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Dimensions of components of tensor products of representations of linear groups with applications to Beurling–Fourier algebras

by

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Abstract. We give universal upper bounds on the relative dimensions of isotypic components of a tensor product of representations of the linear group $\mathrm{GL}(n)$ and universal upper bounds on the relative dimensions of irreducible components of a tensor product of representations of the special linear group $\mathrm{SL}(n)$. This problem is motivated by harmonic analysis problems, and we give some applications to the theory of Beurling–Fourier algebras.

1. Introduction

1.1. The main problem for linear groups $\mathrm{GL}(n)$. In this paper we are interested in the following question: Let λ, μ be two irreducible representations of the linear group $\mathrm{GL}(n)$ and consider the decomposition of their tensor product $\lambda \otimes \mu$ into *isotypic* components. *How big can the dimension of such an isotypic component be?*

For irreducible representations λ, μ, ν we denote by $c^{\nu}_{\lambda,\mu}$ the *Littlewood–Richardson coefficient*, i.e. the multiplicity of the irreducible representation ν in the Kronecker tensor product $\lambda \otimes \mu$. For an irreducible representation ρ we denote by d_ρ its dimension. With these notations, the dimension of the isotypic component $[\nu]$ of $\lambda \otimes \mu$ is equal to $c^{\nu}_{\lambda,\mu} d_\nu$. Our goal will be to give an upper bound for the fraction

$$P_{\lambda,\mu}(\nu) := \frac{c^{\nu}_{\lambda,\mu} d_\nu}{d_\lambda d_\mu}, \tag{1.1}$$

which can be interpreted as the *relative dimension* of the isotypic component $[\nu]$ in $\lambda \otimes \mu$.

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Equation (1.1) defines a probability distribution $P_{\lambda,\mu}$ (called the *Littlewood–Richardson measure*) on irreducible representations. This probability measure can be interpreted as the distribution of a random irreducible component of the Kronecker tensor product $\lambda \otimes \nu$, where each irreducible component is sampled with probability proportional to its dimension. Our problem can therefore be equivalently formulated as finding an upper bound for the atoms of the Littlewood–Richardson measure.

1.2. The main result for linear groups $\mathrm{GL}(n)$**.** The main result of this paper is the following partial answer to the above problem.

THEOREM 1.1. *Let $n \geq 1$ be a fixed integer. There exists a constant C_n such that for any irreducible representations λ, μ, ν of $\mathrm{GL}(n)$ the atom of the Littlewood–Richardson measure is bounded from above as follows:*

$$P_{\lambda,\mu}(\nu) := \frac{c^{\nu}_{\lambda,\mu} d_\nu}{d_\lambda d_\mu} \leq C_n \left(\frac{1}{\lambda_1 - \lambda_n} + \frac{1}{\mu_1 - \mu_n} \right). \tag{1.2}$$

Here, $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$ are the components of the highest weight of λ and μ, respectively. The notations used in the above inequality will be recalled in Section 2. We postpone its proof to Section 5. We will see that this result is optimal in a sense which will be clarified at the end of the paper.

1.3. The main result for special linear groups $\mathrm{SL}(n)$**.** In this paper we are also interested in the analogue of the above problem in the case of the special linear group $\mathrm{SL}(n)$ (the notions of Littlewood–Richardson coefficients and the Littlewood–Richardson measure can be defined in an analogous way to the case of $\mathrm{GL}(n)$). A partial answer is given by the following result, which is a corollary to our main theorem:

COROLLARY 1.2. *Let $n \geq 1$ be a fixed integer. There exists a constant C_n such that for any irreducible representations λ, μ, ν of $\mathrm{SL}(n)$ the atom of the Littlewood–Richardson measure is bounded from above as follows:*

$$P_{\lambda,\mu}(\nu) := \frac{c^{\nu}_{\lambda,\mu} d_\nu}{d_\lambda d_\mu} \leq C_n \left(\frac{1}{\lambda_1 - \lambda_n} + \frac{1}{\mu_1 - \mu_n} \right). \tag{1.3}$$

The notations used in the above inequality will be recalled in Section 2.3, where we will also present its proof.

1.4. The case of unitary groups and special unitary groups. The representation theory of the unitary group $U(n)$ is exactly the same as that of $\mathrm{GL}(n)$, since the restriction map gives a one-to-one map; its inverse is given by analytic continuation. In particular, the correspondence between irreducible representations and highest weights holds also for $U(n)$. For this reason in the formulation of Theorem 1.1 one can replace representations of the linear groups $\mathrm{GL}(n)$ by representations of the unitary groups $U(n)$ and the result holds true without any modifications.

The analogous relationship holds between the representation theory of the special unitary group $\mathrm{SU}(n)$ and the special linear group $\mathrm{SL}(n)$, so that in the formulation of Corollary 1.2 one can replace representations of $\mathrm{SL}(n)$ by representations of $\mathrm{SU}(n)$.

1.5. Applications to Beurling–Fourier algebras. Our paper is motivated by the work of Mahya Ghandehari, Hun Hee Lee, Ebrahim Samei and Nico Spronk [GLSS12] and gives a proof of their conjecture (Condition 1, p. 21). Our main theorem implies that the conjecture is true for any integer $n \geq 2$, whilst it was proved for $n = 3$ in an elementary way in [GLSS12].

In this subsection we briefly describe what Beurling–Fourier algebras are and the implications of our main results for them. See [LS12, LST12] for the details on Beurling–Fourier algebras.

Let G be a compact group and $\widehat{G}$ be the set of equivalence classes of unitary irreducible representations of G. The *Fourier algebra* $A(G)$ of G is defined as

$$A(G) := \Big\{ f \in C(G) : \|f\|_{A(G)} := \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|_1 < \infty \Big\}.$$

Here, $\widehat{f}(\pi)$ denotes the Fourier coefficient given by

$$\widehat{f}(\pi) := \int_G f(x)\overline{\pi}(x)\, dx \in M_{d_\pi}(\mathbb{C})$$

where dx denotes the normalized Haar measure on G; $\overline{\pi}$ denotes the conjugate representation of π; and $\|\cdot\|_1$ is the trace norm. It is well known that the Fourier algebra is actually a Banach algebra under pointwise multiplication.

The Fourier algebra can be defined for any locally compact group (see [Eym64]) and is regarded as one of the most fundamental examples of commutative Banach algebras associated to groups. When the (compact) group G is abelian, $A(G)$ is nothing but the group algebra $L^1(\widehat{G})$ of the Pontryagin dual $\widehat{G}$, so that Fourier algebras are usually called the "dual" object of group algebras. In general, Fourier algebras are quite far away from operator algebras (i.e. norm-closed subalgebras of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$) including C^*-algebras. However, by putting some weights on $A(G)$ for a compact group G we can make weighted versions of $A(G)$ much closer to operator algebras.

We call a function $\omega : \widehat{G} \to [1, \infty)$ a *weight* if

$$\omega(\sigma) \leq \omega(\pi)\omega(\pi') \tag{1.4}$$

for any $\pi, \pi' \in \widehat{G}$ and $\sigma \in \widehat{G}$ appearing as a component of the irreducible decomposition of $\pi \otimes \pi'$.

We define the *Beurling–Fourier algebra* $A(G,\omega)$ by

$$A(G,\omega) := \Big\{ f \in C(G) : \|f\|_{A(G,\omega)} = \sum_{\pi \in \widehat{G}} d_\pi \omega(\pi) \|\widehat{f}(\pi)\|_1 < \infty \Big\}.$$

There is a natural isometry between $A(G)$ and $A(G,\omega)$ (see [LS12] for the details), so that we can endow $A(G,\omega)$ with an operator space structure coming from $A(G)$ (as the predual of the group von Neumann algebra $\mathrm{VN}(G)$) through this isometry. Then from the condition (1.4) one can show that $A(G,\omega)$ is a completely contractive Banach algebra under pointwise multiplication ([LS12]).

Fundamental examples of weights on $\widehat{G}$ are given by the following polynomial dependence on dimensions of the representations. For $\alpha \geq 0$, we define $\omega_\alpha : \widehat{G} \to [1,\infty)$ by

$$\omega_\alpha(\pi) = d_\pi^\alpha \quad (\pi \in \widehat{G}).$$

Clearly ω_α satisfies the condition (1.4), and so it defines a weight on $\widehat{G}$; it is called the *dimension weight of order* α.

In [GLSS12, Theorem 4.11] it has been shown that $A(\mathrm{SU}(n),\omega_\alpha)$ is completely isomorphic to an operator algebra under the assumption that the estimate (1.3) for $\mathrm{SU}(n)$ holds true (this assumption was referred to as [GLSS12, Condition 1]). Since our main result says that the conjecture is indeed true for all $n \geq 2$, this implies the following.

THEOREM 1.3. *Let ω_α be the dimension weight of order $\alpha > d(\mathrm{SU}(n))/2 = (n^2-1)/2$ on $\widehat{\mathrm{SU}(n)}$, $n \geq 2$. Then $A(\mathrm{SU}(n),\omega_\alpha)$ is completely isomorphic to an operator algebra.*

Note that the above result is not true for $U(n)$, $n \geq 2$ (in general, for any compact connected non-simple Lie groups, see [GLSS12, Theorem 4.8]) even though the representations of $U(n)$, $n \geq 2$, satisfy the estimate (1.2).

1.6. Viewpoints of representation theory and random matrix theory. The main result of this paper is also of intrinsic interest in representation theory and also random matrix theory. According to it, the ‘widths’ of representations tell something about the relative dimensions of the Littlewood–Richardson components, namely any irreducible representation appearing in the tensor product cannot have relative dimension too large if the width of both tensored irreducible representations is large enough. This result was known for ‘typical’ irreducible representations (see e.g. [CŚ09]) but here we show that it holds true uniformly, at the expense of a worse, but asymptotically optimal estimate. Thus the difficulty of our main result lies in its uniformity.

Our estimate relies on a combinatorial lemma proved in Section 4 and it turns out that this lemma admits a direct counterpart in random matrix theory of independent interest. We state it as Lemma 3.4.

1.7. Organization of the paper. In Section 2 we recall some notations and facts from representation theory. In Section 3 we give an auxiliary result: a convenient description of the probability distribution of the first coordinate μ_1 of a random representation $\mu = (\mu_1, \mu_2, \dots)$ distributed according to the Littlewood–Richardson measure $P_{\lambda,\mu}$. Following this description, Section 4 gathers the properties of this probability distribution which are necessary in order to prove our main theorem. Section 5 contains the proof of the main theorem, and in Section 6 we explain the sense in which our result is optimal.

2. Representation theory of classical groups

2.1. Representations of $\mathrm{GL}(n)$ and weights. In this article $n \geq 1$ is a fixed integer. We say that λ is a *weight* if $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ is such that $\lambda_1 \geq \cdots \geq \lambda_n$. We denote by $\widehat{\mathrm{GL}(n)}$ the collection of irreducible representations of the linear group $\mathrm{GL}(n)$, up to equivalence. There is a canonical bijective correspondence between $\widehat{\mathrm{GL}(n)}$ and the set of weights which to a representation associates its *highest weight*. In order to simplify the notation we will identify an irreducible representation of $\mathrm{GL}(n)$ with the corresponding weight. We refer to [Ful97] for an extensive treatment of the subject. Throughout the whole paper, we work with the field of complex numbers $\mathbb{C}$. In particular, $\mathrm{GL}(n)$ means the linear group $\mathrm{GL}(n, \mathbb{C})$, and $\mathrm{SL}(n)$ means the special linear group $\mathrm{SL}(n, \mathbb{C})$.

2.2. Kronecker tensor product. If $\rho_1 : \mathrm{GL}(n) \to \operatorname{End} V_1$ and $\rho_2 : \mathrm{GL}(n) \to \operatorname{End} V_2$ are representations of the same group $\mathrm{GL}(n)$, we denote by $\rho_1 \otimes \rho_2 : \mathrm{GL}(n) \to \operatorname{End}(V_1 \otimes V_2)$ their *Kronecker tensor product* given by the diagonal action on simple tensors:

$$((\rho_1 \otimes \rho_2)(g))(v \otimes w) := \rho_1(g)(v) \otimes \rho_2(g)(w)$$

for $g \in \mathrm{GL}(n)$, $v \in V_1$, $w \in V_2$.

2.3. Representations of $\mathrm{SL}(n)$. Here we describe briefly the irreducible representations of the special linear group $\mathrm{SL}(n)$ of matrices of determinant one and their relation to the irreducible representations of $\mathrm{GL}(n)$. It is known (cf. [FH91, Section 15.5]) that any irreducible representation of $\mathrm{GL}(n)$, when restricted to $\mathrm{SL}(n)$, yields again an irreducible representation. Moreover, this map is surjective and its quotient can be precisely described as follows: two representations λ, μ of $\mathrm{GL}(n)$ yield the same representation when restricted to $\mathrm{SL}(n)$ if and only if there exists an integer k such that $\mu + k\mathbf{1} = \lambda$.

Unsurprisingly, the one-dimensional representation given by the determinant is trivial on $\mathrm{SL}(n)$ but non-trivial on $\mathrm{GL}(n)$. Its highest weight is equal to $\mathbf{1} = (1, \ldots, 1)$. The highest weight of the trivial representation is equal to $(0, \ldots, 0)$. As we have seen, they restrict to the same representation of $\mathrm{SL}(n)$.

Put differently, it is possible to parametrize the irreducible representations of $\mathrm{SL}(n)$ as those weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ for which the last component is equal to zero: $\lambda_n = 0$.

We are now ready to show Corollary 1.2, assuming that Theorem 1.1 holds true.

Proof that Theorem 1.1 implies Corollary 1.2. Let λ, μ be (as in Corollary 1.2) representations of $\mathrm{SL}(n)$. We view them as weights with the last components zero: $\lambda_n = 0$, $\mu_n = 0$. These weights give rise to representations of $\mathrm{GL}(n)$ which will be denoted by $\widetilde{\lambda}$, $\widetilde{\mu}$.

The tensor product $\lambda \otimes \mu$ of representations of $\mathrm{SL}(n)$ is nothing else but the restriction of the tensor product $\widetilde{\lambda} \otimes \widetilde{\mu}$ of representations of $\mathrm{GL}(n)$. Furthermore, the decomposition of $\widetilde{\lambda} \otimes \widetilde{\mu}$ into irreducible components gives rise (by restriction) to a decomposition of $\lambda \otimes \mu$ into irreducible components. Any weight $\widetilde{\nu}$ which contributes (with positive multiplicity) to the decomposition of $\widetilde{\lambda} \otimes \widetilde{\mu}$ satisfies

$$\widetilde{\nu}_1 + \cdots + \widetilde{\nu}_n = |\widetilde{\nu}| = |\widetilde{\lambda}| + |\widetilde{\mu}| = \widetilde{\lambda}_1 + \cdots + \widetilde{\lambda}_n + \widetilde{\mu}_1 + \cdots + \widetilde{\mu}_n,$$

which is a constant; thus any two non-equivalent irreducible components of $\widetilde{\lambda} \otimes \widetilde{\mu}$ remain non-equivalent when restricted to $\mathrm{SL}(n)$.

Thus if ν contributes to $\lambda \otimes \mu$ then there exists a unique weight $\widetilde{\nu} = \nu + k\mathbf{1}$ with the property that the isotypic component of $\lambda \otimes \mu$ of type $[\nu]$ corresponds (by restriction) to the isotypic component of type $[\widetilde{\nu}]$ of $\widetilde{\lambda} \otimes \widetilde{\mu}$.

We apply Theorem 1.1 for $\widetilde{\lambda}$, $\widetilde{\mu}$, ν'; equation (1.2) takes the form

$$\frac{c^{\nu}_{\lambda,\mu} d_{\nu}}{d_{\lambda} d_{\mu}} = \frac{c^{\widetilde{\nu}}_{\widetilde{\lambda},\widetilde{\mu}} d_{\widetilde{\nu}}}{d_{\widetilde{\lambda}} d_{\widetilde{\mu}}} \leq C_n \left(\frac{1}{\lambda_1 - \lambda_n} + \frac{1}{\mu_1 - \mu_n} \right).$$

This finishes the proof of Corollary 1.2. ∎

3. The Littlewood–Richardson measure and Gelfand–Tsetlin patterns. In Section 3.1 we recall the definition of *Gelfand–Tsetlin patterns*. As we shall see, patterns provide a concrete model for the Littlewood–Richardson measure (Lemma 3.3(a)). For the purposes of the current paper we do not need this kind of result in full generality; for this reason in Section 3.2 we will state Proposition 3.1 which concerns a simplified setup: the first coordinate of a random weight distributed according to the Littlewood–Richardson measure. This lemma is the key element of the proof of Lemma

4.1 in Section 4, which will be used in the proof of Theorem 1.1 (the main theorem). The remaining part of this section is devoted to the proof of Proposition 3.1.

3.1. Gelfand–Tsetlin patterns. Let λ be a weight. We say that

$$A = (a_l(i))_{l\in\{i,\dots,n-1\},\, i\in\{1,\dots,n-1\}} \in \mathbb{Z}^{n(n-1)/2}$$

is a *Gelfand–Tsetlin pattern* of shape λ (or, briefly, a *pattern*) if the following system of inequalities is satisfied:

$$\begin{array}{ccccccccccc}
a_1(1) & \le & a_2(1) & \le & \cdots & \le & a_{n-2}(1) & \le & a_{n-1}(1) & \le & \lambda_1 \\
\vee\!| & & \vee\!| & & & & \vee\!| & & \vee\!| & & \\
a_2(2) & \le & a_3(2) & \le & \cdots & \le & a_{n-1}(2) & \le & \lambda_2 & & \\
\vee\!| & & \vee\!| & & & & & & & & \\
\vdots & & \vdots & & & & & & & & \\
\vee\!| & & \vee\!| & & & & & & & & \\
a_{n-1}(n-1) & \le & \lambda_{n-1} & & & & & & & & \\
\vee\!| & & & & & & & & & & \\
\lambda_n & & & & & & & & & &
\end{array} \tag{3.1}$$

This system can be represented by an oriented graph $\mathcal{G}$ from Figure 1.

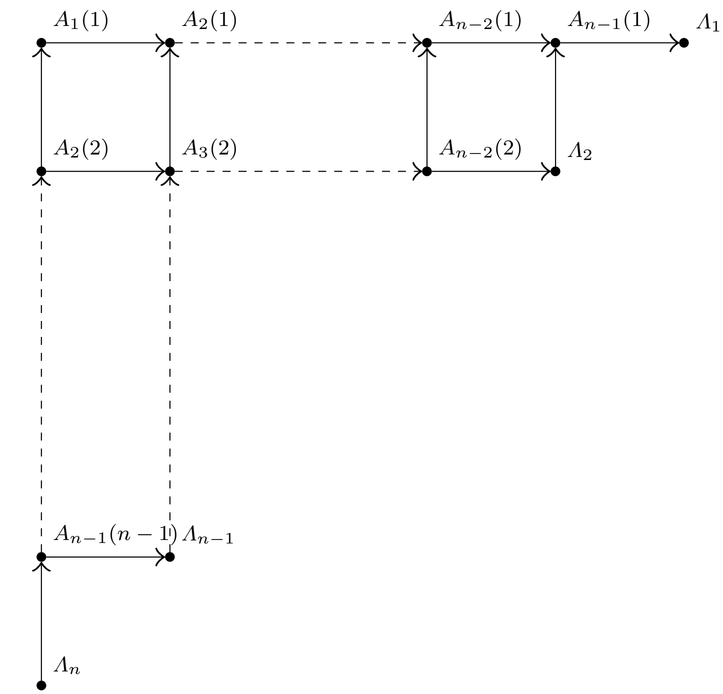

Fig. 1. The oriented graph $\mathcal{G}$ corresponding to (3.1)

The first row of (3.1) will deserve a special attention, and for this reason we will use the simplified notation

$$a_l := a_l(1) \quad \text{for } l \in \{1, \dots, n-1\}.$$

It will also be convenient to define $a_n := \lambda_1$. Analogously, if $B = (b_l(i))$ is a pattern of shape μ we denote

$$b_l := b_l(1) \quad \text{for } l \in \{1, \dots, n-1\}, \qquad b_n := \mu_1.$$

3.2. Concrete realization of the Littlewood–Richardson measure. The following proposition is the key component in the proof of Proposition 4.1. It gives a concrete realization of the first coordinate of a random weight distributed according to the Littlewood–Richardson measure.

PROPOSITION 3.1. *Let λ, μ be weights. Let $A = (a_l(i))$ be a random pattern of shape λ (sampled with the uniform distribution) and let $B = (b_l(i))$ be a random pattern of shape μ (also sampled with the uniform distribution); we assume that A and B are independent.*

Let $\nu = (\nu_1, \dots, \nu_n)$ be a random weight distributed according to the Littlewood–Richardson measure $P_{\lambda,\mu}$; then

$$\nu_1 \overset{\text{dist}}{=} \max_{\substack{k,l \geq 1 \\ k+l=n+1}} (a_k + b_l), \tag{3.2}$$

where $\overset{\text{dist}}{=}$ denotes the equality of distributions of random variables.

We postpone its proof until Section 3.6. The remaining part of the current section is devoted to the preparation to this proof.

3.3. Polynomial representations. *Polynomial* irreducible representations of $\mathrm{GL}(n)$ play a special role. Such a polynomial representation corresponds to a weight $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ are *non-negative* integers. A weight with this property is called a *Young diagram* and can be represented graphically as in Figure 2 (we use the English notation for drawing Young diagrams). Polynomial representations are asso-

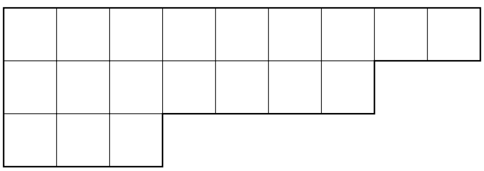

Fig. 2. Young diagram $(9, 7, 3)$

ciated to very rich combinatorial structures related to Young diagrams and *Young tableaux* which we will explore in Section 3.4.

Many problems concerning irreducible representations can be reduced to the special case of irreducible *polynomial* representations. This is also the case for Proposition 3.1; the following lemma gives the details of this reduction.

LEMMA 3.2. *Assume that Proposition* 3.1 *is true under the additional assumption that the weights* λ, μ *are Young diagrams. Then Proposition* 3.1 *is true in general, without that assumption.*

Proof. For $p \in \mathbb{Z}$ we denote by $\mathrm{Det}^p : \mathrm{GL}(n) \to \mathrm{End}(\mathbb{C})$ the one-dimensional representation given by an appropriate power of the determinant:

$$\mathrm{Det}^p(g) := (\det(g))^p \quad \text{for } g \in \mathrm{GL}(n),$$

where the right-hand side should be interpreted as a 1×1 matrix, thus as an endomorphism of the one-dimensional vector space $\mathbb{C} = \mathbb{C}^1$. The representation Det^p is irreducible and corresponds to the highest weight

$$p\mathbf{1} := (p, \dots, p) \in \mathbb{Z}^n.$$

The Kronecker tensor product $\lambda \otimes \mathrm{Det}^p$ of an irreducible representation $\lambda = (\lambda_1, \dots, \lambda_n)$ with Det^p is again an irreducible representation which corresponds to the shifted weight

$$\lambda + p\mathbf{1} := (\lambda_1 + p, \dots, \lambda_n + p).$$

The dimensions of irreducible representations, Littlewood–Richardson coefficients and the Littlewood–Richardson measure are invariant under such shifts:

$$\begin{aligned} d_{\lambda+p\mathbf{1}} &= d_\lambda, \\ c_{\lambda+p\mathbf{1},\mu+q\mathbf{1}}^{\nu+(p+q)\mathbf{1}} &= c_{\lambda,\mu}^{\nu}, \\ P_{\lambda+p\mathbf{1},\mu+q\mathbf{1}}(\nu + (p+q)\mathbf{1}) &= P_{\lambda,\mu}(\nu), \end{aligned}$$

for arbitrary $p, q \in \mathbb{Z}$ and irreducible representations λ, μ, ν of $\mathrm{GL}(n)$.

We use the notations of Proposition 3.1. We denote $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n)$ and set $p := -\lambda_n$ and $q := -\mu_n$ so that the weights $\lambda' := \lambda + p\mathbf{1}$ and $\mu' := \mu + q\mathbf{1}$ are *Young diagrams*. We also set $\nu' = (\nu'_1, \dots, \nu'_n) := (p+q)\mathbf{1} + \nu$. Clearly, since ν is distributed according to the Littlewood–Richardson measure $P_{\lambda,\mu}$ it follows that ν' is distributed according to the Littlewood–Richardson measure $P_{\lambda',\mu'}$.

We define shifted patterns $A' = (a_l(i) + p)$ and $B' = (b_l(i) + q)$. Clearly A' and B' are random patterns of shape λ' and μ' respectively.

We apply Proposition 3.1 to the Young diagrams λ', μ', random weight ν' and random patterns A', B'. It follows that

$$\nu_1 + (p+q) = \nu'_1 \overset{\text{dist}}{=} \max_{\substack{k,l \geq 1 \\ k+l=n+1}} (a'_k + b'_l) = \max_{\substack{k,l \geq 1 \\ k+l=n+1}} ((a_k + p) + (b_l + q)),$$

which shows that Proposition 3.1 holds true for the weights λ and μ as desired. ■

3.4. Young tableaux, Robinson–Schensted–Knuth correspondence and the plactic monoid. We recall some basic notations related to Young tableaux, Robinson–Schensted–Knuth correspondence and the plactic monoid. A good treatment of these topics is given in Part I of the book [Ful97].

3.4.1. *Tableaux.* A *semi-standard tableau* (or, briefly, a *tableau*) T is a filling of the boxes of a given Young diagram λ with letters from the alphabet $\{1, \dots, n\}$ so that the filling is weakly increasing along each row, and strictly increasing down a column (see Figure 3). The value of n will be fixed so we

1	1	1	1	1	2	2	2	3
2	2	2	2	3	3	3		
3	3	3						

Fig. 3. Example of a tableau T in the alphabet $\{1, 2, 3\}$ filling the Young diagram $(9, 7, 3)$ from Figure 2. The corresponding word is given by $w(T) = (3, 3, 3, 2, 2, 2, 2, 3, 3, 3, 1, 1, 1, 1, 1, 2, 2, 2, 3)$.

do not have to specify it for each tableau separately. We also say that the *Young diagram* λ *is the shape of the tableau* T.

For a given tableau T we let $a_l(i)$ be the number of boxes in the ith row of T filled with numbers $\leq l$. It is easy to check that $A = (a_l(i))$ so defined is a pattern; furthermore for any Young diagram λ this gives a bijective correspondence between tableaux of shape λ and patterns of shape λ. In the following we will identify a tableau with the corresponding pattern.

3.4.2. *Words.* A *word* $w = (w_1, \dots, w_\ell)$ is a sequence of elements of the alphabet $\{1, \dots, n\}$. We recall that the *insertion tableau* $P(w)$ of w is defined as the semi-standard tableau obtained by the *Schensted row insertion algorithm* applied iteratively to the letters $w_1, \dots, w_\ell$. For a given tableau T we denote by $w(T)$ the word obtained by reading the entries of T along the lines, from left to right and from bottom to top (see Figure 3). This word has the property that $T = P(w(T))$.

For a word $w = (w_1, \dots, w_\ell)$ we denote by $\mathrm{LI}(w)$ the *length of the longest* (*weakly*) *increasing subsequence* of w, i.e. the length of the longest sequence $i_1 < \dots < i_k \in \{1, \dots, \ell\}$ such that

$$w_{i_1} \leq \dots \leq w_{i_k}.$$

It is well-known that if $\lambda = (\lambda_1, \dots, \lambda_n)$ is the shape of the insertion tableau $P(w)$ then $\mathrm{LI}(w) = \lambda_1$ is equal to the length of the first row of λ.

3.4.3. *Multiplication of tableaux, the plactic monoid and the plactic Littlewood–Richardson rule.* We consider the *free monoid* over the alphabet $\{1, \dots, n\}$, which is just the set of words equipped with multiplication $\cdot$ given by concatenation of words. Let us identify two words w and w' (we write $w \equiv w'$) if and only if the corresponding insertion tableaux are equal: $P(w) = P(w')$. One can show that $w \equiv w'$ and $v \equiv v'$ implies that $w \cdot v \equiv w' \cdot v'$, thus multiplication $\cdot$ is well defined on the equivalence classes of $\equiv$. The set of such equivalence classes of $\equiv$ equipped with multiplication $\cdot$ is called the *plactic monoid.*

The map P gives a bijection between the elements of the plactic monoid and tableaux; thus multiplication in the plactic monoid can be used to define *multiplication of tableaux* which will be denoted by the same symbol $\cdot$. Alternatively, the product $S \cdot T := P(w(S) \cdot w(T))$ of tableaux S and T is defined as the insertion tableau corresponding to concatenation of the words corresponding to the original tableaux.

Recall that the *plactic Schur polynomial* is defined as a formal sum

$$S_\lambda := \sum_T T$$

of all tableaux with shape λ. *The plactic Littlewood–Richardson rule* says that

$$S_\lambda \cdot S_\mu = \sum_\nu c^\nu_{\lambda,\mu} S_\nu, \tag{3.3}$$

where $c^\nu_{\lambda,\mu}$ are the usual Littlewood–Richardson coefficients.

3.4.4. *Involution on tableaux.* Let us consider the antiautomorphism α of the free monoid defined on the generators by $\alpha(i) := n + 1 - i$. Alternatively, α is an involution on words defined by reading the word backwards and by reversing the order in the alphabet. The plactic monoid can be equivalently described as the free monoid divided by the *plactic relations* (*Knuth relations*) satisfied by generators $x, y, z \in \{1, \dots, n\}$:

$$\begin{aligned} y \cdot z \cdot x &= y \cdot x \cdot z \quad \text{if } x < y \le z, \\ x \cdot z \cdot y &= z \cdot x \cdot y \quad \text{if } x \le y < z. \end{aligned}$$

Since α preserves these plactic relations, it gives rise to an antiautomorphism of the plactic monoid.

If we identify the elements of the plactic monoid with tableaux, the antiautomorphism α becomes an involution on the set of tableaux. It can be described explicitly as follows: for a given tableau T we replace each entry i by $\alpha(i) = n + 1 - i$ and we rotate the tableau by angle π, thus obtaining a

						1	1	1
		1	1	1	2	2	2	2
1	2	2	2	3	3	3	3	3

Fig. 4. Skew tableau obtained from the tableau from Figure 3 after rotating by angle π and replacing each entry i by $\alpha(i) = n + 1 - i$.

skew tableau (see Figure 4). After rectifying it (by an application of Schützerberger's *jeu de taquin*), we obtain $\alpha(T)$. Alternatively, $\alpha(T) = P(\alpha(w(T))$. Greene's theorem shows that the involution α maps the set of tableaux of a given shape into itself.

3.5. A concrete model for the Littlewood–Richardson measure. The following lemma is a simple reformulation of well-known combinatorics of representation theory in the language of probability theory. It is the key ingredient for the proof of Proposition 3.1.

LEMMA 3.3. *Let λ, μ be Young diagrams. Let S be a random Young tableau of shape λ and let T be a random Young tableau of shape μ. Assume that S and T are sampled according to the uniform distribution given by their respective shape constraints, and that they are independent. Then*

(a) *the distribution of the shape of the product $S \cdot T$ coincides with the Littlewood–Richardson measure $P_{\lambda,\mu}$;*
(b) *let $\nu = (\nu_1, \dots, \nu_n)$ be a random Young diagram distributed according to the Littlewood–Richardson measure $P_{\lambda,\mu}$; then*

$$\nu_1 \overset{\text{dist}}{=} \max_{\substack{k,l \geq 1 \\ k+l=n+1}} \left(a_k(S) + a_l(T)\right),$$

where $\overset{\text{dist}}{=}$ denotes the equality of distributions of random variables.

Proof. We will identify a probability measure on the set of tableaux with the appropriate formal linear combination of tableaux with coefficients given by appropriate probabilities. The dimension d_λ is equal to the number of tableaux of the shape λ, therefore the normalized plactic Schur polynomial $\frac{1}{d_\lambda} S_\lambda$ can be identified with the uniform probability measure on the set of tableaux of shape λ.

The plactic Littlewood–Richardson rule (3.3) can be equivalently written in the form

$$\left(\frac{1}{d_\lambda} S_\lambda\right) \cdot \left(\frac{1}{d_\mu} S_\mu\right) = \sum_\nu \left(\frac{d_\nu c^\nu_{\lambda,\mu}}{d_\lambda d_\mu}\right) \left(\frac{1}{d_\nu} S_\nu\right).$$

The left-hand side corresponds to the distribution of the random tableau

$S \cdot T$. The right-hand side corresponds to the distribution of the random tableau filling a random Young diagram with distribution $P_{\lambda,\mu}$. By comparing the distribution of the shape of the Young tableaux contributing to both sides of the equality we deduce the first part of the lemma.

Let $\nu = (\nu_1, \ldots, \nu_n)$ be the shape of the tableau $S \cdot T$; from the first part of this lemma it follows that the distribution of ν is given by the Littlewood–Richardson measure $P_{\lambda,\mu}$. Clearly, the length of the first row of ν satisfies

$$\nu_1 = \mathrm{LI}(w(S) \cdot w(T));$$

it follows that

$$\nu_1 = \max_k [\mathrm{LI}(w(S)|_{\{1,\ldots,k\}}) + \mathrm{LI}(w(T)|_{\{k,\ldots,n\}})], \tag{3.4}$$

where $w|_A$ denotes the word w with all letters which do not belong to A omitted. In the following we will analyze the two summands contributing to the right-hand side of (3.4). We start with the first one.

We consider the tableau $S|_{\{1,\ldots,k\}}$ obtained by removing from S all boxes with entries greater than k. Clearly,

$$w(S)|_{\{1,\ldots,k\}} = w(S|_{\{1,\ldots,k\}}).$$

In particular,

$$\mathrm{LI}(w(S)|_{\{1,\ldots,k\}}) = a_k(S) \tag{3.5}$$

is the length of the first row of the tableau $S|_{\{1,\ldots,k\}}$.

We turn now to the second summand on the right-hand side of (3.4). Clearly, for any word w,

$$w|_{\{k,\ldots,n\}} = \alpha(\alpha(w)|_{\{1,\ldots,n+1-k\}}) \quad \text{and} \quad \mathrm{LI}\, w = \mathrm{LI}\, \alpha(w),$$

thus

$$\mathrm{LI}(w|_{\{k,\ldots,n\}}) = \mathrm{LI}(\alpha(w)|_{\{1,\ldots,n+1-k\}}).$$

We define $T' = \alpha(T)$; thus the random tableaux T' and T have the same distribution. We have

$$\mathrm{LI}(w(T)|_{\{k,\ldots,n\}}) = \mathrm{LI}(w(T')|_{\{1,\ldots,n+1-k\}}) = a_{n+1-k}(T'). \tag{3.6}$$

Equations (3.4)–(3.6) finish the proof. ∎

3.6. Proof of Proposition 3.1

Proof of Proposition 3.1. In Lemma 3.2 we showed that it is enough to prove the result under the additional assumption that λ and μ are Young diagrams. We use Lemma 3.3(b) and the fact that there is a bijective correspondence between tableaux and patterns. ∎

3.7. An application to random matrix theory. Below, we state an interesting corollary of Proposition 3.1. This corollary is of purely random matrix nature, but to the best of our knowledge it seems to be new.

COROLLARY 3.4. *Let A, B be independent Hermitian random matrices of the same size $n \times n$. Assume that both the distribution of A and the distribution of B are invariant under unitary conjugation. Then the largest eigenvalue of $A+B$ is a random variable which has the same distribution as*

$$\max_{\substack{k,l\geq 1\\ k+l=n+1}} (a_k + b_l),$$

where a_k (resp. b_k) is the random variable obtained by taking the largest eigenvalue of the upper left $k \times k$ corner of A (resp. B).

We will just sketch the main ideas of the proof and leave the details to the reader.

Sketch of the proof. Without loss of generality we can assume that the eigenvalues of A, B are prescribed. Indeed, if they are random, the proof can be completed by conditioning over the prescribed eigenvalues and a decomposition of measure type argument.

And if the eigenvalues of A, B are prescribed, the result follows from Proposition 3.1 and successive applications of [CŚ09]. Indeed, in [CŚ09, Corollary 5.2] it is proved that if A is a unitarily invariant selfadjoint random matrix and $\lambda^N = (\lambda_1^N \geq \cdots \geq \lambda_n^N)$ is a tuple of sequences of integers such that λ_i^N/N converges to the ith largest eigenvalue of A, then the law of $(a_1, \ldots, a_n)$ is the limit of the laws of $(a_1^N/N, \ldots, a_n^N/N)$ as appearing in Proposition 3.1 and corresponding to weight λ^N. A similar statement holds for a random matrix B and $\mu^N = (\mu_1^N \geq \cdots \geq \mu_n^N)$. It has also been shown in [CŚ09] that the law of the largest eigenvalue of $A+B$ is the limit of the laws of ν_1^N/N, where ν^N is distributed according to the Littlewood–Richardson measure P_{λ^N,ν^N}. We apply Proposition 3.1 to λ^N, μ^N and ν^N and pass to the limit. ■

4. The first row of a random pattern. The main result of this section is the following lemma giving an upper bound on the atoms of the distribution of the first row of a random pattern with a given shape. This proposition is the key to the proof of Theorem 1.1.

PROPOSITION 4.1. *There exists some constant D_n with the following property. Let λ be a weight and let $A = (a_l(i))$ be a random pattern with shape λ. Then for any $x \in \mathbb{Z}$ and $1 \leq k \leq n-1$,*

$$P(a_k = x) \leq D_n \, \frac{1}{\lambda_1 - \lambda_{n+1-k}}.$$

We postpone the proof of Proposition 4.1 until Section 4.3; the remaining part of the current section is a preparation for that proof.

4.1. Taking degeneracy into account. Let the weight λ be fixed. The inequalities (3.1) define a convex polyhedron in the space $\mathbb{R}^{n(n-1)/2}$. For some choices of the weight λ it might happen that this polyhedron is of dimension smaller than the maximal dimension $n(n-1)/2$. This creates some difficulties; in the following, we explain how to avoid them.

Restricting the system of inequalities (3.1) to one row and one column implies that

$$\begin{array}{ccccc} a_l(i) & \leq & \cdots & \leq & \lambda_i \\ \rotatebox{90}{$\leq$} & & & & \\ \vdots & & & & \\ \rotatebox{90}{$\leq$} & & & & \\ \lambda_{n+i-l} & & & & \end{array}$$

In other words, if $\lambda_{n+i-l} = \lambda_i$ then automatically $a_l(i) = \lambda_i$. Such variables are trivial from our viewpoint, thus it is enough to restrict our attention to the index set

$$\mathcal{I} = \{(l,i) : l \in \{i, \dots, n-1\},\ i \in \{1, \dots, n-1\},\ \lambda_{n+i-l} < \lambda_i\}$$

and to study only the variables $(a_l(i) : (l,i) \in \mathcal{I})$. We define $d = |\mathcal{I}|$. The set of solutions to the above system (3.1) of inequalities in integer numbers (respectively, real numbers) will be denoted by $\mathcal{D} \subset \mathbb{Z}^d$ (respectively, by $\mathcal{C} \subset \mathbb{R}^d$). Thus there is a natural bijective correspondence between patterns of shape λ and the elements of $\mathcal{D}$.

We denote by $\widehat{\mathcal{G}}$ the oriented graph $\mathcal{G}$ in which:

- every vertex $A_l(i)$ with $(l,i) \notin \mathcal{I}$ is glued to the vertex Λ_i,
- all pairs of vertices Λ_i and Λ_j are glued together if $\lambda_i = \lambda_j$.

The graph $\widehat{\mathcal{G}}$ encodes all inequalities satisfied by the variables $(a_l(i) : (l,i) \in \mathcal{I})$. The following lemma is elementary.

Lemma 4.2. *The graph $\widehat{\mathcal{G}}$ is acyclic.*

4.2. Continuous versus discrete. Our goal is to understand the uniform measure on $\mathcal{D}$. There is also a simpler object: the uniform measure on $\mathcal{C}$. In the following we investigate how these two measures are related to each other. The following lemma addresses the question of how intersections of $b + I$ with $\mathcal{D}$ and $\mathcal{C}$ are related to each other, where the unit cube I is defined as

$$I = \{(a_l(i)) : |a_l(i)| < 1/2\} \subset \mathbb{R}^d.$$

LEMMA 4.3. *There is some constant $C > 0$ (which depends only on n) with the property that for any weight λ and any lattice point $b \in \mathbb{Z}^d$,*

$$b \in \mathcal{D} \;\Leftrightarrow\; (b+I) \cap \mathcal{D} \neq \emptyset \;\Leftrightarrow\; \mathrm{vol}[(b+I) \cap \mathcal{C}] \geq C \;\Leftrightarrow\; (b+I) \cap \mathcal{C} \neq \emptyset.$$

Proof. Since the lattice point b is the only element of $(b+I) \cap \mathbb{Z}^d$, if $(b+I) \cap \mathcal{D}$ is non-empty then it is equal to $\{b\}$. This explains why the first two conditions are equivalent.

Now we suppose that $b \in \mathcal{D}$. For $m \in \mathbb{Z}$ we denote

$$\mathcal{I}_m = \{(l,i) \in \mathcal{I} : b_l(i) = m\}$$

and we denote by $\mathcal{C}_m \subset \mathbb{R}^{|\mathcal{I}_m|}$ the set of solutions of the system of inequalities (3.1) over variables $a_l(i)$ such that $(l,i) \in \mathcal{I}_m$, subject to the additional requirement that

$$|a_l(i) - m| < 1/2.$$

Since $(b+I) \cap \mathcal{C} = \prod_m \mathcal{C}_m$ (where, on the right hand side of the equality, with the obvious identification of the coordinates, multiplication denotes Cartesian product), it is enough to show that if $\mathcal{I}_m \neq \emptyset$, then $\mathrm{vol}\, \mathcal{C}_m$ is greater than some universal positive constant.

We denote by $\widehat{\mathcal{G}}_m$ the graph $\widehat{\mathcal{G}}$ restricted to the following vertices:

- vertices $A_l(i)$ with $(l,i) \in \mathcal{I}_m$,
- vertices Λ_i with $\lambda_i = m$ (in fact, all such vertices from $\mathcal{G}$ are glued together so they correspond to a single vertex in $\widehat{\mathcal{G}}$).

The graph $\widehat{\mathcal{G}}_m$ encodes all inequalities satisfied by the collection of variables $(a_l(i))$ over (l,i) such that $|a_l(i) - m| < 1/2$.

Since $\widehat{\mathcal{G}}_m$ is acyclic, it is possible to extend it to a linearly ordered set. Let us choose any such linear extension. There are the following two cases:

- The graph $\widehat{\mathcal{G}}_m$ does not contain any vertex Λ_ℓ; then the set of solutions which are compatible with the selected linear order is a simplex with volume
$$\frac{1}{|\mathcal{I}_m|!}.$$
- The graph $\widehat{\mathcal{G}}_m$ contains a vertex Λ_ℓ; let us say that there are p (respectively, q) vertices $A_l(i)$ which are smaller (respectively, greater) than Λ_ℓ with $p + q = |\mathcal{I}_m|$; then the set of solutions which are compatible with the selected linear order is a product of two simplexes with volume
$$\frac{1}{2^{p+q} p! q!}.$$

Note that the simplex obtained by choosing a linear order has a smaller volume than $\mathcal{C}_m$, so that the above cases give us a lower bound. Now this finishes the proof that the first condition implies the third one.

The third condition trivially implies the fourth condition.

Assume that $(b+I)\cap\mathcal{C}\neq\emptyset$. Let a be any element of this set. The system of inequalities (3.1) has a particularly nice form: if a is a solution then also $\operatorname{round}(a)$ is a solution, where 'round' denotes the (coordinatewise) rounding of a real number to the closest integer. On the other hand $\operatorname{round}(a)=b$, therefore $b\in\mathcal{D}$, which finishes the proof that the fourth condition implies the first. ∎

4.3. Proof of Proposition 4.1

Proof of Proposition 4.1. For $x\in\mathbb{Z}$ (resp. $x\in\mathbb{R}$) we denote by $\mathcal{D}^x\subset\mathbb{Z}^{d-1}$ (resp. $\mathcal{C}^x\subset\mathbb{R}^{d-1}$) the set of integer (resp. real) solutions of the system of inequalities (3.1) over variables $a_l(i)$, $(l,i)\in\mathcal{I}$, $(l,i)\neq(k,1)$, subject to the additional requirement that $a_k(1)=x$.

For subsets of $\mathbb{R}^{d-1}$ we denote by vol_{d-1} the usual Lebesgue volume, while for subsets of $\mathbb{Z}^{d-1}$ we denote by vol the counting measure.

Now we fix $x\in\mathbb{Z}$. Lemma 4.3 implies that

$$\int\limits_{|x-y|<1/2}\operatorname{vol}_{d-1}\mathcal{C}^y\,dy=\sum_{b\in\mathcal{D}^x}\operatorname{vol}_d[(b+I)\cap\mathcal{C}]\geq C\operatorname{vol}\mathcal{D}^x.$$

It follows that there exists some y such that

$$\operatorname{vol}_{d-1}\mathcal{C}^y\geq C\operatorname{vol}\mathcal{D}^x. \tag{4.1}$$

It is a simple exercise to check that for $x_0\in\{\lambda_1,\lambda_{n+1-k}\}$ the set $\mathcal{C}^{x_0}$ is non-empty. Let us select the value of x_0 for which

$$|x_0-y|\geq\frac{\lambda_1-\lambda_{n+1-k}}{2}$$

and let us fix some $a\in\mathcal{C}^{x_0}$.

Under the obvious identifications $a\in\mathcal{C}^{x_0}\subset\mathcal{C}\subset\mathbb{R}^d$ and $\mathcal{C}^y\subset\mathcal{C}\subset\mathbb{R}^d$ we can consider the convex cone having vertex a and base $\mathcal{C}^y$. Clearly, $\mathcal{C}$ as a convex set contains this cone. It follows that for $t=(1-\alpha)x_0+\alpha y$ with $0<\alpha<1$ we have

$$\operatorname{vol}_{d-1}\mathcal{C}^t\geq\alpha^{d-1}\operatorname{vol}_{d-1}\mathcal{C}^y,$$

hence

$$\operatorname{vol}_d\mathcal{C}=\int\limits_{\mathbb{R}}\operatorname{vol}_{d-1}\mathcal{C}^z\,dz\geq\frac{\lambda_1-\lambda_{n+1-k}}{2d}\operatorname{vol}_{d-1}\mathcal{C}^y. \tag{4.2}$$

Lemma 4.3 shows that

$$\operatorname{vol}\mathcal{D} \geq \operatorname{vol}_d \mathcal{C}. \tag{4.3}$$

Inequalities (4.1)–(4.3) imply that

$$P(a_k(S) = x) = \frac{\operatorname{vol}\mathcal{D}^x}{\operatorname{vol}\mathcal{D}} \leq \frac{\text{const.}}{\lambda_1 - \lambda_{n+1-k}}. \ \blacksquare$$

5. Proof of the main result

Proof of Theorem 1.1. For a weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ we denote by $\bar{\lambda} = (-\lambda_n, \ldots, -\lambda_1)$ the weight corresponding to the contragredient representation. Since $d_\lambda = d_{\bar{\lambda}}$ and $c^{\nu}_{\lambda,\mu} = c^{\bar{\nu}}_{\bar{\lambda},\bar{\mu}}$, the inequality (1.2) holds for λ, μ, ν if and only if it holds for $\bar{\lambda}, \bar{\mu}, \bar{\nu}$.

Let λ, μ be fixed. By the pigeon-hole principle, there exist $i, j \in \{1, \ldots, n-1\}$ such that

$$\lambda_i - \lambda_{i+1} \geq \frac{\lambda_1 - \lambda_n}{n-1}, \qquad \mu_j - \mu_{j+1} \geq \frac{\mu_1 - \mu_n}{n-1}.$$

For $i' = n - i$ and $j' = n - j$ we have analogous inequalities

$$\bar{\lambda}_{i'} - \bar{\lambda}_{i'+1} \geq \frac{\bar{\lambda}_1 - \bar{\lambda}_n}{n-1}, \qquad \bar{\mu}_{j'} - \bar{\mu}_{j'+1} \geq \frac{\bar{\mu}_1 - \bar{\mu}_n}{n-1}.$$

Since $(i+j)+(i'+j') = 2n$, at least one of the following is true: $i+j \leq n$ or $i' + j' \leq n$. Without loss of generality we will assume that $i + j \leq n$; if this is not the case, simply replace λ, μ, ν by $\bar{\lambda}, \bar{\mu}, \bar{\nu}$.

Let A and B be as in Proposition 3.1. Equation (3.2) implies that

$$P(\nu_1 = x) \leq \sum_{\substack{k,l \geq 1 \\ k+l=n+1}} P(a_k + b_l = x),$$

thus it is enough to find appropriate bounds for the distribution of the sum $a_k + b_l$ for each choice of k and l separately. The latter distribution is a convolution of two probability measures, thus

$$P(a_k + b_l = x) \leq \min\Big(\max_z P(a_k = z), \max_z P(b_l = z)\Big)$$

and it is enough to show that there is such a bound for a_k or for b_l. Clearly,

$$n + 1 - k \geq i + 1 \ \vee \ n + 1 - l \geq j + 1$$

(otherwise $n + 1 = 2n + 2 - (k + l) \leq i + j$ would contradict $i + j \leq n$). We will investigate these two cases separately.

In the first case,

$$\lambda_1 - \lambda_{n+1-k} \geq \lambda_i - \lambda_{i+1} \geq \frac{\lambda_1 - \lambda_n}{n-1}.$$

We apply Lemma 4.1 to obtain

$$P(a_k = z) \le D_n \frac{1}{\lambda_1 - \lambda_{n+1-k}} \le D_n \frac{n-1}{\lambda_1 - \lambda_n}.$$

In the second case,

$$\mu_1 - \mu_{n+1-l} \ge \mu_j - \mu_{j+1} \ge \frac{\mu_1 - \mu_n}{n-1}.$$

We apply Lemma 4.1 for the diagram $\lambda' := \mu$ and $k' = l$ to get

$$P(b_l = z) \le D_n \frac{1}{\mu_1 - \mu_{n+1-l}} \le D_n \frac{n-1}{\mu_1 - \mu_n}.$$

This completes the proof. ■

6. Saturation of the bound. Here we show that our bound is saturated in some natural sense.

PROPOSITION 6.1. *For each n, there exist two sequences (λ_N), (μ_N) of irreducible representations of* $\mathrm{GL}(n)$ (*resp.* $\mathrm{SL}(n)$) *which tend to infinity with the property that the inequality* (1.2) *of Theorem* 1.1 (*resp. inequality* (1.3) *of Corollary* 1.2) *is saturated up to a multiplicative constant that depends only on n and not on N.*

Proof. Take $\lambda = (N, 0, \dots, 0)$ and $\mu = (M, 0, \dots, 0)$. Then it is clear from the Littlewood–Richardson rule that all the ν for which there is a non-zero probability $P_{\lambda,\mu}$ are of the form

$$(A, B, 0, \dots, 0)$$

with the constraints that $A \ge B \ge 0$, $A + B = N + M$, $A \ge \max(N, M)$. There are $\min(N, M)$ choices. By the pigeon-hole principle, at least one of these weights has probability at least

$$\frac{1}{\min(N, M)},$$

which is comparable to the bound obtained in our Corollary 1.2, and therefore also saturates the bound for the main Theorem 1.1. Note that it follows from the proof that the Littlewood–Richardson coefficients appearing in this proof cannot be large. As a matter of fact, one can prove that they are all equal to 1 in this case (but we do not need it in order to complete the proof). ■

The above proposition shows that, for example, if we wanted, for a given N, the inequality

$$P_{\lambda,\mu}(\nu) \le C_n \left(\frac{1}{\lambda_1 - \lambda_n} + \frac{1}{\mu_1 - \mu_n} \right)^{\alpha}$$

to be true for all μ, ν, then necessarily $\alpha \leq 1$, and actually $\alpha = 1$ is the best possible constant.

Note that if the quantifier of Theorem 1.1 is not over *all* choices of μ, ν but just over some nice (possibly infinite) sets of pairs, then it is possible to obtain much better estimates.

As a first example, if in GL(3), one takes the collection $\mu_N = \nu_N = (2N, N, 0)$, it is easy to see that the largest dimension of a Littlewood–Richardson factor that can occur in $\mu_n \otimes \nu_n$ is at most of order N^3, which is less than N^6. However if one in addition allows Littlewood–Richardson coefficients, then one obtains N^5. Here we still saturate Theorem 1.1 but not Corollary 1.2 any more.

As a second example, if one takes in GL(4) the sequence $\mu_N = \nu_N = (3N, 2N, N, 0)$, one can see that the largest dimension of a Littlewood–Richardson summand that can occur in $\mu_n \otimes \nu_n$ is at most of order N^6, which is less than N^{12}. And if one in addition allows Littlewood–Richardson coefficients, then one obtains N^9. Here, we are away from saturation both for Theorem 1.1 and for Corollary 1.2.

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