

Gaussian estimates for Schrödinger perturbations

by

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Abstract. We propose a new general method of estimating Schrödinger perturbations of transition densities using an auxiliary transition density as a majorant of the perturbation series. We present applications to Gaussian bounds by proving an optimal inequality involving four Gaussian kernels, which we call the 4G Theorem. The applications come with honest control of constants in estimates of Schrödinger perturbations of Gaussian-type heat kernels and also allow for specific non-Kato perturbations.

1. Introduction and main results. A Schrödinger perturbation is an addition of an operator of multiplication to a given operator. On the level of inverse operators, the addition results in a resolvent or Duhamel's or a perturbation formula, and under certain conditions it yields von Neumann or perturbation series for the inverse of the perturbation. The subject is very wide, and we intend to touch it in the case when the inverse operator is an evolution semigroup, in fact, a transition density. In this case a convenient and simple setup is that of integral operators on space-time, and the perturbation series has an exponential flavor due to repeated integrations on time simplexes. In this work we propose a general method for pointwise estimates of the series, and we demonstrate its versatility by estimating transition densities of Schrödinger perturbations of heat kernels on \mathbb{R}^d . In an earlier work, Bogdan, Jakubowski and Sydor [6] developed a technique for *sharp* pointwise estimates of Schrödinger perturbations \tilde{p} of transition densities p and more general integral kernels on a state space X by functions $q \geq 0$. The method rests on the assumption

$$(1.1) \quad \int_s^t \int_X p(s, x, u, z) q(u, z) p(u, z, t, y) dz du \leq [\eta + Q(s, t)] p(s, x, t, y),$$

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where $s < t$, $x, y \in X$, $0 \leq \eta < \infty$, and $0 \leq Q(s, u) + Q(u, t) \leq Q(s, t)$ and dz is a measure on X . The left-hand side of (1.1) defines the term $p_1(s, x, t, y)$ in the perturbation series,

$$(1.2) \quad \tilde{p}(s, x, t, y) = \sum_{n=0}^{\infty} p_n(s, x, t, y)$$

(see Section 2 for detailed definitions), and so $p_1(s, x, t, y)/p(s, x, t, y) \leq \eta + Q(s, t)$. The bound is uniform in space and locally uniform in time, and it propagates as follows:

$$(1.3) \quad \begin{aligned} p_n(s, x, t, y) &\leq p_{n-1}(s, x, t, y)[\eta + Q(s, t)/n] \\ &\leq p(s, x, t, y) \prod_{k=1}^n [\eta + Q(s, t)/k]. \end{aligned}$$

Furthermore, if $0 < \eta < 1$, then (1.2) and (1.3) yield

$$(1.4) \quad \tilde{p}(s, x, t, y) \leq p(s, x, t, y) \left(\frac{1}{1 - \eta} \right)^{1+Q(s,t)/\eta},$$

and if $\eta = 0$, then

$$(1.5) \quad \tilde{p}(s, x, t, y) \leq p(s, x, t, y)e^{Q(s,t)}.$$

The above estimates are sharp, i.e., the ratio of the upper bound and (the trivial lower bound) p is bounded locally in time. In fact, as shown by Bogdan, Hansen and Jakubowski [4, Examples 4.3 and 4.5], the exponential factors in (1.4) and (1.5) are very nearly optimal. The estimates also apply to rather general integral kernels on space-time, without assuming the Chapman–Kolmogorov equations. It is now crucial to verify (1.1) for given p and q . To this end we usually try to split and estimate the singularities of the integrand $(u, z) \mapsto p(s, x, u, z)q(u, z)p(u, z, t, y)$ in (1.1). This is straightforward if q satisfies a suitable Kato-type condition and the 3G Theorem holds for p (see [6, Remark 2], (2.10) and the discussion in Sections 2 and 4 below). The latter is the case, e.g., for the transition density of the fractional Laplacian as described by Bogdan, Hansen and Jakubowski [3, Corollary 11], [4, Example 4.13], and for the potential kernel of the stable subordinator [6, Example 2], corresponding to the fact that the functions have power-type asymptotics. However, due to their exponential decay, 3G fails (see (3.4)) for the Gaussian kernels

$$(1.6) \quad g_a(s, x, t, y) = [4\pi(t - s)/a]^{-d/2} \exp(-|y - x|^2/[4(t - s)/a]).$$

Here $d \in \mathbb{N}$, $a > 0$, $s < t$, $x, y \in \mathbb{R}^d$, and we let $g_a(s, x, t, y) = 0$ if $s \geq t$. We observe that for $0 < a < b < \infty$ we have

$$(1.7) \quad g_b(s, x, t, y) \leq (b/a)^{d/2} g_a(s, x, t, y).$$

Motivated by these observations, the results of Zhang [26, 27] and the arguments of Jakubowski and Szczypkowski [17, 18], we propose a more flexible method of estimating Schrödinger perturbations of *transition densities* on X with respect to a measure dz . The method employs an auxiliary transition density p^* as an approximate *majorant* of p substituting for $p(\cdot, \cdot, t, y)$ in (1.1). Namely, we assume that two measurable transition densities satisfy

$$(1.8) \quad p(s, x, t, y) \leq Cp^*(s, x, t, y),$$

with a constant $C \geq 1$. In addition to superadditivity of $Q(s, t) \geq 0$, we also assume that it is right-continuous in s and left-continuous in t , and that

$$(1.9) \quad \int_s^t \int_X p(s, x, u, z)q(u, z)p^*(u, z, t, y) dz du \leq [\eta + Q(s, t)]p^*(s, x, t, y).$$

We write these conditions briefly as $q \in \mathcal{N}(p, p^*, C, \eta, Q)$ (see Definition 2.4 below for details). They allow one to recursively estimate multiple integrals involving p in the perturbation series.

In Section 2 we prove our first main result, which is as follows:

THEOREM 1.1. *If $q \in \mathcal{N}(p, p^*, C, \eta, Q)$ and $0 \leq \eta < 1$, then for all $s < t$, $x, y \in X$ and $0 < \varepsilon < 1 - \eta$ we have*

$$(1.10) \quad \tilde{p}(s, x, t, y) \leq p^*(s, x, t, y) \left(\frac{C}{1 - \eta - \varepsilon} \right)^{1+Q(s,t)/\varepsilon}.$$

REMARK 1.2. Two natural choices are: $\varepsilon = \eta$ if $0 < \eta < 1/2$, and $\varepsilon = (1 - \eta)/2$.

In the second part of the paper we test our methods against Gaussian-type estimates. To this end we first elaborate inequality (4.4) of [26] by giving the best constant in the following estimate involving four different Gaussian kernels (hence 4G).

THEOREM 1.3 (4G). *For $\alpha > 0$, let*

$$L(\alpha) = \max_{\tau \geq \alpha \vee 1/\alpha} \left[\ln(1 + \tau) - \frac{\tau - \alpha}{1 + \tau} \ln(\alpha\tau) \right],$$

let $0 < a < b < \infty$ and

$$M = \left(\frac{b}{b - a} \right)^{d/2} \exp \left(\frac{d}{2} L \left(\frac{a}{b - a} \right) \right).$$

Then

$$(1.11) \quad g_b(s, x, u, z)g_a(u, z, t, y) \leq M[g_{b-a}(s, x, u, z) \vee g_a(u, z, t, y)]g_a(s, x, t, y)$$

for all $s < u < t$ and $x, z, y \in \mathbb{R}^d$. This fails for some $s < u < t$, $x, z, y \in \mathbb{R}^d$ if $M < \left(\frac{b}{b-a}\right)^{d/2} \exp\left(\frac{d}{2}L\left(\frac{a}{b-a}\right)\right)$. Further, we have $\left(\frac{b}{b-a}\right)^{d/2} \exp\left(\frac{d}{2}L\left(\frac{a}{b-a}\right)\right) = (1 - a/b)^{-d}$ if $1/(1 + e^{-1/2}) \leq a/b < 1$.

The proof of Theorem 1.3 is given in Section 3.

Then, in Section 4 we obtain precise Gaussian estimates for the fundamental solution of Schrödinger perturbations of second order parabolic differential operators, recovering and improving existing results, which we discuss there at some length. They follow by considering g_b and g_a of Theorem 1.3 as (multiples of) p and p^* of Theorem 1.1. Most of our discussion in Section 4 is summarized in the following theorem on Borel measurable transition densities p on $X = \mathbb{R}^d$ with the Lebesgue measure dz . To simplify the notation, for $d \geq 3$ and $U: \mathbb{R}^d \rightarrow \mathbb{R}$ we denote

$$(1.12) \quad I_\delta(U) = \sup_{x \in \mathbb{R}^d} \int_{|z-x| < \delta} \frac{|U(z)|}{|z-x|^{d-2}} dz, \quad \delta > 0,$$

and let $c_0 = c_0(d) = \Gamma(d/2 - 1)\pi^{-d/2}/4$.

THEOREM 1.4. *Let $d \geq 1$. Assume that there exist $b > 0$, $\Lambda \geq 1$ and $\lambda \in \mathbb{R}$ such that, for $s < t$,*

$$(1.13) \quad p(s, x, t, y) \leq \Lambda e^{\lambda(t-s)} g_b(s, x, t, y), \quad x, y \in \mathbb{R}^d.$$

Let $0 < a < b$ and $C = \Lambda(b/a)^{d/2}$. If $q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$, then for all $s < t$, $x, y \in \mathbb{R}^d$ and $0 < \varepsilon < 1 - \Lambda\eta$,

$$(1.14) \quad \tilde{p}(s, x, t, y) \leq \left(\frac{C}{1 - \Lambda\eta - \varepsilon}\right)^{1 + \Lambda Q(s,t)/\varepsilon} e^{\lambda(t-s)} g_a(s, x, t, y).$$

Further, if $d \geq 3$, $h > 0$, q is time-independent and $I_{\sqrt{h}}(q) < \infty$, then $q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$ with

$$\eta = bc_0 M I_{\sqrt{h}}(q), \quad Q(s, t) = (t - s) I_{\sqrt{h}}(q) 2M / (|B(0, 1/2)|h).$$

For more details on the role of the term $I_\delta(q)$ see (4.3). Here $|B(0, 1/2)|$ is the volume of the ball with radius $1/2$, M is the optimal constant from (1.11), and the smallness of η may follow from having b small and a proportional to b or choosing h , hence $I_{\sqrt{h}}(q)$, small. This brings about honest control of constants in estimates, which is not available by other existing methods. Our bounds of \tilde{p} are automatically global in time, and we do not need to patch together estimates obtained in small time intervals by means of the Chapman–Kolmogorov equations. Note that, our methods are not restricted to Gaussian-type kernels. Further applications, e.g. to perturbations

of the transition density of the 1/2-stable subordinator, will be given in a forthcoming paper.

In Section 4 we also describe connections to second order differential operators and we identify some of the transition densities \tilde{p} given by (1.2) as left inverses of second order differential operators on space-time: for $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_{s \in \mathbb{R}^d} \int \tilde{p}(s, x, u, z) \times \left(\frac{\partial}{\partial u} + \sum_{i,j=1}^n a_{ij}(u, z) \frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^n b_i(u, z) \frac{\partial}{\partial z_i} + q(u, z) \right) \phi(u, z) dz du = -\phi(s, x).$$

In Section 5 we give miscellaneous methodological comments on super-additivity of Q .

Our inspiration mainly comes from [18] and [6]. Our ideas are also similar to those developed for Gaussian estimates in [26]. In particular, the condition (1.9) for our main Theorem 1.1 may be considered as a generalization of the Main Lemma 4.1 of [26], while the 4G inequality in Theorem 1.3 yields an alternative, synthetic justification of that lemma. Furthermore, the proof of inequality (4.4) of [26] yields 4G, except for the optimal constant M . It is thus of interest that the approach of [26], which was tailor-made for the Gaussian kernel, has a more general context given by Theorem 1.1.

2. Estimates for general transition densities. Let X be an arbitrary set with a σ -algebra \mathcal{M} and a (non-negative) σ -finite measure m defined on \mathcal{M} . To simplify the notation we will write dz for $m(dz)$ in what follows. We also consider the set \mathcal{B} of Borel subsets of \mathbb{R} , and the Lebesgue measure du defined on \mathbb{R} . The *space-time* $\mathbb{R} \times X$ will be equipped with the σ -algebra $\mathcal{B} \times \mathcal{M}$ and the product measure $du dz = du m(dz)$.

We will consider a *measurable transition density* p on space-time, i.e., we assume that $p : \mathbb{R} \times X \times \mathbb{R} \times X \rightarrow [0, \infty]$ is $\mathcal{B} \times \mathcal{M} \times \mathcal{B} \times \mathcal{M}$ -measurable and the Chapman–Kolmogorov equations hold for all $x, y \in X$ and $s < u < t$:

$$(2.1) \quad \int_X p(s, x, u, z) p(u, z, t, y) dz = p(s, x, t, y).$$

Let $q : \mathbb{R} \times X \rightarrow [0, \infty]$ be (non-negative and) $\mathcal{B} \times \mathcal{M}$ -measurable. (All the functions considered below are tacitly assumed to be measurable on their respective domains.) The Schrödinger perturbation \tilde{p} of p by q is defined by the series (1.2), where $p_0(s, x, t, y) = p(s, x, t, y)$,

$$p_1(s, x, t, y) = \int_s^t \int_X p(s, x, u, z) q(u, z) p(u, z, t, y) dz du,$$

and for $n = 2, 3, \dots$,

$$(2.2) \quad p_n(s, x, t, y) = \int_s^t \int_{u_1}^t \cdots \int_{u_{n-1}}^t \int_{(\mathbb{R}^d)^n} p(s, x, u_1, z_1) q(u_1, z_1) \\ \times p(u_1, z_1, u_2, z_2) \cdots q(u_n, z_n) p(u_n, z_n, t, y) dz_n \cdots dz_1 du_n \cdots du_1.$$

By Fubini–Tonelli, for $n = 1, 2, \dots$, we have

$$(2.3) \quad p_n(s, x, t, y) = \int_s^t \int_X p(s, x, u, z) q(u, z) p_{n-1}(u, z, t, y) dz du,$$

and

$$(2.4) \quad p_n(s, x, t, y) = \int_s^t \int_X p_{n-1}(s, x, u, z) q(u, z) p(u, z, t, y) dz du.$$

By [3, Lemma 1], for all $s < u < t$, $x, y \in X$ and $n \in \mathbb{N}_0 = \{0, 1, \dots\}$,

$$(2.5) \quad \sum_{m=0}^n \int_X p_m(s, x, u, z) p_{n-m}(u, z, t, y) dz = p_n(s, x, t, y).$$

By [3, Lemma 2], the Chapman–Kolmogorov equations hold for \tilde{p} . Clearly, $\tilde{p} \geq p$.

REMARK 2.1. The perturbation, say p_V , is given by the same formulae if $V : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ (with complex, in particular negative, values), provided $p|_{|V|}$ is finite. Indeed, p_V then converges absolutely and

$$(2.6) \quad |p_V| \leq p|_{|V|}.$$

A detailed discussion of signed real-valued perturbations of transition densities is given in [3], with a positive lower bound for p_V resulting from Jensen’s inequality. A probabilistic interpretation of p_n and p_V may also be found in [3].

Below we focus on upper bounds of $\tilde{p} = p_q$ for $q \geq 0$ and transition densities p , as defined above. This is a less general setting than that of [6], but within this setting our bound (1.10) holds under more flexible condition (1.9) on p and q . Namely, we consider another (measurable) transition density p^* and $C \geq 1$ such that for all $x, y \in X$ and $s < t$, inequality (1.8) holds. We can estimate the cumulative effect of the integrations involved in (2.2), or (2.3). The following result is an analogue of [18, Lemma 5] and [4, Example 4.5].

LEMMA 2.2. *Let $\theta \geq 0$ and $s_0 < \dots < s_k = t$ be such that*

$$(2.7) \quad \int_s^{s_{i+1}} \int_X p(s, x, u, z) q(u, z) p^*(u, z, s_{i+1}, y) dz du \leq \theta p^*(s, x, s_{i+1}, y)$$

for all $i = 0, \dots, k-1$, $s \in [s_i, s_{i+1}]$ and $x, y \in X$. Then for every $n \in \mathbb{N}_0$,

$$(2.8) \quad p_n(s, x, t, y) \leq \binom{n+k-1}{k-1} \theta^n C^k p^*(s, x, t, y), \quad s \in [s_0, s_1], x, y \in X.$$

Proof. For $k = 1$, the estimate holds for $n = 0$ by (1.8), and then it holds for all $n \geq 1$ by induction, (2.3) and (2.7):

$$\begin{aligned} p_n(s, x, t, y) &= \int_{s, X} \int_s^t p(s, x, u, z) q(u, z) p_{n-1}(u, z, t, y) dz du \\ &\leq \theta^{n-1} C \int_{s, X} \int_s^t p(s, x, u, z) q(u, z) p^*(u, z, t, y) dz du \\ &\leq \theta^n C p^*(s, x, t, y), \quad \text{where } s \in [s_0, s_1], x, y \in X. \end{aligned}$$

If $k \geq 2$, then by (2.5), induction and Chapman–Kolmogorov for p^* ,

$$\begin{aligned} p_n(s, x, t, y) &= \sum_{m=0}^n \int_X p_m(s, x, s_{k-1}, z) p_{n-m}(s_{k-1}, z, t, y) dz \\ &\leq \sum_{m=0}^n \int_X \binom{m+k-2}{k-2} \theta^m C^{k-1} p^*(s, x, s_{k-1}, z) \theta^{n-m} C p^*(s_{k-1}, z, t, y) dz \\ &= \binom{n+k-1}{k-1} \theta^n C^k p^*(s, x, t, y) \quad \text{if } s \in [s_0, s_1], x, y \in X, n \in \mathbb{N}_0. \quad \blacksquare \end{aligned}$$

In passing we note that the assumption and conclusion in the statement of [18, Lemma 5] need a slight strengthening for the induction to work properly: each t_{i+1} in the assumption there should be replaced by τ in $[t_i, t_{i+1}]$, and each t in the conclusion should be replaced by τ in $[t_i, t_{i+1}]$ (then one proceeds as in the proof of Lemma 2.2 above). The correction does not influence applications of Lemma 5 or other results in [18].

We further let $Q: \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ be *regular superadditive*, meaning that

$$(2.9) \quad Q(s, u) + Q(u, t) \leq Q(s, t) \quad \text{if } s < u < t,$$

$Q(s, t) = 0$ if $s \geq t$, $s \mapsto Q(s, t)$ is right-continuous and $t \mapsto Q(s, t)$ is left-continuous. (The continuity assumptions are rather innocuous, as we explain later on in Lemmas 5.2 and 5.3). We see that $t \mapsto Q(s, t)$ is non-decreasing and $s \mapsto Q(s, t)$ is non-increasing. For instance, if μ is a Radon measure on \mathbb{R} , then $Q(s, t) = \mu(\{u \in \mathbb{R} : s < u < t\})$ is regular superadditive. A regular superadditive Q is infinitely decomposable in the following sense:

LEMMA 2.3. *Let $s \leq t$, $k \in \mathbb{N}$ and $\theta \geq 0$ be such that $Q(s, t) \leq k\theta$. Then there exist $s = s_0 \leq s_1 \leq \dots \leq s_k = t$ such that $Q(s_{i-1}, s_i) \leq \theta$ for $i = 1, \dots, k$.*

Proof. We may and do assume that $k > 1$ and $(k - 1)\theta < Q(s, t) \leq k\theta$. Let $s_i = \inf\{u : Q(s, u) \geq i\theta\}$ for $i = 1, \dots, k - 1$. If $s \leq u < s_i$, then $Q(s, u) < i\theta$, and so $Q(s, s_i) \leq i\theta$. If $s_i < u < s_{i+1}$, then $Q(s, u) \geq i\theta$ and $Q(s, u) + Q(u, s_{i+1}) \leq Q(s, s_{i+1}) \leq (i + 1)\theta$, thus $Q(u, s_{i+1}) \leq \theta$. Letting $u \rightarrow s_i$ we obtain $Q(s_i, s_{i+1}) \leq \theta$, which is also true if $s_i = s_{i+1}$. ■

DEFINITION 2.4. We write $q \in \mathcal{N}(p, p^*, C, \eta, Q)$ if $q \geq 0$ is defined (and measurable) on space-time, p and p^* are (measurable) transition densities, $C \geq 1$, $\eta \geq 0$, Q is regular superadditive, and (1.8) and (1.9) hold for all $s < t$ and $x, y \in X$.

The terms η and $Q(s, t)$ of (1.9) propagate differently in estimates of p_n below. We may think about η as giving a bound for instantaneous growth of mass, while Q gives a cap for growth accumulated over time (see [6] and [4] for such insights).

We are in a position to prove our first main result.

Proof of Theorem 1.1. Let $k \in \mathbb{N}$. By Lemma 2.3, there exist $s_0 = s < s_1 < \dots < s_k = t$ such that $Q(s_{i-1}, s_i) \leq Q(s, t)/k$ for $i = 1, \dots, k$. For $\varepsilon \in (0, 1 - \eta)$ we choose $k \in \mathbb{N}$ such that $(k - 1)\varepsilon \leq Q(s, t) < k\varepsilon$. By Lemma 2.2 with $\theta = \eta + Q(s, t)/k$, and by Taylor’s expansion, for all $x, y \in X$ we get

$$\begin{aligned} \tilde{p}(s, x, t, y) &\leq \sum_{n=0}^{\infty} p_n(s, x, t, y) \\ &\leq \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} C^k [\eta + Q(s, t)/k]^n p^*(s, x, t, y) \\ &= \left(\frac{C}{1 - \eta - Q(s, t)/k} \right)^k p^*(s, x, t, y). \quad \blacksquare \end{aligned}$$

Analogous results hold if we state our assumptions and conclusions for s, t in a finite time horizon: $-\infty < t_1 \leq s < t \leq t_2 < \infty$. If Q in Theorem 1.1 is bounded, then $\tilde{p} \leq \text{const} \cdot p^*$ uniformly in time. We may consider q in $\mathcal{N}(p, p^*, C, \eta, Q)$ with $\eta < 1$, $q_1(u, z) = q(u, z) \mathbf{1}_{[0,1]}(u)$, and the bounded superadditive function $Q_1(s, t) = Q((s \vee 0) \wedge 1, (t \wedge 1) \vee 0)$. Then

$$\int_{s, X}^t p(s, x, u, z) q_1(u, z) p^*(u, z, t, y) dz du \leq [\eta + Q_1(s, t)] p^*(s, x, t, y).$$

Thus, by Theorem 1.1, $\tilde{p} \leq cp^*$ uniformly in time. If p^* is not comparable with p in space, then the estimates in Theorem 1.1 cannot be sharp. This is regrettable, but quite common, e.g., in Schrödinger perturbations of Gaussian kernel discussed later on in the paper. The role of p^* is similar to that of f in [4, Theorem 3.2], but the results of [4] do not apply in the present setting if $p \neq p^*$. If (1.9) holds with p^* replaced by p , then we may

take $p^* = p$ and $C = 1$ in (1.8) and Theorem 1.1. However, in this case a more efficient inductive argument of [6] gives better estimates (1.4) and (1.5) above. If $q(u, z) \leq f(u)$, then we may take $Q(s, t) = \int_s^t f(u) du$, $\eta = 0$ and $p = p^*$. In fact, if $q(u, z) = f(u)$, then $p_n(s, x, t, y) = Q(s, t)^n p(s, x, t, y)/n!$ and $\tilde{p}(s, x, t, y) = e^{Q(s,t)} p(s, x, t, y)$.

Less trivial applications of Theorem 1.1 require a detailed study of p and q , and a judicious choice of p^* . In particular, to estimate \tilde{p} for given p , p^* and q , we wish to verify (1.9). This task may be facilitated by splitting the singularities of p and p^* in the integral of (1.9). Various versions of the 3G Theorem are used to this end: see [3], Bogdan and Jakubowski [5], and the “elliptic case” of Cranston, Fabes and Zhao [9], and Hansen [14]. For instance the transition density of the fractional Laplacian enjoys the following 3G inequality:

$$(2.10) \quad p(s, x, u, z) \wedge p(u, z, t, y) \leq cp(s, x, t, y),$$

where $x, z, y \in \mathbb{R}^d$, $s < u < t$ [5, Theorem 4], and this yields

$$\begin{aligned} p(s, x, u, z)p(u, z, t, y) &= [p(s, x, u, z) \vee p(u, z, t, y)][p(s, x, u, z) \wedge p(u, z, t, y)] \\ &\leq c [p(s, x, u, z) + p(u, z, t, y)]p(s, x, t, y). \end{aligned}$$

In this situation we can use $p^* = p$ in Theorem 1.1 to estimate \tilde{p} , provided

$$c \int_{sX}^t \int p(s, x, u, z)q(u, z) dz du + c \int_s^t \int q(u, z)p(u, z, t, y) dz du \leq \eta + Q(s, t).$$

Note that 3G fails for Gaussian kernels, and for such kernels the methods of [6] fall short of optimal known results. This largely motivates the present development. In the next sections we show how to estimate quite general Schrödinger perturbations of Gaussian kernels by means of Theorem 1.1. The application depends on the 4G inequality stated in (1.11) of Theorem 1.3, which partially substitutes for 3G. We note that Theorem 1.3 improves [26, (4.4)], since we give an optimal constant in (1.11). Explicit constants matter in our applications, because we specifically require $\eta < 1$ in Theorem 1.1.

3. Estimates of Gaussian kernels. As usual, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Let $0 < \alpha < \infty$, and

$$\begin{aligned} (3.1) \quad L(\alpha) &= \max_{\tau \geq \alpha \vee 1/\alpha} \left[\ln(1 + \tau) - \frac{\tau - \alpha}{1 + \tau} \ln(\alpha\tau) \right] \\ &= \max_{\tau \geq \alpha \vee 1/\alpha} \left[\ln(1 + 1/\tau) - \ln \alpha + \frac{1 + \alpha}{1 + \tau} \ln(\alpha\tau) \right]. \end{aligned}$$

Clearly, $L(\alpha) < \infty$, and $\tau = \alpha \vee 1/\alpha$ yields $L(\alpha) \geq \ln(1 + \alpha \vee 1/\alpha)$. By an application of calculus, $L(\alpha) = \ln(1 + \alpha)$ if (and only if) $\alpha \geq e^{1/2}$. We let

$$f(\tau, x) = \ln \tau + x^2/\tau, \quad \tau > 0, x \geq 0.$$

LEMMA 3.1. *If $\alpha > 0$, $L = L(\alpha)$, $\xi, \eta \geq 0$ and $\tau > 0$, then*

$$(3.2) \quad f(1 + \tau, \xi + \eta) \leq f(1, \xi) \vee f(\alpha\tau, \eta) + \eta^2/\tau + L.$$

If $L < L(\alpha)$, then the inequality fails for some $\xi, \eta \geq 0$ and $\tau > 0$.

Proof. We first prove the following implication:

$$(3.3) \quad \text{If } \frac{\eta^2}{\alpha\tau} + \ln(\alpha\tau) \leq \xi^2, \quad \text{then } \ln(1 + \tau) \leq \frac{(\tau\xi - \eta)^2}{\tau(1 + \tau)} + L.$$

To this end we consider two special cases:

CASE 1: $\eta^2/(\alpha\tau) + \ln(\alpha\tau) \leq \xi^2$ and $\eta = \tau\xi$.

CASE 2: $\eta^2/(\alpha\tau) + \ln(\alpha\tau) = \xi^2$ and $\eta < \tau\xi$.

Case 1 implies that $(\tau/\alpha - 1)\xi^2 + \ln(\alpha\tau) \leq 0$. This is possible only if $\tau \leq \alpha \vee 1/\alpha$, whence $\ln(1 + \tau) \leq \ln(1 + \alpha \vee 1/\alpha) \leq L(\alpha)$, which proves (3.3).

In Case 2, if $\tau \leq \alpha \vee 1/\alpha$, then $\ln(1 + \tau) \leq \ln(1 + \alpha \vee 1/\alpha) \leq L(\alpha)$ again. For $\tau > \alpha \vee 1/\alpha$ we consider $\xi = \xi(\eta)$ as a function of η , and we have

$$\phi(\eta) := \ln(1 + \tau) - \frac{(\tau\xi - \eta)^2}{\tau(1 + \tau)} \leq L.$$

Indeed, we see that the condition $\eta < \tau\xi$ holds automatically since $\eta^2/\tau^2 \leq \eta^2/(\alpha\tau) = \xi^2 - \ln(\alpha\tau) < \xi^2$. Our assumption now reads $\xi^2 = \eta^2/(\alpha\tau) + \ln(\alpha\tau)$, where $\xi, \eta \geq 0$ and $\tau > \alpha \vee 1/\alpha$. Thus, $\xi' = \eta/(\alpha\tau\xi)$. Note that $\phi(0) = \ln(1 + \tau) - \tau \ln(\alpha\tau)/(1 + \tau) \leq L$. Furthermore,

$$\phi'(\eta) = -2(1 + \tau)^{-1}(\tau\xi - \eta)(\xi' - 1/\tau).$$

We have $\phi'(\eta) = 0$ only if $\xi' = 1/\tau$, or $\xi = \eta/\alpha$, and then $\eta^2/(\alpha\tau) + \ln(\alpha\tau) = \eta^2/\alpha^2$ and $\phi(\eta) = \ln(1 + \tau) - (\tau - \alpha) \ln(\alpha\tau)/(1 + \tau)$. This in fact shows that $L = L(\alpha)$ is sharp in (3.3) (see (3.1)). Furthermore, $\phi'(\eta) \leq 0$ if $\xi' \geq 1/\tau$, or $(\tau/\alpha - 1)\eta^2/\alpha \geq \tau \ln(\alpha\tau)$, in particular if η is large. Thus, ϕ is decreasing for large η , which yields (3.3) in Case 2.

Consider now general ξ, η and $\tau > 0$ in (3.3). If $\eta > \tau\xi$, then decreasing η to $\tau\xi$ strengthens (3.3), so eventually we are done by Case 1. If $\eta < \tau\xi$, then we increase η and strengthen the conclusion in (3.3), falling under Case 1 or 2.

Putting (3.3) differently, $\ln(1 + \tau) + (\xi + \eta)^2/(1 + \tau) \leq \xi^2 + \eta^2/\tau + L$, provided $\eta^2/(\alpha\tau) + \ln(\alpha\tau) \leq \xi^2$. Therefore we have (3.2) under the assumption $f(\alpha\tau, \eta) \leq f(1, \xi)$, and the constant L cannot be improved. In particular, (3.2) holds if $f(1, \xi) = f(\alpha\tau, \eta)$. Decreasing ξ keeps (3.2) valid because $f(1 + \tau, \xi + \eta)$ then decreases too. ■

We note that the 3G inequality fails for g_a defined in (1.6), because

$$(3.4) \quad \frac{g_a(0, 0, t, y) \wedge g_a(t, y, 2t, 2y)}{g_a(0, 0, 2t, 2y)} = \frac{(4\pi t/a)^{-d/2} e^{-|y|^2/(4t/a)}}{(8\pi t/a)^{-d/2} e^{-|y|^2/(2t/a)}} = 2^{d/2} e^{a|y|^2/(4t)}$$

is not bounded in $y \in \mathbb{R}^d$. The next inequality (1.11) between four different instances of the Gaussian kernel substitutes 3G, and so it is coined 4G. We note that [26, proof of (4.4)] yields (1.11) too, although with a rough constant M (see also the first equality on p. 465 in [27] and the last equality on p. 15 in Friedman [12]). We also mention a similar result (with rough constants) for the heat kernel of smooth bounded domains by Riahi [23, Lemma 3.1]. The optimality of the right-hand side of (1.11) is important in view of (1.9) and (1.10), and may be of independent interest. In fact, inspection of our calculations also reveals that $b - a$ in g_{b-a} of (1.11) cannot be replaced by a larger constant.

We are in a position to prove our second main result.

Proof of Theorem 1.3. We have

$$(3.5) \quad \ln g_a(s, x, t, y) = -\frac{d}{2} \ln 4\pi - \frac{d}{2} \ln(t-s) + \frac{d}{2} \ln a - \frac{a|y-x|^2}{4(t-s)}.$$

Considering $\sqrt{2d}x$, $\sqrt{2d}y$ and $\sqrt{2d}z$ instead of x , y and z in (3.5), we see that (1.11) is equivalent to

$$\begin{aligned} & -\ln(u-s) + \ln b - \frac{b|z-x|^2}{u-s} - \ln(t-u) + \ln a - \frac{a|y-z|^2}{t-u} \\ & \leq \frac{2}{d} \ln M + \left[-\ln(u-s) + \ln(b-a) - \frac{(b-a)|z-x|^2}{u-s} \right] \\ & \quad \vee \left[-\ln(t-u) + \ln a - \frac{a|y-z|^2}{t-u} \right] - \ln(t-s) + \ln a - \frac{a|y-x|^2}{t-s}. \end{aligned}$$

We rewrite this using the identity $a + b - a \vee b = a \wedge b$, and we obtain

$$\begin{aligned} & \ln b - \ln(b-a) - \frac{a|z-x|^2}{u-s} + \left[-\ln(u-s) + \ln(b-a) - \frac{(b-a)|z-x|^2}{u-s} \right] \\ & \wedge \left[-\ln(t-u) + \ln a - \frac{a|y-z|^2}{t-u} \right] \leq \frac{2}{d} \ln M - \ln(t-s) + \ln a - \frac{a|y-x|^2}{t-s}. \end{aligned}$$

Adding $\ln(t-u) - \ln a$ to both sides (and moving terms from one side to the other), we have

$$\begin{aligned} & \frac{a|y-x|^2}{t-s} + \ln \frac{t-s}{t-u} \leq \frac{2}{d} \ln M + \ln \frac{b-a}{b} \\ & \quad + \frac{a|y-z|^2}{t-u} \vee \left[\ln \frac{u-s}{t-u} + \ln \frac{a}{b-a} + \frac{(b-a)|z-x|^2}{u-s} \right] + \frac{a|z-x|^2}{u-s}. \end{aligned}$$

We denote $\alpha = a/(b-a)$, $\tau = (u-s)/(t-u)$, $\xi = |y-z|\sqrt{a/(t-u)}$ and $\eta = |z-x|\sqrt{a/(t-u)}$, and observe that $(t-s)/(t-u) = 1 + \tau$. Since $|y-x| \leq |z-x| + |y-z|$, where equality may hold, we see that (1.11) is

equivalent to the following inequality (to hold for all $\tau > 0$ and $\xi, \eta \geq 0$):

$$(3.6) \quad \frac{(\xi + \eta)^2}{1 + \tau} + \ln(1 + \tau) \leq \frac{2}{d} \ln M + \ln \frac{b - a}{b} + \xi^2 \vee \left[\frac{\eta^2}{\alpha\tau} + \ln(\alpha\tau) \right] \frac{\eta^2}{\tau}.$$

We may now use Lemma 3.1. In fact, the constant M in (1.11) is optimal if

$$\frac{2}{d} \ln M + \ln \frac{b - a}{b} = L(\alpha).$$

Considering $\alpha = a/(b - a) \geq e^{1/2}$, we obtain the last statement of the theorem from the comment following (3.1). ■

REMARK 3.2. In applications we usually choose a (smaller than but) close to b , so as to not lose much of Gaussian asymptotics, and in this case the optimality of the simple formula $M = (1 - a/b)^{-d}$ comes as a nice feature of our 4G Theorem.

4. Applications and discussion. In this section we discuss applications of Theorem 1.1 to fundamental solutions of second order parabolic differential operators. Namely, Theorem 1.1, aided by Theorem 1.3, allows making rather singular Schrödinger perturbations of such operators without dramatically changing the magnitude of their fundamental solutions. Most of the estimates given below are known, but our proofs are more synthetic and considerably shorter, and we have explicit constants in the estimates, which may be useful in homogenization problems. We also note that in the case of signed perturbations (not considered here) very precise lower bounds are obtained from Jensen’s inequality for bridges [3] (see also Remark 2.1 above). We begin with a discussion of Kato-type conditions (historical comments are given in Remark 4.6).

Let $d \geq 3$. A Borel function $U: \mathbb{R}^d \rightarrow \mathbb{R}$ is of *Kato class* if (see (1.12) for definition)

$$(4.1) \quad \lim_{\delta \rightarrow 0^+} I_\delta(U) = 0.$$

A typical example is $U(z) = |z|^{-2+\varepsilon}$, where $0 < \varepsilon \leq 2$. By Aizenman and Simon [1, Theorem 4.5], Chung and Zhao [8, Theorem 3.6] or Zhao [28, Theorem 1], (4.1) holds if and only if for every $c > 0$ the following condition is satisfied (see (1.6)):

$$(4.2) \quad \lim_{h \rightarrow 0^+} \sup_{s \in \mathbb{R}, x \in \mathbb{R}^d} \int_s^{s+h} \int_{\mathbb{R}^d} g_c(s, x, u, z) |U(z)| dz du = 0.$$

In fact, there exist $C_0 = C_0(d, c)$ and $C_1 = C_1(d, c)$ such that for all $h > 0$

and U ,

$$(4.3) \quad C_0 I_{\sqrt{h}}(U) \leq \sup_{s \in \mathbb{R}, x \in \mathbb{R}^d} \int_s^{s+h} \int_{\mathbb{R}^d} g_c(s, x, u, z) |U(z)| dz du \leq C_1 I_{\sqrt{h}}(U).$$

The lower bound of (4.3) is given in [1, (4.5)] and [8, Lemma 3.5]. The upper bound can be proved as in [5, Lemma 11], but for the reader's convenience we give a simple, explicit and more flexible argument showing (after Proposition 4.3 below) that in fact in (4.3) we may take

$$(4.4) \quad C_1(d, c) = \Gamma(d/2 - 1) \pi^{-d/2} [c + 2^d d(d - 2)]/4.$$

To this end, for $x \in \mathbb{R}^d$ and $r > 0$, we denote $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$, and we consider $1_{B(0,r)}$, the indicator function of the ball of radius $r > 0$. We call $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$ *radially decreasing* if $f(x_1) \geq f(x_2)$ whenever $|x_1| \leq |x_2|$. We observe the following three auxiliary results.

LEMMA 4.1. *Let $r > 0$, and let $f \geq 0$ be constant on $B(0, r)$ and radially decreasing. Then*

$$f * 1_{B(0,r)} \geq |B(0, r/2)| f,$$

where $|B(0, r/2)|$ denotes the volume of $B(0, r/2)$.

Proof. We have $f * 1_{B(0,r)}(x) = \int_{B(x,r)} f(y) dy$. If $|x| < r$, then

$$f * 1_{B(0,r)}(x) \geq f(0) |B(0, r) \cap B(x, r)| \geq f(0) |B(0, r/2)| = f(x) |B(0, r/2)|.$$

If $|x| \geq r$, then

$$f * 1_{B(0,r)}(x) \geq f(x) |B(0, |x|) \cap B(x, r)| \geq f(x) |B(0, r/2)|. \blacksquare$$

LEMMA 4.2. *Let $0 \leq k \leq K$ be radially decreasing functions and fix $r > 0$. Let $c_1 = \int_{\mathbb{R}^d} k(x) dx$ and $c_2 = K(r, 0, \dots, 0) |B(0, r/2)|$. Let $c_3 = 1$ if $c_2 = 0$ or ∞ , and $c_3 = 1 + c_1/c_2$ otherwise. Then*

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |U(z)| k(x - z) dz du \leq c_3 \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |U(z)| K(x - z) dz.$$

Proof. Define $f(x) = k(x) \wedge k(r, 0, \dots, 0)$. Assume first that $0 < c_2 < \infty$. By Lemma 4.1,

$$\begin{aligned} k &\leq 1_{B(0,r)} K + f \leq 1_{B(0,r)} K + 1_{B(0,r)} * f / |B(0, r/2)| \\ &\leq 1_{B(0,r)} K + (1_{B(0,r)} K) * f / c_2. \end{aligned}$$

The conclusion of the lemma follows from this, because

$$|U| * k \leq |U| * (1_{B(0,r)} K) * (\delta_0 + f/c_2) \leq \sup[|U| * (1_{B(0,r)} K)] (1 + c_1/c_2).$$

If $c_2 = 0$, then the conclusion follows immediately with $c_3 = 1$, since then

$k \leq K = K1_{B(0,r)}$. If $c_2 = \infty$, then we have $K = \infty$ on $B(0, r)$ and the conclusion is trivially true with $c_3 = 1$. ■

PROPOSITION 4.3. *If $c_0 = c_0(d) = \Gamma(d/2 - 1)/(4\pi^{d/2})$, $c, \tau, r > 0$ and $U: \mathbb{R}^d \rightarrow \mathbb{R}$, then*

$$\sup_{s \in \mathbb{R}, x \in \mathbb{R}^d} \int_s^{s+\tau} \int_{\mathbb{R}^d} g_c(s, x, u, z) |U(z)| dz du \leq \left(cc_0 + \frac{\tau}{r^2 |B(0, 1/2)|} \right) I_r(U).$$

Proof. It suffices to take $k(x) = \int_0^\tau g_c(0, 0, u, x) du$ and $K(x) = \int_0^\infty g_c(0, 0, u, x) du = cc_0|x|^{2-d}$ in Lemma 4.2 and observe that $c_1 = \tau$ and $c_2 = |B(0, r/2)|K(r, 0, \dots, 0) = |B(0, 1/2)|cc_0r^2$. ■

Proof of (4.4). $\tau = h$ and $r = \sqrt{h}$ in Proposition 4.3 yield the constant (4.4) in (4.3). ■

By Theorem 1.3 (replacing \vee by $+$) and Proposition 4.3 with $\tau = t - s$ and $r = \sqrt{h}$, for $q: \mathbb{R}^d \rightarrow [0, \infty]$ we have

$$(4.5) \quad \int_s^t \int_{\mathbb{R}^d} g_b(s, x, u, z) q(z) g_a(u, z, t, y) dz du \leq I_{\sqrt{h}}(q) M \left[bc_0 + \frac{2(t-s)}{h|B(0, 1/2)|} \right] g_a(s, x, t, y).$$

We are in a position to summarize part of our discussion as given by Theorem 1.4.

Proof of Theorem 1.4. We consider a (Borel measurable) transition density p on space-time, where the *space* is $X = \mathbb{R}^d$ with Lebesgue measure dz . Let $0 < a < b$ and $p^*(s, x, t, y) = e^{\lambda(t-s)} g_a(s, x, t, y)$. In view of (1.7) we may take $C = (b/a)^{d/2} \Lambda$ in (1.8). If $q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$, then q is in $\mathcal{N}(p, p^*, C, \Lambda\eta, \Lambda Q)$, and the assertion follows from Theorem 1.1. The inequality (4.5) means that a time-independent q is in $\mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$ with

$$\eta = bc_0 M I_{\sqrt{h}}(q), \quad Q(s, t) = (t - s) \frac{2M}{h|B(0, 1/2)|} I_{\sqrt{h}}(q),$$

provided these are finite. ■

We also observe the following characterization of $\mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$.

COROLLARY 4.4. *If $d \geq 3$, $0 < a < b$ and $q: \mathbb{R}^d \rightarrow [0, \infty]$, then $q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$ for some η and Q if and only if $I_{\sqrt{h}}(q) < \infty$ for some (and hence all) $h > 0$.*

Proof. If $q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$, then integrating (1.9) in y , we obtain

$$\sup_{x \in \mathbb{R}^d} \int_0^h \int_{\mathbb{R}^d} g_b(0, x, u, z) q(z) dz du \leq \eta + Q(0, h).$$

By the lower bound of (4.3) we obtain

$$I_{\sqrt{h}}(q) \leq C_0^{-1}(d, b)(\eta + Q(0, h)) < \infty.$$

The converse implication follows from (4.5), which also shows that Q may be taken linear. ■

We now turn to a *parabolic Kato condition*. For $c, h > 0$ and $V : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ we denote

$$N_h^c(V) = \sup_{s,x} \int_s^{s+h} \int_{\mathbb{R}^d} g_c(s, x, u, z) |V(u, z)| dz du + \sup_{t,y} \int_{t-h}^t \int_{\mathbb{R}^d} g_c(u, z, t, y) |V(u, z)| dz du.$$

We say that V is of *parabolic Kato class* if $\lim_{h \rightarrow 0^+} N_h^c(V) = 0$ for every $c > 0$. Considering $V(s, x) = U(x)$ for $s \in \mathbb{R}, x \in \mathbb{R}^d$, we may regard the parabolic Kato class as wider than the (time-independent) Kato class. We note that $N_h^c(V)$ is non-decreasing in h . Obviously, $N_{h_1+h_2}^c(V) \leq N_{h_1}^c(V) + N_{h_2}^c(V)$, hence

$$(4.6) \quad N_{t-s}^c(V) \leq N_h^c(V) + N_h^c(V)(t-s)/h, \quad h > 0.$$

To focus on non-negative Schrödinger perturbations (in this connection see Remark 2.1), we consider, as before, a function $q \geq 0$ on space-time. If $0 < a < b < \infty$, then by the 4G Theorem, there is an explicit constant M' , depending only on d and b/a , such that for all $s < t$ and $x, y \in \mathbb{R}^d$,

$$(4.7) \quad \int_s^t \int_{\mathbb{R}^d} g_b(s, x, u, z) q(u, z) g_a(u, z, t, y) dz du \leq M' N_{t-s}^c(q) g_a(s, x, t, y),$$

where $c = (b-a) \wedge a$. In fact, we may take $M' = \left(\frac{b-a}{a} \vee \frac{a}{b-a}\right)^{d/2} M$, where M is the constant of Theorem 1.3. If $q \geq 0$ belongs to the parabolic Kato class and $0 < a < b < \infty$, then by (4.6) and (4.7) we have

$$(4.8) \quad q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$$

with $Q(s, t) = \beta(t-s)$ and $\beta = \eta/h$, provided $h, \eta > 0$ are such that $N_h^{(b-a) \wedge a}(q) \leq \eta/M'$. Then, with a view to applying Theorem 1.1, we are free to choose arbitrarily small $\eta > 0$ in (4.8), at the expense of having large β .

REMARK 4.5. The condition $q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$ invites a trade-off between η and Q . In particular, it follows from the discussion of (4.5) and (4.8) that there exist arbitrarily small $\eta > 0$ and a linear, possibly large but explicit Q if q is in the Kato class or the parabolic Kato class.

REMARK 4.6. The (time-independent) Kato class was first used to perturb the Laplace operator by Aizenman and Simon [1], and was characterized as smallness with respect to the Laplacian on $L^1(\mathbb{R}^d)$. The parabolic Kato class was proposed for the Gaussian kernel by Zhang [26]. It was then generalized and used by Liskevich and Semenov [19], Liskevich, Vogt and Voigt [20], and Gulisashvili and van Casteren [13]. The condition is related to Miyadera perturbations of the semigroup of the Laplacian on $L^1(\mathbb{R}^d)$ (see Schnaubelt and Voigt [24]). The time-independent Kato class is wider than $L^p(\mathbb{R}^d)$ if $p > d/2$ [1], [8, Chapter 3, Example 2]. Nevertheless, the latter space is quite natural for perturbing Gaussian kernels (see Aronson [2], Dziubański and Zienkiewicz [11], [27, Remark 1.1(b)]). Another Kato-type condition was introduced by Zhang [27] to obtain strict comparability of g and \tilde{g} . As noted in [27, Remark 1.1(c)], the condition may be formulated in terms of *Brownian bridges*. This point of view was later developed in [3] (under the name of the relative Kato condition) and elaborated in [6] to

$$(4.9) \quad \int_s^t \int_X \frac{p(s, x, u, z)p(u, z, t, y)}{p(s, x, t, y)} q(u, z) dz du \leq [\eta + Q(s, t)],$$

where $s < t$, $x, y \in X$, $\eta < \infty$ and $0 \leq Q(s, u) + Q(u, t) \leq Q(s, t)$ (cf. (1.1)). Condition (4.9) indicates why we mention bridges here (see [3] for details). The Kato condition for bridges gives better upper bounds and seems more intrinsic to Schrödinger perturbations than the parabolic Kato condition, but the former may be cumbersome to verify in concrete situations. For the classical Gaussian kernel, (4.9) is stronger than the corresponding parabolic Kato condition with a fixed c (see [3, Lemma 9] for a more general result), and it is rather difficult to explicitly characterize [27, Remark 1.2(a,b)]. This is due to the relatively large values of the integrand in (4.9) for (u, z) in the interval connecting (s, x) and (t, y) . If p satisfies the 3G inequality, $p(s, x, t, y) = p(s, y, t, x)$ and p is a probability transition density, then the parabolic Kato class and the Kato class for bridges coincide. This is the case for the transition density of the fractional Laplacian $\Delta^{\alpha/2}$ with $\alpha \in (0, 2)$ [3, Corollary 11], and the proof of this fact is similar to our application of 4G in (4.7). We emphasize that each transition density p determines its specific Kato classes (either parabolic or for bridges), and detailed analysis is required to manage particularly singular q .

Let $d \geq 3$, $z_1 \in \mathbb{R}^d$, $|z_1| = 1$, and let $B(nz_1, 1/n) \subset \mathbb{R}^d$ be the ball with radius $1/n$ and center nz_1 , $n = 1, 2, \dots$. We define

$$U(z) = \sum_{n=2}^{\infty} n|z - nz_1|^{-1} \mathbf{1}_{B(nz_1, 1/n)}(z), \quad z \in \mathbb{R}^d.$$

(A similar example appears in [1, Appendix 1].) If $\delta > 0$ and $n \geq 1/\delta$, then

$$I_\delta(U) \geq \int_{B(nz_1, 1/n)} n|z - nz_1|^{-d+1} dz = \int_{B(0,1)} |z|^{-d+1} dz.$$

Therefore εU does not belong to the parabolic Kato class for any $\varepsilon > 0$. On the other hand, $I_1(U) < \infty$, and by (4.5), $\varepsilon U \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$ with a *linear* Q and small η , provided ε is sufficiently small (cf. Corollary 4.4 and Theorem 1.1). A similar effect may be obtained for the original U if we instead make b smaller while keeping b/a constant. Since our constant in (4.5) is explicit, so are the resulting upper bounds for \tilde{g}_b . Similar conclusions obtain in the generality of Theorem 1.4, and applications to specific transition densities are presented below.

To summarize this discussion of examples of q manageable by our methods, we let $q(u, z) := U(z) + \varphi(u, z)$, where $\varphi \geq 0$, U is as above, and $f(u) = \sup_{z \in \mathbb{R}^d} \varphi(u, z)$ is finite and increases to infinity as $u \rightarrow \infty$. Such q requires $\eta > 0$ to control U , and a *superlinear* Q to majorize $\int_s^t f(u) du$ (cf. the discussion after the proof of Theorem 1.1).

EXAMPLE 4.7. If $p = g_b$ and $p^* = g_a$, then we take $\Lambda = 1$, $C = (b/a)^{d/2}$ and $\lambda = 0$ in (1.13), and consequently in (1.14). For clarity, a , the coefficient in the exponent of the Gaussian majorant, may be arbitrarily close to b , at the expense of the factor before the majorant in (1.14), and we require that $q \in \mathcal{N}(g_b, g_a, C, \eta, Q)$ with $\eta \in [0, 1)$, which is satisfied, e.g., for q in the Kato class. We thus recover the best results known in this setting [26, proof of Theorem A] with explicit control of constants.

To relate our results to second order differential operators, we let $C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ denote the smooth compactly supported functions on space-time, and recall that for all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,

$$(4.10) \quad \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} p(s, x, u, z) \left[\frac{\partial \phi(u, z)}{\partial u} + \frac{1}{b} \Delta \phi(u, z) \right] dz du = -\phi(s, x).$$

The identity may, for instance, be obtained from integration by parts or by using the Fourier transform in the space variable. By a general result, [6, Lemma 4], the perturbed transition density \tilde{p} corresponds to the Schrödinger-type operator $\frac{1}{b} \Delta + q$ in the same way,

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}(s, x, u, z) \left[\frac{\partial \phi(u, z)}{\partial u} + \frac{1}{b} \Delta \phi(u, z) + q(u, z) \phi(u, z) \right] dz du = -\phi(s, x),$$

provided $q \in \mathcal{N}(g_b, g_a, C, \eta, Q)$ with $\eta \in [0, 1)$, as above.

EXAMPLE 4.8. Let p be the transition density of the one-dimensional Brownian motion with constant unit drift,

$$p(s, x, t, y) = g_1(s, x - s, t, y - t), \quad s < t, x, y \in \mathbb{R}.$$

There are no constants c_1, c_2 such that $p(s, x, t, y) \leq c_1 g_{c_2}(s, x, t, y)$ for all $s < t$ and $x, y \in \mathbb{R}$ (cf. Zhang [25, Remark 1.3]). On the other hand, for each $b \in (0, 1)$ we have

$$p(s, x, t, y) \leq b^{-1/2} \exp\left(\frac{b}{4(1-b)}(t-s)\right) g_b(s, x, t, y), \quad s < t, x, y \in \mathbb{R}.$$

This shows why we may need $\lambda \neq 0$ in (1.13) (see also Norris [22]).

EXAMPLE 4.9. Let $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a function of $(s, x) \in \mathbb{R} \times \mathbb{R}^d$, and

$$Lf = \sum_{i,j=1}^n a_{ij}(s, x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(s, x) \frac{\partial f}{\partial x_i}$$

be a uniformly elliptic operator with bounded uniformly Hölder continuous coefficients b_i and $a_{ij} = a_{ji}$ (see Dynkin [10, Chapter 2, 1.1.A and 1.1.B] for detailed definitions, and [12, Chapter 1] for a wider perspective). Consider the fundamental solution $p(s, x, t, y)$ in the sense of [10, Theorem 1.1] for the parabolic differential operator

$$(4.11) \quad \frac{\partial f}{\partial s} + Lf.$$

By the results of [10, Chapter 2], in particular Theorem 1.1, 1.3.1 and 1.3.3 there, p satisfies our assumptions including (1.13) with $-\infty < t_1 \leq s < t \leq t_2 < \infty$. Therefore the Schrödinger perturbation \tilde{p} of p satisfies (1.14) in the (finite) time horizon $[t_1, t_2]$ if $q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$ and $\eta \in [0, 1/\Lambda)$, as explained after the proof of Theorem 1.1. We thus recover recent results of [20, Theorems 3.10 and 3.12] (see also Remark 4.10 below). We now explain how p and \tilde{p} are related to parabolic operators. For all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$, we have

$$(4.12) \quad \int_s^\infty \int_{\mathbb{R}^d} p(s, x, u, z) \left[\frac{\partial \phi(u, z)}{\partial u} + L\phi(u, z) \right] dz du = -\phi(s, x).$$

In fact, the identity holds if the function $\phi(s, x)$ is bounded, supported in a finite time interval and uniformly Hölder continuous in x , and if the same is true for its first derivative in time and all its derivatives in space up to the second order. Indeed, if we let

$$h(s, x) = \phi(s, x) + \int_s^\infty \int_{\mathbb{R}^d} p(s, x, u, z) \left[\frac{\partial \phi(u, z)}{\partial u} + L\phi(u, z) \right] dz du,$$

then by [10, Chapter 2, 1.3.3] we have $h \equiv 0$, which proves (4.12). By (4.12) and [6, Lemma 4], the perturbed transition density \tilde{p} corresponds to the

Schrödinger-type operator $L + q$ in a similar way: for all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}(s, x, u, z) \left[\frac{\partial \phi(u, z)}{\partial u} + L\phi(u, z) + q(u, z)\phi(u, z) \right] dz du = -\phi(s, x).$$

REMARK 4.10. In this paper, by a fundamental solution we mean the negative of an integral inverse of a given operator acting on space-time (other authors also use the terms heat kernel and Green function). More specifically, our p and \tilde{p} may be considered post-inverses of the respective differential operators acting on $C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ (cf. Example 4.7 above and [27], [23, p. 13]). In the literature on partial differential equations it is also common to consider the pre-inverses, which necessitate sufficient differentiability of the applications of p, \tilde{p} to test functions [12, Theorem I.5.9], [10, (1.12)]. Still differently, if the operator L is in divergence form, then a common notion is that of the weak fundamental solution, related to integration by parts (see Aronson [2, (2), (8), Sections 6 and 7], Cho, Kim and Park [7], and Liskevich and Semenov [19]). It is also customary to study the operator $\partial f / \partial s - Lf$, which is related to (4.11) by time-reversal $s \mapsto -s$ [10, 13], [2, (7.3)]. The setting of (4.11) and (4.12) is most appropriate from the probabilistic point of view: it agrees with time precedence and notation for (measurable) transition densities of Markov processes, which may be conveniently considered as integral operators on space-time.

REMARK 4.11. As we already mentioned, Zhang [27] gives sharp estimates for perturbations of $p = g$. Sharp Gaussian estimates (corresponding to $p^* = p$) are generally not available by our methods if the (plain) Kato condition and 4G should be used to estimate p_1 . Accordingly, [27] assumes an integral condition related to the Brownian bridge to bound p_1 by p . As we explained in Remark 4.6, the Kato condition for bridges is more restrictive than the parabolic Kato condition (a straightforward general approach using bridges is given in [6]).

We now comment on the uniqueness of \tilde{p} . Trivially, \tilde{p} is unique because it is given by (1.2), rather than implicitly. However, in the literature of the subject a departure point for defining \tilde{p} is usually one of the following Duhamel (perturbation/resolvent) formulas:

$$(4.13) \quad \tilde{p}(s, x, t, y) = p(s, x, t, y) + \int_s^t \int_{\mathbb{R}^d} p(s, x, u, z) q(u, z) \tilde{p}(u, z, t, y) du dz,$$

$$(4.14) \quad \tilde{p}(s, x, t, y) = p(s, x, t, y) + \int_s^t \int_{\mathbb{R}^d} \tilde{p}(s, x, u, z) q(u, z) p(u, z, t, y) du dz.$$

In short, $\tilde{p} = p + pq\tilde{p}$ or $\tilde{p} = p + \tilde{p}qp$, depending on whether we consider p and \tilde{p} as pre- or post-inverses, respectively, of the corresponding differential operators (see [6], [4] for notation related to integral kernels). Clearly, (1.2) yields (4.13) and (4.14). Conversely, iterating (4.13) or (4.14) we get (1.2), and uniqueness, if $(pq)^n\tilde{p}$ or $\tilde{p}(qp)^n$ converge to zero as $n \rightarrow \infty$. So is the case with $(pq)^n\tilde{p}$ under the assumptions of Theorem 1.1, provided \tilde{p} is locally in time majorized by a constant multiple of p^* , because then $(pq)^n p^* \rightarrow 0$. We refer to [3, Theorem 2] and [20, Theorem 1.16] for further discussion of the perturbation formula and uniqueness. We also note that in the setting of *bridges*, there is a natural probabilistic Feynman–Kac-type formula for \tilde{p} [3, Section 6], which readily yields (1.2) and uniqueness.

Here is a general argument leading to (4.14). We consider $-p$ and $-\tilde{p}$ as integral operators on space-time and post-inverses of operators \mathcal{L} and $\mathcal{L} + q$, respectively, which in turn act on the same given set of functions. We have $\tilde{p}(\mathcal{L}\phi + q\phi) = -\phi = p\mathcal{L}\phi$, hence $\tilde{p}\psi = p\psi + \tilde{p}qp\psi$, where $\psi = \mathcal{L}\phi$. If the range of \mathcal{L} uniquely determines measures, then we obtain $\tilde{p} = p + \tilde{p}qp$ as integral kernels. This is the case, e.g., in the context of Example 4.9. We finally note that some majorization of \tilde{p} is needed for uniqueness. For instance, both $p = g_b$ and $p(s, x, u, z) = g_b(s, x, u, z) + 2du + b|z|^2$ satisfy (4.10).

5. Miscellanea. Earlier work by Jakubowski [15] and coauthors [6] in slightly different settings does not require continuity assumptions on Q . Namely we call $Q: \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ *superadditive* if

$$(5.1) \quad Q(s, u) + Q(u, t) \leq Q(s, t) \quad \text{for } s < u < t.$$

For convenience we define $Q(s, t) = 0$ if $s \geq t$. We see that $t \mapsto Q(s, t)$ is non-decreasing, $s \mapsto Q(s, t)$ non-increasing, and the following limit exists:

$$(5.2) \quad Q^-(s, t) = \lim_{h \rightarrow 0^+} Q(s + h, t - h).$$

For instance, if μ is a Radon measure on \mathbb{R} and $Q(s, t) = \mu(\{u \in \mathbb{R} : s \leq u < t\})$, then $Q^-(s, t) = \mu(\{u \in \mathbb{R} : s < u < t\})$.

Clearly, $0 \leq Q^-(s, t) \leq Q(s, t)$ and $Q^-(s, u) + Q^-(u, t) \leq Q^-(s, t)$ if $s \leq u \leq t$. We have $Q^{--} = Q^-$. In fact, $Q^-(u, v) \rightarrow Q^-(s, t)$ as $u \rightarrow s^+$, $v \rightarrow t^-$, because if $0 < h < u - s < k$ and $h < t - v < k$, then $Q(s + k, t - k) \leq Q^-(u, v) \leq Q(s + h, t - h)$. In particular, $s \mapsto Q^-(s, t)$ is right-continuous and $t \mapsto Q^-(s, t)$ is left-continuous. We thus obtain the following result:

COROLLARY 5.1. $Q^-(s, t)$ is regular superadditive.

We note that continuous superadditive functions are used in the theory of rough paths by Lyons [21]. There are further similarities due to the role of iterated integrals in time here and in [21], and many differences related to the fact that we require absolute integrability (but see [16]) and also

integrate/average in space (see (1.9) and (2.2)). We also note that methods similar to ours allow one to handle gradient perturbations, which will be discussed in a forthcoming paper (see also [18, 16]).

The next result shows that if Q is (plain) superadditive, then the factor $\eta + Q(s, t)$ in (1.9) may be replaced with $C\eta + CQ^-(s, t)$, where C comes from (1.8) and CQ^- is regular superadditive. Thus, we may ensure *regular* superadditivity at the expense of increasing η and Q .

LEMMA 5.2. *Assume that p and p^* are transition densities, the function $q \geq 0$ is defined (and measurable) on space-time, $\eta \geq 0$, $C \geq 1$, Q is superadditive, and (1.8) and (1.9) hold. Then for all $s < t$ and $x, y \in X$, we have*

$$\int_s^t \int_X p(s, x, u, z)q(u, z)p^*(u, z, t, y) dz du \leq C[\eta + Q^-(s, t)]p^*(s, x, t, y).$$

Proof. For $x, y \in X$, $s < t$, $0 < h < (t-s)/2$, taking $\gamma = \eta + Q(s+h, t-h)$, by Chapman–Kolmogorov and (1.9), we get

$$\begin{aligned} & \int_{s+h}^{t-h} \int_X p(s, x, u, z)q(u, z)p^*(u, z, t, y) dz du = \int_X \int_X p(s, x, s+h, v) \\ & \quad \times \int_{s+h}^{t-h} \int_X p(s+h, v, u, z)q(u, z)p^*(u, z, t-h, w) dz du p^*(t-h, w, t, y) dw dv \\ & \leq \gamma \int_X \int_X p(s, x, s+h, v)p^*(s+h, v, t-h, w)p^*(t-h, w, t, y) dw dv. \end{aligned}$$

Again by Chapman–Kolmogorov, the above equals

$$(5.3) \quad [\eta + Q(s+h, t-h)] \int_X p(s, x, s+h, v)p^*(s+h, v, t, y) dv,$$

which leads to the upper bound $C[\eta + Q(s+h, t-h)]p^*(s, x, t, y)$, by (1.8) and Chapman–Kolmogorov. We then let $h \rightarrow 0^+$, and use (5.2) and the monotone convergence theorem, ending the proof. ■

If p^* is a time-changed p , then we can do even better.

LEMMA 5.3. *Under the assumptions of Lemma 5.2 we have*

$$\int_s^t \int_X p(s, x, u, z)q(u, z)p^*(u, z, t, y) dz du \leq [\eta + Q^-(s, t)]p^*(s, x, t, y)$$

if $t \mapsto p(s, x, t, y)$, $t \in (s, \infty)$, is continuous, p is time-homogeneous: $p(s, x, t, y) = p(s+r, x, t+r, y)$ for $r \in \mathbb{R}$, and $p^*(s, x, t, y) = p(as, x, at, y)$ for some $a > 0$.

Proof. Picking up (5.3), for $s < t$, $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned} & \limsup_{h \rightarrow 0^+} \int_X p(s, x, s+h, v) p^*(s+h, v, t, y) dv \\ &= \limsup_{h \rightarrow 0^+} \int_X p(s, x, s+h, v) p(s+h, v, s+h+s(t-s-h), y) dv \\ &= \limsup_{h \rightarrow 0^+} p^*(s, x, t-h+h/a) = p^*(s, x, t, y). \blacksquare \end{aligned}$$

Lemma 5.3 applies to the Gaussian density, if $p = g_b$ and $p^* = g_a$, where $0 < a < b$. Indeed, $g_b(s, x, t, y) = g_a(as/b, x, at/b, y)$ (see also (1.7)).

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