2) with $b = |\Omega|$, since μ was shown to be an r.i. norm in Lemma 3.1.

When $g = g^* \in C(I_\Omega)$ with $g^*(0+) < \infty$,
$$\lim_{t\to 0+} f^*(t) \operatorname*{csup}_{0<s\le t} s^{1-m/n} g^*(s) = \lim_{t\to|\Omega|-} f^*(t) \operatorname*{csup}_{0<s\le t} s^{1-m/n} g^*(s) = 0,$$
and, thus, integration by parts yields
$$\begin{aligned}\int_0^{|\Omega|} f^*(t)\, d \operatorname*{csup}_{0<s\le t} s^{1-m/n} g^*(s) &= \int_0^{|\Omega|} - \operatorname*{csup}_{0<s\le t} s^{1-m/n} g^*(s)\, df^*(t)\\ &\ge \int_0^1 -t^{1-m/n} g^*(t)\, df^*(t),\end{aligned}$$
whence
$$\mu(f) \ge \tau(f), \qquad f \in \mathfrak{M}_+(I_\Omega).$$
Again,
$$\begin{aligned}&\sup_{\bar\varrho'(S_{n/m}g)\le 1} \int_0^{|\Omega|} - \operatorname*{csup}_{0<s\le t} s^{1-m/n} g^*(s)\, df^*(t)\\
&\quad\lesssim \sup_{\nu(g)\le 1} \int_0^{|\Omega|} -t^{1-m/n} t^{m/n-1} \operatorname*{csup}_{0<s\le t} s^{1-m/n} g^*(s)\, df^*(t)\\
&\quad\lesssim \sup_{\nu(g)\le 1} \int_0^{|\Omega|} -t^{1-m/n} (S_{n/m}g)(t)\, d(t)\\
&\quad\lesssim \sup_{\nu(S_{n/m}g)\le 1} \int_0^{|\Omega|} -t^{1-m/n} (S_{n/m}g)(t)\, df^*(t) \qquad \text{by (3.5)}\\
&\quad\lesssim \sup_{\nu(g)\le 1} \int_0^{|\Omega|} -t^{1-m/n} g^*(t)\, df^*(t)\\
&\quad\lesssim \sup_{\bar\varrho'(S_{n/m}g)\le 1} \int_0^{|\Omega|} -t^{1-m/n} g^*(t)\, df^*(t) \qquad (g = g^* \in \mathfrak{M}_+(I_\Omega))\\
&\quad\lesssim \tau(f).\end{aligned}$$
To get the second line of the last chain of inequalities, we have used the quasiconcavity of $\beta(t) = \sup_{0<s\le t} s^{1-m/n} g^*(s)$, $t \in I_\Omega$. ∎

Proof of Proposition C. In view of Lemma 3.2, σ_ϱ satisfies
$$\sigma_\varrho(f) \approx \sup_{\bar\varrho'(S_{n/m}g)\le 1} \int_0^{|\Omega|} -t^{1-m/n} g^*(t)\, df^*(t) + \int_0^{|\Omega|} f^*(t)\, dt, \tag{3.6}$$
where $f \in \mathfrak{M}_+(\Omega)$ and $g \in C(I_\Omega)$. Define the operator P by

$$(Ph)(t) := t^{-1}\int_0^t h(s)\,ds, \qquad h \in \mathfrak{M}_+(I_\Omega),\ t \in I_\Omega.$$

According to [4, Theorem 3.12], $L_{\sigma_\varrho}(I_\Omega)$ is an interpolation space between $L_{n/(n-m),1}(I_\Omega)$ and $L_\infty(I_\Omega)$, hence Theorem 5.15 in Chapter 3 of [1] ensures

$$P : L_{\sigma_\varrho}(I_\Omega) \to L_{\sigma_\varrho}(I_\Omega).$$

This means we can replace $f^*(t)$ by $f^{**}(t)$ and, indeed, by $t^{-1}\int_0^t f^{**}(s)\,ds$, on the right side of (3.6).

Now, for each ε with $0 < \varepsilon < |\Omega|/2$,

$$\begin{aligned}
&\int_\varepsilon^{|\Omega|-\varepsilon} -t^{1-m/n} g^*(t)\, d\Big[t^{-1}\int_0^t f^{**}(s)\,ds\Big] \\
&\qquad = \int_\varepsilon^{|\Omega|-\varepsilon} -t^{1-m/n} g^*(t)\Big[-t^{-2}\int_0^t f^{**}(s)\,ds + t^{-1}f^{**}(t)\Big]\,dt \\
&\qquad = \int_\varepsilon^{|\Omega|-\varepsilon} t^{-m/n}\Big[t^{-1}\int_0^t [f^{**}(s) - f^*(s)]\,ds\Big] g^*(t)\,dt,
\end{aligned}$$

so

$$\begin{aligned}
&\int_0^{|\Omega|} -t^{1-m/n} g^*(t)\, d\Big[t^{-1}\int_0^t f^{**}(s)\,ds\Big] \\
&\qquad = \int_0^{|\Omega|} t^{-m/n}\Big[t^{-1}\int_0^t [f^{**}(s) - f^*(s)]\,ds\Big] g^*(t)\,dt \\
&\qquad = \int_0^{|\Omega|} t^{-m/n}[f^{**}(t) - f^*(t)]\Big[t^{m/n}\int_t^{|\Omega|} g^*(s)s^{-m/n-1}\,ds\Big]\,dt.
\end{aligned}$$

Again, the operator $R_{n/m}$, defined by

$$(R_{n/m}h)(t) := t^{m/n}\int_t^{|\Omega|} h(s)s^{-m/n-1}\,ds, \qquad h \in \mathfrak{M}_+(I_\Omega),\ t \in I_\Omega,$$

satisfies

$$(R_{n/m}g^*)(t) \le \frac{n}{m}\, g^*(t)$$

and

$$\begin{aligned}
(R_{n/m}g^*)(t/2) &\ge (t/2)^{m/n}\int_{t/2}^t g^*(s)s^{-m/n-1}\,ds \\
&\ge \frac{n}{m}\,[1 - 2^{-m/n}]g^*(t), \qquad g \in \mathfrak{M}_+(I_\Omega),\ t \in I_\Omega.
\end{aligned}$$

We conclude from the foregoing and (3.4) that

$$\begin{aligned}\sigma_\varrho(f) &\approx \sigma_\varrho\Big(t^{-1}\int_0^t f^{**}(s)\,ds\Big)\\
&\approx \sup_{\nu(g)\le 1}\int_0^{|\Omega|} t^{-m/n}[f^{**}(t)-f^*(t)](R_{n/m}g^*)(t)\,dt+\int_0^{|\Omega|} f^*(t)\,dt\\
&\approx \sup_{\nu(R_{n/m}g^*)\le 1}\int_0^{|\Omega|} t^{-m/n}[f^{**}(t)-f^*(t)](R_{n/m}g^*)(t)\,dt+\int_0^{|\Omega|} f^*(t)\,dt\\
&\approx \sup_{\nu(g)\le 1}\int_0^{|\Omega|} t^{-m/n}[f^{**}(t)-f^*(t)]g^*(t)\,dt+\int_0^{|\Omega|} f^*(t)\,dt\\
&\approx \sup_{\bar\varrho'(S_{n/m}g)\le 1}\int_0^{|\Omega|} t^{-m/n}[f^{**}(t)-f^*(t)]g^*(t)\,dt+\int_0^{|\Omega|} f^*(t)\,dt,\end{aligned}$$

with $f\in\mathfrak{M}_+(\Omega)$, $g\in\mathfrak{M}_+(I_\Omega)$, as required. ∎

Our next result is a part of Theorem A which seems to be of independent interest.

THEOREM 3.3. *Let* m, n, Ω, ϱ *and* $\bar\varrho$ *be as in Theorem A. Then*, (1.3) *implies* (1.4).

Proof. As a consequence of Proposition C and (1.3) we have, for $f\in\mathfrak{M}_+(I_\Omega)$,

$$\begin{aligned}\text{(3.7)}\qquad \sigma_\varrho(f) &\approx \sup_{\bar\varrho'(g)\le 1}\int_0^{|\Omega|} t^{-m/n}[f^{**}(t)-f^*(t)]g^*(t)\,dt+\int_0^{|\Omega|} f^*(t)\,dt\\
&\approx (\bar\varrho')_d'(t^{-m/n}[f^{**}(t)-f^*(t)])+\int_0^{|\Omega|} f^*(t)\,dt\\
&\approx \bar\varrho((s^{-m/n}[f^{**}(s)-f^*(s)])^\circ(t))+\int_0^{|\Omega|} f^*(t)\,dt,\end{aligned}$$

by (2.6) and the Principle of Duality.

The definition of the level function ensures

$$\int_0^t s^{-m/n}[f^{**}(s)-f^*(s)]\,ds\le\int_0^t (y^{-m/n}[f^{**}(y)-f^*(y)])^\circ(s)\,ds,$$

from which (2.4) yields

$$\int_0^t s^{-m/n}[f^{**}(s)-f^*(s)]g^{**}(s)\,ds \le \int_0^t (y^{-m/n}[f^{**}(y)-f^*(y)])^\circ(s)g^{**}(s)\,ds,$$

or

$$\int_0^t g^*(s)\int_s^t y^{-m/n}[f^{**}(y)-f^*(y)]\,\frac{dy}{y}\,ds \le \int_0^t g^*(s)\int_s^t (z^{-m/n}[f^{**}(z)-f^*(z)])^\circ(y)\,\frac{dy}{y}\,ds, \tag{3.8}$$

for $f\in\mathfrak{M}_+(\Omega)$, $g\in\mathfrak{M}_+(I_\Omega)$, $t\in I_\Omega$. But, for $f\in\mathfrak{M}_+(\Omega)$,

$$\begin{aligned}
\int_t^{|\Omega|} s^{-m/n}f^{**}(s)\frac{ds}{s} &= \int_t^{|\Omega|} s^{-m/n-2}\int_0^s f^*(y)\,dy\,ds\\
&= \int_0^{|\Omega|} f^*(y)\int_t^{|\Omega|} s^{-m/n-2}\chi_{(0,s)}(y)\,ds\,dy\\
&= \int_0^t f^*(y)\,dy\int_t^{|\Omega|} s^{-m/n-2}\,ds + \int_t^{|\Omega|} f^*(y)\int_y^{|\Omega|} s^{-m/n-2}\,ds\,dy\\
&= \int_t^{|\Omega|} f^*(y)\int_y^{|\Omega|} s^{-m/n-2}\,ds\,dy + \frac{n}{n+m}\,t^{-n/m}t^{-1}\int_0^t f^*(s)\,ds\\
&\quad - \frac{n}{n+m}\,|\Omega|^{-m/n-1}\int_0^t f^*(s)\,ds, \qquad f\in\mathfrak{M}_+(I_\Omega),
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^{|\Omega|} f^*(y)\int_y^{|\Omega|} s^{-m/n-2}\,ds\,dy - \int_t^{|\Omega|} y^{-m/n-1}f^*(y)\,dy\\
&= \frac{n}{n+m}\int_t^{|\Omega|} y^{-m/n-1}f^*(y)\,ds - \frac{n}{n+m}\,|\Omega|^{-m/n-1}\int_t^{|\Omega|} f^*(y)\,dy\\
&\quad - \int_t^{|\Omega|} y^{-m/n-1}f^*(y)\,dy\\
&= -\frac{m}{n+m}\int_t^{|\Omega|} y^{-m/n-1}f^*(y)\,dy - \frac{n}{n+m}\,|\Omega|^{-m/n-1}\int_t^{|\Omega|} f^*(y)\,dy
\end{aligned}$$

$$\geq -\frac{n}{n+m}\,t^{-m/n}f^*(t)-\frac{n}{n+m}\,|\Omega|^{-m/n-1}\int_0^{|\Omega|} f^*(t)\,dt, \qquad f\in\mathfrak{M}_+(I_\Omega).$$

Thus,

$$\int_t^{|\Omega|} s^{-m/n}[f^{**}(s)-f^*(s)]\frac{ds}{s}$$
$$\geq \frac{n}{n+m}\,[f^{**}(t)-f^*(t)]-\frac{2n}{n+m}\,|\Omega|^{-m/n-1}\int_0^{|\Omega|} f^*(t)\,dt,$$

so

$$\begin{aligned}
&\frac{n}{n+m}\,\bar\varrho(t^{-m/n}[f^{**}(t)-f^*(t)])\\
&\leq \bar\varrho\bigg(\int_t^{|\Omega|} s^{-m/n}[f^{**}(s)-f^*(s)]\,\frac{ds}{s}\bigg)+\frac{2n}{n+m}\,|\Omega|\bar\varrho(\chi_{I_\Omega})\int_0^{|\Omega|} f^*(s)\,ds\\
&\lesssim \bar\varrho\bigg(\int_t^{|\Omega|} (y^{-m/n}[f^{**}(y)-f^*(y)])^\circ(s)\,\frac{ds}{s}\bigg)+\int_0^{|\Omega|} f^*(t)\,dt \quad \text{by (3.8) and HLP}\\
&\lesssim \bar\varrho((s^{-m/n}[f^{**}(s)-f^*(s)])^\circ(t))+\int_0^{|\Omega|} f^*(t)\,dt\\
&\lesssim \sigma_\varrho(f), \quad f\in\mathfrak{M}_+(\Omega), \quad \text{by (3.7);}
\end{aligned}$$

here we have used the facts that the operator

$$(Qf)(t):=\int_t^{|\Omega|} f(s)\,\frac{ds}{s}, \qquad f\in\mathfrak{M}_+(I_\Omega),\ t\in I_\Omega,$$

satisfies

$$Q:L_{\bar\varrho}(I_\Omega)\to L_{\bar\varrho}(I_\Omega) \quad \text{if and only if} \quad P:L_{\bar\varrho'}(I_\Omega)\to L_{\bar\varrho'}(I_\Omega),$$

and that

$$L_{\bar\varrho'}(I_\Omega)\in \mathrm{Int}(L_{n/(n-m),\infty}(I_\Omega),L_\infty(I_\Omega)).$$

Since one always has

$$\sigma_\varrho(f)\lesssim \bar\varrho(t^{-m/n}[f^{**}(t)-f^*(t)])+\int_0^{|\Omega|} f^*(t)\,dt, \qquad f\in\mathfrak{M}_+(\Omega),$$

because of (3.7) and $\bar\varrho(h)\geq\varrho(h^\circ)$ (by (2.7) and the HLP Principle), the proof is complete. ■

COROLLARY 3.4. *Let m, n, Ω, ϱ and $\bar\varrho$ be as in Theorem A. Set*

$$\tau(g):=\bar\varrho'(S_{n/m}g^{**}), \qquad g\in\mathfrak{M}_+(\Omega). \tag{3.9}$$

Then, τ is an r.i. norm on $\mathfrak{M}_+(\Omega)$ and

$$\sigma_\varrho(f) \approx \tau'(t^{-m/n}[f^{**}(t) - f^*(t)]) + \int_0^{|\Omega|} f^*(t)\,dt, \qquad f \in \mathfrak{M}_+(\Omega). \tag{3.10}$$

Proof. The functional τ is readily seen to be an r.i. norm such that $L_\varrho(\Omega) \subset L_{\tau'}(\Omega)$. Moreover, by (1.3),

$$\bar{\tau}(S_{n/m}h) \approx \bar{\varrho}'(S_{n/m}(S_{n/m}h)) = \bar{\varrho}'(S_{n/m}h) \approx \bar{\tau}(h), \quad h \in \mathfrak{M}_+(I_\Omega). \tag{3.11}$$

Thus, Theorem 3.3 guarantees

$$\sigma_{\tau'}(f) \approx \tau'(t^{-m/n}[f^{**}(t) - f^*(t)]) + \int_0^{|\Omega|} f^*(t)\,dt, \qquad f \in \mathfrak{M}_+(I_\Omega).$$

But, from Proposition C,

$$\begin{aligned}
\sigma_{\tau'}(f) &= \sup_{\bar{\tau}(S_{n/m}g)\le 1} \int_0^{|\Omega|} t^{-m/n}[f^{**}(t) - f^*(t)]g^*(t)\,dt + \int_0^{|\Omega|} f^*(t)\,dt \\
&\approx \sup_{\bar{\tau}(g)\le 1} \int_0^{|\Omega|} t^{-m/n}[f^{**}(t) - f^*(t)]g^*(t)\,dt + \int_0^{|\Omega|} f^*(t)\,dt \quad \text{by (3.11)} \\
&\approx \sup_{\bar{\varrho}'(S_{n/m}g)\le 1} \int_0^{|\Omega|} t^{-m/n}[f^{**}(t) - f^*(t)]g^*(t)\,dt + \int_0^{|\Omega|} f^*(t)\,dt \\
&\qquad\qquad \text{by (3.9) and (1.3)} \\
&\approx \sigma_\varrho(f), \qquad f \in \mathfrak{M}_+(\Omega),
\end{aligned}$$

and (3.10) follows. ∎

Remark 3.5. Some r.i. norms μ *require* h^* in order to compute $\mu(h)$. Should this prove difficult for the $\mu = \bar{\varrho}$ and $h(t) = t^{-m/n}\,[f^{**}(t) - f^*(t)]$ in (1.4), the first paragraph of the proof of Theorem 3.3, together with (A.1) below, offers an alternative expression, given $P : L_{\bar{\varrho}}(\Omega) \to L_{\bar{\varrho}}(\Omega)$, namely,

$$\sigma_\varrho(f) \approx \bar{\varrho}\Big(\sup_{t\le s<|\Omega|} s^{-1}\int_0^s y^{-m/n}\,[f^{**}(y) - f^*(y)]\,dy\Big) + \int_0^{|\Omega|} f^*(t)\,dt,$$

for $f \in \mathfrak{M}(\Omega)$. Here, the function h to which the norm $\bar{\varrho}$ is applied is its own rearrangement.

4. Proofs of Theorems A and B

Proof of Theorem A. By [4, Corollary 3.14],

$$L_{\overline{\varrho_\sigma}}(I_\Omega) \in \mathrm{Int}(L_1(I_\Omega), L_{n/m,1}(I_\Omega)).$$

Theorem 2.2 then yields

$$S_{n/m} : L_{\overline{\varrho_\sigma}'}(I_\Omega) \to L_{\overline{\varrho_\sigma}'}(I_\Omega), \tag{4.1}$$

and this, by Theorem 3.3, implies

$$\sigma_{\varrho_\sigma}(f) \approx \overline{\varrho_\sigma}(t^{-m/n}[f^{**}(t) - f^*(t)]) + \int_0^{|\Omega|} f^*(t)\,dt, \qquad f \in \mathfrak{M}_+(\Omega). \tag{4.2}$$

Further, Proposition 5.2 in [4] guarantees

$$\varrho_{\sigma_\varrho}(f) \approx \overline{\sigma_\varrho}\Big(\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big), \qquad f \in \mathfrak{M}_+(\Omega). \tag{4.3}$$

When ϱ is optimal in (1.1), $\varrho \approx \varrho_\sigma$, so (1.3) holds, by (4.1), and (4.2) becomes (1.4).

Given (1.3), we have (1.4), in view of Corollary 3.4. We claim that (4.3) and (1.4) together ensure

$$\varrho_{\sigma_\varrho}(f) \approx \varrho(f), \qquad f \in \mathfrak{M}_+(\Omega),$$

and, hence, the optimality of ϱ in (1.1). Indeed, for $f \in \mathfrak{M}_+(\Omega)$,

$$\begin{aligned}
\varrho_{\sigma_\varrho}(f) &\approx \overline{\sigma_\varrho}\Big(\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big) \\
&\approx \bar{\varrho}\Big(t^{-m/n}\Big[t^{-1}\int_0^t\int_s^{|\Omega|} f^*(y)y^{m/n-1}\,dy\,ds - \int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big]\Big) \\
&\quad + \int_0^{|\Omega|}\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\,dt \qquad \text{by (1.4)} \\
&\approx \bar{\varrho}\Big(t^{-m/n-1}\int_0^t f^*(s)s^{m/n}\,ds\Big),
\end{aligned}$$

since

$$t^{-1}\int_0^t\int_s^{|\Omega|} f^*(y)y^{m/n-1}\,dy\,ds = t^{-1}\int_0^t f^*(s)s^{m/n}\,ds + \int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds$$

and

$$\begin{aligned}
\int_0^{|\Omega|}\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\,dt &= \int_0^{|\Omega|} f^*(s)s^{m/n}\,ds \\
&= C\bar{\varrho}\Big(\int_0^{|\Omega|} f^*(s)s^{m/n}\,ds\Big) \qquad (C = \bar{\varrho}(\chi_{I_\Omega})^{-1})
\end{aligned}$$

$$\leq C\bar{\varrho}\bigg(\int_0^{|\Omega|} f^*\bigg(\frac{ts}{|\Omega|}\bigg)s^{m/n}\,ds\bigg) \leq C|\Omega|^{m/n-1}\bar{\varrho}\Big(t^{-m/n-1}\int_0^t f^*(s)s^{m/n}\,ds\Big).$$

The operator

$$f \mapsto t^{-m/n-1}\int_0^t f(s)s^{m/n}\,ds$$

is the associate of the operator $R_{n/m}$ in the proof of Proposition C, and therefore

$$\bar{\varrho}\Big(t^{-m/n-1}\int_0^t f^*(s)s^{m/n}\,ds\Big) \lesssim \bar{\varrho}(f), \qquad f \in \mathfrak{M}_+(\Omega).$$

But

$$t^{-m/n-1}\int_0^t f^*(s)s^{m/n}\,ds \geq \frac{n}{n+m}\,f^*(t), \qquad t \in I_\Omega,$$

whence

$$\bar{\varrho}(f) \approx \bar{\varrho}\Big(t^{-m/n-1}\int_0^t f^*(s)s^{m/n}\,ds\Big) \approx \varrho_{\sigma_\varrho}(f), \qquad f \in \mathfrak{M}_+(\Omega).$$

The proof of the assertion concerning the optimality of σ is similar to the one for ϱ. Thus, if σ is optimal in (1.1), then $\sigma \approx \sigma_\varrho$ and (1.5) holds by [4, Theorem 3.12]; in that case, (1.7) is satisfied.

Given (1.5), Proposition 5.2 in [4] ensures (1.6). Using (4.2) and (1.6), we will obtain

$$\sigma_{\varrho_\sigma}(f) \approx \sigma(f), \qquad f \in \mathfrak{M}_+(\Omega),$$

and thus, the optimality of σ in (1.1). In fact, it suffices to show

$$\sigma_{\varrho_\sigma}(f) \lesssim \sigma(f), \qquad f \in \mathfrak{M}_+(\Omega).$$

Now, if $0 < t < |\Omega|/2$, then

$$\begin{aligned}
&\int_t^{|\Omega|} s^{-m/n}[f^{**}(s) - f^*(s)]\,\frac{ds}{s} \\
&\qquad = \int_t^{|\Omega|} s^{-m/n-2}\int_0^s f^*(y)\,dy\,ds - \int_t^{|\Omega|} s^{-m/n-1}f^*(s)\,ds \\
&\qquad = \int_t^{|\Omega|} s^{-m/n-2}\,ds\int_0^t f^*(y)\,dy + \int_t^{|\Omega|} s^{-m/n-2}\int_t^s f^*(y)\,dy\,ds \\
&\qquad\quad - \int_t^{|\Omega|} s^{-m/n-1}f^*(s)\,ds
\end{aligned}$$

$$\begin{aligned}
&\geq \int_t^{|\Omega|} s^{-m/n-2}\,ds \int_0^t f^*(y)\,dy + \int_t^{|\Omega|} s^{-m/n-2}(s-t)f^*(s)\,ds \\
&\quad - \int_t^{|\Omega|} s^{-m/n-1} f^*(s)\,ds \\
&= \int_t^{|\Omega|} s^{-m/n-2}\,ds \int_0^t f^*(y)\,dy - t\int_t^{|\Omega|} s^{-m/n-2} f^*(s)\,ds \\
&\geq t\int_t^{2t} s^{-m/n-2}\,ds\,[f^{**}(t)-f^*(t)] \\
&\geq \frac{1}{2}\,\frac{n}{n+m}\,t^{-m/n}[f^{**}(t)-f^*(t)],
\end{aligned}$$

while if $|\Omega|/2 \leq t < |\Omega|$, then

$$t^{-m/n}[f^{**}(t)-f^*(t)] \leq \left(\frac{2}{|\Omega|}\right)^{m/n+1} \int_0^{|\Omega|} f^*(t)\,dt, \qquad f \in \mathfrak{M}_+(\Omega).$$

We conclude that when $f \in \mathfrak{M}_+(\Omega)$,

$$\begin{aligned}
\sigma_{\varrho_\sigma}(f) &\approx \overline{\varrho_\sigma}(t^{-m/n}[f^{**}(t)-f^*(t)]) + \int_0^{|\Omega|} f^*(t)\,dt \qquad \text{by (4.2)} \\
&\lesssim \overline{\varrho_\sigma}\Big(\int_t^{|\Omega|} s^{-m/n}[f^{**}(s)-f^*(s)]\,\frac{ds}{s}\Big) + \int_0^{|\Omega|} f^*(t)\,dt \\
&\lesssim \bar{\sigma}\Big(\int_t^{|\Omega|}\Big[\int_s^{|\Omega|} y^{-m/n}[f^{**}(y)-f^*(y)]\,\frac{dy}{y}\Big] s^{m/n-1}\,ds\Big) \\
&\qquad + \int_0^{|\Omega|} f^*(t)\,dt \qquad \text{by (4.3)} \\
&\lesssim \bar{\sigma}\Big(\int_t^{|\Omega|} s^{-m/n}[f^{**}(s)-f^*(s)]s^{m/n-1}\,ds\Big) + \int_0^{|\Omega|} f^*(t)\,dt \\
&= \bar{\sigma}\Big(\int_t^{|\Omega|} s^{-1}\int_0^s f^*(y)\,dy\,\frac{ds}{s} - \int_t^{|\Omega|} f^*(s)\,\frac{ds}{s}\Big) + \int_0^{|\Omega|} f^*(t)\,dt \\
&\lesssim \bar{\sigma}\Big(t^{-1}\int_0^t f^*(s)\,ds - \frac{1}{|\Omega|}\int_t^{|\Omega} f^*(s)\,ds\Big) + \int_0^{|\Omega|} f^*(t)\,dt \\
&\lesssim \bar{\sigma}\Big(t^{-1}\int_0^t f^*(s)\,ds\Big) + \int_0^{|\Omega|} f^*(t)\,dt \lesssim \bar{\sigma}\Big(t^{-1}\int_0^t f^*(s)\,ds\Big),
\end{aligned}$$

since

$$\int_t^{|\Omega|}\int_s^{|\Omega|} h(y)\,\frac{dy}{y}\,s^{m/n-1}\,ds = \int_t^{|\Omega|}\frac{h(y)}{y}\int_t^{y} s^{m/n-1}\,ds\,dy$$
$$= \frac{n}{m}\int_t^{|\Omega|}\frac{h(y)}{y}\,[y^{m/n}-t^{m/n}]\,dy$$
$$\le \frac{n}{m}\int_t^{|\Omega|} h(y)y^{m/n-1}\,dy$$

and

$$\int_t^{|\Omega|} s^{-1}\int_0^{s} h(y)\,dy\,\frac{ds}{s} = \int_t^{|\Omega|} s^{-2}\,ds\int_0^{t} h(y)\,dy + \int_t^{|\Omega|} s^{-2}\int_t^{s} h(y)\,dy\,ds$$
$$\le t^{-1}\int_0^{t} h(y)\,dy + \int_t^{|\Omega|} h(y)\,\frac{dy}{y} - \frac{1}{|\Omega|}\int_t^{|\Omega|} h(y)\,dy, \qquad h \in \mathfrak{M}_+(I_\Omega).$$

Finally, (1.5) and [4, Theorem 3.12] imply, as in the proof of Proposition C, that

$$P : L_{\overline{\sigma}}(I_\Omega) \to L_{\overline{\sigma}}(I_\Omega),$$

which means

$$\overline{\sigma}\Big(t^{-1}\int_0^{t} f^*(s)\,ds\Big) \approx \overline{\sigma}(f), \qquad f \in \mathfrak{M}_+(\Omega). \ ∎$$

Proof of Theorem B. We know the following:

(4.4) $\qquad \varrho_D \lesssim \varrho$, or equivalently, $\varrho' \lesssim \varrho'_D$;

(4.5) $\qquad \varrho' \lesssim \mu$, or equivalently, $\mu' \lesssim \varrho$;

(4.6) $\qquad S_{n/m} : L_{\overline{\varrho}'_D}(I_\Omega) \to L_{\overline{\varrho}'_D}(I_\Omega)$;

(4.7) $\qquad S_{n/m} : L_{\overline{\mu}'}(I_\Omega) \to L_{\overline{\mu}'}(I_\Omega)$.

Now, (4.4) and (4.6) yield

$$\mu(g) = \overline{\varrho}'(S_{n/m}g^{**}) \approx \overline{\varrho}'(S_{n/m}g^{*}) \lesssim \overline{\varrho}'_D(S_{n/m}g^{*}) \lesssim \overline{\varrho}'_D(g^*), \quad g \in \mathfrak{M}_+(\Omega),$$

and, hence, $\varrho_D \lesssim \mu'$. So, keeping (4.5) in mind, we see that

$$\varrho_D \lesssim \mu' \lesssim \varrho.$$

Since $\sigma_{\varrho_D} = \sigma_\varrho$, we conclude $\sigma_{\mu'} = \sigma_\varrho$, that is,

$$W^{m,\mu'}(\Omega) \hookrightarrow L_{\sigma_\varrho}(\Omega),$$

which, in view of (4.7) and Theorem A, means $\mu' \approx \varrho_D$. ∎

5. Examples. We here illustrate Theorem A in the context of Orlicz spaces.

An Orlicz norm is defined in terms of a Young function $A(t) = \int_0^t a(s)\,ds$, with $a(s)$ increasing on $\mathbb{R}_+$, $a(0+) = 0$ and $\lim_{s\to\infty} a(s) = \infty$. Given a domain $\Omega \subset \mathbb{R}^n$, the (Luxemburg) Orlicz (r.i.) norm, ϱ_A, is defined at $f \in \mathfrak{M}_+(I_\Omega)$ by

$$\varrho_A(f) = \inf\biggl\{\lambda > 0 : \int_{I_\Omega} A\biggl(\frac{f(t)}{\lambda}\biggr)\,dt = \int_{I_\Omega} A\biggl(\frac{f^*(t)}{\lambda}\biggr)\,dt \le 1\biggr\}$$

and at $f \in \mathfrak{M}_+(\Omega)$ by

$$\varrho_A(f) = \inf\biggl\{\lambda > 0 : \int_{I_\Omega} A\biggl(\frac{f^*(t)}{\lambda}\biggr)\,dt \le 1\biggr\}.$$

The Köthe norm dual to ϱ_A is equivalent to the Orlicz norm $\varrho_{\widetilde{A}}$, where

$$\widetilde{A}(t) := \int_0^t a^{-1}(s)\,ds, \qquad t > 0,$$

is the Young function complementary to A; in fact,

$$\varrho_{\widetilde{A}}(g) \le \varrho_A'(g) \le 2\varrho_{\widetilde{A}}(g), \qquad g \in \mathfrak{M}_+(I_\Omega).$$

In [3] we determined precisely when $S_{n/m}$ and $T_{n/m}$ are bounded between Orlicz spaces. Theorems B and 5.2 of that paper yield, respectively, Theorems 5.1 and 5.2 below.

THEOREM 5.1. *Let m, n and Ω be as in Theorem A and suppose A is a Young function whose complementary function*, $\widetilde{A}$, *satisfies*

$$\widetilde{A}(t) = 0,\ t \in I_\Omega, \quad \textit{and} \quad L_{\varrho_{\widetilde{A}}}(\Omega) \subsetneq L_{n/(n-m),\infty}(\Omega).$$

Then $\varrho = \varrho_A$ is optimal in (1.1) *for some r.i. norm σ on $\mathfrak{M}(\Omega)$ if and only if*

$$\int_{|\Omega|}^t \frac{\widetilde{A}(s)}{s^{n/(n-m)+1}}\,ds \le \frac{\widetilde{A}(Kt)}{t^{n/(n-m)}}, \qquad t \gg |\Omega|.$$

Moreover, *in that case*,

$$\sigma_{\varrho_A}(f) \approx \varrho_A(t^{-m/n}[f^{**}(t) - f^*(t)]) + \int_0^{|\Omega|} f^*(t)\,dt, \qquad f \in \mathfrak{M}_+(\Omega).$$

THEOREM 5.2. *Let m, n and Ω be as in Theorem A and suppose A is a Young function whose complementary function*, $\widetilde{A}$, *satisfies*

$$\widetilde{A}(t) = 0,\ t \in I_\Omega, \quad \textit{and} \quad L_{n/m,\infty}(\Omega) \subsetneq L_{\varrho_{\widetilde{A}}}(\Omega).$$

Then $\sigma = \varrho_A$ *is optimal in* (1.1) *for some r.i. norm* ϱ *on* $\mathfrak{M}_+(\Omega)$ *if and only if*

$$\int_t^\infty \frac{\widetilde{A}(s)}{s^{n/m+1}}\,ds \le \frac{\widetilde{A}(Kt)}{t^{n/m}}, \qquad t \gg |\Omega|.$$

Moreover, *in that case*,

$$\varrho_\sigma(f) \approx \varrho_A\Big(\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big), \qquad f \in \mathfrak{M}_+(\Omega).$$

Appendix. The following result concerning the level function, f°, of an $f \in \mathfrak{M}_+(I_\Omega)$, was communicated to us by G. Sinnamon.

THEOREM A.1. *For any* $f \in \mathfrak{M}_+(I_\Omega)$, *the function*

$$q(t) := t \sup_{t\le s<1} s^{-1}\int_0^s f(y)\,dy$$

is quasiconcave on I_Ω. *Moreover*,

$$q(t) \le \int_0^t f^\circ(s)\,ds \le 2q(t). \tag{A.1}$$

Proof (A. Gogatishvili). Set $f(s) = 0$ for $s > 1$ so that

$$q(t) = t \sup_{t\le s<\infty} s^{-1}\int_0^s f(y)\,dy, \qquad t \in I_\Omega.$$

Since $q(t)/t$ is clearly nonincreasing, we need only verify that $q(t)$ is nondecreasing to get q quasiconcave on I_Ω. But this is readily seen from

$$t \sup_{t\le s<\infty} s^{-1}\int_0^s f(y)\,dy = \sup_{1\le s<\infty} s^{-1}\int_0^{ts} f(y)\,dy.$$

As $q(t) \ge \int_0^t f(y)\,dy$, the least concave majorant of q dominates $\int_0^t f(y)\,dy$ and hence $\int_0^t f^\circ(s)\,ds$. The least concave majorant of a quasiconcave function $q(t)$ being no greater than $2q(t)$, we have the second of the inequalities in (A.1).

Observe that

$$\int_0^t f^\circ(s)\,ds = \sup_{t_1\le t,\,0<t_2<\infty} \frac{t_2\int_0^{t-t_1} f(s)\,ds + t_1\int_0^{t+t_2} f(s)\,ds}{t_1+t_2}, \qquad 0<t<1.$$

Fix s and t with $t \le s < 1$. Set $t_1 = t$ and $t_2 = s - t$. Then

$$\frac{t_2\int_0^{t-t_1} f(s)\,ds + t_1\int_0^{t+t_2} f(s)\,ds}{t_1+t_2} = \frac{t}{s}\int_0^s f(y)\,dy,$$

whence

$$q(t) \le \int_0^t f^\circ(s)\, ds$$

and we are done. ∎

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