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Uncertainty principles for integral operators

by

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Abstract. The aim of this paper is to prove new uncertainty principles for integral operators $\mathcal{T}$ with bounded kernel for which there is a Plancherel Theorem. The first of these results is an extension of Faris's local uncertainty principle which states that if a nonzero function $f \in L^2(\mathbb{R}^d, \mu)$ is highly localized near a single point then $\mathcal{T}(f)$ cannot be concentrated in a set of finite measure. The second result extends the Benedicks–Amrein–Berthier uncertainty principle and states that a nonzero function $f \in L^2(\mathbb{R}^d, \mu)$ and its integral transform $\mathcal{T}(f)$ cannot both have support of finite measure. From these two results we deduce a global uncertainty principle of Heisenberg type for the transformation $\mathcal{T}$. We apply our results to obtain new uncertainty principles for the Dunkl and Clifford Fourier transforms.

1. Introduction. Uncertainty principles are mathematical results that give limitations on the simultaneous concentration of a function and its Fourier transform. They have implications in two main areas: quantum physics and signal analysis. In quantum physics they tell us that a particle's speed and position cannot both be measured with infinite precision. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. There are many ways to get the statement about concentration precise. The most famous of them are the so-called Heisenberg Uncertainty Principle [31] where concentration is measured by dispersion, and Hardy's Uncertainty Principle [28] where concentration is measured in terms of fast decay. A little less known one consists in measuring concentration in terms of smallness of support. A considerable attention has been devoted recently to discovering new formulations and new contexts for the uncertainty principle (see the surveys [4, 25] and the book [29] for other forms of the uncertainty principle).

Our aim here is to extend the uncertainty principle to a very general class of integral operators. A version of Hardy's Uncertainty Principle for integral operators on $\mathbb{R}^n$ has been proved recently by Cowling, Demange and Sun-

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dari [12]. Their result shows that if the kernel k of an operator T is bounded by the heat kernel $p_t(x,y)$ and the operator is bounded by the heat operator P_t in the sense that $\|Tf\| \le \|P_t f\|$ then $k(x,y) = m(x)p_t(x,y)$ where m is bounded. In this paper, we consider results of a different nature, as concentration is measured either by (generalized) dispersion, like in Heisenberg's Uncertainty Principle, or by the smallness of the support. Moreover, we do not seek for conditions on k which result from concentration but rather the opposite. The transforms under consideration are integral operators $\mathcal{T}$ with polynomially bounded kernels $\mathcal{K}$ and for which there is a Plancherel Theorem. This class includes the usual Fourier transform, the Fourier–Bessel (Hankel) transform, the Fourier–Dunkl transform and the Fourier–Clifford transform as particular cases.

Let us now be more precise. Let Ω, $\widehat{\Omega}$ be two convex cones in $\mathbb{R}^d$ (i.e. $\lambda x \in \Omega$ if $\lambda > 0$ and $x \in \Omega$) with nonempty interiors. We endow them with Borel measures μ and $\widehat{\mu}$. The Lebesgue spaces $L^p(\Omega, \mu)$, $1 \le p \le \infty$, are then defined in the usual way. We assume that the measure μ is absolutely continuous with respect to the Lebesgue measure and has a polar decomposition of the form $d\mu(r\zeta) = r^{2a-1}dr\, Q(\zeta)d\sigma(\zeta)$ where $d\sigma$ is the Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ and $Q \in L^1(\mathbb{S}^{d-1}, d\sigma)$, $Q \ne 0$. Then μ is homogeneous of degree $2a$ in the following sense: for every continuous function f with compact support in Ω and every $\lambda > 0$,

$$\int_\Omega f\left(\frac{x}{\lambda}\right) d\mu(x) = \lambda^{2a} \int_\Omega f(x)\, d\mu(x). \tag{1.1}$$

We define $\widehat{a}$ accordingly for $\widehat{\mu}$, and assume that $\widehat{a} = a$.

Next, let $\mathcal{K} : \Omega \times \widehat{\Omega} \to \mathbb{C}$ be a kernel such that

(1) $\mathcal{K}$ is continuous;
(2) $\mathcal{K}$ is polynomially bounded: $|\mathcal{K}(x,\xi)| \le c_{\mathcal{T}}(1+|x|)^m(1+|\xi|)^{\widehat{m}}$;
(3) $\mathcal{K}$ is homogeneous: $\mathcal{K}(\lambda x, \xi) = \mathcal{K}(x, \lambda\xi)$.

One can then define the integral operator $\mathcal{T}$ on $\mathcal{S}(\Omega)$ by

$$\mathcal{T}(f)(\xi) = \int_\Omega f(x)\mathcal{K}(x,\xi)\, d\mu(x), \qquad \xi \in \widehat{\Omega}. \tag{1.2}$$

For $\rho > 0$, we define the measures $d\mu_\rho(x) = (1+|x|)^\rho d\mu(x)$ and $d\widehat{\mu}_\rho(\xi) = (1+|\xi|)^\rho d\widehat{\mu}(\xi)$. Then $\mathcal{T}$ extends to a continuous operator from $L^1(\Omega, \mu_m)$ to

$$\mathcal{C}_{\widehat{m}}(\widehat{\Omega}) = \left\{ f : \|f\|_{\infty,\widehat{m}} := \sup_{\xi\in\widehat{\Omega}} \frac{|f(\xi)|}{(1+|\xi|)^{\widehat{m}}} < \infty \right\}.$$

Further, if we introduce the *dilation operators* $\mathcal{D}_\lambda, \widehat{\mathcal{D}}_\lambda$ for $\lambda > 0$:

$$\mathcal{D}_\lambda f(x) = \frac{1}{\lambda^a} f\left(\frac{x}{\lambda}\right), \qquad \widehat{\mathcal{D}}_\lambda f(x) = \frac{1}{\lambda^{\widehat{a}}} f\left(\frac{x}{\lambda}\right),$$

then the homogeneity of $\mathcal{K}$ implies

$$\mathcal{T}\mathcal{D}_\lambda = \widehat{\mathcal{D}}_{1/\lambda}\mathcal{T}. \tag{1.3}$$

Also, from the fact that μ and $\widehat{\mu}$ are absolutely continuous with respect to the Lebesgue measure, these dilation operators are continuous from $(0,\infty)\times L^2(\Omega,\mu_\rho)$ to $L^2(\Omega,\mu_\rho)$, resp. from $(0,\infty)\times L^2(\widehat{\Omega},\widehat{\mu}_\rho)$ to $L^2(\widehat{\Omega},\widehat{\mu}_\rho)$.

The integral operators under consideration will be assumed to satisfy some of the following properties that are common for Fourier-like transforms:

(1) $\mathcal{T}$ has an *Inversion Formula*: *When both $f\in L^1(\Omega,\mu_m)$ and $\mathcal{T}(f)\in L^1(\widehat{\Omega},\widehat{\mu}_{\widehat{m}})$, we have $f\in\mathcal{C}_m(\Omega)$ and*

$$f(x)=\mathcal{T}^{-1}[\mathcal{T}(f)](x)=\int_{\widehat{\Omega}}\mathcal{T}(f)(\xi)\overline{\mathcal{K}(x,\xi)}\,d\widehat{\mu}(\xi),\qquad x\in\Omega.$$

(2) *$\mathcal{T}$ satisfies Plancherel's Theorem: For every $f\in\mathcal{S}(\Omega)$,*

$$\|\mathcal{T}(f)\|_{L^2(\widehat{\Omega},\widehat{\mu})}=\|f\|_{L^2(\Omega,\mu)}.$$

In particular, $\mathcal{T}$ extends to a unitary transform from $L^2(\Omega,\mu)$ onto $L^2(\widehat{\Omega},\widehat{\mu})$ (1).

This family of transforms includes for instance the Fourier transform and the Fourier–Dunkl transform. We will also slightly relax the conditions to include the Fourier–Clifford transform. We will here concentrate on uncertainty principles where concentration is measured in terms of dispersion or in terms of smallness of support. Our first result will be the following local uncertainty principle that we state here in the case $m=\widehat{m}=0$ for simplicity (see Theorem 2.1 for the full result):

THEOREM A. *Assume $m=\widehat{m}=0$. Let $\Sigma\subset\widehat{\Omega}$ be a measurable subset of finite measure $0<\widehat{\mu}(\Sigma)<\infty$. Then:*

(1) *If $0<s<a$, there is a constant C such that for all $f\in L^2(\Omega,\mu)$,*

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})}\le C[\widehat{\mu}(\Sigma)]^{\frac{s}{2a}}\big\||x|^s f\big\|_{L^2(\Omega,\mu)}.$$

(2) *If $s>a$, there is a constant C such that for all $f\in L^2(\Omega,\mu)$,*

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})}\le C[\widehat{\mu}(\Sigma)]^{1/2}\|f\|^{1-a/s}_{L^2(\Omega,\mu)}\big\||x|^s f\big\|^{a/s}_{L^2(\Omega,\mu)}.$$

This theorem implies that if f is highly localized in the neighborhood of 0, i.e. the dispersion $\big\||x|^s f\big\|_{L^2(\Omega,\mu)}$ takes a small value, then $\mathcal{T}(f)$ cannot be concentrated in a subset Σ of finite measure. We refer to [24, 41, 42, 43] for the history of these uncertainty inequalities.

Another uncertainty principle which is of particular interest is: *a function f and its integral transform $\mathcal{T}(f)$ cannot both have small support.* In other

(1) The condition $\widehat{a}=a$ is necessary for this to hold.

words we are interested in the following adaptation of a well-known notion from Fourier analysis:

Definition 1.1. Let $S \subset \Omega$ and $\Sigma \subset \widehat{\Omega}$ be measurable subsets. Then:

- (S, Σ) is a *weak annihilating pair* (2) if $\operatorname{supp} f \subset S$ and $\operatorname{supp} \mathcal{T}(f) \subset \Sigma$ implies $f = 0$.
- (S, Σ) is a *strong annihilating pair* if there exists $C = C(S, \Sigma)$ such that for every $f \in L^2(\Omega, \mu)$,

$$\|f\|^2_{L^2(\Omega,\mu)} \le C(\|f\|^2_{L^2(S^c,\mu)} + \|\mathcal{T}(f)\|^2_{L^2(\Sigma^c,\widehat{\mu})}), \tag{1.4}$$

where A^c is the complement of the set A in Ω or $\widehat{\Omega}$. The constant $C(S, \Sigma)$ will be called the *annihilation constant* of (S, Σ).

Of course, every strong annihilating pair is also a weak one. To prove that a pair (S, Σ) is a strong annihilating pair, it is enough to show that there exists a constant $D(S, \Sigma)$ such that for every $f \in L^2(\Omega, \mu)$ supported in S,

$$\|f\|^2_{L^2(\Omega,\mu)} \le D(S, \Sigma)\|\mathcal{T}(f)\|^2_{L^2(\Sigma^c,\widehat{\mu})}. \tag{1.5}$$

The qualitative (or weak) uncertainty principle has been considered in various places [2, 3, 14, 23, 32, 35, 37, 44]. Our main concern here is the quantitative (or strong) uncertainty principle of the form (1.4). In his paper [17], de Jeu proved a quite general uncertainty principle for integral operators with bounded transform. This result states that if S, Σ are sets of sufficiently small measure, then (S, Σ) is a strong annihilating pair. One is thus led to ask whether any pair of sets of finite measure is strongly annihilating.

In the case of the Fourier transform, this was proved by Amrein–Berthier [1] (while the weak counterpart was proved by Benedicks [3]). It is interesting to note that, when $f \in L^2(\mathbb{R}^d)$, the optimal estimate of C, which depends only on Lebesgue's measures $|S|$ and $|\Sigma|$, was obtained by F. Nazarov [40] ($d = 1$), while in higher dimension the question is not fully settled unless either S or Σ is convex (see [34] for the best result to date). For the Fourier–Bessel/Hankel transform, this was done by the authors in [27]. Our main result will be the following adaptation of the Benedicks–Amrein–Berthier uncertainty principle:

Theorem B. *Let $S \subset \Omega$, $\Sigma \subset \widehat{\Omega}$ be measurable subsets with $0 < \mu_{2m}(S)$, $\widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Then there exists a constant $C(S, \Sigma)$ such that for any function $f \in L^2(\Omega, \mu)$,*

$$\|f\|^2_{L^2(\Omega,\mu)} \le C(S, \Sigma)(\|f\|^2_{L^2(S^c,\mu)} + \|\mathcal{T}(f)\|^2_{L^2(\Sigma^c,\widehat{\mu})}).$$

(2) See also the very similar notion of Heisenberg uniqueness pairs [30].

For the Fourier transform the proof of this theorem in stated in [1] where the translation and the modulation operators play a key role. Our theorem includes essentially integral operators for which the translation operator is not explicit (the Dunkl transform for example) or does not behave like the ordinary translation (the Fourier–Bessel transform for example). To do so we will replace translation by dilation and use the fact that the dilates of a $\mathcal{C}_0$-function are linearly independent (see Lemma 3.4).

Finally, from either Theorem A or Theorem B we will deduce the following global uncertainty inequality:

Theorem C. *For $s, \beta > 0$, there exists a constant $C_{s,\beta}$ such that for all $f \in L^2(\Omega, \mu)$,*

$$\big\| |x|^s f \big\|_{L^2(\Omega,\mu)}^{\frac{2\beta}{s+\beta}} \big\| |\xi|^\beta \mathcal{T}(f) \big\|_{L^2(\widehat{\Omega},\widehat{\mu})}^{\frac{2s}{s+\beta}} \geq C_{s,\beta} \|f\|_{L^2(\Omega,\mu)}^2.$$

In particular when $s = \beta = 1$ we obtain a Heisenberg type uncertainty principle for the transformation $\mathcal{T}$.

The structure of this paper is as follows: In the next section we prove the local uncertainty inequality for the transformation $\mathcal{T}$. Section 3 is devoted to our Benedicks–Amrein–Berthier type theorem. In Section 4 we apply our results to the Dunkl and the Clifford Fourier transforms.

Notation. Throughout this paper we denote by $\langle \cdot, \cdot \rangle$ the usual Euclidean inner product in $\mathbb{R}^d$, we write $|x| = \sqrt{\langle x, x \rangle}$ for $x \in \mathbb{R}^d$, and if S is a measurable subset in $\mathbb{R}^d$, we write $|S|$ for its Lebesgue measure.

Furthermore, $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ endowed with the normalized surface measure $d\sigma$.

We write $c(\mathcal{T})$ (resp. $c(s, \mathcal{T})$ etc.) for a constant that depends on the parameters $a, m, \hat{m}$ and $c_{\mathcal{T}}$ defined above (resp. to indicate the dependence on some other parameter $s, \ldots$). These constants may change from line to line.

2. Local uncertainty principles. Local uncertainty inequalities for the Fourier transform were first obtained by Faris [24], and they were subsequently sharpened and generalized by Price and Sitaram [41, 42]. Similar inequalities on Lie groups of polynomial growth were established by Ciatti, Ricci and Sundari [11], based on [43]; they were further extended in [38].

First from the polar decomposition of our measure we remark that

$$\begin{cases} C_1(s) := \displaystyle\int_{\Omega\cap\{|x|\leq 1\}} \frac{d\mu(x)}{|x|^{2s}} < \infty, \quad 0 < s < a, \\ C_2(s) := \displaystyle\int_{\Omega} \frac{d\mu(x)}{(1+|x|)^{2s}} < \infty, \qquad s > a. \end{cases} \tag{2.1}$$

We can now prove the following:

THEOREM 2.1. *Let $\Sigma \subset \widehat{\Omega}$ be a measurable subset of finite measure $0 < \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Then:*

(1) *If $0 < s < a$, then there is a constant $c(s,\mathcal{T})$ such that for all $f \in L^2(\Omega,\mu)$, $\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})}$ is bounded by*

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})} \leq \begin{cases} c(s,\mathcal{T})[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{\frac{s}{2(a+m)}}\big\||x|^s f\big\|_{L^2(\Omega,\mu)} & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) \leq 1, \\ c(s,\mathcal{T})[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{\frac{s}{2a}}\big\||x|^s f\big\|_{L^2(\Omega,\mu)} & \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) > 1. \end{cases} \tag{2.2}$$

(2) *If $a \leq s \leq a+m$, then for every $\varepsilon > 0$ there is a constant $c(s,\mathcal{T},\varepsilon)$ such that for all $f \in L^2(\Omega,\mu)$, $\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})}$ is bounded by*

$$\begin{cases} c(s,\mathcal{T},\varepsilon)[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{\frac{1}{2(1+m/a)}-\varepsilon}\|f\|_{L^2(\Omega,\mu)}^{1-a/s+\varepsilon}\big\||x|^s f\big\|_{L^2(\Omega,\mu)}^{a/s-\varepsilon} \\ \qquad\qquad \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) \leq 1, \\ c(s,\mathcal{T},\varepsilon)[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{1/2-\varepsilon}\|f\|_{L^2(\Omega,\mu)}^{1-a/s+\varepsilon}\big\||x|^s f\big\|_{L^2(\Omega,\mu)}^{a/s-\varepsilon} \\ \qquad\qquad \text{if } \widehat{\mu}_{2\widehat{m}}(\Sigma) > 1. \end{cases}$$

(3) *If $s > m+a$, then there is a constant $c(s,\mathcal{T})$ such that for all $f \in L^2(\Omega,\mu)$, $\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})}$ is bounded by*

$$\begin{cases} c(s,\mathcal{T})[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{1/2}\|f\|_{L^2(\Omega,\mu)}^{1-a/s}\big\||x|^s f\big\|_{L^2(\Omega,\mu)}^{a/s} & \text{if } m=0, \\ c(s,\mathcal{T})[\widehat{\mu}_{2\widehat{m}}(\Sigma)]^{1/2}\|f\|_{L^2(\Omega,\mu_{2s})} & \text{otherwise.} \end{cases} \tag{2.3}$$

Proof. For (1), take $r > 0$ and let $\chi_r = \chi_{\Omega\cap\{|x|\leq r\}}$ and $\tilde{\chi_r} = 1 - \chi_r$. We may then write

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})} = \|\mathcal{T}(f)\chi_\Sigma\|_{L^2(\widehat{\Omega},\widehat{\mu})} \leq \|\mathcal{T}(f\chi_r)\chi_\Sigma\|_{L^2(\widehat{\Omega},\widehat{\mu})} + \|\mathcal{T}(f\tilde{\chi_r})\|_{L^2(\widehat{\Omega},\widehat{\mu})},$$

and Plancherel's Theorem yields

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})} \leq \widehat{\mu}_{2\widehat{m}}(\Sigma)^{1/2}\|\mathcal{T}(f\chi_r)\|_{\infty,\widehat{m}} + \|f\tilde{\chi_r}\|_{L^2(\Omega,\mu)}.$$

Now we have

$$\begin{aligned} \|\mathcal{T}(f\chi_r)\|_{\infty,\widehat{m}} &\leq c_\mathcal{T}\|f\chi_r\|_{L^1(\Omega,\mu_m)} \\ &\leq c_\mathcal{T}\big\||x|^{-s}(1+|x|)^m\chi_r\big\|_{L^2(\Omega,\mu)}\big\||x|^s f\big\|_{L^2(\Omega,\mu)} \\ &\leq c_\mathcal{T}\sqrt{C_1(s)}\,(1+r)^m r^{a-s}\big\||x|^s f\big\|_{L^2(\Omega,\mu)}. \end{aligned}$$

On the other hand,

$$\|f\tilde{\chi_r}\|_{L^2(\Omega,\mu)} \leq \big\||x|^{-s}\tilde{\chi_r}\big\|_{L^\infty(\Omega,\mu)}\big\||x|^s f\big\|_{L^2(\Omega,\mu)} = r^{-s}\big\||x|^s f\big\|_{L^2(\Omega,\mu)},$$

so that

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})} \leq (r^{-s} + 2c_\mathcal{T}\sqrt{C_1(s)}\,(1+r)^m r^{a-s}\widehat{\mu}_{2\widehat{m}}(\Sigma)^{1/2})\big\||x|^s f\big\|_{L^2(\Omega,\mu)}.$$

If $\widehat{\mu}_{2\widehat{m}}(\Sigma) > 1$ we take $r = \widehat{\mu}_{2\widehat{m}}(\Sigma)^{-1/2a} < 1$ (thus $(1+r)^m \le 2^m$) to deduce that there is a constant C depending only on s and $\mathcal{T}$ such that

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})} \le C\widehat{\mu}_{2\widehat{m}}(\Sigma)^{\frac{s}{2a}} \big\||x|^s f\big\|_{L^2(\Omega,\mu)}.$$

If $\widehat{\mu}_{2\widehat{m}}(\Sigma) < 1$ we take $r = \widehat{\mu}_{2\widehat{m}}(\Sigma)^{-1/2(a+m)} > 1$ (thus $(1+r)^m \le 2^m r^m$) to find that there is a constant C depending only on s and $\mathcal{T}$ such that

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})} \le C\widehat{\mu}_{2\widehat{m}}(\Sigma)^{\frac{s}{2(a+m)}} \big\||x|^s f\big\|_{L^2(\Omega,\mu)}.$$

Next, take $0 < \sigma < a \le s \le a+m$, apply (1) with σ replacing s, and then apply the classical inequality

$$\big\||x|^\sigma f\big\|_{L^2(\Omega,\mu)} \le C(\sigma,s)\|f\|_{L^2(\Omega,\mu)}^{1-\sigma/s} \big\||x|^s f\big\|_{L^2(\Omega,\mu)}^{\sigma/s}.$$

As for (3), we write

$$\|\mathcal{T}(f)\|_{L^2(\Sigma,\widehat{\mu})} \le \widehat{\mu}_{2\widehat{m}}(\Sigma)^{1/2}\|\mathcal{T}(f)\|_{\infty,\widehat{m}} \le c_{\mathcal{T}}\widehat{\mu}_{2\widehat{m}}(\Sigma)^{1/2}\|f\|_{L^1(\Omega,\mu_m)}.$$

Moreover

$$\begin{aligned}\|f\|_{L^1(\Omega,\mu_m)}^2 &= \Big(\int_\Omega (1+|x|)^m|f(x)|\,d\mu(x)\Big)^2\\ &= \Big(\int_\Omega (1+|x|)^{-(s-m)}(1+|x|)^s|f(x)|\,d\mu(x)\Big)^2\\ &\le C_2(s-m)\int_\Omega (1+|x|)^{2s}|f(x)|^2\,d\mu(x).\end{aligned}$$

Further, if $m=0$, then this last inequality implies

$$\|f\|_{L^1(\Omega,\mu)}^2 \le 2^{2s}C_2(s)\big(\|f\|_{L^2(\Omega,\mu)}^2 + \big\||x|^s f\big\|_{L^2(\Omega,\mu)}^2\big).$$

Replacing f by $\mathcal{D}_\lambda f$, $\lambda > 0$, in this inequality gives

$$\|f\|_{L^1(\Omega,\mu)}^2 \le 2^{2s}C_2(s)\big(\lambda^{-2a}\|f\|_{L^2(\Omega,\mu)}^2 + \lambda^{2(s-a)}\big\||x|^s f\big\|_{L^2(\Omega,\mu)}^2\big).$$

Minimizing the right hand side over $\lambda > 0$, we obtain the desired result. ∎

We now show that a local uncertainty principle implies a global uncertainty principle type for $\mathcal{T}$. For the sake of simplicity, we will assume that $m = \widehat{m} = 0$. The general case will be treated in the next section.

Corollary 2.2. *Assume that $m = \widehat{m} = 0$. For $s,\beta > 0$, $s \ne a$, there exists a constant $C = C(s,\beta,\mathcal{T})$ such that for all $f \in L^2(\Omega,\mu)$,*

$$\big\||x|^s f\big\|_{L^2(\Omega,\mu)}^{\frac{\beta}{s+\beta}} \big\||\xi|^\beta \mathcal{T}(f)\big\|_{L^2(\widehat{\Omega},\widehat{\mu})}^{\frac{s}{s+\beta}} \ge C\|f\|_{L^2(\Omega,\mu)}. \tag{2.4}$$

Proof. In this proof, we write $B_r = \widehat{\Omega}\cap\{x : |x| \leq r\}$ and $B_r^c = \widehat{\Omega}\setminus B_r$. Let $0 < s < a$ and $\beta > 0$. Then, using Plancherel's Theorem and Theorem 2.1(1),

$$\begin{aligned}\|f\|^2_{L^2(\Omega,\mu)} &= \|\mathcal{T}(f)\|^2_{L^2(\widehat{\Omega},\widehat{\mu})} = \|\mathcal{T}(f)\|^2_{L^2(B_r,\widehat{\mu})} + \|\mathcal{T}(f)\|^2_{L^2(B_r^c,\widehat{\mu})}\\ &\leq c(s,\mathcal{T})\widehat{\mu}(B_r)^{s/a}\big\||x|^s f\big\|^2_{L^2(\Omega,\mu)} + r^{-2\beta}\big\||\xi|^\beta \mathcal{T}(f)\big\|^2_{L^2(\widehat{\Omega},\widehat{\mu})}\\ &\leq c'(s,\mathcal{T})r^{2s}\big\||x|^s f\big\|^2_{L^2(\Omega,\mu)} + r^{-2\beta}\big\||\xi|^\beta \mathcal{T}(f)\big\|^2_{L^2(\widehat{\Omega},\widehat{\mu})}.\end{aligned}$$

The desired result follows by minimizing the right hand side over $r > 0$.

For $s > a$ and $\beta > 0$ we deduce from Plancherel's Theorem and Theorem 2.1(3) that

$$\begin{aligned}\|f\|^2_{L^2(\Omega,\mu)} &= \|\mathcal{T}(f)\|^2_{L^2(\widehat{\Omega},\widehat{\mu})} = \|\mathcal{T}(f)\|^2_{L^2(B_r,\widehat{\mu})} + \|\mathcal{T}(f)\|^2_{L^2(B_r^c,\widehat{\mu})}\\ &\leq c(s,\mathcal{T})^2\big\|f\big\|^{2-2a/s}_{L^2(\Omega,\mu)}\widehat{\mu}(B_r)\big\||x|^s f\big\|^{2a/s}_{L^2(\Omega,\mu)} + \|\mathcal{T}(f)\|^2_{L^2(B_r^c,\widehat{\mu})}.\end{aligned}\tag{2.5}$$

But, using Plancherel's Theorem again,

$$\|\mathcal{T}(f)\|^2_{L^2(B_r^c,\widehat{\mu})} \leq \|\mathcal{T}(f)\|^{2a/s}_{L^2(B_r^c,\widehat{\mu})}\|\mathcal{T}(f)\|^{2-2a/s}_{L^2(\widehat{\Omega},\widehat{\mu})} = \|\mathcal{T}(f)\|^{2a/s}_{L^2(B_r^c,\widehat{\mu})}\|f\|^{2-2a/s}_{L^2(\Omega,\mu)}$$

so that, in (2.5), we may simplify by $\|f\|^{2-2a/s}_{L^2(\Omega,\mu)}$ to obtain

$$\begin{aligned}\|f\|^{2a/s}_{L^2(\Omega,\mu)} &\leq c(s,\mathcal{T})^2\widehat{\mu}(B_r)\big\||x|^s f\big\|^{2a/s}_{L^2(\Omega,\mu)} + \|\mathcal{T}(f)\|^{2a/s}_{L^2(B_r^c,\widehat{\mu})}.\\ &\leq c'(s,\mathcal{T})\; r^{2a}\big\||x|^s f\big\|^{2a/s}_{L^2(\Omega,\mu)} + r^{-2a\beta/s}\big\||\xi|^\beta \mathcal{T}(f)\big\|^{2a/s}_{L^2(\widehat{\Omega},\widehat{\mu})}.\end{aligned}$$

The desired result follows by minimizing the right hand side over $r > 0$. ∎

Inequality (2.4) has been obtained by Cowling and Price [13] for the Fourier transform on $\mathbb{R}^d$ and later generalized in [38] for any pair of positive self-adjoint operators on a Hilbert space. In particular when $s = \beta = 1$ we obtain a version of Heisenberg's Uncertainty Principle for the operator $\mathcal{T}$. Moreover, if the function $f \in L^2(\Omega,\mu)$ is supported in a subset S of finite measure, one can easily obtain bounds on $\mathcal{T}(f)$ that limit the concentration of $\mathcal{T}(f)$ in any small set and may provide lower bounds for the concentration of $\mathcal{T}(f)$ in sufficiently large sets. For instance we have this simple local uncertainty inequality: if f is supported in a set S of finite measure $\mu_{2m}(S) < \infty$, then

$$\begin{aligned}\|\mathcal{T}(f)\|^2_{L^2(\Sigma,\widehat{\mu})} &\leq \widehat{\mu}_{2\widehat{m}}(\Sigma)\|\mathcal{T}(f)\|^2_{\infty,\widehat{m}} \leq c_{\mathcal{T}}^2\widehat{\mu}_{2\widehat{m}}(\Sigma)\|f\|^2_{L^1(\Omega,\mu_m)}\\ &\leq c_{\mathcal{T}}^2\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)\|f\|^2_{L^2(\Omega,\mu)},\end{aligned}\tag{2.6}$$

which implies that the pair (S,Σ) is strongly annihilating provided that $\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma) < c_{\mathcal{T}}^{-2}$. In the next section we will prove this result for arbitrary subsets S and Σ of finite measure.

3. Pairs of sets of finite measure are strongly annihilating. In this section we will show that, if $S \subset \Omega$, $\Sigma \subset \widehat{\Omega}$ are sets of finite measure $0 < \mu_{2m}(S), \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$, then the pair (S, Σ) is strongly annihilating for the operator $\mathcal{T}$. In order to prove this, we will need to introduce a pair of orthogonal projections on $L^2(\Omega, \mu)$ defined by

$$E_S f = \chi_S f, \qquad F_\Sigma = \mathcal{T}^{-1} E_\Sigma \mathcal{T},$$

where $S \subset \Omega$ and $\Sigma \subset \widehat{\Omega}$ are measurable subsets.

We will need the following well-known lemma (see e.g. [27, Lemma 4.1]):

LEMMA 3.1. *If* $\|E_S F_\Sigma\| := \|E_S F_\Sigma\|_{L^2(\Omega,\mu)\to L^2(\Omega,\mu)} < 1$, *then*

$$\|f\|^2_{L^2(\Omega,\mu)} \le (1 - \|E_S F_\Sigma\|)^{-2} (\|E_{S^c} f\|^2_{L^2(\Omega,\mu)} + \|F_{\Sigma^c} f\|^2_{L^2(\Omega,\mu)}). \tag{3.1}$$

Unfortunately, showing that $\|E_S F_\Sigma\| < 1$ is in general difficult. However, the Hilbert–Schmidt norm $\|E_S F_\Sigma\|_{\mathrm{HS}}$ is much easier to compute. Let us illustrate this fact by showing that, if S and Σ are subsets with sufficiently small measure, then the pair (S, Σ) is strongly annihilating. We can deduce this result easily from (2.6), but we give here another proof that we will use later.

LEMMA 3.2. *If* $\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma) < c_{\mathcal{T}}^{-2}$, *then for all* $f \in L^2(\Omega, \mu)$,

$$\|f\|^2_{L^2(\Omega,\mu)} \le \big(1 - c_{\mathcal{T}} \sqrt{\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)}\big)^{-2} (\|f\|^2_{L^2(S^c,\mu)} + \|\mathcal{T}(f)\|^2_{L^2(\Sigma^c,\widehat{\mu})}).$$

Proof. For $f \in L^2(\Omega, \mu)$ we have $|\mathcal{T}(f)(\eta)| \le c_{\mathcal{T}}(1 + |\xi|)^{\widehat{m}} \|f\|_{\infty,m}$, thus if $\widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$, then $\chi_\Sigma(\eta)\mathcal{T}(f)(\eta) \in L^1(\widehat{\Omega}, \widehat{\mu}_{\widehat{m}})$. The Inversion Formula for $\mathcal{T}$ thus gives

$$\begin{aligned} E_S F_\Sigma f(y) &= \chi_S(y) \int_{\widehat{\Omega}} \chi_\Sigma(\eta)\mathcal{T}(f)(\eta)\overline{\mathcal{K}(y,\eta)}\, d\widehat{\mu}(\eta) \\ &= \chi_S(y) \int_{\widehat{\Omega}} \chi_\Sigma(\eta)\Big(\int_{\Omega} f(x)\mathcal{K}(x,\eta)\, d\mu(x)\Big)\overline{\mathcal{K}(y,\eta)}\, d\widehat{\mu}(\eta) \\ &= \int_{\Omega} f(x)\mathcal{N}(x,y)\, d\mu(x), \end{aligned}$$

where

$$\begin{aligned} \mathcal{N}(x,y) &= \chi_S(y) \int_{\widehat{\Omega}} \chi_\Sigma(\eta)\mathcal{K}(x,\eta)\overline{\mathcal{K}(y,\eta)}\, d\widehat{\mu}(\eta) \\ &= \chi_S(y)\overline{\int_{\widehat{\Omega}} \chi_\Sigma(\eta)\mathcal{K}(y,\eta)\overline{\mathcal{K}(x,\eta)}\, d\widehat{\mu}(\eta)} = \chi_S(y)\overline{\mathcal{T}^{-1}[\chi_\Sigma(\cdot)\mathcal{K}(y,\cdot)](x)}. \end{aligned}$$

Here we appealed repeatedly to Fubini's Theorem, which is justified by the fact that $\widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$ and $\mathcal{K}$ is bounded by $c_{\mathcal{T}}(1 + |x|)^m (1 + |\xi|)^{\widehat{m}}$.

This shows that $E_S F_\Sigma$ is an integral operator with kernel $\mathcal{N}$. But, with Plancherel's Theorem,

$$\begin{aligned}\|\mathcal{N}\|^2_{L^2(\Omega,\mu)\otimes L^2(\Omega,\mu)} &= \int_\Omega |\chi_S(y)|^2 \int_\Omega \left|\mathcal{T}^{-1}[\chi_\Sigma(\cdot)\mathcal{K}(y,\cdot)](x)\right|^2 d\mu(x)\, d\mu(y)\\ &= \int_\Omega |\chi_S(y)| \int_{\widehat{\Omega}} |\chi_\Sigma(\eta)\mathcal{K}(y,\eta)|^2\, d\widehat{\mu}(\eta)\, d\mu(y)\\ &\le c_\mathcal{T}^2 \mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)\end{aligned}$$

since $|\mathcal{K}(y,\eta)| \le c_\mathcal{T}(1+|y|)^m(1+|\eta|)^{\widehat{m}}$. It follows that the Hilbert–Schmidt norm of $E_S F_\Sigma$ is bounded:

$$\|E_S F_\Sigma\|_{\mathrm{HS}} = \|\mathcal{N}\|_{L^2(\Omega,\mu)\otimes L^2(\Omega,\mu)} \le c_\mathcal{T}\sqrt{\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)}. \tag{3.2}$$

Now using the fact that $\|E_S F_\Sigma\| \le \|E_S F_\Sigma\|_{\mathrm{HS}}$, we obtain

$$\|E_S F_\Sigma\| \le c_\mathcal{T}\sqrt{\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)}.$$

It follows from Lemma 3.1 that

$$\|f\|^2_{L^2(\Omega,\mu)} \le \left(1 - c_\mathcal{T}\sqrt{\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)}\right)^{-2}\left(\|E_{S^c}f\|^2_{L^2(\Omega,\mu)} + \|F_{\Sigma^c}f\|^2_{L^2(\Omega,\mu)}\right). \tag{3.3}$$

Plancherel's Theorem then gives $\|F_{\Sigma^c}f\|^2_{L^2(\Omega,\mu)} = \|\mathcal{T}(f)\|^2_{L^2(\Sigma^c,\widehat{\mu})}$, which allows us to conclude. ■

Remark 3.3. Let S, Σ be two sets with $\mu_{2m}(S), \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Let $\varepsilon_1, \varepsilon_2 > 0$. Assume that there is a function $f \in L^2(\Omega,\mu)$ with $\|f\|_{L^2(\Omega,\mu)} = 1$ that is ε_1-concentrated on S, i.e. $\|E_{S^c}f\|_{L^2(\Omega,\mu)} \le \varepsilon_1$, and ε_2-bandlimited on Σ for the transformation $\mathcal{T}$, i.e. $\|F_{\Sigma^c}f\|_{L^2(\Omega,\mu)} \le \varepsilon_2$. Then either $\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma) \ge c_\mathcal{T}^{-2}$ or we may apply (3.3) to obtain

$$1 - c_\mathcal{T}\sqrt{\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma)} \le \sqrt{\varepsilon_1^2 + \varepsilon_2^2}.$$

In both cases,

$$\mu_{2m}(S)\widehat{\mu}_{2\widehat{m}}(\Sigma) \ge c_\mathcal{T}^{-2}\left(1 - \sqrt{\varepsilon_1^2+\varepsilon_2^2}\right)^2, \tag{3.4}$$

which is Donoho–Stark's uncertainty inequality for the integral operator $\mathcal{T}$. This inequality improves slightly the result of de Jeu [17]. In the case of the Fourier transform, it goes back to Donoho and Stark [19] in a slightly weaker form and to [33] in the form (3.4).

Before proving our main theorem, we prove the following lemma which results directly from a similar result in [27] for functions in $\mathcal{C}_0(\mathbb{R}^+)$.

Lemma 3.4. *Let f be a function in $L^2(\Omega,\mu)$ and assume that*

$$0 < \mu(\operatorname{supp} f) < \infty.$$

Then the dilates $\{\mathcal{D}_\lambda f\}_{\lambda>0}$ are linearly independent.

Proof. Let $\zeta \in \mathbb{S}^{d-1} \cap \Omega$ and consider

$$f_\zeta(t) = \begin{cases} t^{a-1/2} f(t\zeta) & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

For $\zeta \in \mathbb{S}^{d-1} \cap \Omega^c$, we just define $f_\zeta = 0$. Then there exists ζ such that $f_\zeta \in L^2(\mathbb{R})$ and $0 < |\operatorname{supp} f_\zeta| < \infty$, in particular, $f_\zeta \in L^1(\mathbb{R})$. Indeed, the first property holds for almost every ζ since

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} |f_\zeta(t)|^2 \, dt \, Q(\theta) \, d\sigma(\theta) = \|f\|^2_{L^2(\Omega,\mu)} < \infty.$$

As for the second one, notice that

$$|\operatorname{supp} f_\zeta| \le |[0,1]| + \int_{\operatorname{supp} f_\zeta \cap [1,\infty)} r^{2a-1} \, dr.$$

Integrating with respect to $Q(\zeta) d\sigma(\zeta)$ we get

$$\int_{\mathbb{S}^{d-1} \cap \Omega} |\operatorname{supp} f_\zeta| Q(\zeta) \, d\sigma(\zeta) \le \|Q\|_{L^1(\mathbb{S}^{d-1} \cap \Omega)} + \mu(\operatorname{supp} f \cap \{|x| > 1\}) < \infty.$$

We thus proved that $|\operatorname{supp} f_\zeta| < \infty$ for almost every ζ. Finally, $|\operatorname{supp} f_\zeta| > 0$ on a set of ζ's of positive $d\sigma$ measure, otherwise the support of f would have Lebesgue measure 0, thus μ-measure zero.

Now assume that we have a vanishing linear combination of dilates of f:

$$\sum_{\text{finite}} \alpha_i f(x/\lambda_i) = 0. \tag{3.5}$$

Then, for $t > 0$ and the above ζ,

$$\sum_{\text{finite}} \alpha_i (\lambda_i/t)^{a-1/2} (t/\lambda_i)^{a-1/2} f(t\zeta/\lambda_i) = \frac{1}{t^{a-1/2}} \sum_{\text{finite}} \beta_i f_\zeta(t/\lambda_i) = 0$$

where we have set $\beta_i = \alpha_i \lambda_i^{a-1/2}$. Thus

$$\sum_{\text{finite}} \beta_i f_\zeta(t/\lambda_i) = 0.$$

Taking the Euclidean Fourier transform $\mathcal{F}$, we obtain

$$\sum_{\text{finite}} \beta_i \lambda_i \mathcal{F}(f_\zeta)(\lambda_i x) = 0.$$

But, as $f_\zeta \in L^1(\mathbb{R})$, it follows from Riemann–Lebesgue's Lemma that $\mathcal{F}(f_\zeta)$ is in $\mathcal{C}_0(\mathbb{R})$. It remains to invoke [27, Lemma 2.1] to see that the dilates of $\mathcal{F}(f_\zeta)$ are linearly independent so that the β_i's, and thus the α_i's, are 0. ∎

We can now state our main theorem:

Theorem 3.5. *Let $S \subset \Omega$, $\Sigma \subset \widehat{\Omega}$ be measurable subsets with $0 < \mu_{2m}(S)$, $\widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Then any function $f \in L^2(\Omega, \mu)$ vanishes as soon as*

f is supported in S and $\mathcal{T}(f)$ is supported in Σ. In other words, (S,Σ) is a weak annihilating pair.

Proof. We write $E_S\cap F_\Sigma$ for the orthogonal projection onto the intersection of the ranges of E_S and F_Σ, and we denote by $\operatorname{Im}\mathcal{P}$ the range of a linear operator $\mathcal{P}$.

First we need the following elementary fact on Hilbert–Schmidt operators:

$$\dim(\operatorname{Im}E_S\cap\operatorname{Im}F_\Sigma)=\|E_S\cap F_\Sigma\|_{\mathrm{HS}}^2\le\|E_SF_\Sigma\|_{\mathrm{HS}}^2. \tag{3.6}$$

Since $\mu_{2m}(S),\mu_{2\widehat{m}}(\Sigma)<\infty$, from (3.2) we deduce that

$$\dim(\operatorname{Im}E_S\cap\operatorname{Im}F_\Sigma)<\infty. \tag{3.7}$$

Assume now that there exists $f_0\neq 0$ such that $S_0:=\operatorname{supp}f_0$ and $\Sigma_0:=\operatorname{supp}\mathcal{T}(f_0)$ have both finite measure $0<\mu_{2m}(S_0),\widehat{\mu}_{2\widehat{m}}(\Sigma_0)<\infty$, thus also $\mu(S_0)<\infty$ so that Lemma 3.4 applies.

Next, let S_1 (resp. Σ_1) be a measurable subset of Ω (resp. $\widehat{\Omega}$) of finite measure $0<\mu_{2m}(S_1)<\infty$ (resp. $0<\widehat{\mu}_{2\widehat{m}}(\Sigma_1)<\infty$), such that $S_0\subset S_1$ (resp. $\Sigma_0\subset\Sigma_1$). Since for $\lambda>0$,

$$\mu_{2m}(S_1\cup\lambda S_0)=\|\chi_{\lambda S_0}-\chi_{S_1}\|_{L^2(\Omega,\mu_{2m})}^2+\langle\chi_{\lambda S_0},\chi_{S_1}\rangle_{L^2(\Omega,\mu_{2m})},$$

the function $\lambda\mapsto\mu_{2m}(S_1\cup\lambda S_0)$ is continuous on $\mathbb{R}^+\backslash\{0\}$. The same holds for $\lambda\mapsto\widehat{\mu}_{2\widehat{m}}(\Sigma_1\cup\lambda\Sigma_0)$. From this, one easily deduces that there exists an infinite sequence of distinct numbers $(\lambda_j)_{j=0}^\infty\subset\mathbb{R}^+\backslash\{0\}$ with $\lambda_0=1$ such that, if we write

$$S=\bigcup_{j=0}^\infty\lambda_jS_0\quad\text{and}\quad\Sigma=\bigcup_{j=0}^\infty\frac{1}{\lambda_j}\Sigma_0,$$

then

$$\mu_{2m}(S)<2\mu_{2m}(S_0),\qquad\widehat{\mu}_{2\widehat{m}}(\Sigma)<2\widehat{\mu}_{2\widehat{m}}(\Sigma_0).$$

We next define $f_i=\mathcal{D}_{\lambda_i}f_0$, so that $\operatorname{supp}f_i=\lambda_iS_0\subset S$. Since $\mathcal{T}(f_i)=\lambda_i^{a-\widehat{a}}\widehat{\mathcal{D}}_{1/\lambda_i}\mathcal{T}(f_0)$, we have $\operatorname{supp}\mathcal{T}(f_i)=(1/\lambda_i)\Sigma_0\subset\Sigma$.

As $\operatorname{supp}f_0$ has finite measure, it follows from Lemma 3.4 that $(f_i)_{i=0}^\infty$ are linearly independent vectors belonging to $\operatorname{Im}E_S\cap\operatorname{Im}F_\Sigma$, which contradicts (3.7). ∎

REMARK 3.6. The theorem can be extended to operators $\mathcal{T}$ that take their values in a finite-dimensional Banach algebra.

The proof given here follows roughly the scheme of Amrein–Berthier's proof in [1]. It can obviously be adapted so as to replace dilations by actions of more general groups on measure spaces. The main difficulty would be to prove that this leads to linearly independent functions as in Lemma 3.4.

As we have no specific application in mind, we refrain from stating a more general result.

A simple well-known functional analysis argument allows us to obtain the following improvement (see e.g. [4, Proposition 2.6]):

COROLLARY 3.7. *Let $S \subset \Omega$, $\Sigma \subset \widehat{\Omega}$ be a pair of measurable subsets of finite measure $0 < \mu_{2m}(S)$, $\widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Then there exists a constant $C(S, \Sigma)$ such that for all $f \in L^2(\Omega, \mu)$,*

$$\|f\|^2_{L^2(\Omega,\mu)} \leq C(S,\Sigma)(\|f\|^2_{L^2(S^c,\mu)} + \|\mathcal{T}(f)\|^2_{L^2(\Sigma^c,\widehat{\mu})}).$$

Proof. Assume there is no constant $D(S, \Sigma)$ such that for every function $f \in L^2(\Omega, \mu)$ supported in S,

$$\|f\|^2_{L^2(\Omega,\mu)} \leq D(S,\Sigma)\|\mathcal{T}(f)\|^2_{L^2(\Sigma^c,\widehat{\mu})}.$$

Then there exists a sequence $f_n \in L^2(\Omega, \mu)$ with $\|f_n\|_{L^2(\Omega,\mu)} = 1$ and with support in S such that $\|E_{\Sigma^c}\mathcal{T}(f_n)\|_{L^2(\widehat{\Omega},\widehat{\mu})}$ converges to 0. Moreover, we may assume that f_n is weakly convergent in $L^2(\Omega, \mu)$ with some limit f. As $\mathcal{T}(f_n)(\xi)$ is the scalar product of f_n and $\overline{E_S\mathcal{K}(\cdot, \xi)}$, it follows that $\mathcal{T}(f_n)$ converge to $\mathcal{T}(f)$. Finally, as $|\mathcal{T}(f_n)(\xi)|^2$ is bounded by $c^2_{\mathcal{T}}\mu_{2m}(S)(1+|\xi|)^{2\widehat{m}}$, we may apply Lebesgue's Theorem, thus $E_\Sigma\mathcal{T}(f_n)$ converges to $E_\Sigma\mathcal{T}(f)$ in $L^2(\widehat{\Omega}, \widehat{\mu})$. But we have $\operatorname{supp} f \subset S$ and $\operatorname{supp} \mathcal{T}(f) \subset \Sigma$, so by Theorem 3.5, f is 0, which contradicts the fact that f has norm 1. ∎

Now we will show a global uncertainty type inequality for the transformation $\mathcal{T}$. Following an idea largely exploited by B. Demange [18], we will use Corollary 3.7. The proof here is simpler than that using the local uncertainty principle, but, as Corollary 3.7 does not provide an estimate of the implied constant, this technique does not lead to an estimate of the constant either.

COROLLARY 3.8. *Let $s, \beta > 0$. Then there exists a constant $C = C(s, \beta, a)$ such that for all $f \in L^2(\Omega, \mu)$,*

$$\||x|^s f\|^{\frac{2\beta}{s+\beta}}_{L^2(\Omega,\mu)} \||\xi|^\beta \mathcal{T}(f)\|^{\frac{2s}{s+\beta}}_{L^2(\widehat{\Omega},\widehat{\mu})} \geq C\|f\|^2_{L^2(\Omega,\mu)}. \tag{3.8}$$

Proof. Let $B_1 = \Omega \cap \{x : |x| \leq 1\}$ and $\widehat{B}_1 = \widehat{\Omega} \cap \{\xi : |\xi| \leq 1\}$. Let $B^c_1 = \Omega \setminus B_1$ and $\widehat{B}^c_1 = \widehat{\Omega} \setminus \widehat{B}_1$.

From Corollary 3.7 there exists a constant $C = C(B_1, \widehat{B_1})$ such that

$$\|f\|^2_{L^2(\Omega,\mu)} \leq C(\|f\|^2_{L^2(B^c_1,\mu)} + \|\mathcal{T}(f)\|^2_{L^2(\widehat{B}^c_1,\widehat{\mu})}).$$

It follows that

$$\begin{aligned}\|f\|^2_{L^2(\Omega,\mu)} &\le C\big(\||x|^s f\|^2_{L^2(B_1^c,\mu)} + \||\xi|^\beta \mathcal{T}(f)\|^2_{L^2(\widehat{B}_1^c,\widehat{\mu})}\big)\\ &\le C\big(\||x|^s f\|^2_{L^2(\Omega,\mu)} + \||\xi|^\beta \mathcal{T}(f)\|^2_{L^2(\widehat{\Omega},\widehat{\mu})}\big).\end{aligned}$$

Replacing f by $\mathcal{D}_\lambda f$ in the last inequality we have, by (1.3),

$$\|\mathcal{D}_\lambda f\|^2_{L^2(\Omega,\mu)} \le C\big(\||x|^s \mathcal{D}_\lambda f\|^2_{L^2(\Omega,\mu)} + \||\xi|^\beta \widehat{\mathcal{D}}_{1/\lambda} \mathcal{T}(f)\|^2_{L^2(\widehat{\Omega},\widehat{\mu})}\big), \tag{3.9}$$

which gives

$$\|f\|^2_{L^2(\Omega,\mu)} \le C\big(\lambda^{2s}\||x|^s f\|^2_{L^2(\Omega,\mu)} + \lambda^{-2\beta}\||\xi|^\beta \mathcal{T}(f)\|^2_{L^2(\widehat{\Omega},\widehat{\mu})}\big).$$

The desired result follows by minimizing the right hand side over $\lambda > 0$. ∎

Let us notice that Theorem 3.5 is valid in the L^1-version. More precisely, we have the following proposition:

Proposition 3.9. *Let $S \subset \Omega$, $\Sigma \subset \widehat{\Omega}$ be measurable subsets with $0 < \mu_{2m}(S), \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Suppose $f \in L^1(\Omega, \mu_m)$ (in particular $f \in L^1(\Omega,\mu)$) verifies* $\operatorname{supp} f \subset S$ *and* $\operatorname{supp} \mathcal{T}(f) \subset \Sigma$. *Then $f = 0$.*

Proof. If $f \in L^1(\Omega,\mu_m)$, then $(1+|\xi|)^{-\widehat{m}}\mathcal{T}(f) \in L^\infty(\widehat{\Omega},\widehat{\mu})$. Therefore

$$\begin{aligned}\|\mathcal{T}(f)\|_{L^1(\widehat{\Omega},\widehat{\mu}_{\widehat{m}})} &= \|\chi_\Sigma \mathcal{T}(f)\|_{L^1(\widehat{\Omega},\widehat{\mu}_{\widehat{m}})}\\ &\le \widehat{\mu}_{2\widehat{m}}(\Sigma)\|(1+|\xi|)^{-\widehat{m}}\mathcal{T}(f)\|_{L^\infty(\widehat{\Omega},\widehat{\mu})} < \infty.\end{aligned}$$

This implies that $\mathcal{T}(f) \in L^1(\widehat{\Omega},\widehat{\mu}_{\widehat{m}})$, thus $(1+|x|)^{-m} f \in L^\infty(\Omega,\mu)$. Finally,

$$\begin{aligned}\|f\|^2_{L^2(\Omega,\mu)} &= \int_\Omega (1+|x|)^{-m}|f(x)|\,|f(x)|(1+|x|)^m\,d\mu(x)\\ &\le \|(1+|x|)^{-m} f\|_{L^\infty(\Omega,\mu)}\|f\|_{L^1(\Omega,\mu_m)} < \infty,\end{aligned}$$

hence $f \in L^2(\Omega,\mu)$. By Theorem 3.5 we have $f = 0$. ∎

The same argument as the one used in the proof of Corollary 3.7 gives the following result:

Proposition 3.10. *Let $S \subset \Omega$, $\Sigma \subset \widehat{\Omega}$ be measurable subsets with $0 < \mu_{2m}(S), \widehat{\mu}_{2\widehat{m}}(\Sigma) < \infty$. Then there exists a constant $D(S,\Sigma)$ such that for all functions $f \in L^1(\Omega,\mu)$ supported in S,*

$$\|f\|_{L^1(\Omega,\mu)} \le D(S,\Sigma)\|\mathcal{T}(f)\|_{L^1(\Sigma^c,\widehat{\mu})}.$$

4. Examples

4.1. The Fourier transform and the Fourier–Bessel transform. Let $d\mu(x) = (2\pi)^{-d/2}\,dx$ be the Lebesgue measure and $\mathcal{T} = \mathcal{F}$ be the Fourier transform. For $f \in L^1(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$, the Fourier transform is defined by

$$\mathcal{F}(f)(\xi)=\int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,d\mu(x),\qquad \xi\in\mathbb{R}^d,$$

and is then extended to all $L^2(\mathbb{R}^d,\mu)$ in the usual way. In this case we take $c_{\mathcal{T}}=1$, $a=d/2$ and $m=\widehat{m}=0$. Then (3.4) is Donoho–Stark's theorem [19, 33], Corollary 3.7 is Amrein–Berthier's theorem [1], while the local and the global uncertainty principles for the Fourier transform go back respectively to [41, 42] and [13]. Note that our proof of Theorem 3.5 is inspired by one in [1] where we replace translation by dilation.

If $f(x)=f_0(|x|)$ is a radial function on $\mathbb{R}^d$, then

$$\mathcal{F}(f)(\xi)=\frac{1}{2^{d/2-1}\Gamma(d)}\int_0^\infty f_0(t)j_{d/2-1}(t|\xi|)t^{d-1}\,dt=\mathcal{F}_{d/2-1}(f_0)(|\xi|),$$

where $\mathcal{F}_{d/2-1}$ is the Fourier–Bessel transform of index $d/2-1$. For $\alpha\ge -1/2$, j_α is the Bessel function given by

$$j_\alpha(x)=2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(x)}{x^\alpha}:=\Gamma(\alpha+1)\sum_{n=0}^\infty\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{x}{2}\right)^{2n},$$

where Γ is the gamma function.

We have $|j_\alpha|\le 1$ and if we denote

$$d\mu_\alpha(x)=\frac{1}{2^\alpha\Gamma(\alpha+1)}x^{2\alpha+1}\,dx,$$

then for $f\in L^1(\mathbb{R}^+,\mu_\alpha)\cap L^2(\mathbb{R}^+,\mu_\alpha)$, the Fourier–Bessel (or Hankel) transform is defined by

$$\mathcal{F}_\alpha(f)(\xi)=\int_0^\infty f(x)j_\alpha(x\xi)\,d\mu_\alpha(x),\qquad \xi\in\mathbb{R}^+;$$

it extends to an isometric isomorphism on $L^2(\mathbb{R}^+,\mu_\alpha)$ with $\mathcal{F}_\alpha^{-1}=\mathcal{F}_\alpha$. Theorems A and B have been stated in [27] for this transformation. Moreover we have the following two new results.

THEOREM 4.1 (Donoho–Stark's uncertainty principle for $\mathcal{F}_\alpha$). *Let S, Σ be a pair of measurable subsets of $\mathbb{R}^+$, and $\alpha>-1/2$. If $f\in L^2(\mathbb{R}^+,\mu_\alpha)$ of unit L^2-norm is ε_1-concentrated on S and ε_2-bandlimited on Σ for the Fourier–Bessel transform, then*

$$\mu_\alpha(S)\mu_\alpha(\Sigma)\ge\big(1-\sqrt{\varepsilon_1^2+\varepsilon_2^2}\big)^2\quad\text{and}\quad |S|\,|\Sigma|\ge c_\alpha\big(1-\sqrt{\varepsilon_1^2+\varepsilon_2^2}\big)^2,\tag{4.1}$$

where c_α is a numerical constant that depends only on α.

This result improves the estimate in [39] (which has already improved [51]) showing that, if f of unit L^2-norm is ε_1-concentrated on S

and ε_2-bandlimited on Σ, then

$$|S|\,|\Sigma| \geq c'_\alpha(1-\varepsilon_1-\varepsilon_2)^2.$$

THEOREM 4.2 (Global uncertainty principle for $\mathcal{F}_\alpha$). *For $s,\beta>0$, there exists a constant $C_{s,\beta,\alpha}$ such that for all $f \in L^2(\mathbb{R}^+,\mu_\alpha)$,*

$$\|x^s f\|_{L^2(\mathbb{R}^+,\mu_\alpha)}^{\frac{2\beta}{s+\beta}} \|\xi^\beta \mathcal{F}_\alpha(f)\|_{L^2(\mathbb{R}^+,\mu_\alpha)}^{\frac{2s}{s+\beta}} \geq C_{s,\beta,\alpha}\|f\|_{L^2(\mathbb{R}^+,\mu_\alpha)}^2.$$

The case when $s=\beta=1$ has been established in [5, 47] with the optimal constant $C_{1,1,\alpha}=\alpha+1$.

4.2. The Fourier–Dunkl transform. In this section we will deduce new uncertainty principles for the Dunkl transform. Uncertainty principles for this transformation have been considered in various places, e.g. [45, 49] for a Heisenberg type inequality or [26] for Hardy type uncertainty principles and recently [10, 36] for a generalization and a variant of Cowling–Price's theorem, Beurling's theorem, Miyachi's theorem and Donoho–Stark's uncertainty principle.

Let us fix some notation and present some necessary material on the Dunkl transform. Let G be a finite reflection group on $\mathbb{R}^d$, associated with a root system R, and R_+ the positive subsystem of R (see [16, 21, 48]). We denote by k a nonnegative multiplicity function defined on R with the property that k is G-invariant. We associate with k the index

$$\gamma := \gamma(k) = \sum_{\xi\in R_+} k(\xi) \geq 0$$

and the weight function w_k defined by

$$w_k(x) = \prod_{\xi\in R_+} |\langle \xi, x\rangle|^{2k(\xi)}.$$

Further we introduce the Mehta-type constant c_k by

$$c_k = \Big(\int_{\mathbb{R}^d} e^{-|x|^2/2}\,d\mu_k(x)\Big)^{-1},$$

where (³) $d\mu_k(x) = w_k(x)dx$. Moreover

$$\int_{\mathbb{S}^{d-1}} w_k(x)\,d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+d/2-1}\Gamma(\gamma+d/2)} = d_k.$$

By using the homogeneity of w_k it is shown in [48] that for a radial function $f\in L^1(\mathbb{R}^d,\mu_k)$ the function $\widetilde{f}$ defined on $\mathbb{R}^+$ by $f(x)=\widetilde{f}(|x|)$

(³) We chose here to stick to the notation that is usual in Dunkl analysis rather than that of the previous section in which μ_k is simply denoted by μ.

for all $x \in \mathbb{R}^d$ is integrable with respect to the measure $r^{2\gamma+d-1}dr$. More precisely,

$$\int_{\mathbb{R}^d} f(x)w_k(x)\,dx = \int_{\mathbb{R}^+}\Big(\int_{\mathbb{S}^{d-1}} w_k(ry)\,d\sigma(y)\Big)\widetilde{f}(r)r^{d-1}\,dr = d_k\int_{\mathbb{R}^+}\widetilde{f}(r)r^{2\gamma+d-1}\,dr. \tag{4.2}$$

Introduced by C. F. Dunkl in [20], the *Dunkl operators* T_j, $1 \le j \le d$, on $\mathbb{R}^d$ associated with the reflection group G and the multiplicity function k are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j} + \sum_{\xi\in R_+} k(\xi)\xi_j \frac{f(x)-f(\sigma_\xi(x))}{\langle \xi, x\rangle}, \qquad x \in \mathbb{R}^d,$$

where f is an infinitely differentiable function on $\mathbb{R}^d$, $\xi_j = \langle \xi, e_j\rangle$, $(e_1,\ldots,e_d)$ being the canonical basis of $\mathbb{R}^d$, and σ_ξ denotes the reflection with respect to the hyperplane orthogonal to ξ.

The *Dunkl kernel* $\mathcal{K}_k$ on $\mathbb{R}^d\times\mathbb{R}^d$ has been introduced by C. F. Dunkl in [21]. For $y \in \mathbb{R}^d$ the function $x \mapsto \mathcal{K}_k(x,y)$ can be viewed as the solution on $\mathbb{R}^d$ of the initial value problem

$$T_j u(x,y) = y_j u(x,y), \quad 1 \le j \le d, \qquad u(0,y) = 1.$$

This kernel has a unique holomorphic extension to $\mathbb{C}^d\times\mathbb{C}^d$. M. Rösler [46] has proved the following integral representation for the Dunkl kernel:

$$\mathcal{K}_k(x,z) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle}\,d\mu_x^k(y), \qquad x\in\mathbb{R}^d,\ z\in\mathbb{C}^d,$$

where μ_x^k is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B_{|x|}$. For all $\lambda\in\mathbb{C}$, $z, z' \in \mathbb{C}^d$ and $x, y \in \mathbb{R}^d$ we have (see [46])

$$\mathcal{K}_k(z,z') = \mathcal{K}_k(z',z), \qquad \mathcal{K}_k(\lambda z, z') = \mathcal{K}_k(z,\lambda z'),$$

$$\overline{\mathcal{K}_k(-iy,x)} = \mathcal{K}_k(iy,x), \qquad |\mathcal{K}_k(-iy,x)| \le 1.$$

The *Dunkl transform* $\mathcal{F}_k$ of a function $f \in L^1(\mathbb{R}^d,\mu_k)\cap L^2(\mathbb{R}^d,\mu_k)$, which was introduced by C. F. Dunkl in [22] (see also [16]), is given by

$$\mathcal{F}_k(f)(\xi) := c_k\int_{\mathbb{R}^d}\mathcal{K}_k(-i\xi,x)f(x)\,d\mu_k(x), \qquad \xi\in\mathbb{R}^d;$$

it extends uniquely to an isometric isomorphism on $L^2(\mathbb{R}^d,\mu_k)$ with

$$\mathcal{F}_k^{-1}(f)(\xi) = \mathcal{F}_k(f)(-\xi).$$

The Dunkl transform $\mathcal{F}_k$ provides a natural generalization of the Fourier transform $\mathcal{F}$, to which it reduces in the case $k = 0$, and if $f(x) = \widetilde{f}(|x|)$ is a radial function on $\mathbb{R}^d$, then

$$\mathcal{F}_k(f)(\xi)=\mathcal{F}_{\gamma+d/2-1}(\widetilde{f})(|\xi|),$$

where $\mathcal{F}_{\gamma+d/2-1}$ is the Fourier–Bessel transform of index $\gamma+d/2-1$.

Now if we take $c_{\mathcal{T}}=c_k$, $a=\gamma+d/2$ and $m=\widehat{m}=0$, then from Sections 2 and 3 we obtain a new uncertainty principle for the Dunkl transform $\mathcal{F}_k$.

Theorem 4.3 (Donoho–Stark's uncertainty principle for $\mathcal{F}_k$). *Let S, Σ be a pair of measurable subsets of $\mathbb{R}^d$. If $f\in L^2(\mathbb{R}^d,\mu_k)$ of unit L^2-norm is ε_1-concentrated on S and ε_2-bandlimited on Σ for the Dunkl transform, then*

$$\mu_k(S)\mu_k(\Sigma)\geq c_k^{-2}\big(1-\sqrt{\varepsilon_1^2+\varepsilon_2^2}\big)^2. \tag{4.3}$$

Note that Donoho–Stark's uncertainty principle has recently been proved in [36] for the Dunkl transform, but our inequality (4.3) is a little stronger.

Let us now state how our results translate to the Fourier–Dunkl transform. These results are new to the best of our knowledge.

Theorem 4.4. *Let S, Σ be a pair of measurable subsets of $\mathbb{R}^d$ with finite measure $0<\mu_k(S),\mu_k(\Sigma)<\infty$. Then the following uncertainty principles hold:*

(1) Local uncertainty principle for $\mathcal{F}_k$:

(a) *For $0<s<\gamma+d/2$, there is a constant $c(s,k)$ such that for all $f\in L^2(\mathbb{R}^d,\mu_k)$,*

$$\|\mathcal{F}_k(f)\|_{L^2(\Sigma,\mu_k)}\leq c(s,k)[\mu_k(\Sigma)]^{\frac{s}{2\gamma+d}}\big\||x|^s f\big\|_{L^2(\mathbb{R}^d,\mu_k)}.$$

(b) *For $s>\gamma+d/2$, there is a constant $c'(s,k)$ such that for all $f\in L^2(\mathbb{R}^d,\mu_k)$,*

$$\|\mathcal{F}_k(f)\|_{L^2(\Sigma,\mu_k)}\leq c'(s,k)[\mu_k(\Sigma)]^{1/2}\|f\|_{L^2(\mathbb{R}^d,\mu_k)}^{1-\frac{2\gamma+d}{2s}}\big\||x|^s f\big\|_{L^2(\mathbb{R}^d,\mu_k)}^{\frac{2\gamma+d}{2s}}.$$

(2) Benedicks–Amrein–Berthier's uncertainty principle for $\mathcal{F}_k$: *There exists a constant $C_k(S,\Sigma)$ such that for all $f\in L^2(\mathbb{R}^d,\mu_k)$,*

$$\|f\|^2_{L^2(\mathbb{R}^d,\mu_k)}\leq C_k(S,\Sigma)(\|f\|^2_{L^2(S^c,\mu_k)}+\|\mathcal{F}_k(f)\|^2_{L^2(\Sigma^c,\mu_k)}).$$

(3) Global uncertainty principle for $\mathcal{F}_k$: *For $s,\beta>0$, there exists a constant $C_{s,\beta,k}$ such that for all $f\in L^2(\mathbb{R}^d,\mu_k)$,*

$$\big\||x|^s f\big\|_{L^2(\mathbb{R}^d,\mu_k)}^{\frac{2\beta}{s+\beta}}\big\||\xi|^\beta\mathcal{F}_k(f)\big\|_{L^2(\mathbb{R}^d,\mu_k)}^{\frac{2s}{s+\beta}}\geq C_{s,\beta,k}\|f\|^2_{L^2(\mathbb{R}^d,\mu_k)}.$$

A simple computation shows that

$$c(s,k)=\frac{2\gamma+d}{2\gamma+d-2s}\left[\frac{c_k}{2s}\sqrt{(2\gamma+d-2s)d_k}\right]^{\frac{2\gamma+d}{2s}}$$

and

$$c'(s,k) = c_k\left[\frac{d_k}{2\gamma+d}\left(\frac{s}{\gamma+d/2}-1\right)^{\frac{2\gamma+d}{2s}-1}\Gamma\left(\frac{\gamma+d/2}{s}\right)\Gamma\left(1-\frac{\gamma+d/2}{s}\right)\right]^{1/2}.$$

In the particular case $s = \beta = 1$ for the global uncertainty principle, we recover Heisenberg's inequality for the Dunkl transform but with $C_{1,1,k} \leq \gamma + d/2$, where $\gamma + d/2$ is the optimal constant in the Heisenberg Uncertainty Principle given in [45, 49].

4.3. The Fourier–Clifford transform. Let us now introduce the basics of Clifford analysis that are needed to introduce the Fourier–Clifford transform. Facts used here can be found e.g. in [6, 9]. We also follow as closely as possible the presentation of Clifford analysis from [8, 15].

Throughout this section $d \geq 2$ is a fixed integer, and $d\mu(x) = d\widehat{\mu}(x) = (2\pi)^{-d/2}dx$ is the Lebesgue measure on $\mathbb{R}^d$. We first consider the Clifford algebra $\mathrm{Cl}_{0,d}(\mathbb{C})$ generated by the canonical basis e_j, $j = 1, \ldots, d$. For $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, d\}$ with $j_1 < \cdots < j_k$, we define $e_A = e_{j_1} \cdots e_{j_k}$. The basis of the Clifford algebra is given by $\mathcal{E} = \{e_A : A \subset \{1, \ldots, d\}\}$. The *Clifford algebra* is then the complex vector space generated by $\mathcal{E}$ endowed with the multiplication rule given by

(i) $e_\emptyset = 1$ is the unit element,
(ii) $e_j^2 = -1$, $j = 1, \ldots, d$,
(iii) $e_j e_k + e_k e_j = 0$, $j, k = 1, \ldots, d$, $j \neq k$.

Conjugation is defined by the anti-involution for which $\overline{e_j} = -e_j$, $j = 1, \ldots, d$ with the additional rule $\overline{i} = -i$.

The scalars are then identified with $\mathrm{span}\{e_\emptyset\}$ while we identify a vector $x = (x_1, \ldots, x_d)$ with $\underline{x} = \sum_{j=1}^d e_j x_j$. The product of two vectors splits into a scalar part and a *bivector* part

$$\underline{x}\underline{y} = -\langle \underline{x}, \underline{y}\rangle + \underline{x} \wedge \underline{y} \quad \text{with} \quad \underline{x} \wedge \underline{y} = \sum_{j=1}^d \sum_{k=j+1}^d e_j e_k (x_j y_k - x_k y_j).$$

Note that $\underline{x}^2 = -|x|^2$.

The functions in this section are defined on $\mathbb{R}^d$ and take their values in the Clifford algebra $\mathrm{Cl}_{0,d}(\mathbb{C})$. We can now introduce the so-called *Dirac operator*, a first order vector differential operator defined by

$$\partial_{\underline{x}} = \sum_{j=1}^d \partial_{x_j} e_j.$$

Its square equals, up to a minus sign, the Laplace operator on $\mathbb{R}^d$, $\partial_{\underline{x}}^2 = -\Delta$. The central notion in Clifford analysis is the notion of *monogenicity*, the higher-dimensional analogue of holomorphy: a function is called (*left*)-*monogenic* if $\partial_{\underline{x}} f = 0$.

We will denote by $\mathcal{M}_k$ the space of all *spherical monogenics* of degree k, that is, homogeneous polynomials of degree k that are null-solutions of the Dirac operator. We fix a basis $\{M_k^{(\ell)}\}_{\ell=1,\dots,\dim\mathcal{M}_k}$ of $\mathcal{M}_k$. Further, the Laguerre polynomials are denoted by L_j^α. We then consider the following functions, called the *Clifford–Hermite functions*:

$$\begin{aligned} \psi_{2j,k,\ell}(\underline{x}) &= \gamma_{2j,k,\ell} L_j^{d/2+k-1}(|\underline{x}|^2) M_k^{(\ell)}(\underline{x}) e^{-|\underline{x}|^2/2}, \\ \psi_{2j+1,k,\ell}(\underline{x}) &= \gamma_{2j+1,k,\ell} L_j^{d/2+k}(|\underline{x}|^2) \underline{x} M_k^{(\ell)}(\underline{x}) e^{-|\underline{x}|^2/2}, \end{aligned} \tag{4.4}$$

where $j, k \in \mathbb{N}$ and $\ell \in \{1, \dots, \dim\mathcal{M}_k\}$. Provided the $\gamma_{j,k,\ell}$'s are properly chosen, this is an *orthonormal basis* of $L^2(\mathbb{R}^d)$ (see [7]).

Next, introducing spherical coordinates in $\mathbb{R}^d$: $\underline{x} = r\underline{\omega}$, $r = |\underline{x}| \in \mathbb{R}^+$, $\underline{\omega} \in \mathbb{S}^{d-1}$, we see that the Dirac operator takes the form

$$\partial_{\underline{x}} = \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{x}}\right)$$

where

$$\Gamma = \underline{x} \wedge \partial_{\underline{x}} = -\sum_{j=1}^{d}\sum_{k=j+1}^{d} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j})$$

is the so-called *angular Dirac operator*.

We are now in a position to define the *Clifford–Fourier transform* on $\mathcal{S}(\mathbb{R}^d)$. This can be done in three equivalent ways:

- $\mathcal{F}_\pm[f] = e^{id\pi/4} e^{(i\pi/4)(\Delta - |\underline{x}|^2 \mp 2\Gamma)} f$;
- via an integral kernel

  $$\mathcal{F}_\pm[f](\underline{\eta}) = \int_{\mathbb{R}^d} f(\underline{x}) K_\pm(\underline{x}, \underline{\eta})\, d\mu(\underline{x})$$

  where $K_\pm(\underline{x}, \underline{\eta}) = e^{id\pi/4} e^{i\pi\Gamma_{\underline{\eta}}/2} e^{-i\langle \underline{x}, \underline{\eta}\rangle}$;
- via its eigenfunctions

  $$\begin{aligned} \mathcal{F}_\pm[\psi_{2j,k,\ell}] &= (-1)^{j+k}(\mp 1)^k \psi_{2j,k,\ell}, \\ \mathcal{F}_\pm[\psi_{2j+1,k,\ell}] &= i^d(-1)^{j+1}(\mp 1)^{k+d-1} \psi_{2j,k,\ell}. \end{aligned}$$

The third definition immediately shows that $\mathcal{F}_\pm$ extend to unitary operators on $L^2(\mathbb{R}^d, \mu)$.

The facts that the integral operator definition makes sense on $\mathcal{S}(\mathbb{R}^d)$ and that the kernel of the inverse transform is indeed $\overline{K_\pm(\underline{x}, \underline{\eta})}$ have been proven respectively in [15, Theorem 6.3] and [15, Proposition 3.4].

Finally, the kernel is not known to be polynomially bounded, except when the dimension d is even [15, Theorem 5.3], in which case

$$|K(\underline{x},\underline{\eta})| \leq C(1+|\underline{x}|)^{(d-2)/2}(1+|\underline{\eta}|)^{(d-2)/2}.$$

Thus $m = \widehat{m} = (d-2)/2$, $c_{\mathcal{T}} = C$ and $a = d/2$.

It remains to notice that all results from the first part of the paper extend with no change to Clifford-valued functions. More precisely, we obtain the following results:

THEOREM 4.5. *Let d be even and $d\nu(x) = (1+|x|)^{d-2}d\mu(x)$. Let S, Σ be a pair of measurable subsets of $\mathbb{R}^d$. Then the Clifford–Fourier transform satisfies the following uncertainty principles:*

(1) Donoho–Stark's uncertainty principle for $\mathcal{F}_{\pm}$: *If $f \in L^2(\mathbb{R}^d,\mu)$ of unit L^2-norm is ε_1-concentrated on S and ε_2-bandlimited on Σ for the Clifford–Fourier transform, then*

$$\nu(S)\nu(\Sigma) \geq C^{-2}\big(1-\sqrt{\varepsilon_1^2+\varepsilon_2^2}\big)^2.$$

(2) Local uncertainty principle for $\mathcal{F}_{\pm}$: *If Σ is a subset of finite measure $0 < \nu(\Sigma) < \infty$, then:*

(a) *For $0 < s < d/2$, there is a constant $c(s)$ such that for all $f \in L^2(\mathbb{R}^d,\mu)$,*

$$\|\mathcal{F}_{\pm}(f)\|_{L^2(\Sigma,\widehat{\mu})} \leq \begin{cases} c(s)[\nu(\Sigma)]^{\frac{s}{2(d-1)}}\big\||x|^s f\big\|_{L^2(\mathbb{R}^d,\mu)} & \text{if } \nu(\Sigma)\leq 1,\\ c(s)[\nu(\Sigma)]^{s/d}\big\||x|^s f\big\|_{L^2(\mathbb{R}^d,\mu)} & \text{if } \nu(\Sigma)> 1.\end{cases}$$

- *For $d/2 \leq s \leq d-1$ and every $\varepsilon > 0$, there is a constant $c(s,\varepsilon)$ such that for all $f \in L^2(\mathbb{R}^d,\mu)$, $\|\mathcal{F}_{\pm}(f)\|_{L^2(\Sigma,\widehat{\mu})}$ is bounded by*

$$\begin{cases} c(s,\varepsilon)[\nu(\Sigma)]^{\frac{1}{4(1-1/d)}-\varepsilon}\|f\|_{L^2(\mathbb{R}^d,\mu)}^{1-\frac{d}{2s}+\varepsilon}\big\||x|^s f\big\|_{L^2(\mathbb{R}^d,\mu)}^{\frac{d}{2s}-\varepsilon} \\ \qquad\qquad \text{if } \nu(\Sigma)\leq 1,\\ c(s,\varepsilon)[\nu(\Sigma)]^{1/2-\varepsilon}\|f\|_{L^2(\mathbb{R}^d,\mu)}^{1-\frac{d}{2s}+\varepsilon}\big\||x|^s f\big\|_{L^2(\mathbb{R}^d,\mu)}^{\frac{d}{2s}-\varepsilon} \\ \qquad\qquad \text{if } \nu(\Sigma)> 1.\end{cases}$$

(b) *For $s > d-1$, there is a constant $c'(s)$ such that for all $f \in L^2(\mathbb{R}^d,\mu)$,*

$$\|\mathcal{F}_{\pm}(f)\|_{L^2(\Sigma,\mu)} \leq c'(s)[\nu(\Sigma)]^{1/2}\big\|(1+|x|^s)f\big\|_{L^2(\mathbb{R}^d,\mu)}.$$

(3) Benedicks–Amrein–Berthier's uncertainty principle for $\mathcal{F}_{\pm}$: *If S, Σ are subsets of finite measure $0 < \nu(S), \nu(\Sigma) < \infty$, then there exists a constant $C(S,\Sigma)$ such that for all $f \in L^2(\mathbb{R}^d,\mu)$,*

$$\|f\|_{L^2(\mathbb{R}^d,\mu)}^2 \leq C(S,\Sigma)(\|f\|_{L^2(S^c,\mu)}^2 + \|\mathcal{F}_{\pm}(f)\|_{L^2(\Sigma^c,\mu)}^2).$$

(4) Global uncertainty principle for $\mathcal{F}_{\pm}$: *For $s, \beta > 0$, there exists a constant $C_{s,\beta}$ such that for all $f \in L^2(\mathbb{R}^d, \mu)$,*

$$\big\||x|^s f\big\|_{L^2(\mathbb{R}^d,\mu)}^{\frac{2\beta}{s+\beta}} \big\||\xi|^\beta \mathcal{F}_{\pm}(f)\big\|_{L^2(\mathbb{R}^d,\mu)}^{\frac{2s}{s+\beta}} \geq C_{s,\beta}\|f\|_{L^2(\mathbb{R}^d,\mu)}^2.$$

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