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Unitarily invariant norms related to semi-finite factors

by

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Abstract. Let $\mathcal{M}$ be a semi-finite factor and let $\mathcal{J}(\mathcal{M})$ be the set of operators T in $\mathcal{M}$ such that $T = ETE$ for some finite projection E. We obtain a representation theorem for unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ in terms of Ky Fan norms. As an application, we prove that the class of unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ coincides with the class of symmetric gauge norms on a classical abelian algebra, which generalizes von Neumann's classical 1940 result on unitarily invariant norms on $M_n(\mathbb{C})$. As another application, Ky Fan's dominance theorem of 1951 is obtained for semi-finite factors.

1. Introduction. F. J. Murray and J. von Neumann [12, 13, 14, 21, 22] introduced and studied certain algebras of Hilbert space operators. Those algebras are now called "von Neumann algebras." They are strong-operator closed self-adjoint subalgebras of all bounded linear transformations on a Hilbert space. *Factors* are von Neumann algebras whose centers consist of scalar multiples of the identity operator. Every von Neumann algebra on a separable Hilbert space is a direct sum (or "direct integral") of factors. Thus factors are the building blocks for general von Neumann algebras. Murray and von Neumann [12] classified factors into type $\mathrm{I}_n, \mathrm{I}_\infty, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$ factors. Type I_n and I_∞ factors are full matrix algebras: $M_n(\mathbb{C})$ and $\mathcal{B}(\mathcal{H})$. Type I_n and II_1 factors are called *finite factors*. There is a unique faithful normal tracial state on a finite factor. Factors except type III factors are called *semi-finite factors*. A semi-finite factor admits a faithful normal tracial weight.

The unitarily invariant norms on type I_n factors were introduced by von Neumann [23] for the purpose of metrizing matrix spaces. Von Neumann, together with his associates, established that the class of unitarily invariant norms on type I_n factors coincides with the class of symmetric gauge norms on $\mathbb{C}^n$. These norms have now been variously generalized and utilized in

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several contexts. For example, Schatten [16, 17] defined unitarily invariant norms on two-sided ideals of completely continuous operators in type I_∞ factors; Ky Fan [6] studied Ky Fan norms and obtained his dominance theorem. For historical perspectives and surveys of unitarily invariant norms, see [7, 11, 16, 17, 18, 19].

In [3], a structure theorem for unitarily invariant norms on finite factors is obtained. The main purpose of this paper is to set up a structure theorem for unitarily invariant norms related to semi-finite factors, which has a number of applications. Notably, even for $\mathcal{B}(\mathcal{H})$, this structure theorem is new!

In this paper, a *semi-finite von Neumann algebra* $(\mathcal{M}, \tau)$ means a von Neumann algebra $\mathcal{M}$ with a faithful normal tracial weight τ, and a *Hilbert space* $\mathcal{H}$ means the separable infinite-dimensional complex Hilbert space. If $(\mathcal{M}, \tau)$ is a finite von Neumann algebra, we assume that $\tau(1) = 1$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we assume that $\tau = \mathrm{Tr}$, the classical tracial weight on $\mathcal{B}(\mathcal{H})$. This paper is organized as follows.

In Section 2, we collect some basic facts on the s-numbers of operators in a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$.

In Section 3, we study various norms related to a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$. Let $\mathcal{J}(\mathcal{M})$ be the set of operators T in $\mathcal{M}$ such that $T = ETE$ for some finite projection E. Then $\mathcal{J}(\mathcal{M})$ is a hereditary self-adjoint two-sided ideal of $\mathcal{M}$. If $\mathcal{M}$ is a finite von Neumann algebra, then $\mathcal{J}(\mathcal{M}) = \mathcal{M}$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we simply write $\mathcal{J}(\mathcal{H})$ instead of $\mathcal{J}(\mathcal{B}(\mathcal{H}))$. Note that $\mathcal{J}(\mathcal{H})$ is the set of bounded linear operators T on $\mathcal{H}$ such that both T and T^* are finite rank operators.

A *unitarily invariant* norm $|\!|\!|\cdot|\!|\!|$ on $\mathcal{J}(\mathcal{M})$ is a norm on $\mathcal{J}(\mathcal{M})$ satisfying $|\!|\!|UTV|\!|\!| = |\!|\!|T|\!|\!|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators U, V in $\mathcal{M}$. For a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$, let $\mathrm{Aut}(\mathcal{M}, \tau)$ be the set of $*$-automorphisms of $\mathcal{M}$ preserving τ. A *symmetric gauge* norm $|\!|\!|\cdot|\!|\!|$ on $\mathcal{J}(\mathcal{M})$ is a norm on $\mathcal{J}(\mathcal{M})$ such that $|\!|\!|T|\!|\!| = |\!|\!|\,|T|\,|\!|\!|$ (gauge invariant) and $|\!|\!|\theta(T)|\!|\!| = |\!|\!|T|\!|\!|$ (symmetric) for all operators $T \in \mathcal{J}(\mathcal{M})$ and $\theta \in \mathrm{Aut}(\mathcal{M}, \tau)$. A norm $|\!|\!|\cdot|\!|\!|$ on $\mathcal{J}(\mathcal{M})$ is *normalized* if $|\!|\!|E|\!|\!| = 1$ for a projection E in $\mathcal{M}$ such that $\tau(E) = 1$. We will reserve the notation $\|\cdot\|$ for the operator norm on a von Neumann algebra.

In Section 4, we define and study the normalized Ky Fan norms related to semi-finite von Neumann algebras. To illustrate difficulties one may encounter in studying the unitarily invariant norms related to infinite factors, we point out one example here. The following result plays a key role in the study of unitarily invariant norms on finite factors: if $|\!|\!|\cdot|\!|\!|$ is a normalized unitarily invariant norm on a finite factor $(\mathcal{M}, \tau)$, then

$$\|T\|_1 \le |\!|\!|T|\!|\!| \le \|T\|$$

for all $T \in \mathcal{M}$, where $\|T\|_1 = \tau(|T|)$ (see [3, Corollary 3.31]). However, the above result is not true for infinite factors (see Proposition 4.6).

In Section 5, we study the dual norms of symmetric gauge norms on $\mathcal{J}(\mathcal{M})$. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and let $\|\!|\cdot|\!\|$ be a norm on $\mathcal{J}(\mathcal{M})$. For $T \in \mathcal{J}(\mathcal{M})$, define

$$\|\!|T|\!\|^{\#} = \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}),\ \|\!|X|\!\| \leq 1\}.$$

In that section, we also compute the dual norms of Ky Fan norms and prove that $\|\!|\cdot|\!\|^{\#\#} = \|\!|\cdot|\!\|$ under very general conditions.

A representation theorem (Theorem 6.4) for symmetric gauge norms on $\mathcal{J}(\mathcal{M})$ is set up in Section 6; it is the main result of this paper. In Section 7, we apply the representation theorem to two special cases: factors and abelian von Neumann algebras. In the remaining sections, we give some applications of the representation theorem.

In Section 8, we prove that there is a one-to-one correspondence between unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ for a semi-finite factor $\mathcal{M}$ and symmetric gauge norms on $\mathcal{J}(\mathcal{A})$ for a classical abelian von Neumann algebra $\mathcal{A}$, which generalizes von Neumann's classical result [23] on unitarily invariant norms on type I_n factors. Furthermore, we establish the one-to-one correspondence between the dual norms on $\mathcal{J}(\mathcal{M})$ for a semi-finite factor $\mathcal{M}$ and the dual norms on $\mathcal{J}(\mathcal{A})$, which plays a key role in the study of duality and reflexivity of the completion of $\mathcal{J}(\mathcal{M})$ with respect to unitarily invariant norms. As a quick application, a very simple proof of Ky Fan's dominance theorem for general semi-finite factors is given in Section 9.

For the theory of von Neumann algebras we refer to [2, 10].

In our paper [3] on tracial gauge norms, we mistakenly failed to consider, for a tracial gauge α, the case in which the α-closure $L^{\alpha}(\mathcal{M}, \tau)$ is not the same as the set $\mathcal{L}^{\alpha}(\mathcal{M}, \tau)$ of elements A in the measure-closure of $\mathcal{M}$ with $\alpha(A) < \infty$ (we refer to [1] for the definitions of $L^{\alpha}(\mathcal{M}, \tau)$ and $\mathcal{L}^{\alpha}(\mathcal{M}, \tau)$). This led to incorrect results on dual spaces and reflexivity (Theorems H and I in [3]). Nothing else in that paper was affected by this error. Recently, Yanni Chen [1, Section 11] proved the correct versions. She called a symmetric gauge norm α *strongly continuous* if it is continuous and $L^{\alpha}(\mathcal{M}, \tau) = \mathcal{L}^{\alpha}(\mathcal{M}, \tau)$. She proved that if α is continuous, then the dual space of $L^{\alpha}(\mathcal{M}, \tau)$ is $\mathcal{L}^{\alpha'}(\mathcal{M}, \tau)$, and she demonstrated that $L^{\alpha}(\mathcal{M}, \tau)$ is reflexive if and only if both α and α' are strongly continuous. She also proved that if α is continuous, then $L^{\alpha}(\mathcal{M}, \tau) = \mathcal{L}^{\alpha}(\mathcal{M}, \tau)$ if and only if $L^{\alpha}(\mathcal{M}, \tau)$ is weakly sequentially complete.

2. Preliminaries

2.1. Nonincreasing rearrangements of functions. Throughout this paper, we denote by m the Lebesgue measure on $[0, \infty)$. In the following, a measurable function and a measurable set mean a Lebesgue measurable function and a Lebesgue measurable set, respectively. Let f be a real measurable function on $[0,\infty)$. The *nonincreasing rearrangement*, f^*, of f is defined by

$$f^*(x) = \sup\{y : m(\{f > y\}) > x\}, \quad 0 \le x < \infty. \tag{2.1}$$

We summarize some well-known properties of f^* in the following proposition [8, 20].

PROPOSITION 2.1. *Let $f, g, f_1, f_2, \ldots$ be real measurable functions on $[0,\infty)$, and c be a real number. Then*

(1) *f^* is a nonincreasing, right-continuous function on $[0,\infty)$ such that $f^*(0) = \operatorname{ess\,sup} f(x)$;*
(2) *$(f + c)^* = f^* + c$;*
(3) *$(cf)^* = cf^*$ if $c \ge 0$;*
(4) *if f is a simple function, then so is f^*;*
(5) *if $f(x) \le g(x)$ for almost all x, then $f^*(x) \le g^*(x)$ everywhere;*
(6) *$\|f^* - g^*\|_\infty \le \|f - g\|_\infty$;*
(7) *if $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly, then $\lim_{n\to\infty} f_n^*(x) = f^*(x)$ uniformly;*
(8) *if f_n converges to f in measure, then $\liminf_{n\to\infty} f_n^*(x) \ge f^*(x)$ for every $x \in [0,\infty)$;*
(9) *if f_n converges to f in measure, then $\limsup_{n\to\infty} f_n^*(x) \le f^*(x)$ for every $x \in [0,\infty)$ such that f^* is continuous at x;*
(10) *f and f^* are equi-measurable, i.e., for any real y, $m(\{f > y\}) = m(\{f^* > y\})$;*
(11) *$f^* = g^*$ if and only f and g are equi-measurable;*
(12) *if f and g are bounded functions and $\int_0^\infty f(x)^n\,dx = \int_0^\infty g(x)^n\,dx$ for all $n = 0, 1, 2, \ldots$, then $f^* = g^*$;*
(13) *$\int_0^\infty f(x)\,dx = \int_0^\infty f^*(x)\,dx$ when either integral is well-defined;*
(14) *if f, g are nonnegative measurable functions on $[a, b]$ and f^*, g^* are their respective nonincreasing rearrangements, then $\int_a^b f(x)g(x)\,dx \le \int_a^b f^*(x)g^*(x)\,dx$.*

2.2. s-numbers of operators in type II_∞ factors. In [5], Fack and Kosaki give a rather complete exposition of generalized s-numbers of τ-measurable operators affiliated with semi-finite von Neumann algebras. For the reader's convenience and our purpose, in this section we provide sufficient details on s-numbers of bounded operators in semi-finite von Neumann algebras. We will define the s-numbers using nonincreasing rearrangements, which is implicit in [5]. Recall that a von Neumann algebra $\mathcal{A}$ is called *diffuse* if there is no nonzero minimal projection in $\mathcal{A}$.

Lemma 2.2. *Let $(\mathcal{A},\tau)$ be a separable (i.e., with separable predual) diffuse abelian von Neumann algebra with a faithful normal tracial weight τ on $\mathcal{A}$ such that $\tau(1)=\infty$. Then there is a $*$-isomorphism α from $(\mathcal{A},\tau)$ onto $(L^\infty[0,\infty),\int_0^\infty dx)$ such that $\tau=\int_0^\infty dx\cdot\alpha$.*

Proof. Choose a sequence $\{E_n\}_{n=1}^\infty$ of mutually orthogonal projections in $\mathcal{A}$ such that $\sum_{n=1}^\infty E_n=1$ and $\tau(E_n)=1$ for all n. By [3, Lemma 2.6], there is a $*$-isomorphism α_n from $E_n\mathcal{A}E_n$ onto $L^\infty([n,n+1])$ such that $\tau(E_nTE_n)=\int_n^{n+1}\alpha_n(E_nTE_n)(x)\,dx$ for all $T\in\mathcal{A}$. For $T\in\mathcal{A}$, define

$$\alpha(T)=\sum_{n=1}^\infty \alpha_n(E_nTE_n).$$

Then α is as desired. ∎

Let $\mathcal{M}$ be a type II_∞ factor and let τ be a faithful normal trace on $\mathcal{M}$. For $T\in\mathcal{M}$, there is a separable diffuse abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ containing $|T|$. By Lemma 2.2, there is a $*$-isomorphism α from $(\mathcal{A},\tau)$ onto $(L^\infty[0,\infty),\int_0^\infty dx)$ such that $\tau=\int_0^\infty dx\cdot\alpha$. Let $f=\alpha(|T|)$ and let f^* be the nonincreasing rearrangement of f (see (2.1)). Then the *s-numbers* of T, $\mu_s(T)$, are defined as

$$\mu_s(T)=f^*(s),\qquad 0\le s<\infty.$$

Lemma 2.3. *$\mu_s(T)$ does not depend on $\mathcal{A}$ or α.*

Proof. Let $\mathcal{A}_1$ be another separable diffuse abelian von Neumann subalgebra of $\mathcal{M}$ containing $|T|$ and suppose β is a $*$-isomorphism from $\mathcal{A}_1$ onto $L^\infty[0,\infty)$ such that $\tau=\int_0^\infty dx\cdot\beta$. Let $g=\beta(|T|)$. For $n=0,1,2,\ldots,$ we have $\int_0^\infty f(x)^n\,dx=\tau(|T|^n)=\int_0^\infty g(x)^n\,dx$. Since both f and g are bounded positive functions, by Proposition 2.1(12), $f^*(x)=g^*(x)$ for all $x\in[0,\infty)$. ∎

Corollary 2.4. *For $T\in\mathcal{M}$ and $p\ge 0$, $\tau(|T|^p)=\int_0^\infty \mu_s(T)^p ds$.*

Lemma 2.5. *Let E,F be two projections in $\mathcal{M}$. If $\tau(E^\perp)<\tau(F^\perp)<\infty$, then $\tau(E\wedge F^\perp)>0$.*

Proof. By [10, Vol. 1, p. 119, Proposition 2.5.14], $R(F^\perp E^\perp)=F^\perp-E\wedge F^\perp$, where $R(F^\perp E^\perp)$ is the range projection of $F^\perp E^\perp$. Therefore,

$$\tau(E\wedge F^\perp)=\tau(F^\perp)-\tau(R(F^\perp E^\perp))\ge\tau(F^\perp)-\tau(E^\perp)>0. \ ∎$$

Let $\mathcal{P}(\mathcal{M})$ be the set of projections in $\mathcal{M}$. The following lemma says that the above definition of s-numbers coincides with the definition of s-numbers given by Fack and Kosaki.

Lemma 2.6. *Let M be a type II_∞ factor and τ be a faithful normal trace on M. For $0\le s<\infty$,*

$$\mu_s(T)=\inf\{\|TE\|:E\in\mathcal{P}(\mathcal{M}),\ \tau(E^\perp)=s\}.$$

Proof. By polar decomposition and the definition of $\mu_s(T)$, we may assume that T is positive. Let $\mathcal{A}$ be a separable diffuse abelian von Neumann subalgebra of $\mathcal{M}$ containing T and let α be a $*$-isomorphism from $\mathcal{A}$ onto $L^\infty[0,\infty)$ such that $\tau = \int_0^\infty dx \cdot \alpha$. Let $f = \alpha(T)$ and let f^* be the nonincreasing rearrangement of f. Then $\mu_s(T) = f^*(s)$. By the definition of f^*,

$$m(\{f^* > \mu_s(T)\}) = \lim_{n\to\infty} m\left(\left\{f^* > \mu_s(T) + \frac{1}{n}\right\}\right) \le s$$

and

$$m(\{f^* \ge \mu_s(T)\}) \ge \lim_{n\to\infty} m\left(\left\{f^* > \mu_s(T) - \frac{1}{n}\right\}\right) \ge s.$$

Since f^* and f are equi-measurable, we have $m(\{f > \mu_s(T)\}) \le s$ and $m(\{f \ge \mu_s(T)\}) \ge s$. Therefore, there is a measurable set A of $[0,\infty)$ with $\{f > \mu_s(T)\} \subset [0,\infty) \setminus A \subset \{f \ge \mu_s(T)\}$ such that $m([0,\infty)\setminus A) = s$ and $\|f\chi_A\|_\infty = \mu_s(T)$ and $\|f\chi_B\|_\infty \ge \mu_s(T)$ for every $B \subset [0,\infty)\setminus A$ such that $m(B) > 0$. Let $F = \alpha^{-1}(\chi_A)$. Then $\tau(F^\perp) = s$, $\|TF\| = \|\alpha^{-1}(f\chi_A)\|_\infty = \mu_s(T)$ and $\|TF'\| \ge \mu_s(T)$ for any nonzero subprojection F' of $F^\perp$. This proves that

$$\mu_s(T) \ge \inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}),\ \tau(E^\perp) = s\}.$$

Similarly, for any $\epsilon > 0$, there is a projection $F_\epsilon \in \mathcal{M}$ such that $\tau(F_\epsilon^\perp) = s + \epsilon$, $\|TF_\epsilon\| = \mu_{s+\epsilon}(T)$ and $\|TF'\| \ge \mu_{s+\epsilon}(T)$ for any nonzero subprojection F' of $F_\epsilon^\perp$. Suppose $E \in \mathcal{M}$ is a projection such that $\tau(E^\perp) = s$. By Lemma 2.5, $\tau(E \wedge F_\epsilon^\perp) > 0$. Hence, $\|TE\| \ge \|T(E \wedge F_\epsilon^\perp)\| \ge \mu_{s+\epsilon}(T)$. This proves that $\inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}),\ \tau(E^\perp) = s\} \ge \mu_{s+\epsilon}(T)$. Since $\mu_s(T)$ is right-continuous,

$$\mu_s(T) \le \inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}),\ \tau(E^\perp) = s\}. \blacksquare$$

COROLLARY 2.7. *Let $S, T \in \mathcal{M}$. Then*

$$\mu_s(ST) \le \|S\|\mu_s(T) \quad \textit{for } s \in [0,\infty).$$

We refer to [4, 5] for other interesting properties of s-numbers for operators in type II_∞ factors.

2.3. s-numbers of operators in semi-finite von Neumann algebras. An *embedding* of a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ into another semi-finite von Neumann algebra $(\mathcal{M}_1, \tau_1)$ is a $*$-isomorphism α from $\mathcal{M}$ to $\mathcal{M}_1$ such that $\tau = \tau_1 \cdot \alpha$. Every semi-finite von Neumann algebra can be embedded into a type II_∞ factor.

DEFINITION 2.8. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and $T \in \mathcal{M}$. If α is an embedding of $(\mathcal{M}, \tau)$ into a type II_∞ factor $(\mathcal{M}_1, \tau_1)$, then the *s-numbers* of T are defined as

$$\mu_s(T) = \mu_s(\alpha(T)), \qquad 0 \le s < \infty.$$

Similar to the proof of Lemma 2.3, we can see that $\mu_s(T)$ is well defined, i.e., does not depend on the choice of α or $\mathcal{M}_1$.

Let $T \in (\mathcal{B}(\mathcal{H}), \mathrm{Tr})$ be a finite rank operator, where $\mathcal{H}$ is the separable infinite-dimensional complex Hilbert space and Tr is the classical tracial weight on $\mathcal{B}(\mathcal{H})$. Then $|T|$ is unitarily equivalent to a diagonal operator with diagonal elements $s_1(T) \geq s_2(T) \geq \cdots \geq 0$. In the classical operator theory [7], $s_1(T), s_2(T), \ldots$ are also called the s-numbers of T. It is easy to see that the relation between $\mu_s(T)$ and $s_1(T), s_2(T), \ldots$ is the following:

$$\mu_s(T) = s_1(T)\chi_{[0,1)}(s) + s_2(T)\chi_{[1,2)}(s) + \cdots. \tag{2.2}$$

Since no confusion will arise, we will use both s-numbers for a finite rank operator in $(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$. We refer to [5, 7] for properties of s-numbers for finite rank operators in $(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$.

We end this section with the following definition.

Definition 2.9. Two positive operators S, T in a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ are *equi-measurable* if $\mu_s(S) = \mu_s(T)$ for $0 \leq s < \infty$.

By Proposition 2.1(12) and Corollary 2.4, positive operators S and T in a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ are equi-measurable if and only if $\tau(S^n) = \tau(T^n)$ for all $n = 0, 1, 2, \ldots$.

3. Semi-norms on $\mathcal{J}(\mathcal{M})$.

In this section, $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra with a faithful normal tracial weight τ. Recall that $\mathcal{J}(\mathcal{M})$ is the set of operators T in $\mathcal{M}$ such that $T = ETE$ for some finite projection E. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we simply write $\mathcal{J}(\mathcal{H})$ instead of $\mathcal{J}(\mathcal{B}(\mathcal{H}))$. Note that $\mathcal{J}(\mathcal{H})$ is the set of bounded linear operators T on $\mathcal{H}$ such that both T and T^* are finite rank operators.

3.1. Gauge invariant semi-norms on $\mathcal{J}(\mathcal{M})$

Definition 3.1. *Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra. A semi-norm $\|\!|\cdot|\!\|$ on $\mathcal{J}(\mathcal{M})$ is* gauge invariant *if $\|\!|T|\!\| = \|\!|\,|T|\,|\!\|$ for all $T \in \mathcal{J}(\mathcal{M})$. A semi-norm $\|\!|\cdot|\!\|$ on $\mathcal{J}(\mathcal{M})$ is called* left unitarily invariant *if $\|\!|UT|\!\| = \|\!|T|\!\|$ for all unitary operators U in $\mathcal{M}$ and all T in $\mathcal{J}(\mathcal{M})$.*

Lemma 3.2. *Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and let $\|\!|\cdot|\!\|$ be a left unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$. If $T \in \mathcal{J}(\mathcal{M})$ and $A \in \mathcal{M}$, then $AT \in \mathcal{J}(\mathcal{M})$ and $\|\!|AT|\!\| \leq \|A\| \cdot \|\!|T|\!\|$.*

Proof. Note that by [10, Theorem 6.8.3], $AT \in \mathcal{J}(\mathcal{M})$ if $T \in \mathcal{J}(\mathcal{M})$ and $A \in \mathcal{M}$. We need to prove that if $\|A\| < 1$, then $\|\!|AT|\!\| \leq \|\!|T|\!\|$. Since $\|A\| < 1$, there are unitary operators $U_1, \ldots, U_k$ such that $A = \frac{1}{k}(U_1 + \cdots + U_k)$ (see [9, 15]). Since $\|\!|\cdot|\!\|$ is a left unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$,

$$\|\!|AT|\!\| = \left|\!\left|\!\left| \frac{1}{k}(U_1T + \cdots + U_kT) \right|\!\right|\!\right| \leq \frac{\|\!|U_1T|\!\| + \cdots + \|\!|U_kT|\!\|}{k} \leq \|\!|T|\!\|. \blacksquare$$

Lemma 3.3. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra and let $\|\!|\cdot|\!\|$ be a semi-norm on $\mathcal{J}(\mathcal{M})$. Then $\|\!|\cdot|\!\|$ is gauge invariant if and only if $\|\!|\cdot|\!\|$ is left unitarily invariant.*

Proof. Note that $|UT| = |T|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators U in $\mathcal{M}$. If $\|\!|\cdot|\!\|$ is gauge invariant then $\|\!|\cdot|\!\|$ is left unitarily invariant. Conversely, suppose $\|\!|\cdot|\!\|$ is left unitarily invariant. By Lemma 3.2, $\|\!|T|\!\| = \|\!|V|T|\,|\!\| \le \|\!|\,|T|\,|\!\|$ and $\|\!|\,|T|\,|\!\| = \|\!|V^*T|\!\| \le \|\!|T|\!\|$. Hence, $\|\!|\cdot|\!\|$ is gauge invariant. ∎

Corollary 3.4. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra and let $\|\!|\cdot|\!\|$ be a gauge invariant semi-norm on $\mathcal{J}(\mathcal{M})$. If $T \in \mathcal{J}(\mathcal{M})$ and $0 \le S \le T$, then $S \in \mathcal{J}(\mathcal{M})$ and $\|\!|S|\!\| \le \|\!|T|\!\|$.*

Proof. Since $0 \le S \le T$, there is an operator $A \in \mathcal{M}$ such that $S = AT$ and $\|A\| \le 1$. By Lemmas 3.2 and 3.3, $S \in \mathcal{J}(\mathcal{M})$ and $\|\!|S|\!\| = \|\!|AT|\!\| \le \|A\| \cdot \|\!|T|\!\| \le \|\!|T|\!\|$. ∎

3.2. Unitarily invariant semi-norms on $\mathcal{J}(\mathcal{M})$

Definition 3.5. Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra. A semi-norm $\|\!|\cdot|\!\|$ on $\mathcal{J}(\mathcal{M})$ is *unitarily invariant* if $\|\!|UTV|\!\| = \|\!|T|\!\|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators $U, V \in \mathcal{M}$.

Proposition 3.6. *Let $\|\!|\cdot|\!\|$ be a semi-norm on $\mathcal{J}(\mathcal{M})$. Then the following statements are equivalent:*

(1) *$\|\!|\cdot|\!\|$ is unitarily invariant;*
(2) *$\|\!|\cdot|\!\|$ is gauge invariant and unitarily conjugate invariant, i.e., $\|\!|UTU^*|\!\| = \|\!|T|\!\|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators $U \in \mathcal{M}$;*
(3) *$\|\!|\cdot|\!\|$ is gauge invariant and $\|\!|T|\!\| = \|\!|T^*|\!\|$ for all $T \in \mathcal{J}(\mathcal{M})$;*
(4) *$\|\!|ATB|\!\| \le \|A\| \cdot \|\!|T|\!\| \cdot \|B\|$ for all $A, B \in \mathcal{M}$ and $T \in \mathcal{J}(\mathcal{M})$.*

Proof. (1)⇒(4) is similar to the proof of Lemma 3.2.

(4)⇒(3). Let $T = V|T|$. Then $T^* = |T|V^*$. By (4) and simple arguments, $\|\!|T|\!\| = \|\!|T^*|\!\|$.

(3)⇒(2). By Lemma 3.3 and (3), we have $\|\!|UTU^*|\!\| = \|\!|TU^*|\!\| = \|\!|UT^*|\!\| = \|\!|T^*|\!\| = \|\!|T|\!\|$.

(2)⇒(1). Suppose $\|\!|\cdot|\!\|$ is gauge invariant and unitarily conjugate invariant. Let $U, V \in \mathcal{M}$ be unitary operators and $T \in \mathcal{J}(\mathcal{M})$. By Lemma 3.3, $\|\!|UTV|\!\| = \|\!|V^*VUTV|\!\| = \|\!|VUT|\!\| = \|\!|T|\!\|$. ∎

Corollary 3.7. *Let $\|\!|\cdot|\!\|$ be a unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$ and let E, F be two equivalent projections in $\mathcal{J}(\mathcal{M})$. Then $\|\!|E|\!\| = \|\!|F|\!\|$.*

3.3. Symmetric gauge semi-norms on $\mathcal{J}(\mathcal{M})$

Definition 3.8. Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra and let $\mathrm{Aut}(\mathcal{M},\tau)$ be the set of $*$-automorphisms on $\mathcal{M}$ preserving τ. A semi-norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is called *symmetric* (with respect to τ) if

$$\|\theta(T)\| = \|T\|, \quad \forall T \in \mathcal{J}(\mathcal{M}),\ \theta \in \mathrm{Aut}(\mathcal{M}, \tau);$$

a semi-norm $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ is called a *symmetric gauge semi-norm* if it is both symmetric and gauge invariant on $\mathcal{J}(\mathcal{M})$.

EXAMPLE 3.9. The abelian von Neumann algebra $\mathbb{C}^n$ is a finite von Neumann algebra with the classical tracial state $\tau((x_1, \dots, x_n)) = (x_1 + \dots + x_n)/n$. In this case, $\mathcal{J}(\mathbb{C}^n) = \mathbb{C}^n$. A norm $\|\cdot\|$ on $\mathbb{C}^n$ is a symmetric gauge norm if and only if for every $(x_1, \dots, x_n) \in \mathbb{C}^n$,

- $\|(x_1, \dots, x_2)\| = \|(|x_1|, \dots, |x_n|)\|$ and
- $\|(x_1, \dots, x_n)\| = \|(x_{\pi(1)}, \dots, x_{\pi(n)})\|$ for every permutation π of $\{1, \dots, n\}$.

EXAMPLE 3.10. The abelian von Neumann algebra $l^\infty(\mathbb{N})$ is a semi-finite von Neumann algebra with the classical tracial weight $\tau((x_1, x_2, \dots)) = x_1 + x_2 + \cdots$. It is easy to see that $\mathcal{J}(l^\infty(\mathbb{N})) = c_{00}$ consists of $(x_1, x_2, \dots)$ with $x_n = 0$ except for finitely many n. A norm $\|\cdot\|$ on $\mathcal{J}(l^\infty(\mathbb{N}))$ is a symmetric gauge norm if and only if for every $(x_1, x_2, \ldots) \in c_{00}$,

- $\|(x_1, x_2, \ldots)\| = \|(|x_1|, |x_2|, \ldots)\|$ and
- $\|(x_1, x_2, \ldots)\| = \|(x_{\pi(1)}, x_{\pi(2)}, \ldots)\|$ for every permutation π of $\mathbb{N}$.

EXAMPLE 3.11. The abelian von Neumann algebra $L^\infty[0,1]$ is a finite von Neumann algebra with the classical tracial state $\tau = \int_0^1 dx$. In this case $\mathcal{J}(L^\infty[0,1]) = L^\infty[0,1]$. A norm $\|\cdot\|$ on $L^\infty[0,1]$ is a symmetric gauge norm if and only if for every $f \in L^\infty[0,1]$,

- $\|f\| = \|\,|f|\,\|$ and
- $\|f\| = \|f \circ \phi\|$ for every invertible measure preserving map ϕ of $[0,1]$.

EXAMPLE 3.12. The abelian von Neumann algebra $L^\infty[0,\infty)$ is a semi-finite von Neumann algebra with the classical tracial weight $\tau = \int_0^\infty dx$. A norm $\|\cdot\|$ on $\mathcal{J}(L^\infty[0,\infty))$ is a symmetric gauge norm if and only if for every $f \in \mathcal{J}(L^\infty[0,\infty))$,

- $\|f\| = \|\,|f|\,\|$ and
- $\|f\| = \|f \circ \phi\|$ for every invertible measure preserving map ϕ of $[0,\infty)$.

The following lemma follows from Proposition 3.6.

LEMMA 3.13. *Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and let $\|\cdot\|$ be a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$. Then $\|\cdot\|$ is a unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$.*

3.4. Symmetric gauge norms on $(\mathcal{M}_E, \tau_E)$. In this paper we are interested in symmetric gauge semi-norms on $\mathcal{J}(\mathcal{M})$, where $(\mathcal{M}, \tau)$ is one of the following semi-finite von Neumann algebras:

- $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{Tr}$ on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the separable infinite-dimensional complex Hilbert space;
- $\mathcal{M} = l^\infty(\mathbb{N})$ and $\tau((x_1, x_2, \ldots)) = x_1 + x_2 + \cdots$;
- $\mathcal{M}$ is a type II_∞ factor and τ is a faithful normal tracial weight on $\mathcal{M}$;
- $\mathcal{M} = L^\infty[0, \infty)$ and $\tau = \int_0^\infty dx$.

Note that in each case, $\mathrm{Aut}(\mathcal{M}, \tau)$ acts on $\mathcal{M}$ strongly ergodically in the following sense: for two projections E and F in $\mathcal{M}$ with $\tau(E) \le \tau(F)$, there is a $\theta \in \mathrm{Aut}(\mathcal{M}, \tau)$ such that $\theta(E) \le F$. Furthermore, if $\|\!| \cdot |\!\|$ is a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$, then $\|\!|E|\!\| = \|\!|F|\!\|$. A symmetric gauge semi-norm $\|\!| \cdot |\!\|$ on $\mathcal{J}(M)$ is called a *normalized* symmetric gauge semi-norm if $\|\!|E|\!\| = 1$ whenever $\tau(E) = 1$.

Let $(\mathcal{M}, \tau)$ be one of the above semi-finite von Neumann algebras. For every (nonzero) finite projection E in $\mathcal{M}$, let

$$\mathcal{M}_E = E\mathcal{M}E \quad \text{and} \quad \tau_E(ETE) = \frac{\tau(ETE)}{\tau(E)}.$$

Then $(\mathcal{M}_E, \tau_E)$ is a finite von Neumann algebra satisfying the *weak Dixmier property* (see [3, Definition 3.22]), i.e., for every positive operator $T \in \mathcal{M}_E$, $\tau_E(T)E$ is in the operator norm closure of the convex hull of $\{S \in \mathcal{M}_E :$ S and T are equi-measurable$\}$. So in the following sections we will always assume that $(\mathcal{M}, \tau)$ satisfies the following conditions:

A. $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra such that $\mathrm{Aut}(\mathcal{M}, \tau)$ acts on $\mathcal{M}$ strongly ergodically;

B. for every nonzero finite projection E in $\mathcal{M}$, $(\mathcal{M}_E, \tau_E)$ is a finite von Neumann algebra satisfying the weak Dixmier property.

With the above assumptions, it is easy to show that if E is a finite projection of $\mathcal{M}$, then $\mathrm{Aut}(\mathcal{M}_E, \tau_E)$ acts on $\mathcal{M}_E$ ergodically.

A *simple* operator in a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ is an operator $T = a_1E_1 + \cdots + a_nE_n$, where $E_1, \ldots, E_n$ are mutually orthogonal projections.

LEMMA 3.14. *Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra satisfying conditions* **A** *and* **B** *and let $\|\!| \cdot |\!\|$ be a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$. If $E \in \mathcal{M}$ is a finite projection, then the restriction of $\|\!| \cdot |\!\|$ to $(\mathcal{M}_E, \tau_E)$ is also a symmetric gauge semi-norm on $(\mathcal{M}_E, \tau_E)$.*

Proof. It is obvious that the restriction of $\|\!| \cdot |\!\|$ to $(\mathcal{M}_E, \tau_E)$ is also a gauge semi-norm on $(\mathcal{M}_E, \tau_E)$. Let $\theta \in \mathrm{Aut}(\mathcal{M}_E, \tau_E)$. Define $\|\!|S|\!\|_2 = \|\!|\theta(S)|\!\|$ for $S \in \mathcal{M}_E$. We need to prove $\|\!| \cdot |\!\| = \|\!| \cdot |\!\|_2$ on $\mathcal{M}_E$.

Let $T = a_1E_1 + \cdots + a_nE_n$ be a simple positive operator in $\mathcal{M}_E$, where $E_1 + \cdots + E_n = E$. Then $\theta(T) = a_1\theta(E_1) + \cdots + a_n\theta(E_n)$. Since $\theta \in \mathrm{Aut}(\mathcal{M}_E, \tau_E)$, we have $\tau(E_k) = \tau(\theta(E_k))$ for $1 \le k \le n$. By assumption,

$\mathrm{Aut}(\mathcal{M},\tau)$ acts on $\mathcal{M}$ ergodically. Therefore, there is a $\theta' \in \mathrm{Aut}(\mathcal{M},\tau)$ such that $\theta'(E_k) = \theta(E_k)$ for $1 \le k \le n$. Hence, $\theta'(T) = \theta(T)$. Since $\|\!|\cdot|\!\|$ is a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$, $\|\!|T|\!\| = \|\!|\theta'(T)|\!\| = \|\!|\theta(T)|\!\| = \|\!|T|\!\|_2$. By [3, Corollary 3.6], $\|\!|\cdot|\!\| = \|\!|\cdot|\!\|_2$ on $(\mathcal{M}_E,\tau_E)$. This implies that the restriction of $\|\!|\cdot|\!\|$ to $(\mathcal{M}_E,\tau_E)$ is also a symmetric gauge semi-norm on $(\mathcal{M}_E,\tau_E)$. ∎

The following lemma is [3, Theorem 3.27].

Lemma 3.15. *Let $\mathcal{N}$ be a finite von Neumann algebra with a faithful normal tracial state τ_N. Then $\mathcal{N}$ has the weak Dixmier property if and only if $\mathcal{N}$ satisfies one of the following conditions:*

(1) *$\mathcal{N}$ is finite-dimensional (hence atomic) and for any two nonzero minimal projections $E,F \in \mathcal{N}$, $\tau(E) = \tau(F)$, or equivalently $(\mathcal{N},\tau_{\mathcal{N}})$ can be identified as a von Neumann subalgebra of $(M_n(\mathbb{C}),\tau_n)$ that contains all diagonal matrices;*
(2) *$\mathcal{N}$ is diffuse.*

Corollary 3.16. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra satisfying conditions* **A** *and* **B** *and let $\|\!|\cdot|\!\|$ be a normalized symmetric gauge semi-norm on $\mathcal{M}$. If F is a finite projection in $\mathcal{M}$ such that $\tau(F) \ge 1$, then $\|\!|F|\!\| \ge 1$.*

Proof. Let $E_1 \in \mathcal{M}$ be a finite projection such that $\tau(E_1) = 1$ and $\|\!|E_1|\!\| = 1$. There exists a finite projection $E \in \mathcal{M}$ such that $E_1, F \le E$. By Lemma 3.14, the restriction of $\|\!|\cdot|\!\|$ to $(\mathcal{M}_E,\tau_E)$ is also a symmetric gauge semi-norm on $(\mathcal{M}_E,\tau_E)$. Since $\mathcal{M}_E$ has the weak Dixmier property, there is a projection $F_1 \in \mathcal{M}_E$ such that $F_1 \le F$ and $\tau(F_1) = 1$ by Lemma 3.15. Since $\mathrm{Aut}(\mathcal{M}_E,\tau_E)$ acts on $\mathcal{M}_E$ ergodically, $\|\!|F_1|\!\| = \|\!|E_1|\!\| = 1$. By Corollary 3.4, $\|\!|F|\!\| \ge \|\!|F_1|\!\| = 1$. ∎

Corollary 3.17. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra satisfying conditions* **A** *and* **B***. If $\|\!|\cdot|\!\|$ is a normalized symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$, then $\|\!|\cdot|\!\|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$.*

Proof. Let $T \in \mathcal{J}(\mathcal{M})$. Then there is a finite projection E in $\mathcal{M}$ such that $T = ETE \in (\mathcal{M}_E,\tau_E)$. We may assume that $\tau(E) \ge 1$. By Lemma 3.14, the restriction of $\|\!|\cdot|\!\|$ to $(\mathcal{M}_E,\tau_E)$ is also a symmetric gauge semi-norm. If $T \ne 0$, then by [3, Theorem 3.30] and Corollary 3.16, $\|\!|T|\!\| \ge \tau_E(|T|)\cdot\|\!|E|\!\| > 0$. So $\|\!|\cdot|\!\|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. ∎

Lemma 3.18. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra satisfying conditions* **A** *and* **B***. Suppose $\|\!|\cdot|\!\|_1$ and $\|\!|\cdot|\!\|_2$ are two symmetric gauge norms on $\mathcal{J}(\mathcal{M})$. Then $\|\!|\cdot|\!\|_1 = \|\!|\cdot|\!\|_2$ on $\mathcal{J}(\mathcal{M})$ if $\|\!|T|\!\|_1 = \|\!|T|\!\|_2$ for every simple positive operator T in $\mathcal{J}(\mathcal{M})$ such that $T = a_1E_1 + \cdots + a_nE_n$ and $\tau(E_1) = \cdots = \tau(E_n)$.*

Proof. Suppose $\|\|T\|\|_1 = \|\|T\|\|_2$ for every simple operator T in $\mathcal{J}(\mathcal{M})$. Let $S \in \mathcal{J}(\mathcal{M})$. Then there is a finite projection E in $\mathcal{M}$ such that $S = ESE \in \mathcal{M}_E$. By Lemma 3.14, the restrictions of $\|\|\cdot\|\|_1$ and $\|\|\cdot\|\|_2$ to $(\mathcal{M}_E, \tau_E)$ are symmetric gauge norms. Since $\|\|T\|\|_1 = \|\|T\|\|_2$ for every simple operator T in $\mathcal{M}_E$ such that $T = a_1E_1 + \cdots + a_nE_n$ and $\tau(E_1) = \cdots = \tau(E_n)$, we conclude that $\|\|\cdot\|\|_1 = \|\|\cdot\|\|_2$ by [3, Corollary 4.6]. ∎

Proposition 3.19. *Let $(\mathcal{M}, \tau)$ be a semi-finite factor and let $\|\|\cdot\|\|$ be a norm on $\mathcal{J}(\mathcal{M})$. Then the following conditions are equivalent:*

(1) $\|\|\cdot\|\|$ *is a symmetric gauge norm;*
(2) $\|\|\cdot\|\|$ *is a unitarily invariant norm.*

Proof. (1)⇒(2) is obvious. We only prove (2)⇒(1). We need to prove that for every positive operator $T \in \mathcal{J}(\mathcal{M})$ and $\theta \in \mathrm{Aut}(\mathcal{M}, \tau)$, $\|\|\theta(T)\|\| = \|\|T\|\|$.

Let $S = \theta(T)$. Then $S \in \mathcal{J}(\mathcal{M})$, so there is a finite projection E in $\mathcal{M}$ such that $S, T \in \mathcal{M}_E$. By the spectral decomposition theorem, there is a sequence of simple positive operators $T_n \in \mathcal{M}_E$ such that $S_n = \theta(T_n) \in \mathcal{M}_E$ and $\lim_{n\to\infty} \|T_n - T\| = \lim_{n\to\infty} \|S_n - S\| = 0$. By Lemma 3.3,

$$\|\|T - T_n\|\| \le \|T - T_n\| \cdot \|\|E\|\| \quad \text{and} \quad \|\|S - S_n\|\| \le \|S - S_n\| \cdot \|\|E\|\|.$$

Hence, $\lim_{n\to\infty} \|\|T - T_n\|\| = \lim_{n\to\infty} \|\|S - S_n\|\| = 0$. We need only prove

$$\|\|T_n\|\| = \|\|S_n\|\| \quad \text{for all } n = 1, 2, \ldots.$$

Suppose $T_n = a_1E_1 + \cdots + a_mE_m$. Then $S_n = \theta(T_n) = a_1F_1 + \cdots + a_mF_m$, where $\theta(E_k) = F_k$ for $1 \le k \le m$. Since $\theta \in \mathrm{Aut}(\mathcal{M}, \tau)$, $\tau(E_k) = \tau(F_k)$ for $1 \le k \le m$. Since $\mathcal{M}$ is a factor, there is a unitary operator $U \in \mathcal{M}$ such that $E_k = UF_kU^*$ for $1 \le k \le m$. Therefore, $S_n = UT_nU^*$ and $\|\|T_n\|\| = \|\|S_n\|\|$. ∎

3.5. Semi-norms associated to von Neumann algebras

Definition 3.20. Let $\mathcal{M}$ be a von Neumann algebra (not necessarily semi-finite). A (*generalized*) *semi-norm associated to* $\mathcal{M}$ is a map $\|\|\cdot\|\|$ from $\mathcal{M}$ to $[0, \infty]$ satisfying the following properties:

- $\|\|\lambda T\|\| = |\lambda| \cdot \|\|T\|\|$,
- $\|\|S + T\|\| \le \|\|S\|\| + \|\|T\|\|$

for all $S, T \in \mathcal{M}$ and $\lambda \in \mathbb{C}$. To make the definition nontrivial, we always assume that $0 < \|\|T\|\| < \infty$ for some nonzero $T \in \mathcal{M}$.

Let $\mathcal{I} = \{T \in \mathcal{M} : \|\|T\|\| < \infty\}$. Then $\mathcal{I}$ is called the *domain* of the semi-norm $\|\|\cdot\|\|$.

Definition 3.21. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra. A semi-norm $\|\|\cdot\|\|$ associated to $\mathcal{M}$ is called *gauge invariant* if for all $T \in \mathcal{M}$, $\|\|T\|\| = \|\| \, |T| \, \|\|$; a semi-norm $\|\|\cdot\|\|$ associated to $\mathcal{M}$ is *unitarily invariant* if

$\|\!|UTV|\!\| = \|\!|T|\!\|$ for all $T \in \mathcal{M}$ and unitary operators $U, V \in \mathcal{M}$; a semi-norm $\|\!|\cdot|\!\|$ associated to a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ is called *symmetric* if

$$\|\!|\theta(T)|\!\| = \|\!|T|\!\|, \quad \forall T \in \mathcal{M},\ \theta \in \mathrm{Aut}(\mathcal{M}, \tau);$$

a semi-norm $\|\!|\cdot|\!\|$ associated to $(\mathcal{M}, \tau)$ is called a *symmetric gauge semi-norm* if it is both symmetric and gauge invariant.

Similar to the proof of Proposition 3.6, we can prove the following proposition.

PROPOSITION 3.22. *Let $\|\!|\cdot|\!\|$ be a semi-norm associated to $\mathcal{M}$. Then the following statements are equivalent:*

(1) *$\|\!|\cdot|\!\|$ is unitarily invariant;*
(2) *$\|\!|\cdot|\!\|$ is gauge invariant and unitarily conjugate invariant, i.e., $\|\!|UTU^*|\!\| = \|\!|T|\!\|$ for all $T \in \mathcal{M}$ and unitary operators $U \in \mathcal{M}$;*
(3) *$\|\!|\cdot|\!\|$ is gauge invariant and $\|\!|T|\!\| = \|\!|T^*|\!\|$ for all $T \in \mathcal{M}$;*
(4) *for all operators $T, A, B \in \mathcal{M}$, $\|\!|ATB|\!\| \le \|A\| \cdot \|\!|T|\!\| \cdot \|B\|$.*

COROLLARY 3.23. *Let $\|\!|\cdot|\!\|$ be a semi-norm associated to $\mathcal{M}$. If $S, T \in \mathcal{M}$ and $0 \le S \le T$, then $\|\!|S|\!\| \le \|\!|T|\!\|$.*

COROLLARY 3.24. *Let $\|\!|\cdot|\!\|$ be a unitarily invariant semi-norm associated to $\mathcal{M}$ and let E, F be two equivalent projections in $\mathcal{M}$. Then $\|\!|E|\!\| = \|\!|F|\!\|$.*

LEMMA 3.25. *Let $\|\!|\cdot|\!\|$ be a unitarily invariant semi-norm associated to $\mathcal{M}$ and let $T \in \mathcal{M}$ be a nonzero element such that $\|\!|T|\!\| < \infty$. Then there is a nonzero projection E in $\mathcal{M}$ such that $\|\!|E|\!\| < \infty$.*

Proof. Since $\|\!|\cdot|\!\|$ is unitarily invariant, we may assume $T > 0$. By the spectral decomposition theorem, there exist a $\lambda > 0$ and a nonzero projection E in $\mathcal{M}$ such that $T \ge \lambda E$. By Corollary 3.23, $\|\!|E|\!\| < \infty$. ∎

The following theorem shows that, up to a scale $a > 0$, the operator norm $\|\cdot\|$ is the unique unitarily invariant semi-norm associated to a type III factor.

THEOREM 3.26. *Let $\mathcal{M}$ be a type* III *factor and let $\|\!|\cdot|\!\|$ be a unitarily invariant semi-norm associated to $\mathcal{M}$. Then there exists $a > 0$ such that $\|\!|\cdot|\!\| = a\|\cdot\|$, i.e., $\|\!|T|\!\| = a\|T\|$ for all $T \in \mathcal{M}$.*

Proof. By Lemma 3.25, there is a nonzero projection E in $\mathcal{M}$ such that $\|\!|E|\!\| < \infty$. If $\|\!|E|\!\| = 0$, then $\|\!|1|\!\| = 0$ by Corollary 3.24. By Proposition 3.22, for every T in $\mathcal{M}$, $\|\!|T|\!\| \le \|T\| \cdot \|\!|1|\!\| = 0$. In our definition of semi-norm, we assume that $\|\!|T|\!\| > 0$ for some $T \in \mathcal{M}$. Hence $\|\!|E|\!\| \ne 0$ for some projection E in $\mathcal{M}$. We may assume that $\|\!|E|\!\| = 1$. By Corollary 3.24, $\|\!|F|\!\| = 1$ for every nonzero projection in $\mathcal{M}$. In particular, $\|\!|1|\!\| = 1$. By Proposition 3.22,

for every T in $\mathcal{M}$,
$$|||T||| \leq \|T\| \cdot |||1||| = \|T\|.$$
On the other hand, let $T \in \mathcal{M}$ be a positive operator and $\epsilon > 0$. By the spectral decomposition theorem, there is a nonzero projection F in $\mathcal{M}$ such that $T \geq (\|T\| - \epsilon)F$. By Corollary 3.23,
$$|||T||| \geq (\|T\| - \epsilon) \cdot |||F||| = \|T\| - \epsilon.$$
This proves that $|||T||| = \|T\|$ for every positive operator T in $\mathcal{M}$ and therefore for every T in $\mathcal{M}$. ∎

We end this section with the following lemma.

LEMMA 3.27. *Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra such that* $\mathrm{Aut}(\mathcal{M}, \tau)$ *acts on $\mathcal{M}$ strongly ergodically. If $|||\cdot|||$ is a normalized symmetric gauge semi-norm associated to $\mathcal{M}$ with domain $\mathcal{I}$, then $\mathcal{I} \supseteq \mathcal{J}(\mathcal{M})$ and $|||\cdot|||$ is a normalized symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$.*

Proof. Let E be a finite projection in $\mathcal{M}$ such that $\tau(E) = 1$. Then $|||E||| = 1$. Suppose that F is a finite projection in $\mathcal{M}$ such that $n \leq \tau(F) < n + 1$. Since $\tau(E) \leq \tau(F)$ and $\mathrm{Aut}(\mathcal{M}, \tau)$ acts on $\mathcal{M}$ strongly ergodically, there is a $\theta_1 \in \mathrm{Aut}(\mathcal{M}, \tau)$ such that $\theta_1(E) \leq F$. Let $E_1 = \theta_1(E) \leq F$. If $\tau(F - E_1) \geq \tau(E)$, then there is a $\theta_2 \in \mathrm{Aut}(\mathcal{M}, \tau)$ such that $\theta_2(E) \leq F - E_1$. Let $E_2 = \theta_2(E) \leq F - E_1$. Then $E_1 + E_2 \leq F$. By induction, there are mutually orthogonal finite projections $E_1, \ldots, E_n$ in $\mathcal{M}$ with $\tau(E_1) = \cdots = \tau(E_n) = 1$ such that $E_1 + \cdots + E_n \leq F$. Let $E_{n+1} = F - E_1 - \cdots - E_n$. Now $\tau(F - E_1 - \cdots - E_n) < \tau(E)$. So there is a $\theta \in \mathrm{Aut}(\mathcal{M}, \tau)$ such that $\theta(E_{n+1}) \leq E$. By Corollary 3.23, $|||E_{n+1}||| = |||\theta(E_{n+1})||| \leq |||E||| = 1$. Therefore,
$$|||F||| \leq |||E_1 + \cdots + E_{n+1}||| \leq n + 1.$$
So every finite projection is in $\mathcal{I}$. Hence $\mathcal{I} \supseteq \mathcal{J}(\mathcal{M})$. ∎

4. Ky Fan norms associated to semi-finite von Neumann algebras. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann subalgebra of a type II_∞ factor $(\mathcal{M}_1, \tau_1)$ and let $0 \leq t \leq \infty$. For $T \in \mathcal{M}$, define $|||T|||_{(t)}$, the *Ky Fan t-th norm* of T, by
$$|||T|||_{(t)} = \begin{cases} \|T\|, & t = 0, \\ \dfrac{1}{t}\displaystyle\int_0^t \mu_s(T)\, ds, & 0 < t \leq 1, \\ \displaystyle\int_0^t \mu_s(T)\, ds, & 1 < t \leq \infty. \end{cases}$$

Let $\mathcal{U}(\mathcal{M})$ be the set of unitary operators in $\mathcal{M}$, and $\mathcal{P}(\mathcal{M})$ be the set of projections in $\mathcal{M}$.

Lemma 4.1. *For* $0 < t \le 1$,

$$t|||T|||_{(t)} = \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\}.$$

Proof. First we assume that T is a positive operator. Let $\mathcal{A}$ be a separable diffuse abelian von Neumann subalgebra of $\mathcal{M}_1$ containing T and let α be a $*$-isomorphism from $(\mathcal{A}, \tau_1)$ onto $(L^\infty[0,\infty), \int_0^\infty dx)$ such that $\tau_1 = \int_0^\infty dx \cdot \alpha$. Let $f = \alpha(T)$ and let f^* be the nonincreasing rearrangement of f. Then $\mu_s(T) = f^*(s)$. By the definition of f^* (see (2.1)),

$$m(\{f^* > f^*(t)\}) = \lim_{n\to\infty} m\biggl(\biggl\{f^* > f^*(t) + \frac{1}{n}\biggr\}\biggr) \le t$$

and

$$m(\{f^* \ge f^*(t)\}) \ge \lim_{n\to\infty} m\biggl(\biggl\{f^* > f^*(t) - \frac{1}{n}\biggr\}\biggr) \ge t.$$

Since f^* and f are equi-measurable, we have $m(\{f > f^*(t)\}) \le t$ and $m(\{f \ge f^*(t)\}) \ge t$. Therefore, there is a measurable subset A of $[0,\infty)$ with $\{f > f^*(t)\} \subset A \subset \{f \ge f^*(t)\}$ such that $m(A) = t$. Since f and f^* are equi-measurable, we have $\int_A f(s)\,ds = \int_0^t f^*(s)\,ds$. Let $E' = \alpha^{-1}(\chi_A)$. Then $\tau_1(E') = t$ and

$$\tau_1(TE') = \int\limits_A f(s)\,ds = \int\limits_0^t f^*(s)\,ds = t|||T|||_{(t)}.$$

Hence,

$$t|||T|||_{(t)} \le \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\}.$$

We need to prove that if E is a projection in $\mathcal{M}_1$ with $\tau_1(E) = t$, and $U \in \mathcal{U}(\mathcal{M}_1)$, then

$$t|||T|||_{(t)} \ge |\tau_1(UTE)|.$$

By the Schwarz inequality,

$$|\tau_1(UTE)| = |\tau_1(EUT^{1/2}T^{1/2}E)| \le \tau_1(U^*EUT)^{1/2}\tau_1(ET)^{1/2}.$$

By Corollary 2.4, $\tau_1(ET) = \int_0^1 \mu_s(ET)\,ds$. By Corollary 2.7, $\mu_s(ET) \le \min\{\mu_s(T), \mu_s(E)\|T\|\}$. Note that $\mu_s(E) = 0$ for $s \ge \tau_1(E) = t$. Hence, $\tau_1(ET) \le \int_0^t \mu_s(T)\,ds = t|||T|||_t$. Similarly, $\tau_1(U^*EUT) \le t|||T|||_t$. So $|\tau_1(UTE)| \le t|||T|||_t$. This proves that

$$t|||T|||_{(t)} \ge \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\}.$$

Now we prove the general case. By the polar decomposition theorem, there is an isometry or a co-isometry V in $\mathcal{M}_1$ such that $T = V|T|$.

We first show that if V is an isometry in $\mathcal{M}_1$, then there is a sequence of unitary operators U_n in $\mathcal{M}_1$ that converges to V in the strong operator topology.

To see this, let $\{E_n\}$ be a sequence of mutually orthogonal projections in $\mathcal{M}_1$ such that $\sum_n E_n = I$ and $\tau_1(E_n) = 1$ for all n. Let $F_n = VE_nV^*$. Then F_n is a sequence of mutually orthogonal projections in $\mathcal{M}_1$ such that $\sum_n F_n = VV^*$ and $\tau_1(F_n) = 1$ for all n. Now both $I - \sum_{k=1}^n E_k$ and $I - \sum_{k=1}^n F_k$ are infinite projections in $\mathcal{M}_1$. So there is a partial isometry W_n in $\mathcal{M}_1$ such that the initial space of W_n is $I - \sum_{k=1}^n E_k$ and the final space of W_n is $I - \sum_{k=1}^n F_k$. Define

$$U_n = V\Big(\sum_{k=1}^n E_k\Big) + W_n.$$

Then U_n is a unitary operator in $\mathcal{M}_1$ and U_n converges to V in the strong operator topology.

On the other hand, if V is a co-isometry in $\mathcal{M}_1$, then there is a sequence of unitary operators U_n that converges to V^* in the strong operator topology. So U_n^* converges to V in the weak operator topology. Thus in either case (V is an isometry or co-isometry), there is a sequence of unitary operators in $\mathcal{M}_1$ that converges to V in the weak operator topology.

Now we show that $t|||T|||_{(t)} \le \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1),\, E \in \mathcal{P}(\mathcal{M}_1),\, \tau_1(E) = t\}$. Since

$$\begin{aligned} t|||T|||_{(t)} &= t|||\,|T|\,|||_{(t)} \\ &= \sup\{|\tau_1(U|T|E)| : U \in \mathcal{U}(\mathcal{M}_1),\, E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t\}, \end{aligned}$$

for any $\epsilon > 0$ there is a unitary operator $U \in \mathcal{M}_1$ and $E \in \mathcal{P}(\mathcal{M}_1)$ with $\tau_1(E) = t$ satisfying $t|||T|||_{(t)} \le |\tau_1(U|T|E)| + \epsilon/2$. Let $T = V|T|$, where $V \in \mathcal{M}_1$ is an isometry or a co-isometry. Then $|T| = V^*T$. Since UV^* is an isometry or a co-isometry, by the above arguments, there is a sequence of unitary operators U_n in $\mathcal{M}_1$ that converges to UV^* in the weak operator topology. Thus

$$|\tau_1(U|T|E)| = |\tau_1(UV^*TE)| = \lim_{n\to\infty} |\tau_1(U_nTE)|.$$

Therefore, there is an $N \in \mathbb{N}$ such that $t|||T|||_{(t)} \le |\tau_1(U_NTE)| + \epsilon$. This implies

$$t|||T|||_{(t)} \le \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1),\, E \in \mathcal{P}(\mathcal{M}_1),\, \tau_1(E) = t\}.$$

Next we show the opposite inequality. Let $T = V|T|$ be such that $V \in \mathcal{M}_1$ is an isometry or a co-isometry. Then UV is an isometry or a co-isometry. So there is a sequence of unitary operators U_n in $\mathcal{M}$ that converges to UV in the weak operator topology. Thus

$$|\tau_1(UTE)| = |\tau_1(UV|T|E)| = \lim_{n\to\infty} |\tau_1(U_n|T|E)| \le t|||\,|T|\,|||_{(t)} = t|||T|||_{(t)}. \ \blacksquare$$

Similarly, we can prove the following lemma.

Lemma 4.2. *For* $1 \le t \le \infty$,

$$|||T|||_{(t)} = \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\}.$$

Theorem 4.3. *For* $0 \le t \le \infty$, $|||\cdot|||_{(t)}$ *is a normalized symmetric gauge norm associated to* $(\mathcal{M}, \tau)$.

Proof. By the definition of s-number, $\mu_s(T) = \mu_s(\theta(T))$ for $T \in \mathcal{M}$ and $\theta \in \mathrm{Aut}(\mathcal{M}, \tau)$. To prove that $|||\cdot|||_{(t)}$ is a normalized symmetric gauge norm associated to $(\mathcal{M}, \tau)$, we need only prove the triangle inequality since other parts are obvious.

Let $S, T \in \mathcal{M}$. If $0 < t \le 1$, then by Lemma 4.1,

$$\begin{aligned} t|||S+T|||_{(t)} &= \sup\{|\tau_1(U(S+T)E)| : U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\} \\ &\le \sup\{|\tau_1(USE)| : U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\} \\ &\quad + \sup\{|\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\} \\ &= t|||S|||_{(t)} + t|||T|||_{(t)}. \end{aligned}$$

The proof of the case $t > 1$ is similar. ∎

Corollary 4.4. *Let* $T \in \mathcal{M}$ *and* $\delta > 0$. *If* $|||T|||_{(1)} < \delta$, *then*

$$\tau(\chi_{(\delta,\infty)}(|T|)) \le |||T|||_{(1)}/\delta.$$

Proof. We may assume that $\mathcal{M}$ is a type II_∞ factor and $T \ge 0$. By the proof of Lemma 4.1,

$$|||T|||_{(1)} = \sup\{|\tau(UTE)| : U \in \mathcal{U}(\mathcal{M}),\ E \in \mathcal{P}(\mathcal{M}),\ \tau(E) \le 1\}.$$

If $\tau(\chi_{(\delta,\infty)}(T)) > 1$, then there is a subprojection E of $\chi_{(\delta,\infty)}(T)$ such that $\tau(E) = 1$. Then $TE \ge \delta E$. Hence, $|||T|||_{(1)} \ge \tau(TE) \ge \tau(\delta E) = \delta$. This contradicts the assumption that $|||T|||_{(1)} < \delta$. Therefore, $\tau(\chi_{(\delta,\infty)}(T)) \le 1$. So

$$|||T|||_{(1)} \ge \tau(T\chi_{(\delta,\infty)}(T)) \ge \tau(\delta\chi_{(\delta,\infty)}(T)) \ge \delta\tau(\chi_{(\delta,\infty)}(T)).$$

This implies the corollary. ∎

Proposition 4.5. *Let* $(\mathcal{M}, \tau)$ *be a semi-finite von Neumann algebra and* $T \in (\mathcal{M}, \tau)$. *Then* $|||T|||_{(t)}$ *is a nonincreasing continuous function on* $[0,1]$ *and a nondecreasing continuous function on* $[1,\infty]$.

Proof. Let $0 < t_1 < t_2 \le 1$. Then

$$\begin{aligned} |||T|||_{(t_1)} - |||T|||_{(t_2)} &= \frac{1}{t_1}\int_0^{t_1} \mu_s(T)\,ds - \frac{1}{t_2}\int_0^{t_2} \mu_s(T)\,ds \\ &= \frac{\frac{1}{t_1}\int_0^{t_1} \mu_s(T)\,ds - \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \mu_s(T)\,ds}{t_2(t_2-t_1)} \le 0. \end{aligned}$$

Since $\mu_s(T)$ is right-continuous, $\|T\|_{(t)}$ is a nonincreasing continuous function on $[0,1]$. Since $\mu_s(T) \geq 0$ for $s \in [0,\infty)$, $\|T\|_{(t)}$ is a non-decreasing continuous function on $[1,\infty]$. ■

PROPOSITION 4.6. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra satisfying conditions* **A** *and* **B** *of Section* 3.4, *and let $\|\cdot\|$ be a normalized symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Then for every $T \in \mathcal{J}(\mathcal{M})$,*

$$\|T\|_{(1)} \leq \|T\|.$$

Proof. We can assume that T is a positive operator in $\mathcal{J}(\mathcal{M})$. Then there is a finite projection F in $\mathcal{M}$ such that $T = FTF \in \mathcal{M}_F$. We can assume that $\tau(F) = k$ is a positive integer. By assumption, $(\mathcal{M}_F, \tau_F)$ has the weak Dixmier property. By Lemma 3.15, either $(\mathcal{M}_F, \tau_F)$ is a diffuse von Neumann algebra, or it is $*$-isomorphic to a von Neumann subalgebra of $(M_n(\mathbb{C}), \tau_n)$ that contains all diagonal matrices. In either case, there is a projection E in $\mathcal{M}$ with $E \leq F$ such that $\tau(E) = 1$ and $\|T\|_{(1)} = \|ETE\|_{(1)}$. By Lemma 3.13 and Proposition 3.6, $\|ETE\| \leq \|T\|$. By [3, Corollary 3.36], $\|ETE\|_{(1)} \leq \|ETE\| \leq \|T\|$. ■

EXAMPLE 4.7. The Ky Fan nth norm of a compact operator T in $(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$ is

$$\|T\|_{(n)} = s_1(T) + \cdots + s_n(T),$$

and

$$\|T\|_{(\infty)} = s_1(T) + s_2(T) + \cdots.$$

COROLLARY 4.8. *Let $\|\cdot\|$ be a normalized unitarily invariant norm on $\mathcal{B}(\mathcal{H})$. Then for every $T \in \mathcal{J}(\mathcal{H})$,*

$$s_1(T) \leq \|T\| \leq s_1(T) + s_2(T) + \cdots.$$

Proof. By Proposition 4.6, $s_1(T) = \|T\|_{(1)} \leq \|T\|$. On the other hand, we may assume that T is a positive operator in $\mathcal{J}(\mathcal{H})$. Then T is unitarily equivalent to a diagonal operator $s_1(T)E_1 + \cdots + s_n(T)E_n$. Hence,

$$\|T\| = \|s_1(T)E_1 + \cdots + s_n(T)E_n\| \leq s_1(T) + \cdots + s_n(T). \ ■$$

5. Dual norms of symmetric gauge norms on $\mathcal{J}(\mathcal{M})$. Throughout this section, we assume that $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra satisfying conditions **A** and **B** of Section 3.4. Recall that $\mathcal{J}(\mathcal{M})$ is the subset of $\mathcal{M}$ consisting of operators T in $\mathcal{M}$ such that $T = ETE$ for some finite projection $E \in \mathcal{M}$. Note that for any two operators S, T in $\mathcal{J}(\mathcal{M})$, there is a finite projection F in $\mathcal{M}$ such that $S, T \in \mathcal{M}_F = F\mathcal{M}F$.

5.1. Dual norms. Let $\|\cdot\|$ be a norm on $\mathcal{J}(\mathcal{M})$. For $T \in \mathcal{J}(\mathcal{M})$, define

$$\|T\|^{\#}_{\mathcal{M},\tau} = \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}),\ \|X\| \leq 1\}.$$

When no confusion arises, we simply write $\|\cdot\|^{\#}$ or $\|\cdot\|^{\#}_{\mathcal{M}}$ instead of $\|\cdot\|^{\#}_{\mathcal{M},\tau}$.

Lemma 5.1. *$\|\cdot\|^{\#}$ is a norm on $\mathcal{J}(\mathcal{M})$.*

Proof. Note that if $T \in \mathcal{J}(\mathcal{M})$ is not 0, then $\|T\|^{\#} \geq \tau(TT^*)/\|T^*\| > 0$. It is easy to check that $\|\cdot\|^{\#}$ satisfies the other conditions for a norm. ■

Definition 5.2. $\|\cdot\|^{\#}$ is called the *dual norm* of $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$ with respect to τ.

The following lemma follows simply from the definition of dual norm.

Lemma 5.3. *Let $\|\cdot\|$ be a norm on $\mathcal{J}(\mathcal{M})$ and $\|\cdot\|^{\#}$ be the dual norm on $\mathcal{J}(\mathcal{M})$. Then for $S, T \in \mathcal{J}(\mathcal{M})$,*

$$|\tau(ST)| \leq \|S\| \cdot \|T\|^{\#}.$$

For $T \in \mathcal{M}$, define $\|T\|_1 = \tau(|T|)$. Then $\|T\|_1 = \|T\|_{(\infty)}$. The following corollary is the Hölder inequality for operators in $\mathcal{J}(\mathcal{M})$.

Corollary 5.4. *Let $\|\cdot\|$ be a gauge invariant norm on $\mathcal{J}(\mathcal{M})$ and $\|\cdot\|^{\#}$ be the dual norm. Then for $S, T \in \mathcal{J}(\mathcal{M})$,*

$$\|ST\|_1 \leq \|S\| \cdot \|T\|^{\#}.$$

Proof. Let $ST = V|ST|$ be the polar decomposition. Then $|ST| = V^*ST$. By Lemmas 5.3, 3.2, and 3.3,

$$\|ST\|_1 = \tau(|ST|) = \tau(V^*ST) \leq \|V^*S\| \cdot \|T\|^{\#} \leq \|S\| \cdot \|T\|^{\#}. \ ■$$

Let E be a (nonzero) finite projection in $\mathcal{M}$. Recall that $\mathcal{M}_E = E\mathcal{M}E$ is a finite von Neumann algebra with a faithful normal tracial state τ_E such that $\tau_E(T) = \tau(T)/\tau(E)$ for $T \in \mathcal{M}_E$. If $\|\cdot\|$ is a norm on $\mathcal{M}_E$, the dual norm of $T \in \mathcal{M}_E$ with respect to τ_E is defined by

$$\|T\|^{\#}_{\mathcal{M}_E,\tau_E} = \sup\{|\tau_E(TX)| : X \in \mathcal{M}_E,\ \|X\| \leq 1\}.$$

Lemma 5.5. *Suppose $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. Let E be a nonzero finite projection in $\mathcal{M}$ and $T \in \mathcal{M}_E$. Then*

$$\|T\|^{\#}_{\mathcal{M},\tau} = \tau(E) \cdot \|T\|^{\#}_{\mathcal{M}_E,\tau_E}.$$

Proof. Since $T = ETE$, for every $X \in \mathcal{J}(\mathcal{M})$ we have

$$\tau(TX) = \tau(ETEX) = \tau(ETEEXE) = \tau(E) \cdot \tau_E(ETEEXE).$$

If $\|X\| \leq 1$, then $\|EXE\| \leq \|X\|$ by Proposition 3.6. This implies that

$$\begin{aligned}\|T\|^{\#}_{\mathcal{M},\tau} &= \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}),\ \|X\| \leq 1\} \\ &= \sup\{|\tau(TX)| : X \in \mathcal{M}_E,\ \|X\| \leq 1\} \\ &= \tau(E) \cdot \sup\{|\tau_E(TX)| : X \in \mathcal{M}_E,\ \|X\| \leq 1\} \\ &= \tau(E) \cdot \|T\|^{\#}_{\mathcal{M}_E,\tau_E}. \ ■\end{aligned}$$

The next lemma follows from [3, Propositions 6.5 and 6.6, and Theorem 6.10].

LEMMA 5.6. *Let $\mathcal{N}$ be a finite von Neumann algebra with a faithful normal tracial state $\tau_{\mathcal{N}}$.*

(1) *If $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{N}$, then so is $\|\cdot\|^{\#}_{\mathcal{N},\tau_{\mathcal{N}}}$.*
(2) *If $\|\cdot\|$ is a symmetric gauge norm on $\mathcal{N}$, then so is $\|\cdot\|^{\#}_{\mathcal{N},\tau_{\mathcal{N}}}$. Furthermore, if $\|1\| = 1$, then $\|1\|^{\#}_{\mathcal{N},\tau_{\mathcal{N}}} = 1$.*

Combining Lemmas 5.5 and 5.6, we obtain the following proposition.

PROPOSITION 5.7. *Let $\|\cdot\|$ be a norm on $\mathcal{J}(\mathcal{M})$.*

(1) *If $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$, then so is $\|\cdot\|^{\#}$.*
(2) *If $\|\cdot\|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$, then so is $\|\cdot\|^{\#}$. Furthermore, if $\|\cdot\|$ is a normalized norm, i.e., $\|E\| = 1$ whenever $\tau(E) = 1$, then $\|\cdot\|^{\#}$ is also a normalized norm.*

LEMMA 5.8. *Let $\|\cdot\|$ be a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. If $T = a_1E_1 + \cdots + a_nE_n$ is a positive simple operator in $\mathcal{J}(\mathcal{M})$, then*

$$\|T\|^{\#} = \sup\Big\{\sum_{k=1}^{n} a_k b_k \tau(E_k) : S = b_1E_1 + \cdots + b_nE_n \geq 0,\ \|S\| \leq 1\Big\}.$$

Proof. Let $E = E_1 + \cdots + E_n$. By Lemma 5.5 and [3, Lemma 6.8],

$$\begin{aligned}\|T\|^{\#} &= \tau(E)\cdot\|T\|^{\#}_{\mathcal{M}_E,\tau_E}\\ &= \tau(E)\sup\Big\{\sum_{k=1}^{n} a_k b_k \tau_E(E_k) : S = b_1E_1 + \cdots + b_nE_n \geq 0,\ \|S\| \leq 1\Big\}\\ &= \sup\Big\{\sum_{k=1}^{n} a_k b_k \tau(E_k) : S = b_1E_1 + \cdots + b_nE_n \geq 0,\ \|S\| \leq 1\Big\}. \ \blacksquare\end{aligned}$$

5.2. Dual norms of Ky Fan norms

THEOREM 5.9. *For $T \in \mathcal{J}(\mathcal{H})$ and $k = 1, 2, \ldots, \infty$,*

$$\|T\|^{\#}_{(k)} = \max\Big\{\|T\|, \frac{1}{k}\|T\|_1\Big\},$$

where $\|T\|_{(k)} = s_1(T) + \cdots + s_k(T)$, $\|T\|_1 = \mathrm{Tr}(|T|) = s_1(T) + s_2(T) + \cdots$ and $\frac{1}{\infty} = 0$.

Proof. For $T \in \mathcal{J}(\mathcal{H})$, there is a finite rank projection E such that $T = ETE \in \mathcal{B}(\mathcal{H})_E$. Let $\mathrm{Tr}(E) = n$. Then $\mathcal{B}(\mathcal{H})_E \cong M_n(\mathbb{C})$. First assume $k < \infty$. We may assume that $n \geq k$. Then $\|T\|_{(k)} = k\|T\|_{(k/n),\tau_n}$. By

Lemma 5.5 and [3, Lemma 6.14],

$$\begin{aligned}\|T\|^{\#}_{(k)} &= \mathrm{Tr}(E)\cdot(k\|T\|^{\#}_{(k/n),M_n(\mathbb{C}),\tau_n}) = \frac{n}{k}\max\left\{\frac{k}{n}\|T\|, \|T\|_{1,\tau_n}\right\}\\ &= \max\left\{\|T\|, \frac{1}{k}\|T\|_1\right\}.\end{aligned}$$

If $k=\infty$, then $\|T\|^{\#}_{(\infty)} = \|T\|^{\#}_{(n)}$ by Lemma 5.8. Since $\frac{1}{n}\|T\|_1 \le \|T\|$, we obtain $\|T\|^{\#}_{(\infty)} = \|T\|^{\#}_{(n)} = \max\{\|T\|, \frac{1}{n}\|T\|_1\} = \|T\|$. ∎

THEOREM 5.10. *Let $\mathcal{M}$ be a type II_∞ factor and $0 \le t \le \infty$. Then for all $T \in \mathcal{J}(\mathcal{M})$,*

$$\|T\|^{\#}_{(t)} = \begin{cases} \max\{t\|T\|, \|T\|_1\} & \text{if } 0 \le t \le 1,\\ \max\{\|T\|, \frac{1}{t}\|T\|_1\} & \text{if } 1 < t \le \infty.\end{cases}$$

Proof. Let $T \in \mathcal{J}(\mathcal{M})$ and $0 < t < \infty$. There is a finite projection E in $\mathcal{M}$ such that $T = ETE$ is in $\mathcal{M}_E$. We can assume that $\tau(E) = n > t$. Let $\tau_E(ESE) = \tau(ESE)/\tau(E)$. Then $(\mathcal{M}_E, \tau_E)$ is a type II_1 factor and τ_E is the unique tracial state on $\mathcal{M}_E$. If $0 < t \le 1$, then by Lemma 4.1,

$$\begin{aligned}t\|T\|_{(t)} &= \sup\{|\tau(UTE')| : U \in \mathcal{U}(\mathcal{M}_E),\ E' \in \mathcal{P}(\mathcal{M}_E),\ \tau(E') = t\}\\ &= \tau(E)\cdot\sup\{|\tau_E(UTE')| : U \in \mathcal{U}(\mathcal{M}_E),\ E' \in \mathcal{P}(\mathcal{M}_E),\ \tau_E(E') = t/n\}\\ &= \tau(E)\frac{t}{n}\|T\|_{(t/n),\mathcal{M}_E,\tau_E} = t\|T\|_{(t/n),\mathcal{M}_E,\tau_E},\end{aligned}$$

where $\|T\|_{(t/n),\mathcal{M}_E,\tau_E}$ means the Ky Fan (t/n)th norm of $T \in \mathcal{M}_E$ with respect to the tracial state τ_E.

Hence, $\|T\|_{(t)} = \|T\|_{(t/n),\mathcal{M}_E,\tau_E}$. By Lemma 5.5 and [3, Theorem 6.17],

$$\begin{aligned}\|T\|^{\#}_{(t)} &= \tau(E)\cdot(\|T\|^{\#}_{(t/n),\mathcal{M}_E,\tau_E}) = n\max\left\{\frac{t}{n}\|T\|, \|T\|_{1,\tau_E}\right\}\\ &= \max\{t\|T\|, \|T\|_1\}.\end{aligned}$$

If $1 < t < \infty$, then $\|T\|_{(t)} = t\|T\|_{(t/n),\mathcal{M}_E,\tau_E}$. By Lemma 5.5 and [3, Theorem 6.17],

$$\begin{aligned}\|T\|^{\#}_{(t)} &= \tau(E)\cdot(t\|T\|^{\#}_{(t/n),\mathcal{M}_E,\tau_E}) = \frac{n}{t}\max\left\{\frac{t}{n}\|T\|, \|T\|_{1,\tau_E}\right\}\\ &= \max\left\{\|T\|, \frac{1}{t}\|T\|_1\right\}.\end{aligned}$$

Similar to the proof of Theorem 5.9, $\|T\|^{\#}_{(\infty)} = \|T\|$. ∎

5.3. Second dual norms

THEOREM 5.11. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra satisfying conditions* **A** *and* **B** *of Section* 3.4. *If $\|\cdot\|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$, then so is $\|\cdot\|^{\#}$, and $\|\cdot\|^{\#\#}=\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$.*

Proof. By Proposition 5.7, $\|\cdot\|^{\#}$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Furthermore, both $\|\cdot\|^{\#\#}$ and $\|\cdot\|$ are symmetric gauge norms on $\mathcal{J}(\mathcal{M})$. We need to prove that $\|T\|=\|T\|^{\#\#}$ for every positive operator $T\in\mathcal{J}(\mathcal{M})$. Let E be a finite projection in $\mathcal{M}$ such that $T\in\mathcal{M}_E$. By Lemma 5.5 and [3, Theorem **C**],

$$\begin{aligned}\|T\|^{\#\#}_{\mathcal{M},\tau}&=\sup\{|\tau(TX)|:X\in\mathcal{J}(\mathcal{M}),\ \|X\|^{\#}_{\mathcal{M},\tau}\le 1\}\\&=\sup\{\tau(E)\cdot|\tau_E(TX)|:X\in\mathcal{M}_E,\ \|X\|^{\#}_{\mathcal{M},\tau}\le 1\}\\&=\sup\{|\tau_E(T(\tau(E)X))|:X\in\mathcal{M}_E,\ \|\tau(E)X\|^{\#}_{\mathcal{M}_E,\tau_E}\le 1\}\\&=\|T\|^{\#\#}_{\mathcal{M}_E,\tau_E}=\|T\|_{\mathcal{M}_E,\tau_E}=\|T\|.\ \blacksquare\end{aligned}$$

6. Main result. Throughout this section, we assume that $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra.

LEMMA 6.1. *Let $f(x)=\sum_{k=1}^{n}a_k\chi_{[\alpha_{k-1},\alpha_k)}(x)$, where $a_1\ge\cdots\ge a_n\ge 0$ $(=a_{n+1})$ and $0=\alpha_0<\alpha_1<\cdots<\alpha_n<\infty$. For $T\in\mathcal{M}$, define*

$$\|T\|_f=\int_0^\infty f(s)\mu_s(T)\,ds.$$

Then

$$\|T\|_f=\sum_{k=1}^{n}\min\{\alpha_k,1\}(a_k-a_{k+1})\|T\|_{(\alpha_k)}.$$

Proof. Since $t\|T\|_{(t)}=\int_0^t\mu_s(T)\,ds$ for $0\le t\le 1$ and $\|T\|_{(t)}=\int_0^t\mu_s(T)\,ds$ for $1\le t<\infty$, summation by parts shows that

$$\begin{aligned}\|T\|_f&=\int_0^\infty f(s)\mu_s(T)\,ds\\&=a_1\int_0^{\alpha_1}\mu_s(T)\,ds+a_2\int_{\alpha_1}^{\alpha_2}\mu_s(T)\,ds+\cdots+a_n\int_{\alpha_{n-1}}^{\alpha_n}\mu_s(T)\,ds\\&=\sum_{k=1}^{n}\min\{\alpha_k,1\}(a_k-a_{k+1})\|T\|_{(\alpha_k)}.\ \blacksquare\end{aligned}$$

COROLLARY 6.2. *$\|\cdot\|_f$ is a symmetric gauge norm associated to $\mathcal{M}$ and therefore a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Furthermore, if $\tau(E)=1$ then $\|E\|_f=\int_0^1 f(x)\,dx$.*

LEMMA 6.3. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra and E in $\mathcal{M}$ be a (nonzero) finite projection. Suppose $\mathcal{M}_E$ is a diffuse von Neumann algebra and $T, X \in \mathcal{M}_E$ are positive operators such that $T = a_1E_1 + \cdots + a_nE_n$, $E_1 + \cdots + E_n = E$, and $\tau(E_1) = \cdots = \tau(E_n)$. Then there is a sequence of simple positive operators $X_n \in \mathcal{M}_E$ satisfying the following conditions:*

(1) $0 \le X_1 \le X_2 \le \cdots \le X$ *and hence* $0 \le \mu_s(X_1) \le \mu_s(X_2) \le \cdots \le \mu_s(X)$ *for all* $s \in [0,\infty)$;
(2) $\lim_{n\to\infty} \mu_s(X_n) = \mu_s(X)$ *for almost all* $s \in [0,\infty)$;
(3) *there exists an* $r_n \in \mathbb{N}$ *such that* $T = a_{n,1}E_{n,1} + \cdots + a_{n,r_n}E_{n,r_n}$ *and* $X_n = b_{n,1}F_{n,1} + \cdots + b_{n,r_n}F_{n,r_n}$, *where* $E_{n,1} + \cdots + E_{n,r_n} = F_{n,1} + \cdots + F_{n,r_n} = E$ *and* $\tau(E_{n,i}) = \tau(F_{n,j})$ *for* $1 \le i,j \le r_n$.

Proof. Since $\mathcal{M}_E$ is diffuse, there is a separable diffuse abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}_E$ such that $X \in \mathcal{A}$. Let θ be a $*$-isomorphism from $\mathcal{A}$ onto $L^\infty[0,1]$ such that $\tau_E = \int_0^1 dx \cdot \theta$. Let $f(x) = \theta(X)$. We can choose a sequence of simple functions f_n in $L^\infty[0,1]$ such that $0 \le f_1(x) \le f_2(x) \le \cdots \le f(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all x. Let $X_n = \theta^{-1}(f_n)$. Then $X_n \in \mathcal{M}_E$ and $0 \le X_1 \le X_2 \le \cdots \le X$. By Lemma 2.6,

$$\begin{aligned}\mu_s(X) &= \inf\{\|XF\| : F \in \mathcal{P}(\mathcal{M}),\ \tau(F^\perp) = s\}\\ &= \inf\{\|XF\| : F \in \mathcal{P}(\mathcal{M}_E),\ \tau_E(F^\perp) = s\tau(E)\} = f^*(\tau(E)s),\end{aligned}$$

where f^* is the nonincreasing rearrangement of f. Similarly, $\mu_s(X_n) = f_n^*(\tau(E)s)$, where f_n^* is the nonincreasing rearrangement of f_n. Therefore, we obtain (1) and (2). To obtain (3), we need only take $f_n(x) = \alpha_{n,1}\chi_{I_{n,1}}(x) + \cdots + \alpha_{n,r_n}\chi_{I_{n,r_n}}(x)$ such that $m(I_{n,1}) = \cdots = m(I_{n,r_n}) = \tau_E(E_1)/k_n$ for some $k_n \in \mathbb{N}$. ■

Let $\mathcal{F}$ be the set of nonincreasing, nonnegative, right continuous simple functions f on $[0,\infty)$ with compact support such that $\int_0^1 f(x)\,dx \le 1$. For every $f \in \mathcal{F}$, we have $f(x) = \sum_{k=1}^n a_k \chi_{[\alpha_{k-1},\alpha_k)}(x)$, where $a_1 \ge \cdots \ge a_n \ge 0$ $(= a_{n+1})$ and $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \infty$.

Recall that a normalized norm $|||\cdot|||$ on $\mathcal{J}(\mathcal{M})$ of a semi-finite von Neumann algebra $\mathcal{M}$ is a norm on $\mathcal{J}(\mathcal{M})$ such that $|||E||| = 1$ for some projection E with $\tau(E) = 1$. The following theorem is the main result of this paper.

THEOREM 6.4. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra satisfying conditions* **A** *and* **B** *of Section* 3.4. *If $|||\cdot|||$ is a normalized symmetric gauge norm on $\mathcal{J}(\mathcal{M})$, then there is a subset $\mathcal{F}'$ of $\mathcal{F}$ containing the characteristic function of $[0,1]$ such that for all $T \in \mathcal{J}(\mathcal{M})$,*

$$|||T||| = \sup\{|||T|||_f : f \in \mathcal{F}'\}.$$

Proof. Suppose $|||\cdot|||$ is a normalized symmetric gauge norm on $\mathcal{M}$. Let

$$\mathcal{F}' = \{\mu_s(X) : X \text{ is a simple positive operator in } \mathcal{J}(\mathcal{M}),\ |||X|||^\# \le 1\}.$$

For every positive operator $X \in \mathcal{J}(\mathcal{M})$ such that $|||X|||^{\#} \leq 1$, by Proposition 4.6,

$$\int_0^1 \mu_s(X)\, ds = |||X|||_{(1)} \leq |||X|||^{\#} \leq 1.$$

If E is a projection such that $\tau(E) = 1$, then $|||E|||^{\#} = 1$ by Proposition 5.7. Note that $\mu_s(E) = \chi_{[0,1]}(s)$. Therefore, $\mathcal{F}' \subset \mathcal{F}$ and $\chi_{[0,1]} \in \mathcal{F}'$. For T in $\mathcal{J}(\mathcal{M})$, define

$$|||T|||' = \sup\{|||T|||_f : f \in \mathcal{F}'\}.$$

By Corollary 6.2, $|||\cdot|||'$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. By Lemma 3.18, to prove that $|||\cdot|||' = |||\cdot|||$, we need to prove $|||T|||' = |||T|||$ for every positive simple operator $T \in \mathcal{J}(\mathcal{M})$ such that $T = a_1E_1 + \cdots + a_nE_n$ and $\tau(E_1) = \cdots = \tau(E_n) = c > 0$.

By Lemma 5.8 and Theorem 5.11,

$$|||T||| = \sup\Big\{c\sum_{k=1}^{n} a_kb_k : X = b_1E_1 + \cdots + b_nE_n \geq 0,\ |||X|||^{\#} \leq 1\Big\}.$$

Note that if $X = b_1E_1 + \cdots + b_nE_n$ is a simple positive operator in $\mathcal{J}(\mathcal{M})$ and $|||X|||^{\#} \leq 1$, then $\mu_s(X) \in \mathcal{F}'$ and

$$|||T|||_{\mu_s(X)} = \int_0^\infty \mu_s(X)\mu_s(T)\, ds = c\sum_{k=1}^{n} a_k^* b_k^*,$$

where $\{a_k^*\}$ and $\{b_k^*\}$ are the nondecreasing rearrangements of $\{a_k\}$ and $\{b_k\}$, respectively. By the Hardy–Littlewood–Pólya Theorem [8],

$$\sum_{k=1}^{n} a_kb_k \leq \sum_{k=1}^{n} a_k^* b_k^*.$$

Hence,

$$\begin{aligned}|||T||| &= \sup\Big\{c\sum_{k=1}^{n} a_kb_k : X = b_1E_1 + \cdots + b_nE_n \geq 0,\ |||X|||^{\#} \leq 1\Big\}\\ &\leq \sup\{|||T|||_f : f \in \mathcal{F}'\} = |||T|||'.\end{aligned}$$

Now we need to prove that $|||T|||' \leq |||T|||$. Let $X \in \mathcal{J}(\mathcal{M})$ be a positive simple operator such that $|||X|||^{\#} \leq 1$. We need to show that $|||T|||_{\mu_s(X)} \leq |||T|||$. Since $T, X \in \mathcal{J}(\mathcal{M})$, there is a finite projection $E \in \mathcal{M}$ such that $T, X \in \mathcal{M}_E$.

Since $(\mathcal{M}_E, \tau_E)$ has the weak Dixmier property, by Lemma 3.15, either $\mathcal{M}_E$ is a finite-dimensional von Neumann algebra such that $\tau(F) = \tau(F')$ for any two minimal projections F and F', or $\mathcal{M}_E$ is a diffuse von Neumann algebra. For the first case, both T and X belong to M_E. Therefore, $X =$

$a_1E_1 + \cdots + a_nE_n$ and $T = b_1F_1 + \cdots + b_nF_n$ with $\tau(E_1) = \cdots = \tau(E_n) = \tau(F_1) = \cdots = \tau(F_n) = c > 0$. Thus

$$\|T\|_{\mu_s(X)} = \int_0^\infty \mu_s(X)\mu_s(T)\,ds = c\sum_{k=1}^n a_k^* b_k^*,$$

where $\{a_k^*\}$ and $\{b_k^*\}$ are the nondecreasing rearrangements of $\{a_k\}$ and $\{b_k\}$, respectively. We may assume that $a_1 \geq \cdots \geq a_n$. Let $Y = b_1^*E_1 + \cdots + b_n^*E_n$. Then X and Y are unitarily equivalent in $\mathcal{M}_E$. So $\mu_s(X) = \mu_s(Y)$ and $\|Y\|^\# = \|X\|^\# \leq 1$. Therefore,

$$\begin{aligned}\|T\| &= \sup\Big\{c\sum_{k=1}^n a_k c_k : Z = c_1E_1 + \cdots + c_nE_n \geq 0,\ \|Z\|^\# \leq 1\Big\}\\ &\geq \|T\|_{\mu_s(Y)} = \|T\|_{\mu_s(X)}.\end{aligned}$$

If $\mathcal{M}_E$ is a diffuse von Neumann algebra, by Lemma 6.3 we can construct a sequence of simple positive operators $X_n \in \mathcal{J}(\mathcal{M})$ satisfying the following conditions:

(1) $0 \leq X_1 \leq X_2 \leq \cdots \leq X$ and hence $0 \leq \mu_s(X_1) \leq \mu_s(X_2) \leq \cdots \leq \mu_s(X)$ for all $s \in [0,\infty)$;
(2) $\lim_{n\to\infty} \mu_s(X_n) = \mu_s(X)$ for almost all $s \in [0,\infty)$;
(3) there exists an $r_n \in \mathbb{N}$ such that $T = a_{n,1}E_{n,1} + \cdots + a_{n,r_n}E_{n,r_n}$ and $X_n = b_{n,1}F_{n,1} + \cdots + b_{n,r_n}F_{n,r_n}$, where $E_{n,1} + \cdots + E_{n,r_n} = F_{n,1} + \cdots + F_{n,r_n} = E$ and $\tau(E_{n,i}) = \tau(F_{n,j})$ for $1 \leq i,j \leq r_n$.

By (1) and Corollary 3.4, $\|X_n\|^\# \leq \|X\|^\# \leq 1$ for all $n = 1,2,\ldots.$ We may assume that $a_{n,1} \geq \cdots \geq a_{n,r_n}$ and $b_{n,1} \geq \cdots \geq b_{n,r_n}$. Let $Y_n = b_{n,1}E_{n,1} + \cdots + b_{n,r_n}E_{n,r_n}$. Since $\tau(E_{n,i}) = \tau(F_{n,j})$ for $1 \leq i,j \leq r_n$ and $\mathrm{Aut}(\mathcal{M},\tau)$ acts on $\mathcal{M}$ ergodically, there is a $\theta \in \mathrm{Aut}(\mathcal{M},\tau)$ such that $\theta(E_{n,i}) = F_{n,i}$ for $1 \leq i \leq r_n$. Hence $\theta(Y_n) = X_n$. Since $\|\cdot\|^\#$ is a symmetric gauge norm, $\|Y_n\|^\# = \|X_n\|^\# \leq 1$. By Corollary 5.4,

$$\begin{aligned}\|T\| &\geq \tau(TY_n) = \tau(E_{n,1})\sum_{k=1}^{r_n} a_{n,k}b_{n,k} = \int_0^\infty \mu_s(Y_n)\mu_s(T)\,ds\\ &= \int_0^\infty \mu_s(X_n)\mu_s(T)\,ds = \|T\|_{\mu_s(X_n)}.\end{aligned}$$

By (1), (2) and the monotone convergence theorem,

$$\begin{aligned}\|T\|_{\mu_s(X)} &= \int_0^\infty \mu_s(X)\mu_s(T)\,ds = \lim_{n\to\infty}\int_0^\infty \mu_s(X_n)\mu_s(T)\,ds\\ &= \lim_{n\to\infty}\|T\|_{\mu_s(X_n)} \leq \|T\|. \ \blacksquare\end{aligned}$$

Corollary 6.5. *Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra as in Theorem* 6.4 *and let* $|||\cdot|||$ *be a normalized symmetric gauge norm on* $\mathcal{J}(\mathcal{M})$. *Then* $|||\cdot|||$ *can be extended to a normalized symmetric gauge norm* $|||\cdot|||'$ *associated to* $\mathcal{M}$.

Proof. For $T\in\mathcal{M}$, define $|||T|||'=\max\{|||T|||_f : f\in\mathcal{F}'\}$. Then $|||\cdot|||'$ is an extension of $|||\cdot|||$. ∎

Remark 6.6. In Corollary 6.5, the extension is not unique. Indeed, define $|||\cdot|||$ on $\mathcal{B}(\mathcal{H})$ by $|||T|||=\|T\|$ if T is a finite rank operator and $|||T|||=\infty$ if T is an infinite rank operator. It is easy to see that $|||\cdot|||$ defines a unitarily invariant norm associated to $\mathcal{B}(\mathcal{H})$ such that the restriction of $|||\cdot|||$ on $\mathcal{J}(\mathcal{H})$ is the operator norm.

7. Unitarily invariant norms related to semi-finite factors. As the first application of Theorem 6.4, we set up a structure theorem for unitarily invariant norms related to semi-finite factors. Recall that $\mathcal{F}$ is the set of nonincreasing, nonnegative, right continuous simple functions f on $[0,\infty)$ with compact supports such that $\int_0^1 f(x)\,dx\le 1$.

Theorem 7.1. *Let* $\mathcal{M}$ *be a semi-finite factor and let* $|||\cdot|||$ *be a unitarily invariant norm on* $\mathcal{J}(\mathcal{M})$. *Then there is a subset* $\mathcal{F}'$ *of* $\mathcal{F}$ *containing the characteristic function of* $[0,1]$ *such that for all* $T\in\mathcal{J}(\mathcal{M})$, $|||T|||=\sup\{|||T|||_f : f\in\mathcal{F}'\}$.

Proof. Combine Theorem 6.4 and Proposition 3.19. ∎

The next corollary also follows from Theorem 6.4.

Corollary 7.2. *Let* $|||\cdot|||$ *be a normalized symmetric gauge norm on* $\mathcal{J}(L^\infty[0,\infty))$. *Then there is a subset* $\mathcal{F}'$ *of* $\mathcal{F}$ *containing the characteristic function of* $[0,1]$ *such that for all* $T\in\mathcal{J}(L^\infty[0,\infty))$,

$$|||T|||=\sup\{|||T|||_f : f\in\mathcal{F}'\}.$$

Let $\mathcal{M}=\mathcal{B}(\mathcal{H})$ and $\tau=\mathrm{Tr}$. By the proof of Theorem 6.4, if $f\in\mathcal{F}'$, then $f(s)=\mu_s(X)$ for some finite rank operator $X\in\mathcal{B}(\mathcal{H})$ with $X\ge 0$ and $|||X|||^{\#}\le 1$. Write $\mu_s(X)=s_1(X)\chi_{[0,1)}(s)+s_2(X)\chi_{[1,2)}(s)+\cdots$, where $s_1(X),s_2(X),\ldots$ are the s-numbers of X. Since $\int_0^1\mu_s(X)\,ds\le 1$, $s_1(X)\le 1$. By Lemma 6.1 and simple computations, for every $T\in\mathcal{J}(\mathcal{H})$,

$$|||T|||_{\mu_s(X)}=s_1(X)s_1(T)+s_2(X)s_2(T)+\cdots,$$

where $s_1(T),s_2(T),\ldots$ are the s-numbers of T.

Let

$$\mathcal{G}=\{(a_1,a_2,\ldots):1\ge a_1\ge a_2\ge\cdots\ge 0 \text{ and } a_n=0 \text{ except for finitely many terms}\}.$$

For $(a_1, a_2, \ldots) \in \mathcal{G}$ and $T \in \mathcal{J}(\mathcal{H})$, define

$$\|T\|_{(a_1,a_2,\ldots)} = a_1 s_1(T) + a_2 s_2(T) + \cdots . \tag{7.1}$$

Then $\|T\|_{(a_1,a_2,\ldots)} = \|T\|_f$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$, where

$$f(x) = a_1\chi_{[0,1)}(x) + a_2\chi_{[1,2)}(x) + \cdots .$$

By identifying $\mu_s(X)$ with $(s_1(X), s_2(X), \ldots)$ in $\mathcal{G}$, we obtain the following corollary.

Corollary 7.3. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$. Then there is a subset $\mathcal{G}'$ of $\mathcal{G}$ with $(1, 0, \ldots) \in \mathcal{G}'$ such that for all $T \in \mathcal{J}(\mathcal{H})$,*

$$\|T\| = \sup\{a_1 s_1(T) + a_2 s_2(T) + \cdots : (a_1, a_2, \ldots) \in \mathcal{G}'\},$$

where $s_1(T), s_2(T), \ldots$ are the s-numbers of T.

Similar to the proof of Corollary 7.3, we have the following corollary.

Corollary 7.4. *Let $\|\cdot\|$ be a normalized symmetric gauge norm on $c_{00} = \mathcal{J}(l^\infty(\mathbb{N}))$. Then there is a subset $\mathcal{G}'$ of $\mathcal{G}$ with $(1, 0, \ldots) \in \mathcal{G}'$ such that for all $(x_1, x_2, \ldots) \in c_{00}$,*

$$\|(x_1, x_2, \ldots)\| = \sup\{a_1 x_1^* + a_2 x_2^* + \cdots : (a_1, a_2, \ldots) \in \mathcal{G}'\},$$

where $(x_1^*, x_2^*, \ldots)$ is the nonincreasing rearrangement of $(|x_1|, |x_2|, \ldots)$.

8. Unitarily invariant norms and symmetric gauge norms

Lemma 8.1. *Let θ_1, θ_2 be two embeddings from $(L^\infty[0,\infty), \int_0^\infty dx)$ into a type II_∞ factor $(\mathcal{M}, \tau)$. If $\|\cdot\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$, then $\|\theta_1(f)\| = \|\theta_2(f)\|$ for every positive function $f \in \mathcal{J}(L^\infty[0,\infty))$.*

Proof. For $f \in \mathcal{J}(L^\infty[0,\infty))$, let $\|f\|_1 = \|\theta_1(f)\|$ and $\|f\|_2 = \|\theta_2(f)\|$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are gauge norms on $\mathcal{J}(L^\infty[0,\infty))$. By Lemma 3.18, to prove $\|\cdot\|_1 = \|\cdot\|_2$ on $\mathcal{J}(L^\infty[0,\infty))$, we need to prove $\|f\|_1 = \|f\|_2$ for every simple function f in $\mathcal{J}(L^\infty[0,\infty))$. If f is such a function, then there is a unitary operator U in $\mathcal{M}$ such that $U\theta_1(f)U^* = \theta_2(f)$. Hence $\|f\|_1 = \|f\|_2$. ∎

The following theorem generalizes von Neumann's classical result [23] on unitarily invariant norms on $M_n(\mathbb{C})$.

Theorem 8.2. *There are one-to-one correspondences between*

1. *unitarily invariant norms on $M_n(\mathbb{C})$ and symmetric gauge norms on $\mathbb{C}^n$,*
2. *unitarily invariant norms on a type II_1 factor and symmetric gauge norms on $L^\infty[0,1]$,*
3. *unitarily invariant norms on $\mathcal{J}(\mathcal{H})$ and symmetric gauge norms on $c_{00} = \mathcal{J}(l^\infty(\mathbb{N}))$,*

(4) *unitarily invariant norms on* $\mathcal{J}(\mathcal{M})$ *of a type* II_∞ *factor* $\mathcal{M}$ *and symmetric gauge norms on* $\mathcal{J}(L^\infty[0,\infty))$.

More precisely, let $\mathcal{M}$ *be a semi-finite factor and* $\mathcal{A}$ *be the corresponding abelian von Neumann algebra as above.*

- *If* $\|\!|\cdot|\!\|$ *is a unitarily invariant norm on* $\mathcal{J}(\mathcal{M})$ *and* θ *is an embedding from* $\mathcal{A}$ *into* $\mathcal{M}$, *then the restriction of* $\|\!|\cdot|\!\|$ *to* $\mathcal{J}(\theta(\mathcal{A}))$ *defines a symmetric gauge norm on* $\mathcal{J}(\mathcal{A})$.
- *Conversely, if* $\|\!|\cdot|\!\|'$ *is a symmetric gauge norm on* $\mathcal{J}(\mathcal{A})$ *and* $T \in \mathcal{J}(\mathcal{M})$, *then* $\|\!|\mu_s(T)|\!\|'$ *defines a unitarily invariant norm on* $\mathcal{J}(\mathcal{M})$, *where* $\mu_s(T)$ *is the classical s-number of* T *if* $\mathcal{M} = M_n(\mathbb{C})$ *or* $\mathcal{M} = \mathcal{B}(\mathcal{H})$, *and* $\mu_s(T)$ *is defined as in* [3] *if* $\mathcal{M}$ *is a type* II_1 *factor.*

Proof. We refer to [3, Theorem D] for the proof of cases (1) and (2). We only handle case (4); the proof of case (3) is similar.

We may assume that both norms on $\mathcal{J}(\mathcal{M})$ and $\mathcal{J}(L^\infty[0,\infty))$ are normalized. By the definition of Ky Fan norms, Theorem 4.3 and Lemma 8.1, there is a one-to-one correspondence between Ky Fan tth norms on $\mathcal{J}(\mathcal{M})$ and Ky Fan tth norms on $\mathcal{J}(L^\infty[0,\infty))$. By Theorem 7.1 and Corollary 7.2, there is a one-to-one correspondence between normalized unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ and normalized symmetric gauge norms on $\mathcal{J}(L^\infty[0,\infty))$ as in the theorem. ∎

EXAMPLE 8.3. For $1 \le p \le \infty$, the L^p-norm on $(L^\infty[0,\infty), \int_0^\infty dx)$ defined by

$$\|f\|_p = \begin{cases} \Big(\int_0^\infty |f(x)|^p\,dx\Big)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup} |f|, & p = \infty, \end{cases}$$

is a normalized symmetric gauge norm on $(L^\infty[0,\infty), \int_0^\infty dx)$. By Theorem 8.2, the induced norm for $T \in \mathcal{J}(\mathcal{M})$ of a type II_∞ factor $\mathcal{M}$ defined by

$$\|T\|_p = \begin{cases} (\tau(|T|^p))^{1/p} = \Big(\int_0^1 |\mu_s(T)|^p\,ds\Big)^{1/p}, & 1 \le p < \infty, \\ \|T\|, & p = \infty, \end{cases}$$

is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$. The norms $\{\|\cdot\|_p : 1 \le p \le \infty\}$ are called the L^p-norms on $\mathcal{J}(\mathcal{M})$.

EXAMPLE 8.4. For $1 \le p \le \infty$, the l^p-norm defined on $\mathcal{J}(l^\infty(\mathbb{N}))$ by

$$\|(x_1, x_2, \ldots)\|_p = \begin{cases} (|x_1|^p + |x_2|^p + \cdots)^{1/p}, & 1 \le p < \infty; \\ \sup\{|x_n| : n = 1, 2, \ldots\}, & p = \infty, \end{cases}$$

is a normalized symmetric gauge norm on $\mathcal{J}(l^\infty(\mathbb{N}))$. By Theorem 8.2, the induced norm for T in $\mathcal{J}(\mathcal{H})$ defined by

$$\|T\|_p = \begin{cases} (\tau(|T|^p))^{1/p} = (s_1(T)^p + s_2(T)^p + \cdots)^{1/p}, & 1 \le p < \infty, \\ \|T\|, & p = \infty, \end{cases}$$

is a unitarily invariant norm on $\mathcal{J}(\mathcal{H})$. The norms $\{\|\cdot\|_p : 1 \le p \le \infty\}$ are called the L^p-norms on $\mathcal{J}(\mathcal{H})$.

Theorem 8.2 establishes the one-to-one correspondence between unitarily invariant norms on $\mathcal{J}(\mathcal{M})$ of an infinite semi-finite factor $\mathcal{M}$ and symmetric gauge norms on $\mathcal{J}(\mathcal{A})$ of an abelian von Neumann algebra $\mathcal{A}$. The following theorem further establishes the one-to-one correspondence between the dual norms on $\mathcal{J}(\mathcal{M})$ and the dual norms on $\mathcal{J}(\mathcal{A})$, which plays a key role in the study of duality and reflexivity of the completion of $\mathcal{J}(\mathcal{M})$ with respect to unitarily invariant norms.

THEOREM 8.5. *Let $\mathcal{M}$ be a type II_∞ factor (or $\mathcal{B}(\mathcal{H})$). If $\|\!|\cdot|\!\|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ corresponding to the symmetric gauge norm $\|\!|\cdot|\!\|_1$ on $\mathcal{J}(L^\infty[0,\infty))$ (or $\mathcal{J}(l^\infty(\mathbb{N}))$ respectively) as in Theorem 8.2, then $\|\!|\cdot|\!\|^\#$ is the unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ corresponding to the symmetric gauge norm $\|\!|\cdot|\!\|_1^\#$ on $\mathcal{J}(L^\infty[0,\infty))$ (or $\mathcal{J}(l^\infty(\mathbb{N}))$ respectively) as in Theorem 8.2.*

Proof. We only prove the theorem for type II_∞ factors; the case of type I_∞ factors is similar. Let $\|\!|\cdot|\!\|_2$ be the unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ corresponding to the symmetric gauge norm $\|\!|\cdot|\!\|_1^\#$ on $\mathcal{J}(L^\infty[0,\infty))$ as in Theorem 8.2. By Lemma 3.18, to prove $\|\!|\cdot|\!\|_2 = \|\!|\cdot|\!\|^\#$ on $\mathcal{J}(\mathcal{M})$, we need to prove $\|\!|T|\!\|_2 = \|\!|T|\!\|^\#$ for every simple positive operator $T = a_1E_1 + \cdots + a_nE_n$ in $\mathcal{J}(\mathcal{M})$ such that $\tau(E_1) = \cdots = \tau(E_n) = c$. We may assume that $a_1 \ge \cdots \ge a_n \ge 0$. Then $\mu_s(T) = a_1\chi_{[0,c)}(s) + \cdots + a_n\chi_{[(n-1)c,nc)}(s)$. By Lemma 5.8,

$$\|\!|T|\!\|^\# = \sup\Big\{c\sum_{k=1}^n a_kb_k : X = b_1E_1 + \cdots + b_nE_n \ge 0,\ \|\!|X|\!\| \le 1\Big\}.$$

Since $\sum_{k=1}^n a_kb_k \le \sum_{k=1}^n a_kb_k^*$, where $b_1^*, \ldots, b_n^*$ is the nondecreasing rearrangement of $b_1, \ldots, b_n$, we have

$$\|\!|T|\!\|^\# = \sup\Big\{c\sum_{k=1}^n a_kb_k : X = b_1E_1 + \cdots + b_nE_n \ge 0,\\ b_1 \ge \cdots \ge b_n \ge 0,\ \|\!|X|\!\| \le 1\Big\}.$$

By Theorem 8.2 and Lemma 5.8,

$$\|T\|_2 = \|\mu_s(T)\|_1^{\#}$$
$$= \sup\Big\{c\sum_{k=1}^{n} a_k b_k : g(s) = b_1\chi_{[0,c)}(s) + \cdots + b_n\chi_{[(n-1)c,nc)}(s) \geq 0,\ \|g(s)\|_1 \leq 1\Big\}.$$

Again since $\sum_{k=1}^n a_k b_k \leq \sum_{k=1}^n a_k b_k^*$, we have

$$\|T\|_2 = \|\mu_s(T)\|_1^{\#}$$
$$= \sup\Big\{c\sum_{k=1}^{n} a_k b_k : g(s) = b_1\chi_{[0,c)}(s) + \cdots + b_n\chi_{[(n-1)c,nc)}(s) \geq 0,$$
$$b_1 \geq \cdots \geq b_n \geq 0,\ \|g(s)\|_1 \leq 1\Big\}.$$

Note that if $b_1 \geq \cdots \geq b_n \geq 0$, then $\mu_s(b_1E_1 + \cdots + b_nE_n) = b_1\chi_{[0,c)}(s) + \cdots + b_n\chi_{[(n-1)c,nc)}(s)$. Since $\|\cdot\|$ is the unitarily invariant norm on $\mathcal{J}(\mathcal{M})$ corresponding to the symmetric gauge norm $\|\cdot\|_1$ on $\mathcal{J}(L^\infty[0,\infty))$ as in Theorem 8.2, $\|b_1E_1 + \cdots + b_nE_n\| \leq 1$ if and only if $\|b_1\chi_{[0,c)}(s) + \cdots + b_n\chi_{[(n-1)c,nc)}(s)\|_1 \leq 1$. Therefore, $\|T\|_2 = \|T\|^{\#}$. ∎

Example 8.6. If $p = 1$, let $q = \infty$. If $1 < p < \infty$, let $q = p/(p-1)$. Then the L^q norm on $\mathcal{J}(L^\infty[0,\infty))$ defined in Example 8.3 is the dual norm of the L^p-norm on $\mathcal{J}(L^\infty[0,\infty))$. By Theorem 8.5, the L^q-norm on $\mathcal{J}(\mathcal{M})$ of a type II_∞ factor $\mathcal{M}$ is the dual norm of the L^p-norm on $\mathcal{J}(\mathcal{M})$.

Example 8.7. If $p = 1$, let $q = \infty$. If $1 < p < \infty$, let $q = p/(p-1)$. Then the l^q-norm on $\mathcal{J}(l^\infty(\mathbb{N}))$ defined in Example 8.4 is the dual norm of the l^p-norm on $\mathcal{J}(l^\infty(\mathbb{N}))$. By Theorem 8.5, the L^q-norm on $\mathcal{J}(\mathcal{H})$ is the dual norm of the L^p-norm on $\mathcal{J}(\mathcal{H})$.

9. Ky Fan's dominance theorem. The following theorem generalizes Ky Fan's dominance theorem.

Theorem 9.1. *Let $\mathcal{M}$ be a semi-finite factor and $S, T \in \mathcal{J}(\mathcal{M})$. If $\|S\|_{(t)} \leq \|T\|_{(t)}$ for all Ky Fan t-th norms, $0 \leq t \leq \infty$, then $\|S\| \leq \|T\|$ for all unitarily invariant norms $\|\cdot\|$ on $\mathcal{J}(\mathcal{M})$.*

Proof. Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}$. By Lemma 6.1, $\|S\|_f \leq \|T\|_f$ for every $f \in \mathcal{F}$. By Theorem 7.1, $\|S\| \leq \|T\|$. ∎

Corollary 9.2. *Let $S, T \in \mathcal{J}(\mathcal{H})$. If $\|S\|_{(n)} \leq \|T\|_{(n)}$ for all Ky Fan nth norms, $n = 1, 2, \ldots$, then $\|S\| \leq \|T\|$ for all unitarily invariant norms $\|\cdot\|$ on $\mathcal{J}(\mathcal{H})$.*

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