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# Dual spaces and translation invariant means on group von Neumann algebras

by

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**Abstract.** Let $G$ be a locally compact group. Its dual space, $G^*$, is the set of all extreme points of the set of normalized continuous positive definite functions of $G$. In the early 1970s, Granirer and Rudin proved independently that if $G$ is amenable as discrete, then $G$ is discrete if and only if all the translation invariant means on $L^\infty(G)$ are topologically invariant. In this paper, we define and study $G^*$-translation operators on $\mathrm{VN}(G)$ via $G^*$ and investigate the problem of the existence of $G^*$-translation invariant means on $\mathrm{VN}(G)$ which are not topologically invariant. The general properties of $G^*$ are also investigated.

**1. Introduction.** Let $G$ be a locally compact group, and let $A(G)$, $B(G)$ and $\mathrm{VN}(G)$ be the Fourier algebra, Fourier–Stieltjes algebra and group von Neumann algebra of $G$, respectively, as defined by Eymard [11]. If $G$ is abelian, $A(G)$ can be identified with $L^1(\hat{G})$ via the Fourier transform, $\mathrm{VN}(G)$ can be identified with $L^\infty(\hat{G})$ via the adjoint of the Fourier transform, and $B(G)$ can be identified with $M(\hat{G})$ via the Fourier–Stieltjes transform, where $\hat{G}$ is the dual group of $G$.

The dual space of $G$, $G^*$, is defined to be the set of all extreme points in the set of all continuous positive definite functions on $G$ with norm one (see [6], [5] and [23]). If $G$ is abelian, $G^*$ is just the set of all Dirac measures in $\hat{G}$.

Translation operators are fundamental to the classical theory of $L^1(G)$ and $L^\infty(G)$. Thus, it is natural to search for a non-commutative version of translation operators in $A(G)$ and $\mathrm{VN}(G)$. We find that "generalized" translation operators on $A(G)$ and $\mathrm{VN}(G)$ can be defined by using $G^*$. Note that if $G$ is abelian, the generalized translation operators on $A(G)$ and

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$\mathrm{VN}(G)$ are precisely the usual translation operators on $L^1(\hat{G})$ and $L^\infty(\hat{G})$ under the respective identifications.

The notion of amenability of a group was formulated by von Neumann. Later, Day defined amenability of a locally compact group $G$ by using translation invariant means on $L^\infty(G)$ (see [21], [26]). As mentioned above, $\mathrm{VN}(G)$ can be viewed as the dual object of $L^\infty(G)$. Since we have a non-commutative analogue of translation invariant means on $\mathrm{VN}(G)$, it allows us to define translation invariant means on $\mathrm{VN}(G)$.

Granirer [13] and Rudin [29] proved independently that if $G$ is amenable as discrete, then $G$ is discrete if and only if all the translation invariant means on $L^\infty(G)$ are topologically invariant. However, this is no longer true in general even when $G$ is a compact group (see [7]). As a direct consequence of Granirer–Rudin's Theorem, we have the following observation: if $G$ is abelian, then $G$ is compact if and only if all the translation invariant means on $\mathrm{VN}(G)$ are topologically invariant (see [18]). We shall prove that this result is not true for general locally compact groups. One of the main purposes of this paper is to generalize this result to non-abelian groups under certain assumptions.

Let $H$ be a closed subgroup of $G$, and let $\pi$ be a unitary representation of $G$. It is natural to ask if the restriction of $\pi$ to $H$ is always a direct sum of irreducible representations of $H$. Surprisingly, by making use of a result concerning translation invariant means on $\mathrm{VN}(G)$ and Granirer–Rudin's result mentioned above, we give a negative answer to this question.

There is a one-to-one correspondence between locally compact abelian groups and their dual groups. As a result, properties of a locally compact abelian group $G$ can be recovered from properties of its dual group $\hat{G}$. In the general setting, although dual spaces lack the group structure in general, there is still a close relationship between properties of $G$ and those of $G^*$. In particular, we may characterize compact groups, abelian groups and discrete groups using properties of their dual spaces.

This paper is organized as follows: In Section 2, we provide necessary definitions and notations for the rest of this paper. In Section 3, we study the relationship between generalized translation invariant means on $\mathrm{VN}(G)$ and on $\mathrm{VN}(H)$ where $H$ is a closed subgroup of a locally compact group $G$. In particular, we prove the restriction theorem for generalized translation invariant means on $\mathrm{VN}(G)$. In Section 4, we show that Granirer–Rudin's result does not hold in general. We then generalize Granirer–Rudin's Theorem under various assumptions. In Section 5, we answer a question in representation theory by applying the restriction theorem. In Section 6, we study the properties of dual spaces. We provide characterizations of compact groups, abelian groups and discrete groups using properties of their dual spaces.

In the last section, we generalize some classical results on translation invariant means to the non-commutative setting.

**2. Some preliminaries.** Let $(X,\tau)$ be a topological space, and let $Y$ be a subset of $X$. Denote by $\overline{Y}^{\tau}$ and $Y^o$ the closure of $Y$ and the interior of $Y$, respectively.

Let $E$ be a Banach space. Throughout this paper, $E_1$ and $S_E$ will denote the unit ball and the boundary of the unit ball of $E$, respectively. Let $K$ be a subset of $E$. We denote by $\mathcal{E}(K)$ the set of all extreme points of $K$, and by $\mathrm{co}(K)$ the algebraic convex hull of $K$. Let $E'$ be the Banach dual space of $E$, which consists of all bounded linear functionals on $E$.

In this paper, all groups will be assumed to be locally compact, and $G$ will denote a locally compact group with a fixed Haar measure. Let $f$ be a function on $G$ and $y\in G$. We define the left and right translates of $f$ through $y$ by

$$L_y f(x) = f(y^{-1}x), \qquad R_y f(x) = f(xy).$$

We also write ${}_x f$ and $f_x$ for the functions $f(x\cdot)$ and $f(\cdot x)$, respectively.

Let $\Sigma_G$ be the class of unitary representations of $G$, and let $\lambda_2 : G\to B(L^2(G))$, $[\lambda_2(x)(f)](y) := f(x^{-1}y)$ $(x,y\in G,\ f\in L^2(G))$, be the *left regular representation* of $G$. We will also denote by $\hat{G}$ the class of equivalence classes of irreducible unitary representations of $G$. If $G$ is abelian, $\hat{G}$ is just the dual group of $G$.

Let $G$ be a locally compact group. For any $f\in L^1(G)$, define

$$\|f\|_{C^*(G)} := \sup_{\pi\in\Sigma_G}\|\pi(f)\|.$$

It is easily seen that $\|\cdot\|_{C^*(G)}$ is a $C^*$-norm on $L^1(G)$. Let $C^*(G)$ be the completion of $L^1(G)$ under $\|\cdot\|_{C^*(G)}$. Then $C^*(G)$ is called the *full group $C^*$-algebra* or simply the *group $C^*$-algebra* of $G$.

Let $B(G) := \{x\mapsto\langle\pi(x)\xi,\eta\rangle : \pi\in\Sigma_G,\ \xi,\eta\in\mathcal{H}_\pi\}$ be the *Fourier–Stieltjes algebra* of $G$. It is a commutative Banach algebra with pointwise multiplication and norm given by

$$\|u\|_{B(G)} = \inf\{\|\xi\|\,\|\eta\| : u(x) = \langle\pi(x)\xi,\,\eta\rangle,\ \pi\in\Sigma_G,\ \xi,\eta\in\mathcal{H}_\pi\}.$$

Let $A(G) := \{x\mapsto\langle\lambda_2(x)\xi,\eta\rangle : \xi,\eta\in L^2(G)\}$ be the *Fourier algebra* of $G$. It is well-known that $A(G)$ is a closed ideal of $B(G)$.

Let $P(G)$ be the set of all continuous positive definite functions on $G$, i.e.,

$$P(G) := \Big\{\phi\in B(G) : \int(f^* * f)\phi\ge 0 \text{ for any } f\in L^1(G)\Big\}.$$

It can be shown that $P(G) = \{\langle\pi(\cdot)\xi,\xi\rangle : \pi\in\Sigma_G,\ \xi\in\mathcal{H}_\pi\}$ and that

$\phi(e) = \|\phi\|_{B(G)}$. Let $P_1(G) = S_{B(G)} \cap P(G)$. In other words,

$$P_1(G) = \{\langle \pi(\cdot)\xi, \xi\rangle : \pi \in \Sigma_G,\ \xi \in \mathcal{H}_\pi,\ \|\xi\| = 1\}$$

(see [8]). The dual space $G^*$ is defined to be the set $\mathcal{E}(P_1(G))$ (see [6] and [5] for more details).

For any $f \in L^1(G)$, define

$$\|f\|_{C_r^*} := \|\lambda_2(f)\|.$$

It is easily seen that $\|\cdot\|_{C_r^*(G)}$ is a $C^*$-norm on $L^1(G)$. Let $C_r^*(G)$ be the completion of $L^1(G)$ under $\|\cdot\|_{C_r^*(G)}$. It is called the *reduced group $C^*$-algebra* of $G$. Let $B_r(G) := \{x \mapsto \langle \pi(x)\xi, \eta\rangle : \pi \in \Sigma_G$, $\pi$ is weakly contained in $\lambda_2$, $\xi, \eta \in \mathcal{H}_\pi\}$ be the *reduced Fourier–Stieltjes algebra* of $G$. It is a closed ideal of $B(G)$ and can be regarded as the dual space of $C_r^*(G)$.

Let $\mathrm{VN}(G)$ be the von Neumann algebra generated by the image of $\lambda_2$ in $B(L^2(G))$. It is called the *group von Neumann algebra* of $G$. It was proved by Eymard [11] that $A(G)' = \mathrm{VN}(G)$. For $u \in A(G)$ and $T \in \mathrm{VN}(G)$, define $u \cdot T \in \mathrm{VN}(G)$ by

$$\langle u \cdot T, v\rangle = \langle T, uv\rangle, \qquad v \in A(G).$$

Let $\mathrm{UCB}(\hat{G})$ be the closed linear span of $A(G) \cdot \mathrm{VN}(G)$ in $\mathrm{VN}(G)$. The set of all $T$ in $\mathrm{VN}(G)$ for which the operator from $A(G)$ to $\mathrm{VN}(G)$ given by $u \mapsto u \cdot T$ is weakly compact (resp. compact) is denoted by $\mathrm{WAP}(\hat{G})$ (resp. $\mathrm{AP}(\hat{G})$), *the weakly almost periodic* (resp. *almost periodic*) *functionals in* $\mathrm{VN}(G)$.

Let $\mathbb{C}^G$ be the collection of all complex-valued functions on $G$, and $D(G)$ be any subset of $\mathbb{C}^G$. If $H$ is a closed subgroup of $G$, then we set

$$D(G)|_H := \{f|_H : f \in D(G)\}.$$

**3. Main result: a restriction theorem for invariant means.** We begin with the definition of $G^*$-translation operators.

For any $g^* \in G^*$, the operator $L_{g^*} : A(G) \to A(G)$, $f \mapsto g^* f$, is called the *$G^*$-translation operator on* $A(G)$ via $g^*$.

The Banach adjoint of $L_{g^*}$, $L_{g^*}^t : \mathrm{VN}(G) \to \mathrm{VN}(G)$, is called the *$G^*$-translation operator on* $\mathrm{VN}(G)$ via $g^*$. In this case, we write $g^* \cdot T = L_{g^*}^t(T)$ for any $T \in \mathrm{VN}(G)$.

A subset $E \subseteq A(G)$ (resp. $F \subseteq \mathrm{VN}(G)$) is said to be *$G^*$-translation invariant* if $g^* E \subseteq E$ for any $g^* \in G^*$ (resp. $g^* \cdot F \subseteq F$ for any $g^* \in G^*$).

In addition, a subset $F \subseteq \mathrm{VN}(G)$ is said to be *topologically invariant* if $\phi \cdot F \subseteq F$ for any $\phi \in A(G) \cap P_1(G)$.

A subspace of $\mathrm{VN}(G)$ is said to be *invariant* if it is both topologically invariant and $G^*$-translation invariant.

Note that $C_r^*(G)$, $\mathrm{AP}(\hat{G})$, $\mathrm{WAP}(\hat{G})$ and $\mathrm{UCB}(\hat{G})$ are invariant subspaces of $\mathrm{VN}(G)$.

Let $a_0(G)$ be the closure of the span of $G^*$ in $B(G)$, and let $A_{\mathcal{F}}(G)$ be the $\|\cdot\|_{B(G)}$ closure of $\{x \mapsto \langle \pi(x)\xi, \eta\rangle : \pi$ is a finite-dimensional representation of $G$, $\xi, \eta \in \mathcal{H}_\pi\}$. Let $\hat{G}_{\mathcal{F}}$ be the set of all finite-dimensional irreducible representations of $G$, and let $\pi_F = \bigoplus_{\pi \in \hat{G}_{\mathcal{F}}} \pi$. It is easy to see that $A_{\mathcal{F}}(G) = A_{\pi_F}(G) \subseteq a_0(G)$.

Write $G^*_{\mathcal{F}} = \{x \mapsto \langle \pi(x)\xi, \xi\rangle : \pi \in \hat{G}_{\mathcal{F}},\ \xi \in \mathcal{H}_\pi,\ \|\xi\| = 1\}$.

Let $E$ be an invariant subspace of $\mathrm{VN}(G)$ that is closed under involution and contains $\lambda_2(e)$. Let $m$ be a linear functional on $E$ such that $m(\lambda_2(e)) = 1$. Then:

(a) $m$ is said to be a *topologically invariant mean* if $m(\phi \cdot T) = m(T)$ for any $\phi \in A(G) \cap P_1(G)$ and $T \in E$.
(b) $m$ is said to be a *$G^*$-translation invariant mean* if $m(g^* \cdot T) = m(T)$ for any $g^* \in G^*$ and $T \in E$.
(c) $m$ is said to be an *$\mathcal{F}$-translation invariant mean* if $m(g^* \cdot T) = m(T)$ for any $g^* \in G^*_{\mathcal{F}}$ and $T \in E$.

Let $\mathrm{IM}(\hat{G})$, $\mathrm{FIM}(\hat{G})$ and $\mathrm{TIM}(\hat{G})$ be the sets of all $G^*$-translation invariant means, all $\mathcal{F}$-translation invariant means and all topologically invariant means on $\mathrm{VN}(G)$, respectively. Obviously, $\mathrm{FIM}(\hat{G}) \supseteq \mathrm{IM}(\hat{G})$. Since $A(G)$ is an ideal of $B(G)$, we have $\mathrm{IM}(\hat{G}) \supseteq \mathrm{TIM}(\hat{G})$.

REMARKS 3.1. Assume that $G$ is a locally compact abelian group.

(a) One can show that the operator $L_{g^*} : A(G) \to A(G)$ is just (can be identified with) the translation operator $L_\xi : L^1(\hat{G}) \to L^1(\hat{G})$ via the element $\xi = g^*$ in the dual group $\hat{G} = G^*$.
(b) It is not hard to see that $\mathrm{TIM}(\hat{G})$ is the set of all topological invariant means on $L^\infty(\hat{G})$. Since every irreducible representation is finite-dimensional, $\mathrm{IM}(\hat{G}) = \mathrm{FIM}(\hat{G})$ is the set of all translation invariant means on $L^\infty(\hat{G})$.
(c) It can be deduced easily from Granirer and Rudin's result [13], [29] that $G$ is compact if and only if $\mathrm{FIM}(\hat{G}) = \mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$.
(d) Clearly, if $\mathrm{TIM}(\hat{G}) = \mathrm{FIM}(\hat{G})$, then $\mathrm{TIM}(\hat{G}) = \mathrm{IM}(\hat{G})$.

Since $A(G)$ is an ideal of $B(G)$, there is a natural $B(G)$-action on $\mathrm{VN}(G)$ defined via

$$\langle \psi \cdot T, f\rangle = \langle T, \psi f\rangle$$

where $f \in A(G)$, $T \in \mathrm{VN}(G)$ and $\psi \in B(G)$.

For any $m \in \mathrm{FIM}(\hat{G})$, define

$$B_m(G) = \{u \in B(G) : m(u \cdot T) = u(e)m(T) \text{ for any } T \in \mathrm{VN}(G)\}.$$

The following propositions are crucial to the main result; their proofs are easy.

Proposition 3.2. *Let $G$ be a locally compact group. Then:*

(a) *$B_m(G)$ is a closed subalgebra of $B(G)$ containing $A_{\mathcal{F}}(G)$.*
(b) *$m \in \mathrm{IM}(\hat{G})$ if and only if $a_0(G) \subseteq B_m(G)$.*
(c) *$m \in \mathrm{TIM}(\hat{G})$ if and only if $A(G) \subseteq B_m(G)$.*

Put $B_{\mathrm{IM}}(G) = \bigcap_{m\in \mathrm{IM}(\hat{G})} B_m(G)$ and $B_{\mathrm{FIM}}(G) = \bigcap_{m\in \mathrm{FIM}(\hat{G})} B_m(G)$.

Proposition 3.3. *Let $G$ be a locally compact group. Then:*

(a) *$B_{\mathrm{IM}}(G)$ is a closed subalgebra of $B(G)$ containing $a(G)$.*
(b) *$B_{\mathrm{FIM}}(G)$ is a closed subalgebra of $B(G)$ containing $A_{\mathcal{F}}(G)$.*
(c) *$\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$ if and only if $A(G) \subseteq B_{\mathrm{IM}}(G)$.*
(d) *$\mathrm{FIM}(\hat{G}) = \mathrm{TIM}(\hat{G})$ if and only if $A(G) \subseteq B_{\mathrm{FIM}}(G)$.*
(e) *$\mathrm{FIM}(\hat{G}) = \mathrm{IM}(\hat{G})$ if and only if $a_0(G) \subseteq B_{\mathrm{FIM}}(G)$.*

Let $(X, \Sigma, \mu)$ be a measure space. A Banach space $B$ has the *Radon–Nikodym property* (RNP) *with respect to* $\mu$ if, for every countably additive vector measure $\gamma$ on $(X, \Sigma)$ with values in $B$ which has bounded variation and is absolutely continuous with respect to $\mu$, there is a $\mu$-integrable function $g : X \to B$ such that

$$\gamma(E) = \int_E g\, d\mu$$

for every $E \in \Sigma$.

A Banach space $B$ has the *Radon–Nikodym property* (RNP) if $B$ has RNP with respect to every finite measure.

A locally compact group is a *[Moore]-group* if each of its irreducible unitary representations is finite-dimensional.

A locally compact group $G$ is called an *[AR]-group* if the left regular representation of $G$ is completely reducible. It is known that $G$ is an [AR]-group if and only if $A(G)$ has RNP (see [20], [22] and [31] for more details).

Proposition 3.4. *Let $G$ be a locally compact group.*

(a) *If $G \in$ [AR] (i.e. $A(G)$ has RNP), then $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$.*
(b) *If $G$ is compact, then $\mathrm{FIM}(\hat{G}) = \mathrm{TIM}(\hat{G})$.*
(c) *If $G \in$ [Moore], then $\mathrm{IM}(\hat{G}) = \mathrm{FIM}(\hat{G})$.*

*Proof.* We only prove (a). By [3, Theorem 3] and Proposition 3.3, we have $A(G) \subseteq a_0(G) \subseteq B_{\mathrm{IM}}(G)$. Hence, $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$. ∎

If $H$ is a closed subgroup of $G$, then $A_{\mathcal{F}}(G)|_H \subseteq A_{\mathcal{F}}(H)$.

Let $G$ be a locally compact group. Suppose that $H$ is a closed subgroup of $G$. Let $\Psi : A(G) \to A(H)$ be the restriction map, that is, $u \mapsto u|_H$.

LEMMA 3.5. *Let $\phi \in B(H)$ and $T \in \mathrm{VN}(H)$, and let $\psi \in B(G)$ be such that $\psi|_H = \phi$. Then $\Psi^*(\phi \cdot T) = \psi \cdot \Psi^*(T)$.*

*Proof.* For each $u \in A(G)$, we have

$$\begin{aligned}\langle \Psi^*(\phi \cdot T), u\rangle &= \langle \phi \cdot T, \Psi(u)\rangle = \langle \phi \cdot T, u|_H\rangle = \langle T, \phi u|_H\rangle = \langle T, (\psi u)|_H\rangle \\ &= \langle T, \Psi(\psi u)\rangle = \langle \Psi^*(T), \psi u\rangle = \langle \psi \cdot \Psi^*(T), u\rangle.\end{aligned}$$

Therefore, $\Psi^*(\phi \cdot T) = \psi \cdot \Psi^*(T)$. ∎

THEOREM 3.6. *We have*

$$\Psi^{**}(\mathrm{FIM}(\hat{G})) \supseteq \mathrm{FIM}(\hat{H}) \quad \text{and} \quad \Psi^{**}(\mathrm{TIM}(\hat{G})) = \mathrm{TIM}(\hat{H}).$$

*Proof.* Let $m \in \mathrm{FIM}(\hat{H})$. Put $K = \{M \in \mathrm{VN}(G)^*_+ : \Psi^{**}(M) = m\}$. Since $A(H) \cap P(H)$ is weak*-dense in $\mathrm{VN}(H)^*_+$, there is a net $(m_\alpha) \subseteq A(H) \cap P(H)$ such that $m_\alpha \to^{w^*} m$. Also, note that $\Psi(A(G) \cap P(G)) = A(H) \cap P(H)$ (see [17]). For each $\alpha$, there exists $M_\alpha \in A(G) \cap P(G)$ such that $\Psi(M_\alpha) = m_\alpha$. By passing to a subnet, we may assume that $M_\alpha \to^{w^*} M$ where $M \in \mathrm{VN}(G)^*_+$. Then $m_\alpha = \Psi(M_\alpha) \to^{w^*} \Psi^{**}(M)$. Therefore, $\Psi^{**}(M) = m$, whence $K$ is non-empty. It is easy to check that $K$ is a weak*-compact convex subset of $\mathrm{VN}(G)^*$. For any $g^* \in G^*_{\mathcal{F}}$, define $T_{g^*} : K \to K$ by

$$T_{g^*}(M) = g^* \cdot M \quad \text{where} \quad \langle g^* \cdot M, T\rangle = \langle M, g^* \cdot T\rangle.$$

We need to show that $T_{g^*}$ is well-defined. In fact, for any $T \in \mathrm{VN}(H)$, by using Lemma 3.5, we have

$$\begin{aligned}\langle \Psi^{**}(g^* \cdot M), T\rangle &= \langle g^* \cdot M, \Psi^*(T)\rangle = \langle M, g^* \cdot \Psi^*(T)\rangle \\ &= \langle M, \Psi^*(g^*|_H \cdot T)\rangle = \langle \Psi^{**}(M), g^*|_H \cdot T\rangle \\ &= g^*(e)\langle \Psi^{**}(M), T\rangle = \langle m, T\rangle,\end{aligned}$$

where the second last equality follows from the fact that $A_{\mathcal{F}}(G)|_H \subseteq A_{\mathcal{F}}(H)$. Thus, $\{T_{g^*} : g^* \in G^*_{\mathcal{F}}\}$ is a commuting family of weak*-weak*-continuous affine maps from $K$ to $K$. Therefore, by the Markov–Kakutani fixed point theorem, there is an element $M_0 \in K$ such that $M_0 = g^* \cdot M_0$. Hence, $\Psi^{**}(\mathrm{FIM}(\hat{G})) \supseteq \mathrm{FIM}(\hat{H})$. The last equality can be proved similarly. ∎

COROLLARY 3.7. *Let $G$ be a locally compact group and $H$ a closed subgroup of $G$. Then:*

(a) $B_{\mathrm{FIM}}(G)|_H \subseteq B_{\mathrm{FIM}}(H)$.
(b) *If* $\mathrm{FIM}(\hat{G}) = \mathrm{TIM}(\hat{G})$, *then* $\mathrm{FIM}(\hat{H}) = \mathrm{TIM}(\hat{H})$.

*Proof.* (a) Let $u \in B_{\mathrm{FIM}}(G)$. Then $u|_H \in B(H)$. For any $m \in \mathrm{FIM}(\hat{G})$ and $S \in \mathrm{VN}(H)$, we have

$$\begin{aligned}\langle \Psi^{**}(m), u|_H \cdot S\rangle &= \langle m, \Psi^*(u|_H \cdot S)\rangle = \langle m, u \cdot \Psi^*(S)\rangle \\ &= u(e)\langle m, \Psi^*(S)\rangle = u|_H(e)\langle \Psi^{**}(m), S\rangle.\end{aligned}$$

Therefore, $B_{\mathrm{FIM}}(G)|_H \subseteq B_{\mathrm{FIM}}(H)$.

(b) If $\mathrm{FIM}(\hat{G}) = \mathrm{TIM}(\hat{G})$, then $A(G) \subseteq B_{\mathrm{FIM}}(G)$, and consequently $A(H) = A(G)|_H \subseteq B_{\mathrm{FIM}}(G)|_H$. This implies that $A(H) \subseteq B_{\mathrm{FIM}}(H)$ by (a). Hence, $\mathrm{FIM}(\hat{H}) = \mathrm{TIM}(\hat{H})$. ■

THEOREM 3.8. *Let $G$ be a locally compact group. Suppose that $H$ is a closed subgroup of $G$.*

(a) *If $a_0(H) \subseteq a_0(G)|_H$, then $\Psi^{**}(\mathrm{IM}(\hat{G})) \subseteq \mathrm{IM}(\hat{H})$.*
(b) *If $a_0(G)|_H \subseteq a_0(H)$, then $\Psi^{**}(\mathrm{IM}(\hat{G})) \supseteq \mathrm{IM}(\hat{H})$.*

*Proof.* (a) Let $M \in \mathrm{IM}(\hat{G})$, $\phi \in a_0(H)$ and $T \in \mathrm{VN}(H)$. By assumption, there exists $\psi \in a_0(G)$ such that $\psi|_H = \phi$. Then

$$\begin{aligned}\langle \Psi^{**}(M), \phi \cdot T\rangle &= \langle M, \Psi^*(\phi \cdot T)\rangle = \langle M, \psi \cdot \Psi^*(T)\rangle \\ &= \psi(e)\langle M, \Psi^*(T)\rangle = \phi(e)\langle \Psi^{**}(M), T\rangle.\end{aligned}$$

Therefore, $\Psi^{**}(M)$ is $G^*$-translation invariant. Note that $\Psi^*(\lambda_H(e)) = \lambda_G(e)$. It follows that $\langle \Psi^{**}(M), \lambda_H(e)\rangle = \langle M, \lambda_G(e)\rangle = 1$. The positivity of $\Psi^{**}(M)$ is clear. Hence, $\Psi^{**}(M) \in \mathrm{IM}(\hat{H})$.

(b) Let $m \in \mathrm{IM}(\hat{H})$. Put $K = \{M \in \mathrm{VN}(G)^*_+ : \Psi^{**}(M) = m\}$. Since $A(H) \cap P(H)$ is weak*-dense in $\mathrm{VN}(H)^*_+$, there is a net $(m_\alpha) \subseteq A(H) \cap P(H)$ such that $m_\alpha \to^{w^*} m$. Also, note that $\Psi(A(G) \cap P(G)) = A(H) \cap P(H)$. For each $\alpha$, there exists $M_\alpha \in A(G) \cap P(G)$ such that $\Psi(M_\alpha) = m_\alpha$. By passing to a subnet, we may assume that $M_\alpha \to^{w^*} M$ where $M \in \mathrm{VN}(G)^*_+$. Then $m_\alpha = \Psi(M_\alpha) \to^{w^*} \Psi^{**}(M)$. Therefore, $\Psi^{**}(M) = m$, whence $K$ is non-empty. It is easy to check that $K$ is a weak*-compact convex subset of $\mathrm{VN}(G)^*$. For any $g^* \in G^*$, define $T_{g^*} : K \to K$ by

$$T_{g^*}(M) = g^* \cdot M \quad \text{where} \quad \langle g^* \cdot M, T\rangle = \langle M, g^* \cdot T\rangle.$$

We need to show that $T_{g^*}$ is well-defined. In fact, for any $T \in \mathrm{VN}(H)$, by using Lemma 3.5, we have

$$\begin{aligned}&\langle \Psi^{**}(g^* \cdot M), T\rangle = \langle g^* \cdot M, \Psi^*(T)\rangle = \langle M, g^* \cdot \Psi^*(T)\rangle \\ &\quad = \langle M, \Psi^*(g^*|_H \cdot T)\rangle = \langle \Psi^{**}(M), g^*|_H \cdot T\rangle = g^*(e)\langle \Psi^{**}(M), T\rangle = \langle m, T\rangle\end{aligned}$$

where the second last equality follows from the assumption that $a_0(G)|_H \subseteq a_0(H)$. Thus, $\{T_{g^*} : g^* \in G^*\}$ is a commuting family of weak*-weak*-continuous affine maps from $K$ to $K$. Therefore, by the Markov–Kakutani fixed point theorem, there is an element $M_0 \in K$ such that $M_0 = g^* \cdot M_0$. Hence, $\Psi^{**}(\mathrm{IM}(\hat{G})) \supseteq \mathrm{IM}(\hat{H})$. ■

REMARK 3.9. Note that the proofs of Theorems 3.6 and 3.8(b) are very similar. While one of the key results used in the proof of Theorem 3.6 is that $A_{\mathcal{F}}(G)|_H \subseteq A_{\mathcal{F}}(H)$, the inclusion $a_0(G)|_H \subseteq a_0(H)$ does not always hold (see Section 5). Therefore, this inclusion becomes an assumption of Theorem 3.8(b).

THEOREM 3.10. *Let $G$ be a locally compact group. Suppose that $H$ is a closed subgroup of $G$ such that $a_0(G)|_H \subseteq a_0(H)$. Then:*

(a) $B_{\mathrm{IM}}(G)|_H \subseteq B_{\mathrm{IM}}(H)$.
(b) *If* $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$, *then* $\mathrm{IM}(\hat{H}) = \mathrm{TIM}(\hat{H})$.

*Proof.* (a) Let $u \in B_{\mathrm{IM}}(G)$. Then $u|_H \in B(H)$. For any $m \in \mathrm{IM}(\hat{G})$ and $S \in \mathrm{VN}(H)$, we have

$$\begin{aligned}\langle \Psi^{**}(m), u|_H \cdot S\rangle &= \langle m, \Psi^*(u|_H \cdot S)\rangle \\ &= \langle m, u \cdot \Psi^*(S)\rangle = u(e)\langle m, \Psi^*(S)\rangle = u|_H(e)\langle \Psi^{**}(m), S\rangle.\end{aligned}$$

Therefore, $B_{\mathrm{IM}}(G)|_H \subseteq B_{\mathrm{IM}}(H)$.

(b) If $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$, then $A(G) \subseteq B_{\mathrm{IM}}(G)$, and consequently $A(H) = A(G)|_H \subseteq B_{\mathrm{IM}}(G)|_H$. This implies that $A(H) \subseteq B_{\mathrm{IM}}(H)$ by (a). Hence, $\mathrm{IM}(\hat{H}) = \mathrm{TIM}(\hat{H})$. ∎

**4. Generalizations of Granirer–Rudin's Theorem.** The fact that $\mathrm{IM}(\hat{G}) \neq \mathrm{TIM}(\hat{G})$ for any non-compact abelian group $G$ is a direct consequence of Granirer–Rudin's Theorem [13], [29]. However, it is impossible to remove the commutativity condition, as shown below.

EXAMPLE 4.1. If $G$ is a non-compact [AR]-group (say, the "$ax + b$" group or Fell's group), then $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$ by Proposition 3.4. Therefore, unlike the abelian case, there is a non-compact group $G$ such that $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$.

The major purpose of this section is to provide some sufficient conditions on $G$ for $\mathrm{IM}(\hat{G}) \neq \mathrm{TIM}(\hat{G})$.

We first recall the definitions of some important classes of locally compact groups.

Let $G$ be a locally compact group. Then $G$ is called a *[SIN]-group* if it has a base for the neighborhood system at the identity consisting of compact neighborhoods which are invariant under all inner automorphisms of $G$.

A $C^*$-algebra $A$ is said to be *CCR* if $\pi(f)$ is a compact operator for every $f \in A$ and irreducible $^*$-representation $\pi$ of $A$. A group $G$ is called a *[CCR]-group* if $C^*(G)$ is CCR.

A unitary $^*$-representation $\pi$ of $G$ is *primary* if the center of $C(\pi) = \{T \in B(\mathcal{H}_\pi) : T\pi(x) = \pi(x)T$ for any $x \in G\}$ consists of scalar multiples of $I$. And $G$ is said to be a *[Type 1]-group* if every primary representation of $G$ is a direct sum of copies of some irreducible representations.

We say that a locally compact group $G$ is *almost abelian* if it has an abelian subgroup of finite index.

A locally compact group $G$ is called a *central group* if the quotient group $G/Z$ is compact where $Z$ is the center of $G$. An *almost connected* group is a locally compact group $G$ such that the quotient group $G/G_e$ is compact where $G_e$ is the connected component of $e$ in $G$. For more details, see [25, Chapter 12].

For more results of [SIN], [CCR] and [Type 1]-groups, we refer the readers to [25].

The following theorem is the main result in this section.

THEOREM 4.2. *Let $G$ be a locally compact group. Suppose that one of the following conditions holds:*

(a) *$G = G_1 \times G_2$ where $G_1$ is [Type 1] and $G_2$ is any locally compact group such that $\mathrm{IM}(\hat{G}_i) \neq \mathrm{TIM}(\hat{G}_i)$ for some $i = 1, 2$.*
(b) *The center of $G$, $Z(G)$, is non-compact.*
(c) *$G$ is a [Moore]-group which has a closed subgroup $H$ such that $\mathrm{IM}(\hat{H}) \neq \mathrm{TIM}(\hat{H})$.*
(d) *$G$ is a [CCR]-group which has an open subgroup $H$ such that $\mathrm{IM}(\hat{H}) \neq \mathrm{TIM}(\hat{H})$.*

*Then $\mathrm{IM}(\hat{G}) \neq \mathrm{TIM}(\hat{G})$ (i.e. there exists a $G^*$-translation invariant mean on $\mathrm{VN}(G)$ which is not topologically invariant).*

We need the following preparation before proving the main result. In particular, we will provide different sufficient conditions for the inclusion $a_0(G)|_H \subseteq a_0(H)$ to hold.

If $G$ has the $H$-extension property where $H$ is a closed subgroup of $G$ (i.e. $B(G)|_H = B(H)$), then $H^* \subseteq G^*|_H$ (see [16, Proposition 2, p. 275]). Therefore, $a_0(H) \subseteq a_0(G)|_H$.

LEMMA 4.3. *Let $G$ be a [Moore]-group and $H$ a closed subgroup of $G$. Then $a_0(H) = a_0(G)|_H$.*

*Proof.* Every [Moore]-group is a [SIN]-group, so it has the extension property. If $g^* \in G^*$, then $g^*(x) = \langle \pi(x)\epsilon, \epsilon\rangle$ where $\pi \in \hat{G}$ and $\epsilon \in \mathcal{H}_\pi$, $\|\epsilon\| = 1$. Since $G$ is a [Moore]-group, it follows that $\dim(\pi|_H) = \dim(\pi) < \infty$. Therefore, $g^*|_H \in A_{\mathcal{F}}(H) \subseteq a_0(H)$. ∎

LEMMA 4.4. *Let $G$ be a locally compact group and $H$ an open subgroup. Let $\pi$ be a unitary representation of $G$. For any $f \in L^1(H)$, define $\dot{f} \in L^1(G)$ by $\dot{f}(x) = f(x)$ if $x \in H$, and $\dot{f}(x) = 0$ otherwise. Then $\pi(\dot{f}) = \pi|_H(f)$ for any $f \in L^1(H)$, and $\|\dot{f}\|_{C^*(G)} = \|f\|_{C^*(H)}$ for any $f \in L^1(H)$. Hence, the map $L^1(H) \to L^1(G)$, $f \mapsto \dot{f}$, extends to a $C^*$-algebra monomorphism $\Phi : C^*(H) \to C^*(G)$.*

*Proof.* Let $\xi, \eta \in \mathcal{H}_\pi$. Then

$$\langle \pi(\dot{f})\xi, \eta\rangle = \int_G \dot{f}(x)\langle \pi(x)\xi, \eta\rangle\, dx = \int_H f(x)\langle \pi|_H(x)\xi, \eta\rangle\, dx = \langle \pi|_H(f)\xi, \eta\rangle.$$

Therefore, $\pi(\dot{f}) = \pi|_H(f)$ for any $f \in L^1(H)$. For the last statement, note that every unitary representation of $H$ induces a unitary representation of $G$. Hence,

$$\|\dot{f}\|_{C^*(G)} = \sup\{\|\pi(\dot{f})\| : \pi \in \Sigma_G\}$$
$$= \sup\{\|\pi|_H(f)\| : \pi \in \Sigma_G\} = \sup\{\|\pi(f)\| : \pi \in \Sigma_H\} = \|f\|_{C^*(H)}. \blacksquare$$

LEMMA 4.5. *Let $G$ be a [CCR]-group and $H$ an open subgroup of $G$. Then $a_0(H) = a_0(G)|_H$.*

*Proof.* Since $G \in$ [CCR], we have $\pi(C^*(G)) \subseteq \mathcal{K}(\mathcal{H}_\pi)$ for any irreducible representation $\pi$. Thus, $\pi|_H(C^*(H)) = \Phi(C^*(H)) \subseteq \mathcal{K}(\mathcal{H}_\pi)$ where $\Phi$ is defined in Lemma 4.4. By [9, Proposition 5.4.13], $\pi|_H$ is a direct sum of irreducible representations of $H$. Hence, $a_0(H) \supseteq a_0(G)|_H$. ∎

Let $u : G_1 \to \mathbb{C}$ and $v : G_2 \to \mathbb{C}$ be functions. Define $u \otimes v : G_1 \times G_2 \to \mathbb{C}$ by $u \otimes v(x, y) = u(x)v(y)$.

LEMMA 4.6. *Let $G_1$ be a [Type 1]-group and $G_2$ any locally compact group. Let $G = G_1 \times G_2$. Identify $G_1$ as $G_1 \times \{e_2\}$ and $G_2$ as $\{e_1\} \times G_2$. Then $a_0(G_1) = a_0(G)|_{G_1}$ and $a_0(G_2) = a_0(G)|_{G_2}$.*

*Proof.* Note that $\hat{G} \to \hat{G}_1 \times \hat{G}_2$, $\pi \mapsto \pi_1 \otimes \pi_2$, is a bijection. We have

$$a_0(G) = \{(x, y) \mapsto \langle \pi_1 \otimes \pi_2(x, y)\xi, \eta\rangle : \xi, \eta \in \mathcal{H}_{\pi_1 \otimes \pi_2},\ \pi_1 \in \hat{G}_1,\ \pi_2 \in \hat{G}_2\}.$$

It follows that

$$a(G) = \overline{\operatorname{span}(a_0(G_1) \otimes a_0(G_2))}^{\|\cdot\|_{B(G)}}. \blacksquare$$

LEMMA 4.7. *Let $G$ be a locally compact group, and $Z$ be the center of $G$. Then $a_0(G)|_Z \subseteq a_0(Z)$.*

*Proof.* If $\pi$ is an irreducible representation of $G$, then $\pi|_Z$ is a multiple of $\rho$ where $\rho$ is an irreducible representation of $Z$. In particular, $\rho$ is completely irreducible. ∎

We are now ready to prove the main theorem.

*Proof of Theorem 4.2.* (a) Suppose that $\mathrm{IM}(\widehat{G_1 \times G_2}) = \mathrm{TIM}(\widehat{G_1 \times G_2})$. Then $\mathrm{IM}(\hat{G}_i) = \mathrm{TIM}(\hat{G}_i)$ for all $i = 1, 2$ by Theorem 3.10 and Lemma 4.6.

(b) Suppose that $\mathrm{TIM}(\hat{G}) = \mathrm{IM}(\hat{G})$. Then by Theorem 3.10 and Lemma 4.7, $\mathrm{TIM}(\hat{Z}) = \mathrm{IM}(\hat{Z})$ where $Z$ is the center of $G$. Therefore, by Granirer–Rudin's Theorem (see [13] and [29]), we conclude that $Z$ is compact.

(c) This follows from Theorem 3.10 and Lemma 4.3.

(d) This follows from Theorem 3.10 and Lemma 4.5. ∎

The following result should be well known. However, we can also prove it independently by just using the results of the previous section.

COROLLARY 4.8. *Let $G$ be a locally compact group. Suppose that one of the following conditions holds:*

(a) *$G = G_1 \times G_2$ where $G_1$ is a non-compact locally compact abelian group and $G_2$ any locally compact group.*
(b) *$G$ is a [Moore]-group which has a non-compact connected subgroup $H$.*
(c) *$G$ is a [Moore]-group which has a non-compact abelian closed subgroup $H$.*
(d) *$G$ is a [CCR]-group which has a non-compact abelian open subgroup $H$.*
(e) *$G$ is a non-compact connected [SIN]-group.*
(f) *$G$ is a non-compact almost connected [Moore]-group.*
(g) *$G$ is a non-compact almost abelian group.*
(h) *$G$ is a non-compact central group.*

*Then* $\mathrm{IM}(\hat{G}) \neq \mathrm{TIM}(\hat{G})$ (*i.e. there exists a $G^*$-translation invariant mean on* $\mathrm{VN}(G)$ *which is not topologically invariant*).

*Proof.* (a) Follows from Granirer–Rudin's Theorem and Theorem 3.10(a).

(b) Follows from Granirer–Rudin's Theorem and Theorem 3.10(b).

(c) Since $H$ is also a connected [Moore]-group, we have $H = V \times K$ where $V$ is a vector group and $K$ is a compact group (see [25, 12.6.6]). Since $H$ is non-compact, $V$ is not trivial. It follows from Granirer–Rudin's Theorem and (a) that $\mathrm{IM}(\hat{H}) \neq \mathrm{TIM}(\hat{H})$. The proof is completed by using Theorem 3.10(c).

(d) Follows from Granirer–Rudin's Theorem and Theorem 3.10(d).

(e) Note that a connected [SIN]-group is a [Moore]-group. The proof is similar to that of (b).

(f) If $G$ is an almost connected [Moore]-group, then $G = V \times_\eta K$ where $K$ is compact, $V$ is a vector group and $\eta(K)$ is finite (see [25, 12.6.6]). If $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$, then $\mathrm{IM}(\hat{V}) = \mathrm{TIM}(\hat{V})$ by Theorem 3.10, which implies that $V$ is trivial. Consequently, $G$ is compact.

(g) Note that $G$ is a [Moore]-group. Let $H$ be such an abelian subgroup of $G$. If $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$, then $\mathrm{IM}(\hat{H}) = \mathrm{TIM}(\hat{H})$ by Theorem 3.10. Consequently, $H$ is compact. Therefore, $G$ is compact since $H$ is of finite index.

(h) Assume that $G$ is central (i.e. $G/Z(G)$ is compact). Then $Z(G)$ is compact by (b). Hence, $G$ is compact. ∎

The following result concerning [AR]-groups is a direct consequence of the main theorem.

Corollary 4.9. *The center of an [AR]-group is always compact. In particular, if $A(G)$ has RNP, then $A(Z)$ has RNP where $Z$ is the center of $G$.*

*Proof.* Let $G$ be an [AR]-group. Then $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$. The result follows immediately from Theorem 4.2(b). ∎

For $\mathcal{F}$-translation invariant means, we have a stronger result.

THEOREM 4.10. *If $G$ has a non-compact closed abelian subgroup, then*

$$\mathrm{FIM}(\hat{G}) \neq \mathrm{TIM}(\hat{G}).$$

*Proof.* Follows directly from Granirer–Rudin's Theorem and Corollary 3.7. ∎

COROLLARY 4.11. *Let $G$ be a [SIN]-group such that $\mathrm{FIM}(\hat{G}) = \mathrm{TIM}(\hat{G})$. Then every closed connected subgroup of $G$ is compact.*

*Proof.* Similar to the proof of Corollary 4.8(a). ∎

REMARKS 4.12. Let $G$ be the "ax+b"-group, $H = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) : a > 0\right\}$ and $N = \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{R}\right\}$.

Surprisingly, these groups provide a lot of counter-examples in the study of functorial properties of invariant means on $\mathrm{VN}(G)$.

(a) Since $G$ is an [AR]-group, $\mathrm{IM}(\hat{G}) = \mathrm{TIM}(\hat{G})$ by Proposition 3.4. However, $\mathrm{IM}(\hat{H}) \neq \mathrm{TIM}(\hat{H})$ as $H$ is a non-compact abelian group by Granirer–Rudin's Theorem.
(b) By Theorem 4.10, $\mathrm{FIM}(\hat{G}) \neq \mathrm{TIM}(\hat{G})$, but $\mathrm{FIM}(\hat{H}) = \mathrm{TIM}(\hat{H})$ since $H$ is abelian.
(c) Note that $\mathrm{FIM}(\hat{H}) = \mathrm{TIM}(\hat{H}) \neq \mathrm{IM}(\hat{H})$ and $G/N \cong H$. We have $\mathrm{FIM}(\widehat{G/N}) = \mathrm{TIM}(\widehat{G/N}) \neq \mathrm{IM}(\widehat{G/N})$.

We obtain a lot of examples by using the above corollary. Say, if $G$ is $\mathrm{GL}_n(\mathbb{R})$ or the Heisenberg group, then $\mathrm{TIM}(\hat{G}) \neq \mathrm{IM}(\hat{G})$.

## 5. An application in representation theory

QUESTION. Let $H$ be a closed subgroup of $G$, and let $\pi \in \hat{G}$. Is $\pi|_H$ completely reducible in general? (i.e. is $\pi|_H$ a sum of irreducible unitary representations of $H$?)

We have the following observation:

Let $G$ be a locally compact group such that $\mathrm{TIM}(\hat{G}) = \mathrm{IM}(\hat{G})$. If $H$ is closed subgroup of $G$ such that $\mathrm{TIM}(\hat{H}) \neq \mathrm{IM}(\hat{H})$, then $a_0(G)|_H$ is not a subset of $a_0(H)$. Hence, there exists an irreducible representation $\pi$ of $G$ such that $\pi|_H$ is not completely reducible.

Recall that the infinite-dimensional representations of the "ax+b"-group are given by:

$$[\pi_+(a,b)g](s) = a^{1/2}e^{2\pi ibs}g(as) \quad (a > 0,\ b \in \mathbb{R},\ g \in L^2((0,\infty), ds));$$

$$[\pi_-(a,b)g](s) = a^{1/2}e^{2\pi ibs}g(as) \quad (a > 0,\ b \in \mathbb{R},\ g \in L^2((-\infty,0), ds)).$$

The following gives a negative answer to our question. In its proof we will use Theorem 3.10 and Granirer–Rudin's result.

PROPOSITION 5.1. *Let $G$ be the "ax+b"-group, and $H$ be the subgroup of $G$ defined in Remarks* 4.12. *Then $a_0(G)|_H$ is not a subset of $a_0(H)$. Moreover, $\pi_+|_H$ and $\pi_-|_H$ are not completely reducible.*

*Proof.* Since $\hat{G}$ consists of all characters, $\pi_+$ and $\pi_-$, we conclude that $\pi_+|_H$ and $\pi_-|_H$ cannot both be written as direct sums of irreducible representations of $H$. Let $U : L^2((0,\infty), ds) \to L^2((-\infty,0), ds)$ be defined by

$$Ug = \tilde{g} \quad \text{where} \quad \tilde{g}(x) = g(-x).$$

By direct calculation, we can prove that $U\pi_+(a,b) = \pi_-(a,-b)U$ for any $a > 0$, $b \in \mathbb{R}$. Thus $U\pi_+|_H = \pi_-|_H U$, whence $\pi_+|_H$ and $\pi_-|_H$ are unitarily equivalent. Therefore, neither of them can be written as a direct sum of irreducible representations of $H$. ∎

**6. Some general properties of dual spaces.** The purpose of this section is to discuss how the properties of $G$ are related to those of $G^*$. In particular, we give some characterizations of discrete, abelian and compact groups by using properties of their dual spaces. Some of the results presented in this section will be used in the next section.

We start by recalling the following definition. Recall that a net $(\mu_\alpha)$ in $M(G)$ is said to *converge strictly* to $\mu$ if

$$\|g * (\mu_\alpha - \mu)\| + \|(\mu_\alpha - \mu) * g\| \to 0 \quad \text{for any } g \in L^1(G).$$

REMARK 6.1. Let $\delta_G := \{\delta_x : x \in G\}$ be the set of all Dirac measures on $G$. Then its convex hull $\mathrm{co}(\delta_G)$ is strictly dense in $M(G)_1^+$, and hence $l^1(G)$ is also strictly dense in $M(G)$.

Inspired by the above classical definition of strict topology for $M(G)$, we define the strict topology of $B(G)$ analogously:

A net $(f_\alpha)$ in $B(G)$ is said to *converge strictly* to $f$ if

$$\|(f_\alpha - f) \cdot g\| \to 0 \quad \text{for any } g \in A(G).$$

The following lemma is an analogue of the remark above, which will be particularly useful.

LEMMA 6.2. *The convex hull of the dual space,* $\mathrm{co}(G^*)$, *is strictly dense in $P_1(G)$. Hence, the linear span of the dual space,* $\mathrm{span}(G^*)$, *is strictly dense in $B(G)$.*

*Proof.* Note that $\mathrm{co}(G^*) = \mathrm{co}(\mathcal{E}(P_1(G)))$ is weak*-dense in $P_1(G)$ and the strict topology coincides with the weak*-topology in $B(G)_1$ by [15, Theorem B2]. Therefore, $\mathrm{co}(G^*)$ is strictly dense in $P_1(G)$. The rest is trivial. ∎

The proposition below can be proved by applying the previous lemma. This is an analogue of [12, Proposition 2.43], which also motivated the author to study $G^*$.

PROPOSITION 6.3. *Let $I$ be a closed subalgebra of $A(G)$. If $I$ is $G^*$-translation invariant, then $I$ is an ideal of $A(G)$. Suppose, in addition, that $u \in \overline{uA(G)}$ for any $u \in A(G)$. Then $I$ is $G^*$-translation invariant if and only if $I$ is an ideal of $A(G)$.*

*Proof.* Let $\phi \in A(G) \cap P_1(G)$, $f \in I$, and let $(\phi_\alpha)$ be a net in $\mathrm{co}(G^*)$ such that $\phi_\alpha \to \phi$ in the strict topology of $B(G)$. Note that $\phi_\alpha \cdot f \in I$ for each $\alpha$. Since $I$ is norm-closed, it follows that $\phi \cdot f \in I$.

Conversely, let $f \in I$ and $g^* \in G^*$. Then $g^* f \in A(G)$. By assumption, there exists a net $(e_\alpha) \subseteq A(G)$ such that $(g^* f)e_\alpha \to g^* f$. However, $(g^* f)e_\alpha = (g^* e_\alpha)f \in A(G)I \subseteq I$. Therefore, $g^* f \in I$. ∎

Now, we are ready to give some characterizations of discrete groups using properties of $G^*$.

**6.1. Discrete groups.** The following lemma should be well-known; we give the proof for the sake of completeness.

LEMMA 6.4. *Let $X$ be a locally compact Hausdorff space. If $C_0(X)$ has an $n$-dimensional ideal, then $X$ has at least $n$ distinct discrete points. Moreover, if such an $n$-dimensional ideal exists, then it is the linear span of $\delta_{x_1}, \ldots, \delta_{x_n}$ where $x_1, \ldots, x_n \in X$.*

*Proof.* Let $I$ be an $n$-dimensional ideal in $C_0(X)$. Define

$$\Omega := \{x \in G : f(x) \neq 0 \text{ for some } f \in I\}.$$

Clearly, $|\Omega| \geq n$. It remains to show that $|\Omega| \leq n$. Suppose that $x_1, \ldots, x_{n+1} \in \Omega$ are distinct. For each $i \in \{1, \ldots, n+1\}$, by Urysohn's Lemma, there exist $f_i \in I$ and $g_i \in C_0(G)$ such that

$$f_i(x_i) \neq 0 \quad \text{and} \quad g_i(x_j) = \delta_{ij}, \qquad 1 \leq i, j \leq n+1.$$

Let $c_i \in C$ be such that $\sum_{i=1}^{n+1} c_i g_i f_i = 0$. Then for each $j \in \{1, \ldots, n+1\}$,

$$\sum_{i=1}^{n+1} c_i g_i f_i(x_j) = c_j f_j(x_j) = 0, \quad \text{so} \quad c_j = 0.$$

Therefore, $\{g_i f_i\}_{i=1}^{n+1}$ is a linearly independent subset of $I$. This is a contradiction.

The proof of the latter part of the lemma is clear. ∎

The following assertion about $G^*$-translation invariant elements of $A(G)$ and $B(G)$ will be useful in the next section.

THEOREM 6.5. *Let $G$ be a locally compact group. Then the following statements are equivalent:*

(a) *$G$ is discrete.*
(b) *There exists a non-zero $f \in A(G)$ which is $G^*$-translation invariant.*
(c) *There exists a non-zero $f \in B(G)$ which is $G^*$-translation invariant.*

*Moreover, if such an $f$ exists, then $f = c\delta_e$ for some $c \in \mathbb{C}$.*

*Proof.* (a)⇒(c) is clear. If (c) holds, choose $u \in A(G)$ such that $uf \neq 0$. Then $\{uf\}$ is a $G^*$-translation invariant subset in $A(G)$.

Now, suppose that (b) holds. Let $I = \mathbb{C}f \subseteq A(G)$. Let $h \in A(G)$ and let $\{\sum c_\alpha g_\alpha^*\}$ be a net in $\mathrm{span}(G^*)$ which converges to $h$ strictly. Then

$$h \cdot f = \lim_\alpha \sum c_\alpha (g_\alpha^* \cdot f) = \Big(\lim_\alpha \sum c_\alpha\Big) f,$$

so $I$ is a one-dimensional ideal in $A(G)$, which implies that $G$ is discrete.

For the last statement, without loss of generality, we may assume that $f \in A(G)$. Now, $I$ is also an ideal in $C_0(G)$. So $I = \langle \delta_x \rangle$ for some $x \in G$ by Lemma 6.4. It follows that $f = c\delta_x$ for some $c \in \mathbb{C}$. Without loss of generality, assume that $c = 1$. For any $y \in G$, $\pi \in \hat{G}$ and $\xi \in \mathcal{H}_\pi$, $\|\xi\| = 1$, we have $\langle \pi(y)\xi, \xi\rangle \delta_x(y) = \delta_x(y)$. Suppose that $x \neq e$. Pick $\pi_0 \in \hat{G}$ such that $\pi_0(x) \neq \pi_0(e)$. Thus, we have $\langle \pi_0(x)\xi, \xi\rangle = 1$. By Cauchy–Schwarz's inequality, it follows that $\pi_0(x)\xi = \xi$. Hence, $\pi_0(x)$ is the identity map, which is a contradiction. ∎

Remark 6.6. By Lemma 6.4, it is not hard to see that the following statements are equivalent:

(a) $G$ is discrete.
(b) $A(G)$ has a non-zero finite-dimensional ideal.
(c) $C_0(G)$ has a non-zero finite-dimensional ideal.

The following lemma gives a characterization of $G^*$-translation invariant elements of $\mathrm{VN}(G)$. For the definition of the support of an element of $\mathrm{VN}(G)$, the basic reference is [11, Chapter 4].

Theorem 6.7. *Let $T$ be a non-zero element in* $\mathrm{VN}(G)$. *Then the following statements are equivalent:*

(a) *$g^* \cdot T = T$ for all $g^* \in G^*$.*
(b) *$\phi \cdot T = T$ for all $\phi \in P_1(G)$.*
(c) *$\phi \cdot T = T$ for all $\phi \in A(G) \cap P_1(G)$.*
(d) *$T = c\lambda_2(e)$ for some non-zero constant $c \in \mathbb{C}$.*

*Proof.* (a)⇒(b): For any $\phi \in P_1(G)$, there exists a net $(e_\alpha)$ in $\mathrm{co}(G^*)$ such that $e_\alpha \to \phi$ strictly. Observe that $e_\alpha \cdot T = T$. So, for any $u \in A(G)$,

$$\langle T, u\rangle = \langle e_\alpha \cdot T, u\rangle = \langle T, e_\alpha \cdot u\rangle \to \langle T, \phi \cdot u\rangle.$$

Thus, $T = \phi \cdot T$.

(b)⇒(c) is clear.

Suppose that (c) holds. By [11, Proposition 4.4.8], we have $\operatorname{supp}(T) = \operatorname{supp}(\phi \cdot T) \subseteq \operatorname{supp}(\phi) \cap \operatorname{supp}(T)$. It follows that $\operatorname{supp}(T) \subseteq \operatorname{supp}(\phi)$ for any $\phi \in A(G) \cap P_1(G)$. However, for any $x \neq e \in G$, there exists $f$ in $A(G) \cap P_1(G)$ such that $x$ lies outside $\operatorname{supp}(f)$. Therefore, $\operatorname{supp}(T) = \{e\}$. Hence (a) follows by [11, Theorem 4.4.9]. ∎

The following proposition is a consequence of Theorem 6.5, which gives another characterization of discrete groups.

PROPOSITION 6.8. *Let $G$ be a locally compact group. Then the following statements are equivalent:*

(a) *$G$ is discrete.*
(b) *$P_1(G)$ is weak*-compact.*
(c) *$B_r(G) \cap P_1(G)$ is weak*-compact.*

*Hence, if $G^*$ is weak*-compact, then $G$ is discrete.*

*Proof.* If $G$ is discrete, then

$$P_1(G) = \{\phi \in B(G) : \phi(e) = \langle \phi, \delta_e \rangle = 1 = \|\phi\|\}$$

is clearly weak*-compact.

Suppose that (b) holds. For each $g^* \in G^*$, define

$$T_{g^*} : P_1(G) \to P_1(G) \quad \text{by} \quad T_{g^*}(\phi) = g^* \cdot \phi.$$

Then $\{T_{g^*} : g^* \in G^*\}$ is a commuting family of continuous affine maps on $P_1(G)$. By the Markov–Kakutani fixed point theorem, there exists $\phi_0$ in $P_1(G)$ such that $g^* \cdot \phi_0 = \phi_0$. Thus, $G$ is discrete by Theorem 6.5.

The proof of the equivalence of (a) and (c) is similar. ∎

This gives another proof of the following theorem which appears in [24].

COROLLARY 6.9. *Let $G$ be a locally compact group. Then the following statements are equivalent:*

(a) *$G$ is discrete.*
(b) *$C^*(G)$ is unital.*
(c) *$C^*_r(G)$ is unital.*

*Proof.* Note that if $A$ is a unital $C^*$-algebra, its state space is weak*-compact. However, the state spaces of $C^*(G)$ and $C^*_r(G)$ are $P_1(G)$ and $B_r(G) \cap P_1(G)$, respectively. The result therefore follows from Proposition 6.8. ∎

Note that $C^*(G)$ and $C^*_r(G)$ are $B(G)$-bimodules, and hence they are $G^*$-translation invariant.

COROLLARY 6.10. *Let $G$ be a locally compact group. Then the following statements are equivalent:*

(a) *$G$ is discrete.*
(b) *There exists a non-zero $T \in C^*(G)$ which is $G^*$-translation invariant.*
(c) *There exists a non-zero $T \in C^*_r(G)$ which is $G^*$-translation invariant.*

*Moreover, if such a $T$ exists, then $T = c\delta_e$ for some $c \in \mathbb{C}$.*

*Proof.* This follows from Corollary 6.9 and Theorem 6.7. ∎

Next, we will give some characterizations of abelian groups using properties of $G^*$.

**6.2. Abelian groups.** Let $A$ be a $C^*$-algebra. Denote by $(A^*)^1_+$ the state space of $A$.

LEMMA 6.11. *Let $A$ be a $C^*$-algebra. Then $A$ is commutative if and only if, for any $a \in A$, the norm of $a$ is given by*

$$\|a\| = \sup\{|\langle a, f\rangle| : f \in (A^*)^1_+\}.$$

*Proof.* Suppose that $A$ is non-abelian. Then there exists $a \in A$ such that $\|a\| = 1$ and $a^2 = 0$ (see [4, II.6.4.14]). Then for any state $f$ on $A$,

$$\begin{aligned}|f(a)|^2 &\le \sqrt{f(a^*a)f(aa^*)} \le \frac{f(a^*a + aa^*)}{2} \le \frac{\|a^*a + aa^*\|}{2} \\ &= \frac{\max(\|a^*a\|, \|aa^*\|)}{2} = \frac{1}{2}\end{aligned}$$

where the second last equality follows from the fact that $a^*a$ and $aa^*$ are orthogonal. Thus, $|f(a)| \le 1/\sqrt{2}$ for any state $f$ on $A$. ∎

In fact, $G$ is abelian only when $G^*$ has the following extraordinary properties from the non-commutative point of view:

THEOREM 6.12. *Let $G$ be a locally compact group. The following are equivalent:*

(a) *$G$ is abelian.*
(b) *Given any $g^* \in G^*$, we have $\|g^* \cdot \phi\| = \|\phi\|$ for all $\phi \in B(G)$.*
(c) *Given any $g^* \in G^*$, we have $\|g^* \cdot \phi\| = \|\phi\|$ for all $\phi \in A(G)$.*
(d) *For any $g^* \in G^*$, we have $g^* \cdot (T_1T_2) = (g^* \cdot T_1)(g^* \cdot T_2)$ for any $T_1, T_2 \in \mathrm{VN}(G)$.*
(e) *For any $g^* \in G^*$, we have $\|g^* \cdot T\| = \|T\|$ for any $T \in \mathrm{VN}(G)$.*
(f) *The relative topology of $G^*$ inherited from the norm topology of $B(G)$ is discrete.*
(g) *The set of all extreme points of $B(G)_1$ is $\mathbb{T}G^* = \{\lambda g^* : \lambda \in \mathbb{T}, g^* \in G^*\}$.*
(h) *The weak*-closed convex hull of $\mathbb{T}G^*$ is $B(G)_1$.*
(i) $|\langle TT^*, g^*\rangle| = |\langle T, g^*\rangle|^2$ *for any $T \in C^*(G)$ and $g^* \in G^*$.*
(j) $\|T\| = \sup\{|\langle T, g^*\rangle| : g^* \in G^*\}$ *for any $T \in C^*(G)$.*

*Proof.* If $G$ is abelian, then the implications (a)$\Rightarrow$(b)$\Rightarrow$(c) are obvious.

If (c) holds, for each $g^* \in G^*$, let $L_{g^*} : A(G) \to A(G)$, $L_{g^*}(f) = g^* \cdot f$. Then $L_{g^*}$ is clearly a bounded multiplier on $A(G)$. By using a similar idea as applied in the proofs of [27, Lemmas 1, 2], one can show that $L_{g^*}$ is an isometric linear isomorphism (onto). Therefore, $L_{g^*}(S_{A(G)}) = S_{A(G)}$. It follows that $\|g^* \cdot T\| = \|T\|$ for any $T \in \mathrm{VN}(G)$. Hence, (c)$\Rightarrow$(e).

Now, we show that (a), (b), (c) and (e) are equivalent by proving that (e)$\Rightarrow$(a).

Suppose that (e) holds. Since $g^* \cdot \lambda_2(x) = g^*(x)\lambda_2(x)$ for each $x \in G$, we have

$$|g^*(x)| = \|g^* \cdot \lambda_2(x)\| = \|\lambda_2(x)\| = 1.$$

It follows that $g^* \cdot \bar{g^*} = |(g^*)^2| = 1$. So, $G$ is abelian by [1].

Next, we show that (a)$\Leftrightarrow$(d); only the backward implication needs proof. Suppose that (d) holds. For any $f_1, f_2 \in C_c(G)$, we have

$$\lambda_2(g^*(f_1 * f_2)) = g^* \cdot \lambda_2(f_1 * f_2) = g^* \cdot \lambda_2(f_1) \circ g^* \cdot \lambda_2(f_2) = \lambda_2(g^* f_1 * g^* f_2).$$

However, for any $x \in G$,

$$g^*(x) f_1 * f_2(x) = \int_G g^*(x) f_1(y) f_2(y^{-1}x)\, dy$$

and

$$(g^* f_1) * (g^* f_2)(x) = \int_G g^*(y) f_1(y) g^*(y^{-1}x) f_2(y^{-1}x)\, dy.$$

Since $\lambda_2$ is a faithful representation, it follows that

$$\langle g^*(x), f_1 L_x \breve{f_2}\rangle = \langle g^* L_x \bar{g^*}, f_1 L_x \breve{f_2}\rangle$$

for each $x \in G$ where $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $L^\infty(G)$ and $L^1(G)$. Since $g^*$ is continuous, $g^*(x) = g^*(y) g^*(y^{-1}x)$ for any $x, y \in G$. In particular, $g^* \cdot \bar{g^*} = 1$. Hence, $G$ is abelian (see [1]).

Next, we have to show that (a) and (f) are equivalent. If $G$ is abelian, then $G^* = \delta_{\hat{G}}$ and $B(G) = M(\hat{G})$. Since $\|\delta_x - \delta_y\| = 2$ whenever $x, y \in \hat{G}$ and $x \neq y$, the forward direction follows. Conversely, suppose that $G$ is not abelian. Then there exists $\pi \in \hat{G}$ such that $\dim \mathcal{H}_\pi > 1$. Let $\eta_1, \eta_2 \in \mathcal{H}_\pi$ with $\|\eta_1\| = \|\eta_2\| = 1$ be such that $\eta_1$, $\eta_2$ are linearly independent. Put $\epsilon_n = (\eta_1 + \eta_2/n)/\|\eta_1 + \eta_2/n\|$. Then $\epsilon_n$ and $\eta_1$ are linearly independent and $\epsilon_n \to \eta_1$. Hence, $\langle \pi(x)\eta_1, \eta_1\rangle \neq \langle \pi(x)\epsilon_n, \epsilon_n\rangle$ for each $n \in \mathbb{N}$ and $x \in G$. However, for each $f \in C^*(G)$,

$$\begin{aligned}\|\langle \pi(f)\eta_1, \eta_1\rangle - \langle \pi(f)\epsilon_n, \epsilon_n\rangle\| &\le \|\langle \pi(f)(\eta_1 - \epsilon_n), \eta_1\rangle + \langle \pi(f)(\epsilon_n - \eta_1), \epsilon_n\rangle\| \\ &\le 2\|f\|_{C^*(G)}\|\eta_1 - \epsilon_n\| \to 0.\end{aligned}$$

Consequently, the relative topology of $G^*$ inherited from the norm topology of $B(G)$ is non-discrete.

By the Krein–Milman Theorem, it is easy to see that (a)⇒(g)⇒(h)⇒(j); and (a)⇒(i) is also clear. Note that $\|T\| = \sup\{|\langle T, g^*\rangle| : g^* \in G^*\}$ for any $T \in C^*(G)_{\mathrm{sa}}$ (see Lemma 6.2 and [30, 1.5.4]). If (i) holds, then

$$\|T\| = \|TT^*\|^{1/2} = \sup\{|\langle TT^*, g^*\rangle|^{1/2} : g^* \in G^*\} = \sup\{|\langle T, g^*\rangle| : g^* \in G^*\}.$$

Therefore, it remains to show (j)⇒(a). Note $\|T\| = \sup\{|\langle T, h\rangle| : h \in \mathrm{co}(G^*)\}$ for any $T \in C^*(G)$ and $\mathrm{co}(G^*) = \mathrm{co}(\mathcal{E}(P_1(G)))$ is weak*-dense in $P_1(G)$. It follows that $\|T\| = \sup\{|\langle T, h\rangle| : h \in P_1(G)\}$ for any $T \in C^*(G)$. Consequently, $C^*(G)$ is commutative (by Lemma 6.11), hence $G$ is abelian. ■

Finally, we give a characterization of compact groups using properties of $G^*$.

**6.3. Compact groups.** Note that $\delta_G \subseteq M(G)$ is discrete in the relative norm topology since $\|\delta_x - \delta_y\| = 2$ whenever $x \neq y$. In particular, $G$ and $(\delta_G, \|\cdot\|_{M(G)})$ are homeomorphic if $G$ is discrete. We are going to prove the non-commutative analogue of this phenomenon.

For any $\pi \in \hat{G}$, write $G^*_\pi = \{\langle \pi(\cdot)\xi, \xi\rangle : \xi \in \mathcal{H}_\pi,\ \|\xi\| = 1\}$. If $G$ is abelian, then $G^*_\pi$ is always a singleton for any $\pi \in \hat{G}$.

The following proposition gives a characterization of compact groups by using properties of $G^*$:

Proposition 6.13. *Let $G$ be a separable group. The following statements are equivalent:*

(a) *$G$ is compact.*
(b) *The identity map* $\mathrm{id} : (G^*, w^*) \to (G^*, \|\cdot\|)$ *is continuous* (*a homeomorphism*).
(c) *The interior of $G^*_\pi$ is non-empty for each $\pi \in \hat{G}$.*

*Proof.* (a)⇒(b): If $G$ is compact, then the weak*-topology and the norm topology coincides on $S_{B(G)}$ ([15, Corollary 2, p. 463]).

(b)⇒(c): Let $g_0^* \in G^*_\pi$. By assumption, there exists a weak*-open set $U$ containing $g_0^*$ such that

$$U \subseteq \{g^* \in G^* : \|g^* - g_0^*\| < 2\}.$$

However, $\{g^* \in G^* : \|g^* - g_0^*\| < 2\} \subseteq G^*_\pi$ (see [10, 2.12.1]). Therefore, $(G^*_\pi)^o$ is non-empty.

(c)⇒(a): Note the natural map $q : G^* \to \hat{G}$ is open [10, Theorem 3.4.11], and $\{\pi\} = q((G^*_\pi)^o)$ by the definition of $q$ and the assumption that $(G^*_\pi)^o$ is non-empty. It follows that the hull-kernel topology on $\hat{G}$ is discrete, hence $G$ is compact (see [2]). ■

## 7. General properties of $G^*$-translation invariant means in $\mathrm{VN}(G)$.

In this section we will study the non-commutative analogues of classical results about translation invariant means on $L^\infty(G)$. We will discuss the general properties of $G^*$-translation invariant means and $\mathcal{F}$-translation invariant means. These notions were defined at the beginning of Section 3.

REMARK 7.1. In fact, $\mathrm{VN}(G)$ always has a topological invariant mean (see [24, Theorem 4]). We also notice that the set of all $G^*$-translation invariant means on $\mathrm{VN}(G)$ is a weak*-compact convex subset in $\mathrm{VN}(G)^*$, and $A(G) \cap P_1(G)$ is weak*-dense in the set of all means in $\mathrm{VN}(G)$ (see [14]).

Recall that in Section 4, we have the following definition: a subspace of $\mathrm{VN}(G)$ is said to be *invariant* if it is topologically invariant and $G^*$-translation invariant.

PROPOSITION 7.2. *Let $E$ be an invariant closed subspace of* $\mathrm{VN}(G)$ *which is closed under involution and contains $\lambda_2(e)$. Then every topologically invariant mean on $E$ is $G^*$-translation invariant. Hence, for any locally compact group, $E$ has a mean which is $\mathcal{F}$-translation invariant and $G^*$-translation invariant. Furthermore, if $G$ is non-discrete, then* $\mathrm{VN}(G)$ *has uncountably many $G^*$ (hence $\mathcal{F}$)-translation invariant means.*

*Proof.* Let $m$ be a topological invariant mean on $E$. For any $g^* \in G^*$, $T \in E$ and $\phi \in A(G) \cap P_1(G)$, we have

$$m(g^* \cdot T) = m(\phi \cdot (g^* \cdot T)) = m((\phi \cdot g^*) \cdot T) = m(T).$$

Therefore, $m$ is $G^*$-translation invariant. Note that $G^*_{\mathcal{F}} \subseteq G^*$. The rest follows from the remark above. ∎

LEMMA 7.3. *The following statements are equivalent:*

(a) *$G$ is discrete.*
(b) *There is a bounded linear functional on $C_r^*(G)$ which is $G^*$-translation invariant.*
(c) *There is a bounded linear functional on $C_r^*(G)$ which is topologically invariant.*

*Proof.* (a)⇒(c)⇒(b) is clear. Let $\phi \in B_r(G)$ be such that $\langle \phi, T \rangle = \langle \phi, g^* \cdot T \rangle = \langle g^* \cdot \phi, TT \rangle$ for any $g^* \in G^*$ and $T \in C_r^*(G)$. Then $g^* \cdot \phi = \phi$. Note that $\phi \in B_r(G) \subseteq B(G)$. So, $G$ is discrete by Theorem 6.5. ∎

PROPOSITION 7.4. *Let $G$ be a non-discrete locally compact group, and let $M$ be a $G^*$-translation invariant mean on* $\mathrm{VN}(G)$. *Then the restriction of $M$ on $C_r^*(G)$ is always zero.*

*Proof.* Let $m = M|_{C_r^*(G)}$. Assume that $m \neq 0$. Clearly, $m$ is positive and $G^*$-translation invariant on $C_r^*(G)$. Therefore, $n = m/\|m\|$ is a $G^*$-translation invariant mean on $C_r^*(G)$, which contradicts Lemma 7.3. ∎

Since all topological means are $G^*$-translation invariant (Proposition 7.2), we thus provide another proof of [24, Theorem 12].

Corollary 7.5. *Let $G$ be a locally compact group. Then $G$ is discrete if and only if there is a $G^*$-translation invariant mean on* $\mathrm{VN}(G)$ *belonging to $A(G) \cap P_1(G)$ (or $A(G)$).*

*Proof.* If $G$ is discrete, then $\delta_e \in A(G) \cap P_1(G)$. Hence, $m(T) := \langle \delta_e, T\rangle$ defines a $G^*$-translation invariant mean on $\mathrm{VN}(G)$. Conversely, if there is $f \in A(G) \cap P_1(G)$ such that $\langle f, T\rangle = \langle f, g^* \cdot T\rangle$ for any $g^* \in G^*$ and $T \in \mathrm{VN}(G)$, then $f = g^* \cdot f$. So, $G$ is discrete. ∎

Theorem 7.6. *If $A(G)$ has an approximate identity, then every $G^*$-translation invariant mean on* $\mathrm{UCB}(\hat{G})$ *is topologically invariant.*

*Proof.* Let $m$ be a $G^*$-translation invariant mean on $\mathrm{UCB}(\hat{G})$, and let $S = u \cdot T \in \mathrm{UCB}(\hat{G})$ where $T \in \mathrm{VN}(G)$ and $u \in A(G)$. As the functional $A(G) \to \mathbb{C}$, $f \mapsto m(f \cdot S)$, is continuous, there exists $T_0 \in \mathrm{VN}(G)$ such that $m(f \cdot S) = \langle T_0, f\rangle$. Since $m$ is $G^*$-translation invariant, for any $g^* \in G^*$ we have

$$\langle g^* \cdot T_0, f\rangle = \langle T_0, g^* \cdot f\rangle = m(g^* \cdot (f \cdot S)) = m(f \cdot S) = \langle T, f\rangle.$$

That is, $g^* \cdot T_0 = T_0$. By Theorem 6.7, $T_0 = c\lambda_2(e)$ for some constant $c \neq 0$. It follows that $m(f \cdot S) = c$ for any $f \in A(G) \cap P_1(G)$ and $S \in A(G) \cdot \mathrm{VN}(G)$. By assumption, $A(G)$ has an approximate identity $\{e_\alpha\}$. So, we have

$$m(f \cdot S) = \lim_\alpha m((f \cdot e_\alpha) \cdot S) = \lim_\alpha m(e_\alpha \cdot S) = m(S).$$

However, $A(G) \cdot \mathrm{VN}(G)$ is a norm-dense subset of $\mathrm{UCB}(\hat{G})$. Hence we conclude that $m$ is topologically invariant. ∎

Corollary 7.7. *If $G$ is a compact group, then every $G^*(\mathcal{F})$-translation invariant mean on* $\mathrm{VN}(G)$ *is topologically invariant.*

*Proof.* Note that $G$ is amenable, $G^*_{\mathcal{F}} = G^*$ and $\mathrm{VN}(G) = \mathrm{UCB}(\hat{G})$ under the assumption. ∎

Recall that $\mathrm{WAP}(\hat{G})$, the weakly almost periodic functionals in $\mathrm{VN}(G)$ is the set of all $T$ in $\mathrm{VN}(G)$ for which the operator from $A(G)$ to $\mathrm{VN}(G)$ given by $u \mapsto u \cdot T$ is weakly compact. It is proved by Granrier [14] that $\mathrm{WAP}(\hat{G})$ has a unique topologically invariant mean.

Proposition 7.8. $\mathrm{WAP}(\hat{G})$ *has a unique $G^*$-translation invariant mean.*

*Proof.* The proof is the same as that of [14, Theorem 1]. ∎

Lemma 7.9. *Let $\phi \in A(G) \cap P_1(G)$. If $m$ is a topologically invariant mean on* $\mathrm{UCB}(\hat{G})$*, then $m'$ is a topologically invariant mean on* $\mathrm{VN}(G)$*, where $m'$ is given by $m'(T) = m(\phi \cdot T)$. Furthermore, $m'$ is independent of the choice of $\phi$.*

*Proof.* Let $T_0 \in \mathrm{VN}(G)$. Define $F \in A(G)^*$ by $F(\psi) = m(\psi \cdot T_0)$. Now, for any $\psi \in A(G)$ and $\varphi \in A(G) \cap P_1(G)$, we have

$$(\varphi \cdot F)(\psi) = F(\varphi \cdot \psi) = m(\varphi \cdot \psi \cdot T_0) = m(\psi \cdot T_0) = F(\psi).$$

So, by Theorem 6.7,

$$F(\psi) = m(\psi \cdot T_0) = \langle c\lambda_2(e), \psi\rangle = c\psi(e).$$

In particular, $m(\varphi \cdot T_0) = c$ for any $\varphi \in A(G) \cap P_1(G)$. Thus, $m'$ is independent of the choice of $\phi$. It is routine to check that $m'$ is a topologically invariant mean on $\mathrm{VN}(G)$. ■

PROPOSITION 7.10. *There is a bijection between the set of topologically invariant mean on* $\mathrm{UCB}(\hat{G})$ *and the set of topologically invariant means on* $\mathrm{VN}(G)$.

*Proof.* If $m$ is a topologically invariant mean on $\mathrm{UCB}(\hat{G})$, then for any $T \in \mathrm{UCB}(\hat{G})$, we have

$$m'|_{\mathrm{UCB}(\hat{G})}(T) = m(\phi \cdot T) = m(T) \quad \text{for all } \phi \in A(G) \cap P_1(G).$$

On the other hand, if $m$ is a topologically invariant mean on $\mathrm{VN}(G)$, then for any $T \in \mathrm{VN}(G)$, we have

$$(M|_{\mathrm{UCB}(\hat{G})})'(T) = M|_{\mathrm{UCB}(\hat{G})}(\phi \cdot T) = M(T) \quad \text{for all } \phi \in A(G) \cap P_1(G). \ ■$$

COROLLARY 7.11. *Suppose that $A(G)$ has an approximate identity. Then $G$ is discrete if and only if there exists a unique $G^*$-translation* (*topologically*) *invariant mean on* $\mathrm{UCB}(\hat{G})$.

*Proof.* Note that $G$ is discrete if and only if $\mathrm{VN}(G)$ has a unique topologically invariant mean (see [19, Theorem 11] and [28, Corollary 4.11]). The result thus follows from the last proposition and Theorem 7.6. ■

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