Embeddings of Besov spaces of logarithmic smoothness

by

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Dedicated to Professor Albrecht Pietsch
on the occasion of his 80th birthday

Abstract. This paper deals with Besov spaces of logarithmic smoothness $B_{p,r}^{0,b}$ formed by periodic functions. We study embeddings of $B_{p,r}^{0,b}$ into Lorentz–Zygmund spaces $L_{p,q}(\log L)_\beta$. Our techniques rely on the approximation structure of $B_{p,r}^{0,b}$, Nikol'skiĭ type inequalities, extrapolation properties of $L_{p,q}(\log L)_\beta$ and interpolation.

1. Introduction. Besov spaces of generalized smoothness arise in the solution of some natural questions. Among other things, they are useful in fractal analysis and a related spectral theory (see the book by Triebel [32] and the references given there), as well as in probability theory and in the theory of stochastic processes (see the paper by Farkas and Leopold [18]). Besov spaces of smoothness near zero are distinguished elements of the class above.

Already in 1979, DeVore, Riemenschneider and Sharpley [14] introduced spaces $B_{p,r}^{0,b}$ with zero classical smoothness and logarithmic smoothness with exponent b. They defined them by means of the modulus of continuity. Similar spaces $B_{p,r}^{(0,b)}$ were considered later by Merucci [22] and Cobos and Fernandez [10] but following the Fourier-analytical approach. The precise relationship between these two kinds of spaces has not been established yet. The first results in this direction can be found in the recent report of Triebel [33, Section 3].

Triebel has also studied in [33] embeddings of those spaces into Lorentz–Zygmund spaces $L_{p,q}(\log L)_\beta$. For spaces $B_{p,r}^{0,b}$, this problem was already considered by Caetano, Gogatishvili and Opic [6] who proved sharp embeddings for $1 \le p < \infty$ and $1 \le r \le q \le \infty$. Here $b + 1/r > 0$ and

[193]

$\beta = b + 1/r + 1/\max\{p, q\} - 1/q$. See also [7] where compact embeddings are studied. Techniques of [6] are related to Kolyada's inequality which estimates from below the modulus of continuity in terms of non-increasing rearrangements of functions. The method used in [33] to establish embeddings relies on expansions in terms of the Haar basis and the description of Lorentz–Zygmund spaces as extrapolation spaces.

In this paper we consider Besov spaces of logarithmic smoothness formed by periodic functions and we prove similar embeddings to those of [6] which cover the full range of indices if $\beta > 0$, that is, for $0 < p < \infty$ and $0 < r \le q \le \infty$. Our methods are based on the description of Besov spaces as approximation spaces and are simpler than those in [6].

As was shown by Pietsch [27, 28] (see also [5, 24, 26]) the theory of approximation spaces is very flexible and produces interesting results on spaces of functions, sequences and operators. In particular, Pietsch established in [27, p. 126] (see also [29, 6.7.8]) embeddings between classical Besov spaces by means of the Nikol'skiĭ inequality for trigonometric polynomials. In the present paper, we use the description of Besov spaces of logarithmic smoothness as limiting approximation spaces in the sense of Cobos and Milman [12] and Fehér and Grässler [19] to derive embeddings into Lorentz–Zygmund spaces $L_{p,q}(\log L)_\beta$. A key role in our arguments is played by the Nikol'skiĭ inequality and a variant of it. Extrapolation properties of spaces $L_{p,q}(\log L)_\beta$ are important in the case $\beta > 0$, while we use interpolation to deal with the case $\beta \le 0$.

In the subsequent paper [8], the authors have given another approach to study embeddings of spaces $B_{p,r}^{0,b}$ into Lorentz–Zygmund spaces.

2. Limiting approximation spaces and Besov spaces. Let $(X, \|\cdot\|_X)$ be a quasi-Banach space and let $(G_n)_{n \in \mathbb{N}_0}$ be a sequence of subsets of X satisfying the following conditions:

(2.1) $G_0 = \{0\}$,

(2.2) $G_n \subseteq G_{n+1}$ for any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

(2.3) $G_n + G_m \subseteq G_{n+m}$ for any $n, m \in \mathbb{N}$.

For any $f \in X$, we put

$$E_n(f) = E_n(f; X) = \inf\{\|f - g\|_X : g \in G_{n-1}\}, \quad n \in \mathbb{N},$$

for the error of best approximation of f by the elements of G_{n-1}.

Let $0 < q \le \infty$ and $\gamma \in \mathbb{R}$. The (limiting) approximation space $X_q^{(0,\gamma)} = (X, G_n)_q^{(0,\gamma)}$ consists of all $f \in X$ having a finite quasi-norm

$$\|f\|_{X_q^{(0,\gamma)}} = \Big(\sum_{n=1}^{\infty} \big((1+\log n)^{\gamma} E_n(f)\big)^q n^{-1}\Big)^{1/q} \quad \text{if } 0 < q < \infty,$$

$$\|f\|_{X_q^{(0,\gamma)}} = \sup_{n\geq 1}\{(1+\log n)^{\gamma} E_n(f)\} \quad \text{if } q = \infty.$$

See [13, 12, 19]. Note that if $\gamma < -1/q$ then $(\sum_{n=1}^{\infty}(1+\log n)^{\gamma q}n^{-1})^{1/q} < \infty$ and so $X_q^{(0,\gamma)} = X$. Hence, the case of interest is $\gamma \geq -1/q$.

Put $\mu_n = 2^{2^n}$, $n = 0, 1, 2, \ldots$. It is shown in [13, 19] that $X_q^{(0,\gamma)}$ is formed by all those $f \in X$ such that there is a representation $f = \sum_{n=0}^{\infty} g_n$ with $g_n \in G_{\mu_n}$ and

$$\Big(\sum_{n=0}^{\infty}(2^{n(\gamma+1/q)}\|g_n\|_X)^q\Big)^{1/q} < \infty.$$

Moreover

$$(2.4) \quad \|f\|_{(0,\gamma),q} = \inf\Big\{\Big(\sum_{n=0}^{\infty}(2^{n(\gamma+1/q)}\|g_n\|_X)^q\Big)^{1/q} : f = \sum_{n=0}^{\infty} g_n,\ g_n \in G_{\mu_n}\Big\}$$

is a quasi-norm equivalent to $\|\cdot\|_{X_q^{(0,\gamma)}}$.

Let $\mathbb{T} = [0, 2\pi)$ be the unit circle. For $0 < p \leq \infty$, we write L_p for the Lebesgue space of all (classes of) real-valued 2π-periodic measurable functions f such that

$$\|f\|_{L_p} = \Big(\int_0^{2\pi} |f(e^{ix})|^p\, dx\Big)^{1/p} < \infty$$

with the obvious modification if $p = \infty$. Given $0 < r \leq \infty$ and $b \geq -1/r$, the Besov space $B_{p,r}^{0,b}$ consists of all functions $f \in L_p$ having a finite quasi-norm

$$\|f\|_{B_{p,r}^{0,b}} = \|f\|_{L_p} + \Big(\int_0^1 \big((1+|\log t|)^b \omega(f,t)_p\big)^r \frac{dt}{t}\Big)^{1/r}$$

where $\omega(f, t)_p$ is the modulus of continuity

$$\omega(f,t)_p = \sup_{|h|\leq t} \|\Delta_h f\|_{L_p}$$

and

$$\Delta_h f(x) = f(x+h) - f(x).$$

In order to describe $B_{p,r}^{0,b}$ as an approximation space, we take $X = L_p$ and $G_n = T_n$, the subset of all trigonometric polynomials of order n,

$$T_n = \Big\{\sum_{|k|\leq n} c_k e^{ikx} : c_{-k} = \overline{c_k}\Big\}.$$

In what follows, if W, Z are non-negative quantities depending on certain parameters, we write $W \lesssim Z$ if there is a constant $c > 0$ independent of the

parameters in W and Z such that $W \leq cZ$. If $W \lesssim Z$ and $Z \lesssim W$, we write $W \sim Z$.

LEMMA 2.1. *Let* $0 < p, r \leq \infty$ *and* $b \geq -1/r$. *Then we have, with equivalence of quasi-norms,*

$$B_{p,r}^{0,b} = (L_p, T_n)_r^{(0,b)}.$$

Proof. If $1 \leq p, r \leq \infty$ this formula was established in [14, Corollary 7.1] by using weak type interpolation ideas. Next we check the case of the other values of parameters with the help of Jackson and Bernstein-type inequalities.

Assume that $0 < p < 1$. According to Ivanov [20] and Storozhenko, Krotov and Osval'd [31], for any $f \in L_p$ we have

$$E_n(f) \leq c\omega\left(\frac{\pi}{n}, f\right)_p \quad \text{where } c = c(p).$$

Hence

$$
\begin{aligned}
\|f\|_{(L_p)_r^{(0,b)}} &= \left(\sum_{n=1}^{\infty}[(1 + \log n)^b E_n(f)]^r n^{-1}\right)^{1/r} \\
&\lesssim \left(\sum_{n=1}^{\infty}\left[(1 + \log n)^b \omega\left(\frac{\pi}{n}, f\right)_p\right]^r n^{-1}\right)^{1/r} \\
&\lesssim \left(\omega(\pi, f)_p^r + \int_1^{\infty}\left[(1 + \log t)^b \omega\left(\frac{\pi}{t}, f\right)_p\right]^r \frac{dt}{t}\right)^{1/r} \\
&\lesssim \|f\|_{L_p} + \left(\int_0^1\left[\left(1 + \log\frac{\pi}{t}\right)^b \omega(t, f)_p\right]^r \frac{dt}{t}\right)^{1/r} \\
&\sim \|f\|_{B_{p,r}^{0,b}}.
\end{aligned}
$$

To establish the converse inequality, we first recall that it was also shown by Ivanov [20] and Storozhenko, Krotov and Osval'd [31] that

$$\omega\left(\frac{\pi}{n}, f\right)_p \leq \frac{c}{n}\left(\sum_{k=1}^{n} k^{p-1} E_k(f)^p\right)^{1/p}.$$

Therefore

$$
\begin{aligned}
\|f\|_{B_{p,r}^{0,b}} &\lesssim \|f\|_{L_p} + \left(\sum_{n=0}^{\infty}[(1 + n)^b \omega(2^{-n}, f)_p]^r\right)^{1/r} \\
&\lesssim \|f\|_{L_p} + \left(\sum_{n=0}^{\infty}\left[(1 + n)^b 2^{-n}\left(\sum_{k=1}^{2^n} k^{p-1} E_k(f)^p\right)^{1/p}\right]^r\right)^{1/r}
\end{aligned}
$$

$$\sim \|f\|_{L_p} + \Big(\sum_{n=0}^{\infty} \Big[(1+n)^b 2^{-n} \Big(\sum_{\nu=0}^{n} 2^{\nu p} E_{2^\nu}(f)^p \Big)^{1/p} \Big]^r \Big)^{1/r}$$

$$= \|f\|_{L_p} + \Big(\sum_{n=0}^{\infty} \Big[(1+n)^{bp} 2^{-np} \sum_{\nu=0}^{n} 2^{\nu p} E_{2^\nu}(f)^p \Big]^{r/p} \Big)^{1/r}$$

$$\lesssim \|f\|_{L_p} + \Big(\sum_{n=0}^{\infty} ((1+n)^b E_{2^n}(f))^r \Big)^{1/r}$$

where we have used in the last inequality a variant of Hardy's inequality (see [26, Lemma 3.10, p. 70]). Consequently,

$$\|f\|_{B_{p,r}^{0,b}} \lesssim \Big(\sum_{n=0}^{\infty} [(1+n)^b E_{2^n}(f)]^r \Big)^{1/r}$$

$$\sim \Big(\sum_{n=1}^{\infty} [(1+\log n)^b E_n(f)]^r n^{-1} \Big)^{1/r}$$

$$= \|f\|_{(L_p)_r^{(0,b)}}. \quad \blacksquare$$

3. Lorentz–Zygmund spaces and Nikol'skiĭ inequality. Let $0 < p < \infty, 0 < q \le \infty$ and $-\infty < \beta < \infty$. The Lorentz–Zygmund space $L_{p,q}(\log L)_\beta$ on $\mathbb{T}$ is formed by all (classes of) measurable functions f on $\mathbb{T}$ having a finite quasi-norm

$$\|f\|_{L_{p,q}(\log L)_\beta} = \Big(\int_0^{2\pi} [t^{1/p}(1+|\log t|)^\beta f^*(t)]^q \frac{dt}{t} \Big)^{1/q}.$$

Here f^* is the non-increasing rearrangement of f given by

$$f^*(t) = \inf \{ s > 0 : m(\{x \in \mathbb{T} : |f(e^{ix})| > s\}) \le t \}.$$

See [2, 16]. Observe that if $p = q$, then $L_{p,q}(\log L)_\beta$ is just the Zygmund space $L_p(\log L)_\beta$. In particular, for $\beta = 0$ we obtain the Lebesgue space L_p. If $\beta = 0$ but $p \neq q$, we get the Lorentz function space $L_{p,q}$.

The *Nikol'skiĭ inequality* for trigonometric polynomials (see [23, 3.4.3] and [1]) says that there is a universal constant $c > 1$ such that

(3.1) $\|g\|_{L_q} \le cn^{1/p-1/q}\|g\|_{L_p}$ for all $0 < p \le q \le \infty$, $g \in T_n$ and $n \in \mathbb{N}$.

This inequality has been extended to Lorentz spaces by Sherstneva [30] (see also [15]).

Next we establish a result in this direction but involving Lorentz–Zygmund spaces.

LEMMA 3.1. *Let $0 < q < p < \infty$, $d > 1/p - 1/q$ and let $\mu_n = 2^{2^n}$. There is a constant $c > 0$ such that*

$$\|g\|_{L_{p,q}(\log L)_d} \le c2^{n(1/q-1/p+d)}\|g\|_{L_p} \quad \text{for all } g \in T_{\mu_n} \text{ and } n \in \mathbb{N}_0.$$

Proof. It is not hard to check that

(3.2) $\|f\|_{L_{p,\infty}} \leq (q/p)^{1/q}\|f\|_{L_{p,q}}, \quad f \in L_{p,q}.$

Moreover, by [4, Theorem 5.1.2], we have

$$\|h\|_{L_\infty} \leq e^{4ns}h^*(s) \quad \text{for all } s \in (0, \pi/2] \text{ and } h \in T_n.$$

This inequality yields

(3.3) $\|h\|_{L_{p,\infty}} \geq s^{1/p}e^{-4ns}\|h\|_{L_\infty} \quad \text{for all } s \in (0, \pi/2] \text{ and } h \in T_n.$

Indeed,

$$\|h\|_{L_{p,\infty}} = \sup_{0 < t < 2\pi} t^{1/p}h^*(t) \geq s^{1/p}h^*(s) \geq s^{1/p}e^{-4ns}\|h\|_{L_\infty}.$$

We proceed now with the inequality of the statement. Take any $n \in \mathbb{N}_0$ and $g \in T_{\mu_n}$. Let $s = 2^{-2^n}$ and $\rho(t) = (1 + |\log t|)^d$. We have

$$\|g\|_{L_{p,q}(\log L)_d}^q = \int_0^s t^{q/p-1}\rho(t)^q g^*(t)^q \ dt + \int_s^{2\pi} t^{q/p-1}\rho(t)^q g^*(t)^q \ dt$$

$$= I_1 + I_2.$$

Using [16, Proposition 3.4.33/(v)] and (3.3), (3.2), we derive

$$I_1 \leq \|g\|_{L_\infty}^q \int_0^s t^{q/p-1}\rho(t)^q \ dt$$

$$\sim \|g\|_{L_\infty}^q s^{q/p}\rho(s)^q \leq e^{4\mu_n sq}\rho(s)^q\|g\|_{L_p}^q$$

$$= e^{4q}(1 + |\log 2^{-2^n}|)^{dq}\|g\|_{L_p}^q$$

$$\sim 2^{ndq}\|g\|_{L_p}^q.$$

As for I_2, we write

$$I_2 = \int_s^{2\pi} (g^*(t)^p)^{q/p}(\rho(t)^\lambda t^{-1})^{1-q/p} \ dt$$

where $\lambda = q(1 - q/p)^{-1} = (1/q - 1/p)^{-1}$. By the Hölder inequality, we get

$$I_2 \leq \|g\|_{L_p}^q \Big(\int_s^{2\pi} \rho(t)^\lambda t^{-1} \ dt \Big)^{1-q/p}.$$

We estimate the integral by splitting it into two sets. Clearly,

$$\int_1^{2\pi} \rho(t)^\lambda t^{-1} \ dt = c_1 < \infty.$$

Furthermore, since $d\lambda > -1$, we have

$$\int_s^1 (1 - \log t)^{d\lambda}t^{-1} \ dt \lesssim (1 - \log s)^{1+d\lambda}.$$

Taking into account that $s = 2^{-2^n}$, we obtain

$$I_2 \lesssim \|g\|_{L_p}^q (1 - \log s)^{(1+d\lambda)(1-q/p)}$$
$$\sim 2^{n(1/q-1/p+d)q}\|g\|_{L_p}^q.$$

Consequently,

$$\|g\|_{L_{p,q}(\log L)_d} \lesssim 2^{n(1/q-1/p+d)}\|g\|_{L_p}. \quad \blacksquare$$

4. Embeddings. By the construction of Besov spaces $B_{p,r}^{0,b}$, we have $B_{p,r}^{0,b} \hookrightarrow L_p$. Next we will improve this embedding with the help of Lorentz–Zygmund spaces. Note that

$$L_{p,q}(\log L)_\beta \hookrightarrow L_p$$

if either $q \leq p$ and $\beta \geq 0$, or $p < q$ and $\beta > 1/p - 1/q$ (see [16, Theorem 3.4.45]).

It was shown by Edmunds and Triebel [17, 2.6.2, pp. 69–73] that Zygmund spaces $L_p(\log L)_\beta$ can be described in terms of simple Lebesgue spaces. For Lorentz–Zygmund spaces a similar result holds but now in terms of Lorentz spaces. In particular, for $\beta > 0$ the result reads as follows (see Karadzhov and Milman [21, Theorems 4.4 and 4.7] or Cobos, Fernández-Cabrera, Manzano and Martínez [11, Corollary 3.3]).

Let $0 < p < \infty$, $0 < q \leq \infty$, $\beta > 0$ and $j_0 \in \mathbb{N}$ be such that for all $j \in \mathbb{N}$ with $j \geq j_0$ we have $1/p^{\nu_j} = 1/p - 1/2^j > 0$. Then $L_{p,q}(\log L)_\beta$ consists of all measurable functions f on $\mathbb{T}$ which can be represented as

$$(4.1) \qquad f = \sum_{j=j_0}^{\infty} f_j, \qquad f_j \in L_{p^{\nu_j},q},$$

such that

$$(4.2) \qquad \left(\sum_{j=j_0}^{\infty} 2^{j\beta q}\|f_j\|_{L_{p^{\nu_j},q}}^q\right)^{1/q} < \infty.$$

Moreover, the infimum of the expression in (4.2) taken over all admissible representations (4.1) is a quasi-norm equivalent to $\|\cdot\|_{L_{p,q}(\log L)_\beta}$.

Next we establish the embedding results.

THEOREM 4.1. *Let* $0 < p < \infty$, $0 < r \leq q \leq \infty$, $b + 1/r > 0$ *and let* $\beta = b + 1/r + 1/\max\{p,q\} - 1/q > 0$. *Then*

$$B_{p,r}^{0,b} \hookrightarrow L_{p,q}(\log L)_\beta.$$

Proof. By Lemma 2.1 we know that $B_{p,r}^{0,b} = (L_p, T_n)_r^{(0,b)}$, so we can work with the quasi-norm $\|\cdot\|_{(0,b),r}$ defined in (2.4). We distinguish two cases.

If $q \leq p$, then $\beta = b + 1/r + 1/p - 1/q$ and $\beta > 0$ by assumption. Let $j_0 \in \mathbb{N}$ be such that $1/p^{\nu_j} > 0$ for all $j \geq j_0$. If $j \geq j_0 + 1$, it follows from

[16, Proposition 3.4.4] that

$$(4.3) \qquad \|h\|_{L_{p^{\nu_j},q}} \le c_j \|h\|_{L_{p^{\nu_{j-1}}}} \qquad \text{for all } h \in L_{p^{\nu_{j-1}}}$$

where

$$
\begin{aligned}
c_j &= (2\pi)^{1/p^{\nu_j}-1/p^{\nu_{j-1}}} \left[\frac{p^{\nu_j}(p^{\nu_{j-1}}-q)}{q(p^{\nu_{j-1}}-p^{\nu_j})} \right]^{1/q-1/p^{\nu_{j-1}}} \\
&= (2\pi)^{1/2^j} \left[\frac{p-q}{pq} 2^j + 2 \right]^{1/q-1/p+1/2^{j-1}} \\
&\sim 2^{j(1/q-1/p)}.
\end{aligned}
$$

By (2.4) there is a constant $c > 0$ such that given any $f \in B_{p,r}^{0,b}$ we can find a representation $f = \sum_{j=0}^\infty g_j$ with $g_j \in T_{\mu_j}$ and

$$\left(\sum_{j=0}^\infty (2^{j(b+1/r)} \|g_j\|_{L_p})^r \right)^{1/r} \le c \|f\|_{B_{p,r}^{0,b}}.$$

Using (4.3) and the Nikol'skiĭ inequality (3.1), we derive

$$
\begin{aligned}
\|g_{j-j_0-1}\|_{L_{p^{\nu_j},q}} &\lesssim 2^{j(1/q-1/p)} 2^{2^{j-j_0-1} 2^{-j+1}} \|g_{j-j_0-1}\|_{L_p} \\
&\sim 2^{j(1/q-1/p)} \|g_{j-j_0-1}\|_{L_p}.
\end{aligned}
$$

Therefore, since $r \le q$, we obtain

$$
\begin{aligned}
\|f\|_{L_{p,q}(\log L)_\beta} &\lesssim \left(\sum_{j=j_0+1}^\infty 2^{j\beta q} \|g_{j-j_0-1}\|_{L_{p^{\nu_j},q}}^q \right)^{1/q} \\
&\lesssim \left(\sum_{j=j_0+1}^\infty 2^{j\beta q} 2^{j(1/q-1/p)q} \|g_{j-j_0-1}\|_{L_p}^q \right)^{1/q} \\
&\lesssim \left(\sum_{j=0}^\infty 2^{j(b+1/r)r} \|g_j\|_{L_p}^r \right)^{1/r} \\
&\lesssim \|f\|_{B_{p,r}^{0,b}}.
\end{aligned}
$$

Suppose now $p < q$, so $\beta = b + 1/r$. Let $j_0 \in \mathbb{N}$ be such that $p < p^{\nu_j} < q$ for all $j \ge j_0$. According to [16, Proposition 3.4.4],

$$(4.4) \qquad \|h\|_{L_{p^{\nu_j},q}} \le C \|h\|_{L_{p^{\nu_{j-1}}}} \qquad \text{for all } h \in L_{p^{\nu_{j-1}}}$$

where now C is independent of $j \in \mathbb{N}$ with $j \ge j_0 + 1$. Take any $f \in B_{p,r}^{0,b}$ and choose a representation $f = \sum_{j=0}^\infty g_j$ with $g_j \in T_{\mu_j}$ and

$$\left(\sum_{j=0}^\infty (2^{j(b+1/r)} \|g_j\|_{L_p})^r \right)^{1/r} \le c \|f\|_{B_{p,r}^{0,b}}.$$

By (4.4) and the Nikol'skiĭ inequality (3.1), we get

$$\|g_{j-j_0-1}\|_{L_{p^{\nu_j},q}} \lesssim \|g_{j-j_0-1}\|_{L_{p^{\nu_{j-1}}}} \lesssim \|g_{j-j_0-1}\|_{L_p}.$$

Consequently,

$$\|f\|_{L_{p,q}(\log L)_\beta} \lesssim \Big(\sum_{j=j_0+1}^{\infty} 2^{j(b+1/r)q} \|g_{j-j_0-1}\|_{L_{p^{\nu_j},q}}^q \Big)^{1/q}$$

$$\lesssim \Big(\sum_{j=j_0+1}^{\infty} 2^{j(b+1/r)q} \|g_{j-j_0-1}\|_{L_p}^q \Big)^{1/q}$$

$$\lesssim \Big(\sum_{j=0}^{\infty} 2^{j(b+1/r)r} \|g_j\|_{L_p}^r \Big)^{1/r}$$

$$\lesssim \|f\|_{B_{p,r}^{0,b}}.$$

This completes the proof. ∎

If $\beta \leq 0$ then the description of $L_{p,q}(\log L)_\beta$ in terms of Lorentz spaces is of a different type (see [11, Corollary 3.3]). For this reason we have to use another approach. We start with the case $0 < r \leq 1$.

THEOREM 4.2. *Let* $1 \leq q < p < \infty$, $0 < r \leq 1$, $b + 1/r > 0$ *and* $\beta = b + 1/r + 1/p - 1/q$. *Then*

$$B_{p,r}^{0,b} \hookrightarrow L_{p,q}(\log L)_\beta.$$

Proof. First note that under the present assumptions, the quasi-norm of $L_{p,q}(\log L)_\beta$ is equivalent to a norm (see [16, Lemma 3.4.39]). Take any $f \in B_{p,r}^{0,b}$ and choose a representation $f = \sum_{j=0}^{\infty} g_j$ with $g_j \in T_{\mu_j}$ and

$$\Big(\sum_{j=0}^{\infty} (2^{j(b+1/r)} \|g_j\|_{L_p})^r \Big)^{1/r} \leq c\|f\|_{B_{p,r}^{0,b}}.$$

Using Lemma 3.1, we derive

$$\|f\|_{L_{p,q}(\log L)_\beta} \lesssim \sum_{j=0}^{\infty} \|g_j\|_{L_{p,q}(\log L)_\beta}$$

$$\lesssim \sum_{j=0}^{\infty} 2^{j(b+1/r)} \|g_j\|_{L_p}$$

$$\lesssim \Big(\sum_{j=0}^{\infty} (2^{j(b+1/r)} \|g_j\|_{L_p})^r \Big)^{1/r}$$

$$\lesssim \|f\|_{B_{p,r}^{0,b}}. \quad\blacksquare$$

Next we deal with the case $1 < r \leq q$. For this, we will use the real interpolation method (see [3]).

THEOREM 4.3. *Let* $1 < r \leq q < p < \infty$, $b + 1/r > 0$ *and* $\beta = b + 1/r + 1/p - 1/q$. *Then*
$$B_{p,r}^{0,b} \hookrightarrow L_{p,q}(\log L)_\beta.$$

Proof. Take b_0, b_1 such that
$$0 < b_0 + 1 < b + 1/r < b_1 + 1$$
and let $0 < \theta < 1$ with
$$b + 1/r = (1 - \theta)(b_0 + 1) + \theta(b_1 + 1) = (1 - \theta)b_0 + \theta b_1 + 1.$$
According to Theorem 4.2, we have the continuous embeddings
$$B_{p,1}^{0,b_i} \hookrightarrow L_{p,q}(\log L)_{b_i + 1 + 1/p - 1/q}, \qquad i = 0, 1.$$
Moreover, by [19, Theorem 5], we have
$$(B_{p,1}^{0,b_0}, B_{p,1}^{0,b_1})_{\theta,r} = B_{p,r}^{0,\gamma}$$
where $\gamma = (1 - \theta)b_0 + \theta b_1 + 1 - 1/r = b$, and according to [25, Example 6.1], if $\tau_i = b_i + 1 + 1/p - 1/q, i = 0, 1$, then
$$(L_{p,q}(\log L)_{\tau_0}, L_{p,q}(\log L)_{\tau_1})_{\theta,r} \hookrightarrow (L_{p,q}(\log L)_{\tau_0}, L_{p,q}(\log L)_{\tau_1})_{\theta,q}$$
$$= L_{p,q}(\log L)_\tau$$
where $\tau = (1 - \theta)\tau_0 + \theta\tau_1 = b + 1/r + 1/p - 1/q = \beta$. Consequently,
$$B_{p,r}^{0,\gamma} \hookrightarrow L_{p,q}(\log L)_\beta. \ \blacksquare$$

Note that if $\beta = 0$ then Theorems 4.2 and 4.3 give the following embedding into Lorentz spaces.

COROLLARY 4.4. *Let* $1 < p < \infty$, $0 < r < \infty$ *and* $-1/r < b \leq \min\{-1/p, 1 - 1/r - 1/p\}$. *Set* $1/q = b + 1/r + 1/p$. *Then*
$$B_{p,r}^{0,b} \hookrightarrow L_{p,q}.$$

We close the paper with a direct consequence of the previous embeddings and [9, Corollary 3.6].

COROLLARY 4.5. *Assume that either*
$$0 < p < \infty, \ 0 < r \leq q \leq \infty, \ b + 1/r > 0, \ \beta = b + \frac{1}{r} + \frac{1}{\max\{p, q\}} - \frac{1}{q} > 0,$$
or
$$1 \leq q < p < \infty, \quad 0 < r < \infty, \quad r \leq q, \quad b + 1/r > 0, \quad \beta = b + \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$
If $\delta < \beta$ *then the embedding*
$$B_{p,r}^{0,b} \hookrightarrow L_{p,q}(\log L)_\delta$$
is compact.

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