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A counter-example in singular integral theory

by

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Abstract. An improvement of a lemma of Calderón and Zygmund involving singular spherical harmonic kernels is obtained and a counter-example is given to show that this result is best possible. In a particular case when the singularity is $O(|\log r|)$, let $f \in C^1(\mathbb{R}^N\backslash\{0\})$ and suppose f vanishes outside of a compact subset of $\mathbb{R}^N, N \geq 2$. Also, let $k(x)$ be a Calderón–Zygmund kernel of spherical harmonic type. Suppose $f(x) = O(|\log r|)$ as $r \to 0$ in the L^p-sense. Set

$$F(x) = \int_{\mathbb{R}^N} k(x-y)f(y)\,dy \quad \forall x \in \mathbb{R}^N\backslash\{0\}.$$

Then $F(x) = O(\log^2 r)$ as $r \to 0$ in the L^p-sense, $1 < p < \infty$. A counter-example is given in $\mathbb{R}^2$ where the increased singularity $O(\log^2 r)$ actually takes place. This is different from the situation that Calderón and Zygmund faced.

1. Introduction. We will operate in real N-dimensional Euclidean space, $\mathbb{R}^N, N \geq 2$, and use the following notation:

$$\begin{gathered} x = (x_1, \dots, x_N), \qquad y = (y_1, \dots, y_N), \\ \alpha x + \beta y = (\alpha x_1 + \beta y_1, \dots, \alpha x_N + \beta y_N), \\ x \cdot y = x_1 y_1 + \dots + x_N y_N, \qquad |x| = (x \cdot x)^{1/2}. \end{gathered}$$

Also, $B(x, \rho)$ will designate the open ball centered at x with radius ρ. Then $B(x,\rho)\backslash\{x\}$ is the corresponding ball punctured at x.

$k(x)$ will designate a Calderón–Zygmund kernel of spherical harmonic type. So

$$k(x) = \frac{P_n(x)}{|x|^{n+N}} \quad \text{for } x \neq 0 \tag{1.1}$$

where $P_n(x)$ is a spherical harmonic of order n, i.e., a homogeneous polynomial of degree n, $n \geq 1$, which is also a harmonic function. (For more about Calderón–Zygmund kernels of spherical harmonic type, see [3, Chapter 2].)

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Furthermore, $\eta(r)$ will be a positive continuous strictly decreasing function with

$$\text{(1.2)} \qquad \text{(i) } \eta \in C((0,1)) \quad \text{and} \quad \text{(ii) } \eta(r) \to \infty \ \text{ as } r \to 0.$$

In what follows, f will designate a function of the following nature:

$$\text{(1.3)} \qquad f \in L^p(\mathbb{R}^N) \quad \text{with } f(x) = 0 \text{ a.e. for } |x| > 1$$

where $1 < p < \infty$.

With $|x| = r$, we will say $f(x) = O(\eta(r))$ as $r \to 0$ in the L^p-sense provided

$$\text{(1.4)} \qquad \left(\frac{1}{r^N} \int_{B(0,r)} |f(x)|^p \, dx\right)^{1/p} = O(\eta(r)) \quad \text{as } r \to 0.$$

We will say $f(x) = O(\eta(r) + \int_r^1 \eta(\rho)\, d\rho)$ as $r \to 0$ in the L^p-sense provided

$$\text{(1.5)} \qquad \left(\frac{1}{r^N} \int_{B(0,r)} |f(x)|^p \, dx\right)^{1/p} = O\Big(\eta(r) + \int_r^1 \eta(\rho)\, d\rho\Big) \quad \text{as } r \to 0.$$

Likewise, we have $f(x) = o(\eta(r))$ as $r \to 0$ in the L^p-sense if we replace O by o in the above definition.

It follows from [**?**, Theorem 1] that if $k(x)$ is defined as in (1.1) and $f(x)$ is as in (1.4), then

$$\text{(1.6)} \qquad F(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} k(x-y) f(y)\, dy \quad \text{exists a.e. in } \mathbb{R}^N,$$

and $F \in L^p(\mathbb{R}^N)$ with

$$\text{(1.7)} \qquad \|F\|_{L^p(\mathbb{R}^N)} \le C_p^* \|f\|_{L^p(\mathbb{R}^N)}$$

where C_p^* is a constant independent of f.

It is our intention here to prove the following singularity theorem about $F(x)$ defined by the singular integral in (1.6).

THEOREM. *With $k(x)$ and $\eta(r)$ as in* (1.1) *and* (1.2), *respectively, and $N \ge 2$, suppose that $f \in L^p(\mathbb{R}^N)$, where $1 < p < \infty$, and $f = 0$ a.e. for $|x| > 1$. Suppose also that*

(i) $\left(\frac{1}{r^N} \int_{B(0,r)} |f(x)|^p \, dx\right)^{1/p} = O(\eta(r))$ *as* $r \to 0$,

(ii) $F(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} k(x-y) f(y)\, dy$ *a.e. in* $\mathbb{R}^N$.

Then

$$(1.8)\qquad \left(\frac{1}{r^N}\int\limits_{B(0,r)} |F(x)|^p\,dx\right)^{1/p} = O\left(\eta(r) + \int\limits_r^1 \frac{\eta(s)}{s}\,ds\right) \qquad \text{as } r\to 0.$$

Also, if O is replaced by o in (i), *then O can be replaced by o in* (1.8).

In §3 we will give a counter-example to show that the above theorem is a best possible result, i.e., $\int_r^1 (\eta(s)/s)\,ds$ cannot be dropped in (1.8).

The above theorem is an improvement of Lemma 5.1 in [2] where, in this context, Calderón and Zygmund deal only with the case $\eta(r) = r^{-\alpha}, 0 < \alpha \le N/p$. In this case, when $\eta(r) = 1/r^\alpha$, the integral in (1.8) adds nothing new. We are concerned here with $\eta(r) = |\log r|$ for example. In this case, the integral in (1.8) becomes $\log^2 r$, showing that the singularity is actually increased. Our counter-example will show that the singularity $\log^2 r$ actually takes place. So the bound in our theorem is sharp.

The above theorem is also true in dimension $N = 1$. In this case, $k(s)$ becomes the familiar Hilbert kernel

$$k(s) = \frac{\operatorname{sgn} s}{|s|} \qquad \text{for } s\in\mathbb{R} \text{ and } s\neq 0.$$

2. Proof of Theorem. We first note that by [**?**], F defined by (ii) in the theorem is also in $L^p(\mathbb{R}^N)$.

We make the following definitions:

$$F_{1\rho}(x) = \lim_{\varepsilon\to 0}\int\limits_{|y|\le\rho,\, y\in\mathbb{R}^N\setminus B(x,\varepsilon)} k(x-y)f(y)\,dy,$$

$$F_{2\rho}(x) = \lim_{\varepsilon\to 0}\int\limits_{|y|\ge\rho,\, y\in\mathbb{R}^N\setminus B(x,\varepsilon)} k(x-y)f(y)\,dy,$$

and

$$f_\rho(y) = \begin{cases} f(y) & \text{for } |y|\le\rho,\\ 0 & \text{for } |y|>\rho.\end{cases}$$

Thus,

$$F_{1\rho}(x) = \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^N\setminus B(x,\varepsilon)} k(x-y)f_\rho(y)\,dy.$$

Applying [**?**, Theorem 1, p. 289], we see that

$$\left(\frac{1}{\rho^N}\int\limits_{B(0,\rho/2)} |F_{1\rho}(x)|^p\,dx\right)^{1/p} \le \frac{1}{\rho^{N/p}} C_p^* \|f_\rho\|_{L^p(\mathbb{R}^N)},$$

where C_p^* is a constant depending only on k and p. Therefore, from (i) in the hypothesis of the theorem,

$$\left(\frac{1}{\rho^N}\int_{B(0,\rho/2)}|F_{1\rho}(x)|^p\,dx\right)^{1/p}\le C_p^* O(\eta(\rho)). \tag{2.1}$$

Next, from (1.1), we see that $|k(x)|\le \mathrm{const}/|x|^N$ for $x\neq 0$. Hence,

$$|k(x-y)|\le \frac{C_1}{|x-y|^N}\le \frac{2^N C_1}{|y|^N} \tag{2.2}$$

for $|x|\le \rho/2$ and $|y|\ge \rho$, where C_1 is a constant depending only on k. Therefore

$$\begin{aligned}\int_{|y|\ge\rho}|k(x-y)|\,|f(y)|\,dy &\le 2^N C_1\int_{|y|\ge\rho}\frac{|f(y)|}{|y|^N}\,dy\\ &\le 2^N C_1\int_\rho^1 s^{-N}\int_{|y|=s}|f(y)|\,dS(y)\,ds\end{aligned}$$

for $|x|\le\rho/2$. Consequently, after integrating by parts, we obtain

$$\int_{|y|\ge\rho}|k(x-y)|\,|f(y)|\,dy\le 2^N C_1\Big[\|f\|_{L^1(\mathbb{R}^N)}+N\int_\rho^1 s^{-(N+1)}\Big(\int_{|y|\le s}|f(y)|\,dy\Big)\,ds\Big].$$

Since $\int_{|y|\le s}|f(y)|\,dy\le C_2|s^N|^{1/q}(\int_{|y|\le s}|f(y)|^p\,dy)^{1/p}$ where $1/p+1/q=1$ and C_2 is a constant, we see from (i) applied to this last formula that

$$\int_{|y|\ge\rho}|k(x-y)|\,|f(y)|\,dy\le C_3\|f\|_{L^1(\mathbb{R}^N)}+C_3\int_\rho^1 s^{-1}\eta(s)\,ds \tag{2.3}$$

where C_3 is a constant and $|x|\le\rho/2$. Therefore,

$$\begin{aligned}\frac{1}{\rho^N}\int_{B(0,\rho/2)}|F_{2\rho}(x)|^p\,dx &\le \frac{1}{\rho^N}\int_{B(0,\rho/2)}\Big[C_3\|f\|_{L^1(\mathbb{R}^N)}+C_3\int_\rho^1 s^{-1}\eta(s)\,ds\Big]^p\,dx\\ &\le C_4\Big[C_3\|f\|_{L^1(\mathbb{R}^N)}+C_3\int_\rho^1 s^{-1}\eta(s)\,ds\Big]^p.\end{aligned}$$

where C_4 is another constant.

Using Minkowski's inequality, we see from this last computation combined with (2.1) that

$$\left(\frac{1}{\rho^N}\int_{B(0,\rho/2)}|F(x)|^p\,dx\right)^{1/p}\le C_5\Big(\eta(\rho)+\int_\rho^1 s^{-1}\eta(s)\,ds\Big)$$

for $0<\rho<1/2$, where C_5 is a constant. This establishes the first part of the theorem.

To prove the second part of the theorem where O is replaced by o in (i) we observe first that (2.1) can be replaced with

$$\left(\frac{1}{\rho^N}\int_{B(0,\rho/2)}|F_{1\rho}(x)|^p\,dx\right)^{1/p}\le C_p^* o(\eta(\rho)) \tag{2.4}$$

and next that (2.3) can be replaced with

$$\int_{|y|\ge\rho}|k(x-y)|\,|f(y)|\,dy\le C_3\|f\|_{L^1(\mathbb{R}^N)}+C_3\int_\rho^1 s^{-1}g(s)\eta(s)\,ds$$

where $g(s)$ is a positive continuous bounded strictly increasing function on $(0,1)$ with

$$\lim_{s\to 0} g(s)=0. \tag{2.5}$$

Consequently, we see as before that

$$\frac{1}{\rho^N}\int_{B(0,\rho/2)}|F_{2\rho}(x)|^p\,dx\le C_4\Big[C_3\|f\|_{L^1(\mathbb{R}^N)}+C_3\int_\rho^1 s^{-1}g(s)\eta(s)\,ds\Big]^p. \tag{2.6}$$

It follows from (1.2) and (2.5) that

$$\int_\rho^1 s^{-1}g(s)\eta(s)\,ds=o(1)\Big[\eta(\rho)+\int_\rho^1 s^{-1}\eta(s)\,ds\Big]$$

as $\rho\to 0$. Therefore, from (2.6) we obtain

$$\left(\frac{1}{\rho^N}\int_{B(0,\rho/2)}|F_{2\rho}(x)|^p\,dx\right)^{1/p}\le o(1)\Big[\eta(\rho)+\int_\rho^1 s^{-1}\eta(s)\,ds\Big].$$

(2.4) in conjunction with this last inequality shows that

$$\left(\frac{1}{\rho^N}\int_{B(0,\rho/2)}|F(x)|^p\,dx\right)^{1/p}\le o(1)\Big[\eta(\rho)+\int_\rho^1 s^{-1}\eta(s)\,ds\Big]\quad\text{as } \rho\to 0,$$

and the proof of the Theorem is complete. ∎

3. A counter-example. In this section, we will give a counter-example to show that, in general, the integral on the right-hand side in (1.8) cannot be dropped. In other words, in some cases the singularity that occurs in the singular integral actually increases.

For simplicity, we will give our example in dimension $N=2$. A similar example holds in dimensions $N\ge 3$.

From the start, we will assume that $\eta(r)$ meets both conditions in (1.2) and also the following two conditions:

$$\lim_{r\to 0} \frac{\eta(r)}{\int_r^1 (\eta(s)/s)\,ds} = 0 \tag{3.1(i)}$$

and

(3.1(ii)) for $p > 1$, $\lim_{r\to 0} r(\eta(r))^p = 0$, and also there exists r^* with $0 < r^* < 1$ such that $r(\eta(r))^p$ is an increasing function in $(0, r^*)$.

Condition (1.2) as well as both conditions (3.1(i)) and (3.1(ii)) are met by the function $\eta(r) = |\log r|$.

For our singular integral kernel $k(x)$ of spherical harmonic type with $N = 2$, we take

$$k(x) = \frac{x_1 x_2}{|x|^4} \qquad \text{for } x \neq 0. \tag{3.2}$$

Also, we set

$$\xi(x) = \frac{x_1 x_2}{|x|^2} \qquad \text{for } x \neq 0, \tag{3.3}$$

and

$$f(x) = \begin{cases} \xi(x)\eta(|x|) & \text{for } 0 < |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases} \tag{3.4}$$

We see from (1.2) and (3.1(ii)) that $f(x)$ defined by (3.4) is in $L^p(\mathbb{R}^2)$ and

$$\left(\frac{1}{r^2} \int\limits_{B(0,r)} |f(x)|^p\,dx \right)^{1/p} \leq (2\pi)^{1/p}\eta(r) \qquad \text{for } 0 < r < r^*.$$

So to show that our counter-example works, all we have to do is show that for the singular integral defined using (3.2) and (3.4), the term $\int_r^1 (\eta(s)/s)\,ds$ cannot be eliminated in (1.8).

In order to do this, we observe that for fixed x,

$$k(x-y)\xi(y) \text{ is a continuous function for } y \neq x \text{ and } y \neq 0.$$

Also,

$$\exists M > 0\colon \quad |k(x-y)\xi(y)| \leq M \qquad \text{for } |y| = 1 \text{ and } |x| \leq 1/2.$$

As a consequence of these last two facts and (3.2) and (3.3), with $dS(y)$ representing the natural measure on $|y| = 1$ and B any positive measure subset of $|y| = 1$, we have

$$\lim_{x\to 0} \int\limits_B k(x-y)\xi(y)\,dS(y) = \int\limits_B k(-y)\xi(y)\,dS(y) = \int\limits_B \frac{y_1^2 y_2^2}{|y|^6}\,dS(y) > 0. \tag{3.5}$$

Next, we label the quadrants in the plane in the usual manner, i.e.,

$$\mathrm{Quad}_1 = \left\{x : x = (r\cos\theta, r\sin\theta),\ 0 < \theta < \frac{\pi}{2},\ 0 < r < \infty\right\},$$
$$\mathrm{Quad}_2 = \left\{x : x = (r\cos\theta, r\sin\theta),\ \frac{\pi}{2} < \theta < \pi,\ 0 < r < \infty\right\},$$
$$\mathrm{Quad}_3 = \left\{x : x = (r\cos\theta, r\sin\theta),\ \pi < \theta < \frac{3\pi}{2},\ 0 < r < \infty\right\},$$
$$\mathrm{Quad}_4 = \left\{x : x = (r\cos\theta, r\sin\theta),\ \frac{3\pi}{2} < \theta < 2\pi,\ 0 < r < \infty\right\}.$$

We will refer to Quad_1 and Quad_3 as opposite quadrants. Likewise, Quad_2 and Quad_4 are opposite quadrants. So for example, x and y are in opposite quadrants if $x \in \mathrm{Quad}_4$ and $y \in \mathrm{Quad}_2$.

Next, we define

$$A_i = \{y : |y| = 1,\ y \notin \mathrm{Quad}_i\}$$

for $i = 1, \dots, 4$. It follows from (3.5) and this last definition that

$$\exists \rho_i > 0\colon \quad \int_{A_i} k(x-y)\xi(y)\,dS(y) > 0 \quad \text{for } 0 < |x| < \rho_i \tag{3.6}$$

for $i = 1, \dots, 4$. We set

$$r_0 = \min(1/2, \rho_1, \rho_2, \rho_3, \rho_4), \tag{3.7}$$

and claim that the following fact holds where Q stands for one of the four quadrants: With $x \in Q$ and $|x| < r_0 s$ where $0 < s < 1$,

$$\int_{|y|=s} k(x-y)\xi(y)\,dS(y) \geq \int_{|y|=s,\, y\in Q^*} k(x-y)\xi(y)\,dS(y), \tag{3.8}$$

where Q^* is the quadrant opposite to Q.

To establish (3.8), assume

$$x \in \mathrm{Quad}_j \quad \text{and} \quad |x| < r_0 s \quad \text{for some } 0 < s < 1, \tag{3.9}$$

and let

$$\mathrm{Quad}_i \ \text{be the quadrant opposite to } \mathrm{Quad}_j. \tag{3.10}$$

So $Q = \mathrm{Quad}_j$ and $Q^* = \mathrm{Quad}_i$.

Set $\nu = x/s$ and $\zeta = y/s$. Then, with A_i as defined above,

$$\int_{|y|=s,\, y/s\in A_i} k(x-y)\xi(y)\,dS(y) = \int_{|\zeta|=1,\, \zeta\in A_i} k(s(\nu-\zeta))\xi(s\zeta)s\,dS(\zeta)$$
$$= \frac{1}{s}\int_{|\zeta|=1,\, \zeta\in A_i} k(\nu-\zeta)\xi(\zeta)\,dS(\zeta) > 0$$

by (3.9), because $|\nu| < r_0$ where r_0 is defined in (3.7) above. Since $Q^* = \mathrm{Quad}_i$, claim (3.8) follows from this last computation.

For $x \in Q$ and $y \in Q^*$, we observe that both

$$y_1(y_1 - x_1) > 0 \quad \text{and} \quad y_2(y_2 - x_2) > 0. \tag{3.11}$$

Also, it follows from (3.2) and (3.3) that

$$k(x-y)\xi(y) = \frac{y_1(y_1 - x_1)y_2(y_2 - x_2)}{|x-y|^4|y|^2}$$

for $x \neq y$ and $y \neq 0$. Hence, we conclude from (3.11) that

$$k(x-y)\xi(y) > 0 \tag{3.12}$$

for $x \in Q$ and $y \in Q^*$.

Next, we are going to shrink the interval of integration slightly in the integral on the right-hand side of (3.8). We do this as follows. Set

$$\gamma = \sin\frac{\pi}{24}$$

and

$$\Gamma = \{y \in Q^* : |y| = s, |\,y_1| \geq \gamma s, \text{ and } |y_2| \geq \gamma s\}. \tag{3.13}$$

An easy computation shows that

$$\int_\Gamma dS(y) = s\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = \frac{5\pi}{12}s. \tag{3.14}$$

We also claim that

$$y_1(y_1 - x_1) \geq \gamma^2 s^2 \quad \text{and} \quad y_2(y_2 - x_2) \geq \gamma^2 s^2 \tag{3.15}$$

for $x \in Q$ and $y \in \Gamma$.

We establish claim (3.15) for $y_1(y_1 - x_1)$. A similar computation works for $y_2(y_2 - x_2)$. First of all, we note that for $x \in Q$ and $y \in \Gamma$,

$$|y_1 - x_1| = |y_1| + |x_1| \geq |y_1| \geq \gamma s,$$

and consequently from (3.11) that

$$y_1(y_1 - x_1) = |y_1|\,|y_1 - x_1| \geq \gamma s \gamma s = \gamma^2 s^2,$$

which establishes claim (3.15).

Using the fact that for $x \in Q$, $y \in \Gamma$, with $|x| \leq r_0 s$,

$$|y - x| \leq |y| + |x| \leq (1 + r_0)s,$$

we see from (3.15) that

$$\begin{aligned} k(x-y)\xi(y) &= \frac{y_1(y_1 - x_1)y_2(y_2 - x_2)}{|x-y|^4|y|^2} \\ &\geq \frac{\gamma^4 s^4}{(1+r_0)^4 s^4 s^2} = \frac{\gamma^4}{(1+r_0)^4}s^{-2}. \end{aligned}$$

Hence, from (3.8) and (3.12), we obtain

$$\begin{aligned}\int\limits_{|y|=s} k(x-y)\xi(y)\,dS(y) &\geq \int\limits_{\Gamma} k(x-y)\xi(y)\,dS(y)\\ &\geq \frac{\gamma^4}{(1+r_0)^4}\int\limits_{\Gamma} s^{-2}\,dS(y)\\ &\geq \frac{\gamma^4}{(1+r_0)^4}\,\frac{5\pi}{12}s^{-1},\end{aligned}$$

where we have also used (3.14).

Thus,

$$\int\limits_{|y|=s} k(x-y)\xi(y)\,dS(y) \geq C^{\Diamond}s^{-1}$$

for $|x| \leq r_0 s$ where

$$C^{\Diamond} = \frac{\gamma^4}{(1+r_0)^4}\,\frac{5\pi}{12} \quad \text{and} \quad \gamma = \sin\frac{\pi}{24}. \tag{3.16}$$

In view of this last inequality, we conclude from (3.4) that

$$\int\limits_{|y|\geq\rho} k(x-y)f(y)\,dy \geq C^{\Diamond}\int\limits_{\rho}^{1}\eta(s)s^{-1}\,ds \tag{3.17}$$

for $0<\rho<1$, where $|x| \leq r_0\rho$ and $C^{\Diamond}$ is the constant given in (3.16).

To complete our example, we define

$$F_{1r}(x) = \int\limits_{|y|\leq r} k(x-y)f(y)\,dy$$

and

$$F_{2r}(x) = \int\limits_{|y|\geq r} k(x-y)f(y)\,dy$$

for $0<r<1$. These two definitions show that

$$\begin{aligned}&\Big(\frac{1}{r^2}\int\limits_{B(0,r_0r)} |F(x)|^p\,dx\Big)^{1/p} \\ &\quad\geq \Big(\frac{1}{r^2}\int\limits_{B(0,r_0r)} |F_{2r}(x)|^p\,dx\Big)^{1/p} - \Big(\frac{1}{r^2}\int\limits_{B(0,r_0r)} |F_{1r}(x)|^p\,dx\Big)^{1/p}.\end{aligned} \tag{3.18}$$

For the first integral on the right of (3.18), we can use (3.17) to obtain

$$\begin{aligned}\frac{1}{r^2}\int\limits_{B(0,r_0r)}|F_{2r}(x)|^p\,dx &= \frac{1}{r^2}\int\limits_{B(0,r_0r)}\Big|\int\limits_{|y|\geq r}k(x-y)f(y)\,dy\Big|^p\,dx\\ &\geq \frac{1}{r^2}\int\limits_{B(0,r_0r)}\Big|C^{\Diamond}\int\limits_r^1\eta(s)s^{-1}\,ds\Big|^p\,dx\\ &\geq \pi r_0^2\Big(C^{\Diamond}\int\limits_r^1\eta(s)s^{-1}\,ds\Big)^p.\end{aligned}$$

Therefore,

$$\Big(\frac{1}{r^2}\int\limits_{B(0,r_0r)}|F_{2r}(x)|^p\,dx\Big)^{1/p}\geq(\pi r_0^2)^{1/p}C^{\Diamond}\int\limits_r^1\eta(s)s^{-1}\,ds. \tag{3.19}$$

To handle the second integral on the right-hand side of (3.18), define

$$f_r(x)=\begin{cases}f(x) & \text{for } |x|\leq r,\\ 0 & \text{for } |x|>r.\end{cases}$$

Consequently, $(r^{-2}\int_{B(0,r_0r)}|F_{1r}(x)|^p\,dx)^{1/p}$ is bounded above by

$$\frac{1}{r^{2/p}}\Big(\int\limits_{\mathbb{R}^2}\Big|\int\limits_{\mathbb{R}^2}k(x-y)f_r(y)\,dy\Big|^p\,dx\Big)^{1/p}.$$

By (1.7) above, this last expression is in turn bounded by

$$r^{-2/p}C_p^*\|f_r\|_{L^p(\mathbb{R}^2)}.$$

On the other hand, by (3.1(ii)) for r small,

$$r^{-2/p}\|f_r\|_{L^p(\mathbb{R}^2)}\leq r^{-2/p}\Big(\int\limits_{B(0,r)}|\xi(y)\eta(|y|)|^p\,dy\Big)^{1/p}\leq(2\pi)^{1/p}\eta(r).$$

We conclude that for r small,

$$\Big(\frac{1}{r^2}\int\limits_{B(0,r_0r)}|F_{1r}(x)|^p\,dx\Big)^{1/p}\leq(2\pi)^{1/p}C_p^*\eta(r).$$

This, in conjunction with (3.18) and (3.19), leads to

$$\Big(\frac{1}{r^2}\int\limits_{B(0,r_0r)}|F(x)|^p\,dx\Big)^{1/p}\geq(\pi r_0^2)^{1/p}C^{\Diamond}\int\limits_r^1\eta(s)s^{-1}\,ds-(2\pi)^{1/p}C_p^*\eta(r)$$

for r small. But then it follows from (3.1(i)) that there exists $r^{**}>0$ such that

$$\Big(\frac{1}{r^2}\int\limits_{B(0,r_0r)}|F(x)|^p\,dx\Big)^{1/p}\geq\frac{1}{2}(\pi r_0^2)^{1/p}C^{\Diamond}\int\limits_r^1\eta(s)s^{-1}\,ds$$

for $0<r<r^{**}$.

This last inequality shows that the integral $\int_r^1 \eta(s)s^{-1}\,ds$ cannot be eliminated in (1.8) in the statement of the theorem, and our counter-example is complete.

The counter-example we just presented to show that the bound in our theorem is sharp makes use of a singular spherical harmonic kernel of order two. Hence, it is an even kernel. The question arises as to whether we can get a counter-example using an odd singular spherical harmonic kernel. In particular, in dimension $N = 2$, will the technique we presented work for the familiar Riesz kernel, $k(x) = x_1/|x|^3$? The answer is affirmative, and we give a brief outline how one proceeds in this case.

With $k(x) = x_1/|x|^3$ and $\xi(x) = -x_1/|x|$, set

$$f(x) = \begin{cases} \xi(x)\eta(|x|) & \text{for } 0 < |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

It then follows that the key inequalities presented above continue to hold. This is seen to be true for the inequalities in (3.5), (3.6), (3.8), (3.12), and (3.17). The rest of the computations in the proof work as before. So our counter-example is also valid for the odd singular Riesz kernel.

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