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Crossed products by Hilbert pro-C^*-bimodules

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Abstract. We define the crossed product of a pro-C^*-algebra A by a Hilbert A-A pro-C^*-bimodule and we show that it can be realized as an inverse limit of crossed products of C^*-algebras by Hilbert C^*-bimodules. We also prove that under some conditions the crossed products of two Hilbert pro-C^*-bimodules over strongly Morita equivalent pro-C^*-algebras are strongly Morita equivalent.

1. Introduction. The crossed product construction, in its various forms and generalizations, has proved to be one of the most important ideas in operator algebras, both for internal structure theory and for applications. Crossed products go back on the one hand to statistical mechanics, where they were called "covariance algebras", and on the other hand to the group measure space constructions of Murray and von Neumann (see [W]). A crossed product C^*-algebra is a C^*-algebra A together with a locally compact group G of automorphisms of A. When $A = \mathbb{C}$, the crossed product C^*-algebra is the well known group C^*-algebra. There is a vast literature on crossed products of C^*-algebras (see e.g. [W]), but the corresponding theory in the context of non-normed topological algebras has still a long way to go.

Crossed products of pro-C^*-algebras under inverse limit actions of locally compact groups were considered first by Phillips [P2] and secondly by Joiţa whose main results are included in the monograph [J2]. If X is a direct limit of a sequence $\{K_n\}_n$ of compact spaces, then $C(X)$, the vector space of all continuous complex-valued functions on X, is a unital commutative pro-C^*-algebra with the topology given by the family of C^*-seminorms $\{p_n\}_n$, $p_n(f) = \sup\{|f(x)|; x \in K_n\}$. A homeomorphism $h : X \to X$ such that $h(K_n) = K_n$ for all n induces a pro-C^*-isomorphism $\alpha : C(X) \to C(X)$, $\alpha(f) = f \circ h$, such that $p_n(\alpha(f)) = p_n(f)$ for all $f \in C(X)$ and for all n. An automorphism α of a pro-C^*-algebra $A[\tau_\Gamma]$ such that $p_\lambda(\alpha(a)) = p_\lambda(a)$ for all $a \in A[\tau_\Gamma]$ and $p_\lambda \in \Gamma$ induces an inverse limit action of the integers $\mathbb{Z}$

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on A, and an action β, called the dual action, of the circle $\mathbb{T}$ on the crossed product $A \times_\alpha \mathbb{Z}$ of A by α, the fixed point algebra of β being $A[\tau_\Gamma]$.

Given an action of $\mathbb{T}$ on a pro-C^*-algebra $A[\tau_\Gamma]$, it is natural to ask when $A[\tau_\Gamma]$ is isomorphic to the crossed product of a pro-C^*-algebra by an automorphism. To answer this question in the case of C^*-algebras, B. Abadie, S. Eilers and R. Exel [AEE] introduced the notion of crossed products by Hilbert C^*-bimodules, thus generalizing the notion of C^*-crossed products by automorphisms.

In this paper we extend the construction of a crossed product by Hilbert C^*-bimodules to the context of pro-C^*-algebras. In Section 3, we introduce the notion of a covariant representation of a Hilbert pro-C^*-bimodule X over a pro-C^*-algebra $A[\tau_\Gamma]$ on a pro-C^*-algebra $B[\tau_{\Gamma'}]$, and define the crossed product of $A[\tau_\Gamma]$ by X as the universal object with respect to the covariant representations of X. We show that the crossed product of $A[\tau_\Gamma]$ by X is isomorphic to a pro-C^*-algebra whose topology is given by a family of C^*-seminorms having the same index set as the family of C^*-seminorms that give the topology of $A[\tau_\Gamma]$. Also we show that the crossed product of a pro-C^*-algebra $A[\tau_\Gamma]$ by an automorphism can be regarded as the crossed product of $A[\tau_\Gamma]$ by a Hilbert pro-C^*-bimodule.

In Section 4, we show that an inverse limit action α of $\mathbb{T}$ on a pro-C^*-algebra $A[\tau_\Gamma]$ is semi-saturated (that is, $A[\tau_\Gamma]$ is generated by the fixed point algebra A_0 of α and the first spectral subspace A_1) if and only if $A[\tau_\Gamma]$ is isomorphic to the crossed product of A_0 by A_1.

In Section 5, we show that if X and Y are pro-C^*-bimodules over the pro-C^*-algebras $A[\tau_\Gamma]$, $B[\tau_{\Gamma'}]$ respectively and if A and B are strongly Morita equivalent and the Hilbert pro-C^*-bimodules $X \otimes_A E$ and $E \otimes_B Y$ are isomorphic, where E is an imprimitivity Hilbert A-B pro-C^*-bimodule, then the pro-C^*-algebras $A \times_X \mathbb{Z}$ and $B \times_Y \mathbb{Z}$ are strongly Morita equivalent. This is a generalization of [AEE, Theorem 4.2].

2. Preliminaries. Throughout this paper all vector spaces are considered over the field $\mathbb{C}$ of complex numbers and all topological spaces are assumed to be Hausdorff.

A *pro-C^*-algebra* $A[\tau_\Gamma]$ is a complete topological $*$-algebra for which there exists an upward directed family Γ of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$ defining the topology τ_Γ. Other terms that have been used for a pro-C^*-algebra are: locally C^*-algebra (A. Inoue), b^*-algebra (C. Apostol) and LMC*-algebra (G. Lassner, K. Schmüdgen).

For a pro-C^*-algebra $A[\tau_\Gamma]$, and every $\lambda \in \Lambda$, the quotient normed $*$-algebra $A_\lambda = A/\mathcal{N}_\lambda$, where $\mathcal{N}_\lambda = \{a \in A;\, p_\lambda(a) = 0\}$, is already complete, hence a C^*-algebra in the norm $\|a+\mathcal{N}_\lambda\|_{A_\lambda} = p_\lambda(a)$, $a \in A$ (C. Apostol). The canonical map from A to A_λ is denoted by π_λ^A. For $\lambda, \mu \in \Lambda$ with $\lambda \ge \mu$, there is a canonical surjective C^*-morphism $\pi^A_{\lambda\mu} : A_\lambda \to A_\mu$ such that $\pi^A_{\lambda\mu}(a+\mathcal{N}_\lambda) = a+\mathcal{N}_\mu$ for all $a \in A$. The Arens–Michael decomposition gives us a representation of $A[\tau_\Gamma]$ as an inverse limit of C^*-algebras: $A[\tau_\Gamma] = \lim_{\leftarrow\lambda} A/\mathcal{N}_\lambda$, up to a topological $*$-isomorphism. A *pro-C^*-morphism* is a continuous $*$-morphism Φ from a pro-C^*-algebra $A[\tau_\Gamma]$ to another pro-C^*-algebra $B[\tau_{\Gamma'}]$. We refer the reader to [F] for further information about pro-C^*-algebras.

Let Λ be an upward directed index set and $\mathcal{H}_\lambda$, $\lambda \in \Lambda$, a family of Hilbert spaces such that $\mathcal{H}_\mu \subseteq \mathcal{H}_\lambda$ and $\langle\cdot,\cdot\rangle_\mu = \langle\cdot,\cdot\rangle_\lambda|_{\mathcal{H}_\mu}$ for all $\lambda,\mu \in \Lambda$ with $\lambda \geq \mu$. Then $\{\mathcal{H}_\lambda; i_{\lambda\mu};\ \lambda,\mu \in \Lambda \text{ with } \lambda \geq \mu\}$, where $i_{\lambda\mu}$ is the natural embedding of $\mathcal{H}_\mu$ in $\mathcal{H}_\lambda$, is a direct system of Hilbert spaces. $\mathcal{H} = \lim_{\lambda\to} \mathcal{H}_\lambda$ endowed with the inductive limit topology is called a *locally Hilbert space*. Let $L(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H};\ T|_{\mathcal{H}_\lambda} \in L(\mathcal{H}_\lambda)\}$. If $T \in L(\mathcal{H})$, then the map $T^* : \mathcal{H} \to \mathcal{H}$ such that $T^*|_{\mathcal{H}_\lambda} = (T|_{\mathcal{H}_\lambda})^* \in L\,(\mathcal{H}_\lambda)$ for all $\lambda \in \Lambda$ is an element in $L(\mathcal{H})$, called the *adjoint* of T. In this way, $L(\mathcal{H})$ is a pro-C^*-algebra with respect to the topology given by the family of C^*-seminorms $\{p_{\lambda,L(\mathcal{H})}\}_{\lambda\in\Lambda}$, where $p_{\lambda,L(\mathcal{H})}(T) = \|T|_{\mathcal{H}_\lambda}\| = \sup\{\|T|_{\mathcal{H}_\lambda}(\xi)\|;\ \xi \in \mathcal{H}_\lambda,\ \|\xi\| \leq 1\}$ [F].

Here we recall some basic facts from [J1] and [Z] regarding Hilbert pro-C^*-modules and Hilbert pro-C^*-bimodules respectively.

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra and X a linear space that is also a right A-module. Let $\langle\cdot,\cdot\rangle_A$ be a right A-valued inner product on X, $\mathbb{C}$-linear in the second variable and conjugate linear in the first variable, with the following properties:

(1) $\langle x,x\rangle_A \geq 0$ and $\langle x,x\rangle_A = 0$ if and only if $x = 0$,
(2) $(\langle x,y\rangle_A)^* = \langle y,x\rangle_A$,
(3) $\langle x,ya\rangle_A = \langle x,y\rangle_A a$.

Then X is called a *right Hilbert pro-C^*-module over A* (or just a *Hilbert A-module*) if endowed with the family of seminorms $\{p^A_\lambda\}_{\lambda\in\Lambda}$, with $p^A_\lambda(x) = p_\lambda(\langle x,x\rangle_A)^{1/2}$, $x \in X$, is a complete locally convex space. A Hilbert A-module is *full* if the pro-C^*-subalgebra of $A[\tau_\Gamma]$ generated by $\{\langle x,y\rangle_A;\ x,y \in X\}$ coincides with $A[\tau_\Gamma]$.

A *left Hilbert pro-C^*-module X over a pro-C^*-algebra $A[\tau_\Gamma]$* is defined in the same way, where for instance (3) becomes now ${}_A\langle ax,y\rangle = a({}_A\langle x,y\rangle)$ for all $x,y \in X$ and $a \in A$ and completeness is required with respect to the family of seminorms $\{{}^Ap_\lambda\}_{\lambda\in\Lambda}$, where ${}^Ap_\lambda(x) = p_\lambda({}_A\langle x,x\rangle)^{1/2}$, $x \in X$.

In case X is a left Hilbert pro-C^*-module over $A[\tau_\Gamma]$ and a right Hilbert pro-C^*-module over $B[\tau_{\Gamma'}]$ ($\tau_{\Gamma'}$ is given by the family of C^*-seminorms $\{q_\lambda\}_{\lambda\in\Lambda}$) such that the following relations hold:

- ${}_A\langle x,y\rangle z = x\langle y,z\rangle_B$ for all $x,y,z \in X$,
- $q^B_\lambda(ax) \leq p_\lambda(a)q^B_\lambda(x)$ and ${}^Ap_\lambda(xb) \leq q_\lambda(b)\,{}^Ap_\lambda(x)$ for all $x \in X$, $a \in A$, $b \in B$ and for all $\lambda \in \Lambda$,

we say that X is *a Hilbert A-B pro-C^*-bimodule*.

A Hilbert A-B pro-C^*-bimodule X is *full* if it is full as a right and as a left Hilbert pro-C^*-module.

Two Hilbert A-B pro-C^*-bimodules X and Y are *isomorphic* if there is a topological isomorphism $\Phi : X \to Y$ such that $\langle \Phi(x_1), \Phi(x_2)\rangle_B = \langle x_1, x_2\rangle_B$ and ${}_A\langle \Phi(x_1), \Phi(x_2)\rangle = {}_A\langle x_1, x_2\rangle$ for all $x_1, x_2 \in X$.

Let Λ be an upward directed set and $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}; \chi_{\lambda\mu}; \sigma_{\lambda\mu};\ \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ an inverse system of Hilbert C^*-bimodules, that is:

- $\{A_\lambda; \pi_{\lambda\mu};\ \lambda, \mu \in \Lambda,\ \lambda \geq \mu\}$ and $\{B_\lambda; \chi_{\lambda\mu};\ \lambda, \mu \in \Lambda,\ \lambda \geq \mu\}$ are inverse systems of C^*-algebras;
- $\{X_\lambda; \sigma_{\lambda\mu};\ \lambda, \mu \in \Lambda,\ \lambda \geq \mu\}$ is an inverse system of Banach spaces;
- for each $\lambda \in \Lambda$, X_λ is a Hilbert A_λ-B_λ C^*-bimodule;
- we have $\langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y)\rangle_{B_\mu} = \chi_{\lambda\mu}(\langle x, y\rangle_{B_\lambda})$ and ${}_{A_\mu}\langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y)\rangle = \pi_{\lambda\mu}({}_{A_\lambda}\langle x, y\rangle)$ for all $x, y \in X_\lambda$ and for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$;
- $\sigma_{\lambda\mu}(x)\chi_{\lambda\mu}(b) = \sigma_{\lambda\mu}(xb)$, $\pi_{\lambda\mu}(a)\sigma_{\lambda\mu}(x) = \sigma_{\lambda\mu}(ax)$ for all $x \in X_\lambda$, $a \in A_\lambda$, $b \in B_\lambda$ and for all $\lambda, \mu \in \Lambda$ such that $\lambda \geq \mu$.

Let $A = \lim_{\leftarrow\lambda} A_\lambda$, $B = \lim_{\leftarrow\lambda} B_\lambda$ and $X = \lim_{\leftarrow\lambda} X_\lambda$. Then X has a structure of a Hilbert A-B pro-C^*-bimodule with

$$\langle (x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}\rangle_B = (\langle x_\lambda, y_\lambda\rangle_{B_\lambda})_{\lambda\in\Lambda},$$
$${}_A\langle (x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}\rangle = ({}_{A_\lambda}\langle x_\lambda, y_\lambda\rangle)_{\lambda\in\Lambda}.$$

Let X be a Hilbert A-B pro-C^*-bimodule. Then, for each $\lambda \in \Lambda$, ${}^A p_\lambda(x) = q_\lambda^B(x)$ for all $x \in X$, and the normed space $X_\lambda = X/N_\lambda^B$, where $N_\lambda^B = \{x \in X;\ q_\lambda^B(x) = 0\}$, is complete in the norm $\|x + N_\lambda^B\|_{X_\lambda} = q_\lambda^B(x)$, $x \in X$. Moreover, X_λ has a canonical structure of a Hilbert A_λ-B_λ C^*-bimodule with $\langle x + N_\lambda^B, y + N_\lambda^B\rangle_{B_\lambda} = \langle x, y\rangle_B + \ker q_\lambda$ and ${}_{A_\lambda}\langle x + N_\lambda^B, y + N_\lambda^B\rangle = {}_A\langle x, y\rangle + \ker p_\lambda$ for all $x, y \in X$. The canonical surjection from X to X_λ is denoted by σ_λ^X. For $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, there is a canonical surjective linear map $\sigma_{\lambda\mu}^X : X_\lambda \to X_\mu$ such that $\sigma_{\lambda\mu}^X(x + N_\lambda^B) = x + N_\mu^B$ for all $x \in X$. Then $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}^A; \sigma_{\lambda\mu}^X; \pi_{\lambda\mu}^B;\ \lambda, \mu \in \Lambda,\ \lambda \geq \mu\}$ is an inverse system of Hilbert C^*-bimodules in the above sense and the Hilbert A-B pro-C^*-bimodules X and $\lim_{\leftarrow\lambda} X_\lambda$ are isomorphic.

Let X and Y be Hilbert pro-C^*-modules over B. A morphism $T : X \to Y$ of right modules is *adjointable* if there is another morphism of modules $T^* : Y \to X$ such that $\langle Tx_1, x_2\rangle_B = \langle x_1, T^* x_2\rangle_B$ for all $x_1, x_2 \in X$. The vector space $L_B(X, Y)$ of all adjointable module morphisms from X to Y has a structure of locally convex space under the topology given by the family of seminorms $\{q_{\lambda, L_B(X,Y)}\}_{\lambda\in\Lambda}$, where $q_{\lambda, L_B(X,Y)}(T) = \sup\{q_\lambda^B(Tx);\ x \in X,\ q_\lambda^B(x) \leq 1\}$. For $x \in X$ and $y \in Y$, the map $\theta_{y,x} : X \to Y$ given by $\theta_{y,x}(z) = y\langle x, z\rangle_B$ is an adjointable module morphism and the closed subspace of $L_B(X, Y)$ generated by $\{\theta_{y,x};\ x \in X$ and $y \in Y\}$ is denoted by $K_B(X, Y)$;

its elements are usually called *compact operators*. For $Y = X$, $L_B(X) = L_B(X,X)$ is a pro-C^*-algebra with $(L_B(X))_\lambda = L_{B_\lambda}(X_\lambda)$ for each $\lambda \in \Lambda$, and $K_B(X) = K_B(X,X)$ is a closed two-sided $*$-ideal of $L_B(X)$ with $(K_B(X))_\lambda = K_{B_\lambda}(X_\lambda)$ for each $\lambda \in \Lambda$.

The *dual* of the Hilbert B-module X, $X^* = K_B(X,B)$ has a natural structure of a Hilbert pro-C^*-module over $K_B(X)$. If X is a full Hilbert A-B pro-C^*-bimodule, then the pro-C^*-algebras A and $K_B(X)$ are isomorphic, and so X^* can be regarded as a Hilbert pro-C^*-module over A. Moreover X^* is a Hilbert B-A pro-C^*-bimodule.

Let E be an A-B Hilbert pro-C^*-bimodule. The direct sum $E \oplus B$ of the Hilbert B-modules E and B has a natural structure of a right Hilbert B-module. Moreover, for each $\lambda \in \Lambda$, the right Hilbert B_λ-module $(E \oplus B)_\lambda$ can be identified with $E_\lambda \oplus B_\lambda$. Then the pro-$C^*$-algebras $K_B(E \oplus B)$ and $\lim_{\leftarrow \lambda} K_{B_\lambda}(E_\lambda \oplus B_\lambda)$ can be identified. For each $\lambda \in \Lambda$, the C^*-algebra $K_{B_\lambda}(E_\lambda \oplus B_\lambda)$ can be identified with the C^*-algebra of all matrices of the form

$$\begin{bmatrix} a_\lambda & \xi_\lambda \\ \widetilde{\eta}_\lambda & b_\lambda \end{bmatrix}, \qquad a_\lambda \in A_\lambda,\ b_\lambda \in B_\lambda,\ \xi_\lambda \in E_\lambda,\ \widetilde{\eta}_\lambda \in (E_\lambda)^*,$$

and then $K_B(E \oplus B)$ can be realized by the pro-C^*-algebra of all matrices of the form

$$\begin{bmatrix} a & \xi \\ \widetilde{\eta} & b \end{bmatrix}, \qquad a \in A,\ b \in B,\ \xi \in E,\ \widetilde{\eta} \in E^*.$$

This pro-C^*-algebra is denoted by $\mathcal{L}(E)$ and called the *linking algebra* of E. Moreover, $\mathcal{L}(E) = \lim_{\leftarrow \lambda} \mathcal{L}(E_\lambda)$.

3. Crossed products by Hilbert pro-C^*-bimodules. Let $A[\tau_\Gamma]$ be a pro-C^*-algebra with $\Gamma = \{p_\lambda;\ \lambda \in \Lambda\}$ a defining family of C^*-seminorms, X a Hilbert A-A pro-C^*-bimodule, and $B[\tau_{\Gamma'}]$ a pro-C^*-algebra with $\Gamma' = \{q_i;\ i \in I\}$ a defining family of C^*-seminorms.

Definition 3.1. *A covariant representation* of a Hilbert A-A pro-C^*-bimodule X on a pro-C^*-algebra B is a pair (φ_X, φ_A) consisting of a pro-C^*-morphism $\varphi_A : A \to B$ and a map $\varphi_X : X \to B$ which satisfies the following relations:

(1) $\varphi_X(xa) = \varphi_X(x)\varphi_A(a)$ for all $x \in X$ and $a \in A$;
(2) $\varphi_X(ax) = \varphi_A(a)\varphi_X(x)$ for all $x \in X$ and $a \in A$;
(3) $\varphi_X(x)^*\varphi_X(y) = \varphi_A(\langle x, y\rangle_A)$ for all $x, y \in X$;
(4) $\varphi_X(x)\varphi_X(y)^* = \varphi_A({}_A\langle x, y\rangle)$ for all $x, y \in X$.

The covariant representation (φ_X, φ_A) is *nondegenerate* if φ_A is nondegenerate, that is, $[\varphi_A(A)B] = B$, $[\varphi_X(X)B] = B$ and $[\varphi_X(X)^*B] = B$.

If φ_A is nondegenerate and X is full, then (φ_X, φ_A) is nondegenerate. Indeed,

$$B = [\varphi_A(A)B] = [\varphi_A({}_A\langle X, X\rangle)B] = [\varphi_X(X)\varphi_X(X)^*B] \subseteq [\varphi_X(X)B] \subseteq B$$

and so $[\varphi_X(X)B] = B$. Similarly also $[\varphi_X(X)^*B] = B$.

If (φ_X, φ_A) is a covariant representation of a Hilbert A-A pro-C^*-bimodule X on a pro-C^*-algebra B, then φ_X is continuous, since for each $i \in I$, there is $\lambda(i) \in \Lambda$ such that

$$\begin{aligned} q_i(\varphi_X(x))^2 &= q_i(\varphi_X(x)^*\varphi_X(x)) = q_i(\varphi_A(\langle x, x\rangle_A)) \le p_{\lambda(i)}(\langle x, x\rangle_A) \\ &= p^A_{\lambda(i)}(x)^2 \end{aligned}$$

for all $x \in X$.

Relation (3) implies that φ_X is linear. Also, (3) implies (1). Indeed,

$$\begin{aligned} &q_i(\varphi_X(xa) - \varphi_X(x)\varphi_A(a))^2 \\ &\quad= q_i\big((\varphi_X(xa)^* - \varphi_A(a^*)\varphi_X(x)^*)(\varphi_X(xa) - \varphi_X(x)\varphi_A(a))\big) \\ &\quad= q_i\big(\varphi_A(\langle xa, xa\rangle_A) - \varphi_A(\langle xa, x\rangle_A)\varphi_A(a) - \varphi_A(a^*)\varphi_A(\langle x, xa\rangle_A) \\ &\qquad+ \varphi_A(a^*)\varphi_A(\langle x, x\rangle_A)\varphi_A(a)\big) = 0 \end{aligned}$$

for all $x \in X$, $a \in A$ and $q_i \in \Gamma'$. Similarly, (4) implies (2).

PROPOSITION 3.2. *For every Hilbert A-A pro-C^*-bimodule X, there exists a covariant representation of X.*

Proof. For each $\lambda \in \Lambda$, there is a covariant representation $(\varphi_{X_\lambda}, \varphi_{A_\lambda})$ of (X_λ, A_λ) on $L(H_\lambda)$, the C^*-algebra of all bounded linear operators on a Hilbert space H_λ (see [AEE, Proposition 2.3]). Let $\mathcal{H}_\lambda = \bigoplus_{\mu\le\lambda} H_\mu$ for each $\lambda \in \Lambda$. Then $\mathcal{H} = \lim_{\lambda\to} \mathcal{H}_\lambda$ is a locally Hilbert space. For each $a \in A$, the family $(\varphi_A^\lambda(a))_\lambda$, where $\varphi_A^\lambda(a)$ is the bounded linear operator on $\mathcal{H}_\lambda$ given by

$$\varphi_A^\lambda(a)\Big(\bigoplus_{\mu\le\lambda} \xi_\mu\Big) = \bigoplus_{\mu\le\lambda} \varphi_{A_\mu}(a + \mathcal{N}_\mu)\xi_\mu,$$

is an inductive system of bounded linear operators, and the map $a \mapsto \varphi_A(a) = \lim_{\lambda\to} \varphi_A^\lambda(a)$ from A to $L(\mathcal{H})$ is a pro-C^*-morphism (see [F, Theorem 8.5] or [I]).

Let $x \in X$. For each $\lambda \in \Lambda$, the linear map $\varphi_X^\lambda(x) : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ defined by

$$\varphi_X^\lambda(x)\Big(\bigoplus_{\mu\le\lambda} \xi_\mu\Big) = \bigoplus_{\mu\le\lambda} \varphi_{X_\mu}(x + N_\mu^A)\xi_\mu$$

is a bounded linear operator, since

$$\Big\|\varphi_X^\lambda(x)\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big)\Big\|^2 = \sum_{\mu\le\lambda}\|\varphi_{X_\mu}(x+N_\mu^A)\xi_\mu\|^2 \le \sum_{\mu\le\lambda} p_\mu^A(x)^2\|\xi_\mu\|^2$$
$$\le p_\lambda^A(x)^2\sum_{\mu\le\lambda}\|\xi_\mu\|^2 = p_\lambda^A(x)^2\Big\|\bigoplus_{\mu\le\lambda}\xi_\mu\Big\|^2$$

for all $\xi_\mu \in H_\mu$, $\mu,\lambda\in\Lambda$, $\mu\le\lambda$. It is easy to check that $(\varphi_X^\lambda(x))_\lambda$ is an inductive system of bounded linear operators, and so $\varphi_X(x) = \lim_{\lambda\to}\varphi_X^\lambda(x)$ is an element in $L(\mathcal{H})$. In this way, we obtain a map φ_X from X to $L(\mathcal{H})$. Moreover,

$$\begin{aligned}
\varphi_X(x)^*\varphi_X(y)\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big) &= \varphi_X^\lambda(x)^*\varphi_X^\lambda(y)\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big)\\
&= \bigoplus_{\mu\le\lambda}\varphi_{X_\mu}(x+N_\mu^A)^*\varphi_{X_\mu}(y+N_\mu^A)\xi_\mu\\
&= \bigoplus_{\mu\le\lambda}\varphi_{A_\mu}(\langle x,y\rangle_A+\mathcal{N}_\mu)\xi_\mu\\
&= \varphi_A^\lambda(\langle x,y\rangle_A)\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big) = \varphi_A(\langle x,y\rangle_A)\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_X(x)\varphi_X(y)^*\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big) &= \varphi_X^\lambda(x)\varphi_X^\lambda(y)^*\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big)\\
&= \bigoplus_{\mu\le\lambda}\varphi_{X_\mu}(x+{}^AN_\mu)\varphi_{X_\mu}(y+{}^AN_\mu)^*\xi_\mu\\
&= \bigoplus_{\mu\le\lambda}\varphi_{A_\mu}({}_A\langle x,y\rangle+\mathcal{N}_\mu)\xi_\mu\\
&= \varphi_A^\lambda({}_A\langle x,y\rangle)\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big) = \varphi_A({}_A\langle x,y\rangle)\Big(\bigoplus_{\mu\le\lambda}\xi_\mu\Big)
\end{aligned}$$

for all $x,y\in X$, $\xi_\mu\in H_\mu$, $\mu,\lambda\in\Lambda$, $\mu\le\lambda$. Therefore, (φ_X,φ_A) is a covariant representation of (X,A) on $L(\mathcal{H})$. ■

DEFINITION 3.3. Let X be a Hilbert A-A pro-C^*-bimodule. The crossed product of A by X is a pro-C^*-algebra, denoted by $A\times_X\mathbb{Z}$, together with a covariant representation (i_X, i_A) of (X,A) on $A\times_X\mathbb{Z}$ with the property that for any covariant representation (φ_X,φ_A) of (X,A) on a pro-C^*-algebra $B[\tau_{\Gamma'}]$, there is a unique pro-C^*-morphism $\Phi: A\times_X\mathbb{Z}\to B$ such that $\Phi\circ i_X = \varphi_X$ and $\Phi\circ i_A=\varphi_A$.

Proposition 3.4.

(1) *The crossed product of A by X is unique up to an isomorphism of pro-C^*-algebras.*
(2) *The pro-C^*-subalgebra of $A \times_X \mathbb{Z}$ generated by the range of i_X and the range of i_A coincides with $A \times_X \mathbb{Z}$.*
(3) *If X is full, then (i_X, i_A) is nondegenerate.*

Proof. (1) Suppose that there is another pro-C^*-algebra $B[\tau_{\Gamma'}]$ and a covariant representation (j_X, j_A) of (X, A) on B with the property that for any covariant representation (φ_X, φ_A) of (X, A) on a pro-C^*-algebra $C[\tau_{\Gamma''}]$, there is a unique pro-C^*-morphism $\Psi : B \to C$ such that $\Psi \circ j_X = \varphi_X$ and $\Psi \circ j_A = \varphi_A$. Then from the universal property of the crossed product $A \times_X \mathbb{Z}$, there is a unique pro-C^*-morphism $\Phi : A \times_X \mathbb{Z} \to B$ such that $\Phi \circ i_A = j_A$ and $\Phi \circ i_X = j_X$.

On the other hand, from the universal property of B, there is a unique pro-C^*-morphism $\Psi : B \to A \times_X \mathbb{Z}$ such that $\Psi \circ j_X = i_X$ and $\Psi \circ j_A = i_A$. Then, since $\Psi \circ \Phi \circ i_X = i_X$ and $\Psi \circ \Phi \circ i_A = i_A$, from the uniqueness condition of the universal property of $A \times_X \mathbb{Z}$, we have $\Psi \circ \Phi = \mathrm{id}_{A \times_X \mathbb{Z}}$. Similarly, since $\Phi \circ \Psi \circ j_X = j_X$ and $\Phi \circ \Psi \circ j_A = j_A$, from the uniqueness condition of the universal property of B, we have $\Phi \circ \Psi = \mathrm{id}_B$. Therefore, Φ is a pro-C^*-isomorphism.

(2) Let ι be the inclusion of the pro-C^*-subalgebra pro-$C^*(i_X(X), i_A(A))$ of $A \times_X \mathbb{Z}$ generated by $i_X(X)$ and $i_A(A)$. Then (i_X^0, i_A^0), with $i_X^0(x) = i_X(x)$ for all $x \in X$ and $i_A^0(a) = i_A(a)$ for all $a \in A$, is a covariant representation of (X, A) on pro-$C^*(i_X(X), i_A(A))$. We show that pro-$C^*(i_X(X), i_A(A))$ with the covariant representation (i_X^0, i_A^0) is a universal object for the covariant representations of (X, A). Indeed, if (φ_X, φ_A) is a covariant representation of (X, A) on a pro-C^*-algebra B, then there is a unique pro-C^*-morphism $\Phi : A \times_X \mathbb{Z} \to B$ such that $\Phi \circ i_X = \varphi_X$ and $\Phi \circ i_A = \varphi_A$. Let $\Psi = \Phi \circ \iota$. Clearly, Ψ : pro-$C^*(i_X(X), i_A(A)) \to B$ is a pro-C^*-morphism, $\Psi \circ i_X^0 = \varphi_X$ and $\Psi \circ i_A^0 = \varphi_A$. If $\widetilde{\Psi}$: pro-$C^*(i_X(X), i_A(A)) \to B$ is another pro-C^*-morphism such that $\widetilde{\Psi} \circ i_X^0 = \varphi_X$ and $\widetilde{\Psi} \circ i_A^0 = \varphi_A$, then $(\Psi - \widetilde{\Psi}) \circ i_X^0 = 0$ and $(\Psi - \widetilde{\Psi}) \circ i_A^0 = 0$, whence we deduce that $\Psi = \widetilde{\Psi}$, and by (1), the pro-C^*-algebras $A \times_X \mathbb{Z}$ and pro-$C^*(i_X(X), i_A(A))$ are isomorphic.

(3) Let $\{e_i\}_i$ be an approximate unit for A. Then for each $x \in X$, $i_A(e_i) i_X(x) = i_X(e_i x)$ and $i_A(e_i) i_X(x)^* = i_X(x e_i)^*$ for all $i \in I$, and since the nets $\{e_i x\}_i$ and $\{x e_i\}_i$ converge to x, the nets $\{i_A(e_i) i_X(x)\}_i$ and $\{i_A(e_i) i_X(x)^*\}_i$ converge to $i_X(x)$ respectively $i_X(x)^*$. Therefore, $\big[i_A(A) \, A \times_X \mathbb{Z}\big] = A \times_X \mathbb{Z}$ and since X is full, (i_X, i_A) is nondegenerate. ∎

Let Λ be an upward directed index set and $\{A_\lambda; X_\lambda; \pi_{\lambda\mu}; \sigma_{\lambda\mu};\ \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ an inverse system of Hilbert C^*-bimodules, $A = \lim_{\leftarrow\lambda} A_\lambda$ and

$X = \lim_{\leftarrow\lambda} X_\lambda$. For $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, $(i_{X_\mu} \circ \sigma_{\lambda\mu}, i_{A_\mu} \circ \pi_{\lambda\mu})$ is a covariant representation of (X_λ, A_λ) on $A_\mu \times_{X_\mu} \mathbb{Z}$. Indeed, we have

$$(i_{X_\mu} \circ \sigma_{\lambda\mu})(x)^*(i_{X_\mu} \circ \sigma_{\lambda\mu})(y) = i_{X_\mu}(\sigma_{\lambda\mu}(x))^* i_{X_\mu}(\sigma_{\lambda\mu}(y))$$
$$= i_{A_\mu}(\langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y)\rangle_{A_\mu}) = i_{A_\mu}(\pi_{\lambda\mu}(\langle x, y\rangle_{A_\lambda})) = i_{A_\mu} \circ \pi_{\lambda\mu}(\langle x, y\rangle_{A_\lambda})$$

and

$$(i_{X_\mu} \circ \sigma_{\lambda\mu})(x)(i_{X_\mu} \circ \sigma_{\lambda\mu})(y)^* = i_{X_\mu}(\sigma_{\lambda\mu}(x)) i_{X_\mu}(\sigma_{\lambda\mu}(y))^*$$
$$= i_{A_\mu}({}_{A_\mu}\langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y)\rangle) = i_{A_\mu}(\pi_{\lambda\mu}({}_{A_\lambda}\langle x, y\rangle)) = i_{A_\mu} \circ \pi_{\lambda\mu}({}_{A_\lambda}\langle x, y\rangle)$$

for all $x, y \in X_\lambda$. Then, by [AEE, Definition 2.4], there is a unique C^*-morphism $\chi_{\lambda\mu} : A_\lambda \times_{X_\lambda} \mathbb{Z} \to A_\mu \times_{X_\mu} \mathbb{Z}$ such that $\chi_{\lambda\mu} \circ i_{X_\lambda} = i_{X_\mu} \circ \sigma_{\lambda\mu}$ and $\chi_{\lambda\mu} \circ i_{A_\lambda} = i_{A_\mu} \circ \pi_{\lambda\mu}$. Moreover, if the maps $\pi_{\lambda\mu}$ and $\sigma_{\lambda\mu}$ are surjective, then $\chi_{\lambda\mu}$ is surjective.

Let $\lambda, \mu, \eta \in \Lambda$ with $\lambda \geq \mu \geq \eta$. We have

$$(\chi_{\mu\eta} \circ \chi_{\lambda\mu}) \circ i_{X_\lambda} = \chi_{\mu\eta} \circ (\chi_{\lambda\mu} \circ i_{X_\lambda}) = (\chi_{\mu\eta} \circ i_{X_\mu}) \circ \sigma_{\lambda\mu} = i_{X_\eta} \circ \sigma_{\mu\eta} \circ \sigma_{\lambda\mu}$$
$$= i_{X_\eta} \circ \sigma_{\lambda\eta}$$

and

$$(\chi_{\mu\eta} \circ \chi_{\lambda\mu}) \circ i_{A_\lambda} = \chi_{\mu\eta} \circ (\chi_{\lambda\mu} \circ i_{A_\lambda}) = (\chi_{\mu\eta} \circ i_{A_\mu}) \circ \pi_{\lambda\mu} = i_{A_\eta} \circ \pi_{\mu\eta} \circ \pi_{\lambda\mu}$$
$$= i_{A_\eta} \circ \pi_{\lambda\eta}.$$

Then, by [AEE, Definition 2.4], $\chi_{\mu\eta} \circ \chi_{\lambda\mu} = \chi_{\lambda\eta}$. Therefore, $\{A_\lambda \times_{X_\lambda} \mathbb{Z};\ \chi_{\lambda\mu};\ \lambda, \mu \in \Lambda \text{ with } \lambda \geq \mu\}$ is an inverse system of C^*-algebras.

PROPOSITION 3.5. *Let $\{A_\lambda; X_\lambda; \pi_{\lambda\mu}; \sigma_{\lambda\mu};\ \lambda, \mu \in \Lambda,\ \lambda \geq \mu\}$ be an inverse system of Hilbert C^*-bimodules such that the canonical projections $\pi_\lambda : A \to A_\lambda$ and $\sigma_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$ are all surjective, where $A = \lim_{\leftarrow\lambda} A_\lambda$ and $X = \lim_{\leftarrow\lambda} X_\lambda$. Then there is a covariant representation (i_X, i_A) of (X, A) on $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$ with the property that for any covariant representation (φ_X, φ_A) of (X, A) on a pro-C^*-algebra $B[\tau_{\Gamma'}]$, there is a unique pro-C^*-morphism Φ from $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$ to B such that $\Phi \circ i_X = \varphi_X$ and $\Phi \circ i_A = \varphi_A$. Moreover, $p_{\lambda, \lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}}(i_A(a)) = p_\lambda(a)$ for all $a \in A$ and $p_{\lambda, \lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}}(i_X(x)) = p_\lambda^A(x)$ for all $x \in X$.*

Proof. From $\chi_{\lambda\mu} \circ i_{X_\lambda} = i_{X_\mu} \circ \sigma_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, we deduce that $\{i_{X_\lambda}\}_{\lambda \in \Lambda}$ is an inverse system of linear maps, and from $\chi_{\lambda\mu} \circ i_{A_\lambda} = i_{A_\mu} \circ \pi_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, we deduce that $\{i_{A_\lambda}\}_{\lambda \in \Lambda}$ is an inverse system of C^*-morphisms. Let $i_X = \lim_{\leftarrow\lambda} i_{X_\lambda}$ and $i_A = \lim_{\leftarrow\lambda} i_{A_\lambda}$. Then i_A is a pro-C^*-morphism from A to $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$, and (i_X, i_A) is a covariant representation of (X, A) on $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$, since

$$i_X((x_\lambda)_{\lambda\in\Lambda})^* i_X((y_\lambda)_{\lambda\in\Lambda}) = (i_{X_\lambda}(x_\lambda)^* i_{X_\lambda}(y_\lambda))_{\lambda\in\Lambda} = (i_{A_\lambda}(\langle x_\lambda, y_\lambda\rangle_{A_\lambda}))_{\lambda\in\Lambda}$$
$$= i_A((\langle x_\lambda, y_\lambda\rangle_{A_\lambda})_{\lambda\in\Lambda}) = i_A(\langle (x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}\rangle_A)$$

and

$$\begin{aligned} i_X((x_\lambda)_{\lambda\in\Lambda})i_X((y_\lambda)_{\lambda\in\Lambda})^* &= (i_{X_\lambda}(x_\lambda)i_{X_\lambda}(y_\lambda)^*)_{\lambda\in\Lambda} = (i_{A_\lambda}({}_{A_\lambda}\langle x_\lambda, y_\lambda\rangle))_{\lambda\in\Lambda} \\ &= i_A(({}_{A_\lambda}\langle x_\lambda, y_\lambda\rangle)_{\lambda\in\Lambda}) = i_A({}_A\langle (x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}\rangle) \end{aligned}$$

for all $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda} \in X$.

Let $\lambda \in \Lambda$. Then, by [AEE, Corollary 2.10],

$$p_{\lambda, \lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}}(i_A((a_\lambda)_{\lambda\in\Lambda})) = \|i_{A_\lambda}(a_\lambda)\|_{A_\lambda \times_{X_\lambda} \mathbb{Z}} = \|a_\lambda\|_{A_\lambda} = p_\lambda((a_\lambda)_{\lambda\in\Lambda})$$

for all $(a_\lambda)_{\lambda\in\Lambda} \in A$, and

$$p_{\lambda, \lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}}(i_X((x_\lambda)_{\lambda\in\Lambda})) = \|i_{X_\lambda}(x_\lambda)\|_{A_\lambda \times_{X_\lambda} \mathbb{Z}} = \|x_\lambda\|_{X_\lambda} = p^A_\lambda((x_\lambda)_{\lambda\in\Lambda})$$

for all $x \in X$.

We note that $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$ is generated by $i_X(X)$ and $i_A(A)$. This comes as a consequence of [M, Chapter III, Theorem 3.1]. Indeed, we have

$$\begin{aligned} \text{pro-}C^*(i_A(A), i_X(X)) &= \lim_{\leftarrow\lambda} \overline{\chi_\lambda(\text{pro-}C^*(i_A(A), i_X(X)))} \\ &= \lim_{\leftarrow\lambda} C^*(\chi_\lambda(i_A(A)), \chi_\lambda(i_X(X))) \\ &= \lim_{\leftarrow\lambda} C^*(i_{A_\lambda}(\pi_\lambda(A)), i_{X_\lambda}(\sigma_\lambda(X))) \\ &= \lim_{\leftarrow\lambda} C^*(i_{A_\lambda}(A_\lambda), i_{X_\lambda}(X_\lambda)) = \lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z} \end{aligned}$$

where χ_λ, $\lambda \in \Lambda$, are the canonical projections from $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$ to $A_\lambda \times_{X_\lambda} \mathbb{Z}$.

Let (φ_X, φ_A) be a covariant representation of (X, A) on a pro-C^*-algebra $B[\tau_{\Gamma'}]$ with $\Gamma' = \{q_i; i \in I\}$ a defining family of C^*-seminorms. For $i \in I$, there is $\lambda(i) \in \Lambda$ such that $q_i(\varphi_A(a)) \le p_{\lambda(i)}(a)$ for all $a \in A$ and $q_i(\varphi_X(x)) \le p^A_{\lambda(i)}(x)$ for all $x \in X$. Then there are a C^*-morphism $\varphi_{A_{\lambda(i)}} : A_{\lambda(i)} \to B_i$ such that $\varphi_{A_{\lambda(i)}} \circ \pi_{\lambda(i)} = \pi^B_i \circ \varphi_A$, and a continuous linear map $\varphi_{X_{\lambda(i)}} : X_{\lambda(i)} \to B_i$ such that $\varphi_{X_{\lambda(i)}} \circ \sigma_{\lambda(i)} = \pi^B_i \circ \varphi_X$. It is easy to check that $(\varphi_{X_{\lambda(i)}}, \varphi_{A_{\lambda(i)}})$ is a covariant representation of $(X_{\lambda(i)}, A_{\lambda(i)})$ on B_i. By [AEE, Definition 2.4], there is a unique C^*-morphism $\phi_{\lambda(i)} : A_{\lambda(i)} \times_{X_{\lambda(i)}} \mathbb{Z} \to B_i$ such that $\phi_{\lambda(i)} \circ i_{X_{\lambda(i)}} = \varphi_{X_{\lambda(i)}}$ and $\phi_{\lambda(i)} \circ i_{A_{\lambda(i)}} = \varphi_{A_{\lambda(i)}}$. Let $\Phi_i = \phi_{\lambda(i)} \circ \chi_{\lambda(i)}$. Clearly, Φ_i is a continuous $*$-morphism from $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$ to B_i, and from

$$\begin{aligned} (\pi^B_{ij} \circ \Phi_i)(i_X((x_\mu)_{\mu\in\Lambda})) &= (\pi^B_{ij} \circ \phi_{\lambda(i)} \circ \chi_{\lambda(i)})((i_{X_\mu}(x_\mu))_{\mu\in\Lambda}) \\ &= (\pi^B_{ij} \circ \phi_{\lambda(i)})(i_{X_{\lambda(i)}}(x_{\lambda(i)})) = \pi^B_{ij}(\varphi_{X_{\lambda(i)}}(x_{\lambda(i)})) \\ &= \pi^B_{ij}(\pi^B_i(\varphi_X((x_\mu)_{\mu\in\Lambda}))) = \pi^B_j(\varphi_X((x_\mu)_{\mu\in\Lambda})) \\ &= \varphi_{X_{\lambda(j)}}(\sigma_{\lambda(j)}((x_\mu)_{\mu\in\Lambda})) = \phi_{\lambda(j)}(i_{X_{\lambda(j)}} \circ \sigma_{\lambda(j)}((x_\mu)_{\mu\in\Lambda})) \\ &= \phi_{\lambda(j)} \circ \chi_{\lambda(j)}(i_X((x_\mu)_{\mu\in\Lambda})) = \Phi_j(i_X((x_\mu)_{\mu\in\Lambda})) \end{aligned}$$

for all $(x_\mu)_{\mu\in\Lambda} \in X$ and

$$\begin{aligned}(\pi_{ij}^B \circ \Phi_i)(i_A((a_\mu)_{\mu\in\Lambda})) &= (\pi_{ij}^B \circ \phi_{\lambda(i)} \circ \chi_{\lambda(i)})((i_{A_\mu}(a_\mu))_{\mu\in\Lambda}) \\ &= (\pi_{ij}^B \circ \phi_{\lambda(i)})(i_{A_{\lambda(i)}}(a_{\lambda(i)})) = \pi_{ij}^B(\varphi_{A_{\lambda(i)}}(a_{\lambda(i)})) \\ &= \pi_{ij}^B(\pi_i^B(\varphi_A((a_\mu)_{\mu\in\Lambda}))) = \pi_j^B(\varphi_A((a_\mu)_{\mu\in\Lambda})) = \Phi_j(i_A((a_\mu)_{\mu\in\Lambda}))\end{aligned}$$

for all $(a_\mu)_{\mu\in\Lambda} \in A$, and taking into account the first part of the proof, we deduce that $\pi_{ij}^B \circ \Phi_i = \Phi_j$ for all $i, j \in I$ with $i \geq j$. Then there is a continuous $*$-morphism $\Phi : \lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z} \to B$ such that $\pi_i^B \circ \Phi = \Phi_i$ for all $i \in I$. Moreover, $\Phi \circ i_X = \varphi_X$, since

$$\pi_i^B((\Phi \circ i_X)((x_\mu)_{\mu\in\Lambda})) = \Phi_i(i_X((x_\mu)_{\mu\in\Lambda})) = \pi_i^B(\varphi_X((x_\mu)_{\mu\in\Lambda}))$$

for all $(x_\mu)_{\mu\in\Lambda} \in X$ and for all $i \in I$. Also $\Phi \circ i_A = \varphi_A$, since

$$\pi_i^B((\Phi \circ i_A)((a_\mu)_{\mu\in\Lambda})) = \Phi_i(i_A((a_\mu)_{\mu\in\Lambda})) = \pi_i^B(\varphi_A((a_\mu)_{\mu\in\Lambda}))$$

for all $(a_\mu)_{\mu\in\Lambda} \in A$ and $i \in I$. The morphism Φ with these properties is unique, since $i_A(A)$ and $i_X(X)$ generate $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$. ∎

COROLLARY 3.6. *Let $\{A_\lambda; X_\lambda; \pi_{\lambda\mu}; \sigma_{\lambda\mu};\ \lambda, \mu \in \Lambda,\ \lambda \geq \mu\}$ be an inverse system of Hilbert C^*-bimodules such that the canonical projections $\pi_\lambda : A \to A_\lambda$ and $\sigma_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$ are all surjective, where $A = \lim_{\leftarrow\lambda} A_\lambda$ and $X = \lim_{\leftarrow\lambda} X_\lambda$. Then:*

(1) *$A \times_X \mathbb{Z}$ is isomorphic to $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$.*
(2) *A and X are embedded in $A \times_X \mathbb{Z}$.*

LEMMA 3.7. *Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two pro-C^*-algebras. If $\varphi : A \to B$ is a pro-C^*-isomorphism, then we may suppose that Γ and Γ' have the same index set. Moreover, $\varphi = \lim_{\leftarrow\lambda} \varphi_\lambda$, where $\varphi_\lambda : A_\lambda \to B_\lambda$, $\lambda \in \Lambda$, are C^*-isomorphisms.*

Proof. We show that the family $\{p_\lambda \circ \varphi^{-1};\ \lambda \in \Lambda\}$ of continuous C^*-seminorms on B gives the topology on B. Indeed, for each $p_\lambda \in \Gamma$, there is $q_{i(\lambda)} \in \Gamma'$ such that

$$(p_\lambda \circ \varphi^{-1})(b) = p_\lambda(\varphi^{-1}(b)) \leq q_{i(\lambda)}(b)$$

for all $b \in B$. Also for each $q_i \in \Gamma'$, there is $p_{\lambda(i)} \in \Gamma$ such that

$$q_i(b) = q_i(\varphi(\varphi^{-1}(b))) \leq p_{\lambda(i)}(\varphi^{-1}(b)) = (p_{\lambda(i)} \circ \varphi^{-1})(b)$$

for all $b \in B$. Moreover, for each $\lambda \in \Lambda$, there is a C^*-isomorphism $\varphi_\lambda : A_\lambda \to B_\lambda$, where $A_\lambda = A/\ker p_\lambda$ and $B_\lambda = B/\ker(p_\lambda \circ \varphi^{-1})$, such that $\varphi_\lambda \circ \pi_\lambda^A = \pi_\lambda^B \circ \varphi$ and $\varphi = \lim_{\leftarrow\lambda} \varphi_\lambda$. ∎

PROPOSITION 3.8. *Let $A[\tau_\Gamma]$ be a pro-C^*-algebra and X a Hilbert A-A pro-C^*-bimodule. Then there is a family of C^*-seminorms which gives the topology on $A \times_X \mathbb{Z}$ having the same index set as the family of C^*-seminorms* *which gives the topology on A, and moreover, for each $\lambda \in \Lambda$, the C^*-algebras $(A \times_X \mathbb{Z})_\lambda$ and $A_\lambda \times_{X_\lambda} \mathbb{Z}$ are isomorphic.*

Proof. We have seen that $\{A_\lambda; X_\lambda; \pi^A_{\lambda\mu}; \sigma^X_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ is an inverse system of Hilbert C^*-bimodules such that the canonical projections $\pi^A_\lambda : A \to A_\lambda$ and $\sigma^X_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, are all surjective. By Corollary 3.6, the pro-C^*-algebras $A \times_X \mathbb{Z}$ and $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$ are isomorphic and so by Lemma 3.7, there is a family of C^*-seminorms which gives the topology on $A \times_X \mathbb{Z}$ and has the same index set as the family of C^*-seminorms which gives the topology on A. Since the canonical surjections $\chi_\lambda : \lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z} \to A_\lambda \times_{X_\lambda} \mathbb{Z}$, $\lambda \in \Lambda$, are surjective, for each $\lambda \in \Lambda$ the C^*-algebras $(A \times_X \mathbb{Z})_\lambda$ and $A_\lambda \times_{X_\lambda} \mathbb{Z}$ are isomorphic. ∎

EXAMPLE 3.9. Let $A[\tau_\Gamma]$ be a pro-C^*-algebra and α an automorphism of $A[\tau_\Gamma]$ such that $p_\lambda(\alpha(a)) = p_\lambda(a)$ for all $a \in A$, $\lambda \in \Lambda'$, where Λ' is a cofinal subset of Λ. Then the pro-C^*-algebras $A \times_{X_\alpha} \mathbb{Z}$ and $A \times_\alpha \mathbb{Z}$ are isomorphic.

Indeed, if α is an automorphism of $A[\tau_\Gamma]$ as above, then $(A, \alpha, \mathbb{Z})$ is a pro-C^*-dynamical system with the action of $\mathbb{Z}$ on A given by $n \mapsto \alpha^n$.

Let $X_\alpha = \{\xi_x; x \in A\}$. Then X_α is a Hilbert A-A pro-C^*-bimodule. The bimodule structure is defined as $\xi_x a = \xi_{xa}$, respectively $a\xi_x = \xi_{\alpha^{-1}(a)x}$, and the inner products are defined as $\langle \xi_x, \xi_y \rangle_A = x^* y$, respectively ${}_A\langle \xi_x, \xi_y \rangle = \alpha(xy^*)$.

As in the case of C^*-algebras, if (u, φ) is a nondegenerate covariant representation of $(A, \alpha, \mathbb{Z})$ on a pro-C^*-algebra B, then $(\varphi_{X_\alpha}, \varphi_A)$, where $\varphi_A = \varphi$ and $\varphi_{X_\alpha}(\xi_x) = u_1 \varphi(x)$, is a nondegenerate covariant representation of (X_α, A) on B.

Conversely, if $(\varphi_{X_\alpha}, \varphi_A)$ is a nondegenerate covariant representation of (X_α, A) on a pro-C^*-algebra B, then the map $u : B \to B$ defined by $u(\varphi_A(a)b) = \varphi_{X_\alpha}(\xi_a)b$ is a unitary operator, and (u, φ), where $\varphi = \varphi_A$ and $n \mapsto u_n = u^n$ with $u_0 = \mathrm{id}_B$, is a nondegenerate covariant representation of $(A, \alpha, \mathbb{Z})$ on B. Using these facts and the universal property of the crossed product of pro-C^*-algebras [J3, Theorem 2.4], we deduce that the pro-C^*-algebras $A \times_{X_\alpha} \mathbb{Z}$ and $A \times_\alpha \mathbb{Z}$ are isomorphic.

4. An application. Let α be an action of $\mathbb{T}$ on a pro-C^*-algebra A, let $A_0 = \{a \in A;\ \alpha_z(a) = a \text{ for all } z \in \mathbb{T}\}$ be the fixed point algebra of α, and let $A_1 = \{a \in A;\ \alpha_z(a) = za \text{ for all } z \in \mathbb{T}\}$ be the first spectral subspace of α. Clearly, A_1 has a natural structure of an A_0-A_0 Hilbert pro-C^*-bimodule. We will show that an inverse limit action α of the unit circle on a pro-C^*-algebra A is semi-saturated if and only if A is isomorphic to the crossed product of A_0 by A_1.

REMARK 4.1. Let α be an inverse limit action of $\mathbb{T}$ on a pro-C^*-algebra A, that is, $\alpha_z = \lim_{\leftarrow\lambda} \alpha^\lambda_z$ for each $z \in \mathbb{T}$, where α^λ, $\lambda \in \Lambda$, are actions of $\mathbb{T}$ on A_λ. Then $\{(A_\lambda)_0; \pi^A_{\lambda\mu}|_{(A_\lambda)_0}; \lambda \geq \mu; \lambda, \mu \in \Lambda\}$ is an inverse system of C^*-algebras and $\{(A_\lambda)_1; \pi^A_{\lambda\mu}|_{(A_\lambda)_1}; \lambda \geq \mu; \lambda, \mu \in \Lambda\}$ is an inverse system of Hilbert $(A_\lambda)_0$-$(A_\lambda)_0$ C^*-bimodules. It is easy to check that $A_0 = \lim_{\leftarrow\lambda}(A_\lambda)_0$ and $A_1 = \lim_{\leftarrow\lambda}(A_\lambda)_1$. Furthermore, $\{C^*((A_\lambda)_0, (A_\lambda)_1); \pi^A_{\lambda\mu}|_{C^*((A_\lambda)_0,(A_\lambda)_1)}\}$ is an inverse system of C^*-algebras and *the pro-C^*-algebras* $\lim_{\leftarrow\lambda} C^*((A_\lambda)_0, (A_\lambda)_1)$ *and pro-$C^*(A_0, A_1)$ are isomorphic.*

Definition 4.2. An action α of $\mathbb{T}$ on a pro-C^*-algebra A is *semi-saturated* if the pro-C^*-subalgebra pro-$C^*(A_0, A_1)$ of A generated by A_0 and A_1 coincides with A.

Lemma 4.3. *Let $\alpha_z = \lim_{\leftarrow\lambda} \alpha^\lambda_z$ be an inverse limit action of $\mathbb{T}$ on a pro-C^*-algebra A. Then α is semi-saturated if and only if α^λ, $\lambda \in \Lambda$, are semi-saturated.*

Proof. Suppose that α is semi-saturated. Then

$$\begin{aligned} A &= \text{pro-}C^*(A_0, A_1) \\ &\qquad [\text{M, Chapter III, Theorem 3.1}] \\ &= \lim_{\leftarrow\lambda} \overline{\pi^A_\lambda(\text{pro-}C^*(A_0, A_1))} = \lim_{\leftarrow\lambda} C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1)) \\ &\subseteq \lim_{\leftarrow\lambda} C^*((A_\lambda)_0, (A_\lambda)_1) \subseteq \lim_{\leftarrow\lambda} A_\lambda = A. \end{aligned}$$

and so $A = \lim_{\leftarrow\lambda} C^*((A_\lambda)_0, (A_\lambda)_1) = \lim_{\leftarrow\lambda} C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1))$. From this fact, and taking into account that the maps $\pi^A_\lambda : A \to A_\lambda$, $\lambda \in \Lambda$, are all surjective and $C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1)) \subseteq C^*((A_\lambda)_0, (A_\lambda)_1) \subseteq A_\lambda$, $\lambda \in \Lambda$, we deduce that $C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1)) = C^*((A_\lambda)_0, (A_\lambda)_1) = A_\lambda$ for all $\lambda \in \Lambda$, and so α^λ, $\lambda \in \Lambda$, are semi-saturated.

Conversely, suppose that $\alpha^\lambda, \lambda \in \Lambda$, are semi-saturated. Then we have $C^*((A_\lambda)_0, (A_\lambda)_1) = A_\lambda$ for all $\lambda \in \Lambda$, and by Remark 4.1, the pro-C^*-algebras pro-$C^*(A_0, A_1)$ and A are isomorphic. ∎

Remark 4.4. Let $\alpha_z = \lim_{\leftarrow\lambda} \alpha^\lambda_z$, $z \in \mathbb{T}$, be an inverse limit action of $\mathbb{T}$ on a pro-C^*-algebra A. For each $\lambda \in \Lambda$, $\pi^A_\lambda(A_0) \subseteq (A_\lambda)_0$, $\pi^A_\lambda(A_1) \subseteq (A_\lambda)_1$ and $\alpha^\lambda_z(C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1))) \subseteq C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1))$ for all $z \in \mathbb{T}$.

Therefore, for each $\lambda \in \Lambda$, $z \mapsto \alpha^\lambda_z|_{C^*(\pi^A_\lambda(A_0),\pi^A_\lambda(A_1))}$ is an action of $\mathbb{T}$ on $C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1))$. Moreover, the fixed point algebra and the first spectral subspace of α^λ restricted to $C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1))$ are $\pi^A_\lambda(A_0)$, respectively $\pi^A_\lambda(A_1)$, and so $\alpha^\lambda|_{C^*(\pi^A_\lambda(A_0),\pi^A_\lambda(A_1))}$ is semi-saturated. Therefore, by [AEE, Theorem 3.1], there is a C^*-morphism $\psi_\lambda : \pi^A_\lambda(A_0) \times_{\pi^A_\lambda(A_1)} \mathbb{Z} \to C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1))$ such that $\psi_\lambda \circ i_{\pi^A_\lambda(A_1)} = \tau^\lambda_1$ and $\psi_\lambda \circ i_{\pi^A_\lambda(A_0)} = \tau^\lambda_0$, where τ^λ_1 and τ^λ_0 are the inclusions of $\pi^A_\lambda(A_1)$, respectively $\pi^A_\lambda(A_0)$, in $C^*(\pi^A_\lambda(A_0), \pi^A_\lambda(A_1))$. It is not difficult to check that $(\psi_\lambda)_\lambda$ is an inverse system of C^*-isomorphisms and then $\psi = \lim_{\leftarrow\lambda} \psi_\lambda$ is a pro-C^*-isomorphism from $\lim_{\leftarrow\lambda} \pi_\lambda^A(A_0) \times_{\pi_\lambda^A(A_1)} \mathbb{Z}$ onto $\lim_{\leftarrow\lambda} C^*(\pi_\lambda^A(A_0), \pi_\lambda^A(A_1))$, such that

$$\psi \circ i_{A_1} = \lim_{\leftarrow\lambda} \psi_\lambda \circ i_{\pi_\lambda^A(A_1)} = \lim_{\leftarrow\lambda} \tau_1^\lambda = \tau_1,$$
$$\psi \circ i_{A_0} = \lim_{\leftarrow\lambda} \psi_\lambda \circ i_{\pi_\lambda^A(A_0)} = \lim_{\leftarrow\lambda} \tau_1^\lambda = \tau_0,$$

where τ_1 and τ_0 are the inclusions of A_1, respectively A_0, in pro-$C^*(A_0, A_1)$.

The following theorem is a generalization of [AEE, Theorem 3.1].

THEOREM 4.5. *Let α be an inverse limit action of $\mathbb{T}$ on a pro-C^*-algebra A. Then α is semi-saturated if and only if there is a pro-C^*-isomorphism $\varphi : A_0 \times_{A_1} \mathbb{Z} \to A$ such that $\varphi \circ i_{A_0} = \tau_{A_0}$ and $\varphi \circ i_{A_1} = \tau_{A_1}$, where τ_{A_0} and τ_{A_1} are the inclusions of A_0, respectively A_1, in A.*

Proof. Since $\{\pi_\lambda^A(A_1); \pi_\lambda^A(A_0); \pi_{\lambda\mu}^A|_{A_1}; \pi_{\lambda\mu}^A|_{A_0}\}$ is an inverse system of Hilbert C^*-bimodules such that the canonical projections are all surjective, $A_0 = \lim_{\leftarrow\lambda} \pi_\lambda^A(A_0)$ and $A_1 = \lim_{\leftarrow\lambda} \pi_\lambda^A(A_1)$, by Proposition 3.5, $A_0 \times_{A_1} \mathbb{Z} = \lim_{\leftarrow\lambda} \pi_\lambda^A(A_0) \times_{\pi_\lambda^A(A_1)} \mathbb{Z}$, $i_{A_1} = \lim_{\leftarrow\lambda} i_{\pi_\lambda^A(A_1)}$ and $i_{A_0} = \lim_{\leftarrow\lambda} i_{\pi_\lambda^A(A_0)}$.

Suppose that α is semi-saturated. Then $A = \lim_{\leftarrow\lambda} C^*(\pi_\lambda^A(A_0), \pi_\lambda^A(A_1))$ (see the proof of Lemma 4.3), and so the pro-C^*-isomorphism ψ constructed in Remark 4.4 satisfies the required conditions.

Conversely, suppose that there is a pro-C^*-isomorphism

$$\varphi : \lim_{\leftarrow\lambda} \pi_\lambda^A(A_0) \times_{\pi_\lambda^A(A_1)} \mathbb{Z} \to A$$

such that $\varphi \circ i_{A_0} = \tau_{A_0}$ and $\varphi \circ i_{A_1} = \tau_{A_1}$. Then

$$\varphi \circ \psi^{-1} : \lim_{\leftarrow\lambda} C^*(\pi_\lambda^A(A_0), \pi_\lambda^A(A_1)) \to A$$

is a pro-C^*-isomorphism such that $\varphi \circ \psi^{-1} \circ \tau_0 = \tau_{A_0}$ and $\varphi \circ \psi^{-1} \circ \tau_1 = \tau_{A_1}$ (see Remark 4.4), and so α is semi-saturated, since pro-$C^*(A_0, A_1) = \lim_{\leftarrow\lambda} C^*(\pi_\lambda^A(A_0), \pi_\lambda^A(A_1))$. ∎

5. Morita equivalence. We recall that two pro-C^*-algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ are *strongly Morita equivalent*, written $A \sim_M B$, if there is a full Hilbert B-module E such that the pro-C^*-algebras A and $K_B(E)$ are isomorphic [J5]. The full Hilbert B-module E has a natural structure of a full Hilbert A-B pro-C^*-bimodule and it is called an *imprimitivity Hilbert A-B pro-C^*-bimodule.*

REMARK 5.1. (1) Suppose that E is an imprimitivity Hilbert A-B pro-C^*-bimodule. Since $K_B(E) = \lim_{\leftarrow i} K_{B_i}(E_i)$, by Lemma 3.7, we may suppose that Γ and Γ' have the same index set. Moreover, the C^*-algebras A_λ and $K_{B_\lambda}(E_\lambda)$ are isomorphic. Then $A_\lambda \sim_M B_\lambda$ and E_λ is an imprimitivity Hilbert A_λ-B_λ C^*-bimodule for each $\lambda \in \Lambda$.

(2) Let X and Y be two Hilbert A-A pro-C^*-bimodules and $\Phi : X \to Y$ be an isomorphism of Hilbert pro-C^*-bimodules. Since $X = \lim_{\leftarrow\lambda} X_\lambda$, $Y = \lim_{\leftarrow\lambda} Y_\lambda$, $\langle \Phi(x_1), \Phi(x_2)\rangle_A = \langle x_1, x_2\rangle_A$, respectively ${}_A\langle \Phi(x_1), \Phi(x_2)\rangle = {}_A\langle x_1, x_2\rangle$ for all $x_1, x_2 \in X$ and $\lambda \in \Lambda$, there is a bijective map $\Phi_\lambda : X_\lambda \to Y_\lambda$ such that

$$\langle \Phi_\lambda(\sigma_\lambda^X(x_1)), \Phi_\lambda(\sigma_\lambda^X(x_2))\rangle_{A_\lambda} = \langle \sigma_\lambda^X(x_1), \sigma_\lambda^X(x_2)\rangle_{A_\lambda},$$
$${}_{A_\lambda}\langle \Phi_\lambda(\sigma_\lambda^X(x_1)), \Phi_\lambda(\sigma_\lambda^X(x_2))\rangle = {}_{A_\lambda}\langle \sigma_\lambda^X(x_1), \sigma_\lambda^X(x_2)\rangle$$

for all $x_1, x_2 \in X$. Therefore, for each $\lambda \in \Lambda$, the Hilbert C^*-bimodules X_λ and Y_λ are isomorphic.

Suppose that $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ are two pro-C^*-algebras such that the families of C^*-seminorms which give the topologies on A and B have the same index set, say Λ. Let X be a Hilbert A-A pro-C^*-bimodule, and Y a Hilbert A-B pro-C^*-bimodule. For $\Phi : A \to L_B(Y)$, $\Phi(a)y = ay$, the completion of the algebraic tensor product $X \otimes_A Y$ of X and Y over A with respect to the topology given by the B-valued inner product $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_B = \langle y_1, \langle x_1, x_2\rangle_A y_2\rangle_B$ for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ becomes a right Hilbert module over B, with the B -module action given by $(x \otimes y)b = x \otimes yb$ for all $b \in B$, $x \in X$, $y \in Y$[J1]. Moreover, for each $\lambda \in \Lambda$, $(X \otimes_A Y)_\lambda = X_\lambda \otimes_{A_\lambda} Y_\lambda$ (the map $\psi_\lambda : (X \otimes_A Y)_\lambda \to X_\lambda \otimes_{A_\lambda} Y_\lambda, \psi_\lambda(\sigma_\lambda^{X\otimes_A Y}(x \otimes y)) = \sigma_\lambda^X(x) \otimes \sigma_\lambda^Y(y)$ is an isomorphism of Hilbert C^*-modules).

For each $\lambda \in \Lambda$, since X_λ is a Hilbert A_λ-A_λ C^*-bimodule and Y_λ is a Hilbert A_λ-B_λ C^*-bimodule, $X_\lambda \otimes_{A_\lambda} Y_\lambda$ is a Hilbert A_λ-B_λ C^*-bimodule with the structure of A_λ-B_λ module given by

$$(\sigma_\lambda^X(x) \otimes \sigma_\lambda^Y(y))\pi_\lambda^B(b) = \sigma_\lambda^X(x) \otimes \sigma_\lambda^Y(yb),$$
$$\pi_\lambda^A(a)(\sigma_\lambda^X(x) \otimes \sigma_\lambda^Y(y)) = \sigma_\lambda^X(ax) \otimes \sigma_\lambda^Y(y),$$

the B_λ-valued inner product $\langle \sigma_\lambda^X(x_1) \otimes \sigma_\lambda^Y(y_1), \sigma_\lambda^X(x_2) \otimes \sigma_\lambda^Y(y_2)\rangle_{B_\lambda}$ is given by

$$\left\langle \sigma_\lambda^Y(y_1), \langle \sigma_\lambda^X(x_1), \sigma_\lambda^X(x_2)\rangle_{A_\lambda} \sigma_\lambda^Y(y_2)\right\rangle_{B_\lambda}$$

and the A_λ-valued inner product ${}_{A_\lambda}\langle \sigma_\lambda^X(x_1) \otimes \sigma_\lambda^Y(y_1), \sigma_\lambda^X(x_2) \otimes \sigma_\lambda^Y(y_2)\rangle$ is given by

$${}_{A_\lambda}\left\langle \sigma_\lambda^X(x_1)\, {}_{A_\lambda}\langle \sigma_\lambda^Y(y_1), \sigma_\lambda^Y(y_2)\rangle, \sigma_\lambda^X(x_2)\right\rangle$$

for all $a \in A, b \in B$, $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$. Therefore $X \otimes_A Y$ is a Hilbert A-B pro-C^*-bimodule and $(X \otimes_A Y)_\lambda = X_\lambda \otimes_{A_\lambda} Y_\lambda$ for each $\lambda \in \Lambda$.

Theorem 5.2. *Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two pro-C^*-algebras, X a Hilbert A-A pro-C^*-bimodule, and Y a Hilbert B-B pro-C^*-bimodule. If A and B are strongly Morita equivalent and if the Hilbert pro-C^*-bimodules $X \otimes_A E$ and $E \otimes_B Y$ are isomorphic, where E is an imprimitivity Hilbert A-B pro-C^*-bimodule, then the pro-C^*-algebras $A \times_X \mathbb{Z}$ and $B \times_Y \mathbb{Z}$ are strongly Morita equivalent.*

Proof. By Remark 5.1(1), we can suppose that Γ and Γ' have the same index set, say Λ, and E_λ is an imprimitivity Hilbert A_λ-B_λ C^*-bimodule. By Remark 5.1(2) the Hilbert C^*-bimodules $X_\lambda \otimes_{A_\lambda} E_\lambda$ and $E_\lambda \otimes_{B_\lambda} Y_\lambda$ are isomorphic, for each $\lambda \in \Lambda$. Then, by [AEE, Lemma 4.1] there are a faithful representation $(\theta_\lambda, H_\lambda)$ of the linking algebra of E_λ, $\mathcal{L}(E_\lambda)$, maps $\theta_{X_\lambda}: X_\lambda \to L(H_\lambda)$ and $\theta_{Y_\lambda}: Y_\lambda \to L(H_\lambda)$ such that $(\theta_\lambda|_{A_\lambda}, \theta_{X_\lambda})$ and $(\theta_\lambda|_{B_\lambda}, \theta_{Y_\lambda})$ are faithful covariant representations of the Hilbert C^*-bimodules X_λ and Y_λ and $\theta_{X_\lambda}(X_\lambda)\theta_\lambda(E_\lambda) = \theta_\lambda(E_\lambda)\theta_{Y_\lambda}(Y_\lambda)$.

For $\lambda \in \Lambda$, let $\mathcal{H}_\lambda = \bigoplus_{\mu \le \lambda} H_\mu$. It is easy to check that the map $\varphi_\lambda : \mathcal{L}(E_\lambda) \to L(\mathcal{H}_\lambda)$ defined by

$$\varphi_\lambda(\pi_\lambda^{\mathcal{L}(E)}(c))\Big(\bigoplus_{\mu \le \lambda} \xi_\mu\Big) = \bigoplus_{\mu \le \lambda} \theta_\mu(\pi_\mu^{\mathcal{L}(E)}(c))\xi_\mu$$

is a faithful representation of $\mathcal{L}(E_\lambda)$. Moreover $(\varphi_\lambda|_{A_\lambda}, \varphi_{X_\lambda})$ and $(\varphi_\lambda|_{B_\lambda}, \varphi_{Y_\lambda})$, where

$$\varphi_{X_\lambda}: X_\lambda \to L(\mathcal{H}_\lambda), \quad \varphi_{X_\lambda}(\sigma_\lambda^X(x))\Big(\bigoplus_{\mu \le \lambda} \xi_\mu\Big) = \bigoplus_{\mu \le \lambda} \theta_{X_\mu}(\sigma_\mu^X(x))\xi_\mu,$$
$$\varphi_{Y_\lambda}: Y_\lambda \to L(\mathcal{H}_\lambda), \quad \varphi_{Y_\lambda}(\sigma_\lambda^Y(y))\Big(\bigoplus_{\mu \le \lambda} \xi_\mu\Big) = \bigoplus_{\mu \le \lambda} \theta_{Y_\mu}(\sigma_\mu^Y(y))\xi_\mu$$

are covariant representations of (X_λ, A_λ) and (Y_λ, B_λ) and $\varphi_{X_\lambda}(X_\lambda)\varphi_\lambda(E_\lambda) = \varphi_\lambda(E_\lambda)\varphi_{Y_\lambda}(Y_\lambda)$.

Let $\mathcal{H} = \lim_{\lambda\to} \mathcal{H}_\lambda$ and $L(\mathcal{H})$ be the pro-C^*-algebra of all continuous linear operators on $\mathcal{H}$ ([F, I]). For $c \in \mathcal{L}(E)$, the map $\varphi(c): \mathcal{H} \to \mathcal{H}$, defined by $\varphi(c)(\xi_\lambda) = \varphi_\lambda(\pi_\lambda^{\mathcal{L}(E)}(c))(\xi_\lambda)$ for all $\xi_\lambda \in \mathcal{H}_\lambda$ and $\lambda \in \Lambda$ is an element in $L(\mathcal{H})$. In this way we obtain an injective pro-C^*-morphism $\varphi : \mathcal{L}(E) \to L(\mathcal{H})$ such that $p_{\lambda,L(\mathcal{H})}(\varphi(c)) = \|\varphi_\lambda(\pi_\lambda^{\mathcal{L}(E)}(c))\|_{L(\mathcal{H}_\lambda)} = p_{\lambda,\mathcal{L}(E)}(c)$ for all $c \in \mathcal{L}(E)$ and $\lambda \in \Lambda$. For $x \in X$ the map $\varphi_X(x) : \mathcal{H} \to \mathcal{H}$, defined by $\varphi_X(x)(\xi_\lambda) = \varphi_{X_\lambda}(\sigma_\lambda^X(x))(\xi_\lambda)$ for all $\xi_\lambda \in \mathcal{H}_\lambda$ and $\lambda \in \Lambda$, is an element in $L(\mathcal{H})$. Thus we obtain an injective map $\varphi_X : X \to L(\mathcal{H})$. Since

$$\begin{aligned}
{}_{L(\mathcal{H})}\langle \varphi_X(x_1), \varphi_X(x_2)\rangle(\xi_\lambda) &= \varphi_X(x_1)(\varphi_X(x_2))^*(\xi_\lambda) \\
&= \varphi_{X_\lambda}(x_1)(\varphi_{X_\lambda}(x_2))^*(\xi_\lambda) =_{L(\mathcal{H}_\lambda)} \langle \varphi_{X_\lambda}(x_1), \varphi_{X_\lambda}(x_2)\rangle(\xi_\lambda) \\
&= \varphi_\lambda|_{A_\lambda}({}_{A_\lambda}\langle \sigma_\lambda^X(x_1), \sigma_\lambda^X(x_2)\rangle)(\xi_\lambda) = \varphi|_A({}_A\langle x_1, x_2\rangle)(\xi_\lambda)
\end{aligned}$$

and in a similar way

$$\langle \varphi_X(x_1), \varphi_X(x_2)\rangle_{L(\mathcal{H})}(\xi_\lambda) = \varphi|_A(\langle x_1, x_2\rangle_A)(\xi_\lambda)$$

for all $x_1, x_2 \in X$, $\xi_\lambda \in \mathcal{H}_\lambda$, and $\lambda \in \Lambda$, we see that $(\varphi|_A, \varphi_X)$ is a covariant representation of (X, A). In a similar way, we define an injective map $\varphi_Y : Y \to L(\mathcal{H})$ such that $(\varphi|_B, \varphi_Y)$ is a covariant representation of (Y, B). Moreover, $\varphi_X(X)\varphi(E) = \varphi(E)\varphi_Y(Y)$. Since $p_{\lambda,L(\mathcal{H})}(\varphi(c)) = p_{\lambda,\mathcal{L}(E)}(c)$,

$p_{\lambda,L(\mathcal{H})}(\varphi_X(x)) = p_\lambda^A(x)$ and $p_{\lambda,L(\mathcal{H})}$ $(\varphi_Y(y)) = p_\lambda^B(y)$ for all $c \in \mathcal{L}(E)$, $x \in X$, $y \in Y$ and $\lambda \in \Lambda$, we can identify $\mathcal{L}(E)$, A, B, X, Y and E with their images in $L(\mathcal{H})$.

Let

$$W = \overline{\operatorname{span}\left\{\begin{bmatrix} X & XE \\ YE^* & Y \end{bmatrix}\right\}},$$

where E^* is the dual module of E. Clearly, W is an $\mathcal{L}(E)$-$\mathcal{L}(E)$ Hilbert pro-C^*-bimodule with respect to the structure of $L(\mathcal{H})$ and $W = \lim_{\leftarrow\lambda} W_\lambda$, where

$$W_\lambda = \overline{\pi_\lambda^{L(\mathcal{H})}(W)} = \overline{\operatorname{span}\left\{\begin{bmatrix} X_\lambda & X_\lambda E_\lambda \\ Y_\lambda E_\lambda^* & Y_\lambda \end{bmatrix}\right\}} \quad \text{for each } \lambda \in \Lambda.$$

Since $\{\mathcal{L}(E_\lambda); W_\lambda; \pi_{\lambda\mu}^{L(\mathcal{H})}|_{\mathcal{L}(E_\lambda)}; \pi_{\lambda\mu}^{L(\mathcal{H})}|_{W_\lambda};\ \lambda, \mu \in \Lambda,\ \lambda \geq \mu\}$ is an inverse system of Hilbert C^*-bimodules such that the canonical projections are all surjective, by Corollary 3.6 and Proposition 3.8, $\mathcal{L}(E) \times_W \mathbb{Z}$ is isomorphic to $\lim_{\leftarrow\lambda} \mathcal{L}(E_\lambda) \times_{W_\lambda} \mathbb{Z}$ and $(\mathcal{L}(E) \times_W \mathbb{Z})_\lambda = \mathcal{L}(E_\lambda) \times_{W_\lambda} \mathbb{Z}$.

For each $\lambda \in \Lambda$, since $\mathcal{L}(E_\lambda), A_\lambda, B_\lambda, X_\lambda, Y_\lambda$ and E_λ can be identified with their images in $L(\mathcal{H}_\lambda)$, and $X_\lambda E_\lambda = E_\lambda Y_\lambda$, by the proof of [AEE, Theorem 4.2], the C^*-algebras $\mathcal{P}_\lambda(\mathcal{L}(E_\lambda) \times_{W_\lambda} \mathbb{Z})\mathcal{P}_\lambda$ and $A_\lambda \times_{X_\lambda} \mathbb{Z}$ are isomorphic, where

$$\mathcal{P}_\lambda = \begin{bmatrix} 1_{M(A_\lambda)} & 0 \\ 0 & 0 \end{bmatrix}$$

can be seen as a projection in the multiplier algebra $M(\mathcal{L}(E_\lambda) \times_{W_\lambda} \mathbb{Z})$ of the C^*-algebra $\mathcal{L}(E_\lambda) \times_{W_\lambda} \mathbb{Z}$. Then

$$\mathcal{P} = \begin{bmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{bmatrix}$$

can be seen as a projection in $M(\mathcal{L}(E) \times_W \mathbb{Z})$ (see [P1]) and

$$\mathcal{P}(\mathcal{L}(E) \times_W \mathbb{Z})\mathcal{P} = \lim_{\leftarrow\lambda} \mathcal{P}_\lambda(\mathcal{L}(E_\lambda) \times_{W_\lambda} \mathbb{Z})\mathcal{P}_\lambda.$$

Therefore, the pro-C^*-algebras $\mathcal{P}(\mathcal{L}(E) \times_W \mathbb{Z})\mathcal{P}$ and $\lim_{\leftarrow\lambda} A_\lambda \times_{X_\lambda} \mathbb{Z}$ are isomorphic.

In the same manner, we show that the pro-C^*-algebras $\mathcal{Q}(\mathcal{L}(E) \times_W \mathbb{Z})\mathcal{Q}$ and $\lim_{\leftarrow\lambda} B_\lambda \times_{Y_\lambda} \mathbb{Z}$ are isomorphic, where

$$\mathcal{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1_{M(B)} \end{bmatrix}.$$

Since $\mathcal{P}$ and $\mathcal{Q}$ are full complementary projections in $M(\mathcal{L}(E) \times_W \mathbb{Z})$, by [J4, Theorem 9], the pro-C^*-algebras $\lim_{\leftarrow\lambda} A_\lambda \times_{Y_\lambda} \mathbb{Z}$ and $\lim_{\leftarrow\lambda} B_\lambda \times_{Y_\lambda} \mathbb{Z}$ are strongly Morita equivalent, and so $A \times_X \mathbb{Z} \sim_M B \times_Y \mathbb{Z}$. ∎

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References

[AEE] B. Abadie, S. Eilers and R. Exel, *Morita equivalence for crossed products by Hilbert C^*-bimodules*, Trans. Amer. Math. Soc. 350 (1998), 3043–3054.

[F] M. Fragoulopoulou, *Topological Algebras with Involution*, North-Holland Math. Stud. 200, Elsevier, Amsterdam, 2005.

[I] A. Inoue, *Locally C^*-algebras*, Mem. Fac. Sci. Kyushu Univ. Ser. A 25 (1971), 197–235.

[J1] M. Joiţa, *Hilbert Modules over Locally C^*-Algebras*, Bucharest Univ. Press, Bucureşti, 2006.

[J2] M. Joiţa, *Crossed Products of Locally C^*-Algebras*, Editura Academiei Române, Bucureşti, 2007.

[J3] M. Joiţa, *Crossed products of pro-C^*-algebras and Hilbert pro-C^*-modules*, Bull. Malays. Math. Sci. Soc., to appear.

[J4] M. Joiţa, *On the linking algebra of Hilbert modules and Morita equivalence of locally C^*-algebras*, Surveys Math. Appl. 1 (2006), 23–32.

[J5] M. Joiţa, *Morita equivalence for locally C^*-algebras*, Bull. London Math. Soc. 36 (2004), 802–810.

[M] A. Mallios, *Topological Algebras. Selected Topics*, North-Holland, 1986.

[P1] N. C. Phillips, *Inverse limits of C^*-algebras*, J. Operator Theory 19 (1988), 159–195.

[P2] N. C. Phillips, *Representable K-theory for σ-C^*-algebras*, K-Theory 3 (1989), 441–478.

[W] D. P. Williams, *Crossed Products of C^*-algebras*, Math. Surveys Monogr. 134, Amer. Math. Soc., 2007.

[Z] I. Zarakas, *Hilbert pro-C^*-bimodules and applications*, Rev. Roumaine Math. Pures Appl. 57 (2012), 289–310.

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