CONSEQENCES OF THE TORSION NONCOMMUTATIVE BORSUK-ULAM THEOREM

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Borsuk-Ulam theorem (1933):

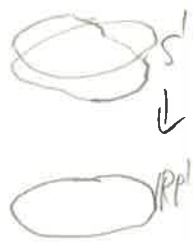
$$\boxed{\forall n \in \mathbb{N} \ \nexists \text{ cont. equiv. } (\mathbb{Z}/2\mathbb{Z})^{*n+1} \to (\mathbb{Z}/2\mathbb{Z})^{*n}}$$

Progress:

① Generalized from $\mathbb{Z}/2\mathbb{Z}$ to any (non-trivial) finite group F (see Matoušek).

Locally trivial setting, finite trivializing (Schwartz?) index.

Remark A: $\prod_{\mathbb{N}} S^1 \longrightarrow \prod_{\mathbb{N}} \mathbb{RP}^1$



is a $\prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ (Cantor set) compact principal bundle which is not locally trivial.

$\prod$

<u>Remark B</u>: <u>Piecewise triviality</u> (finite closed trivializing covering) is more general than local triviality (open trivializing covering). Let C be the Cantor set with the group structure $\prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. Glue $[0,1]\times C$ with $[0,1]\times C$ over $C\times C$ via

$$C\times C \ni (g,h) \longmapsto (g,gh) \in C\times C.$$

Then

$$\begin{array}{c} [0,1]\times C \coprod\limits_{C\times C} [0,1]\times C \\ \Big\downarrow \\ [0,1] \coprod\limits_{C} [0,1] \end{array}$$

(bubble space)

is a piecewise trivial principal C-bundle which is not locally trivial. (Baum, H., Matthes, Szymański)

<u>Remark C</u>: Replacing $[0,1]$ by D and C by $U(1)$ we obtain the <u>Hopf fibration</u>

$$S^3 = D\times U(1) \mathop{\sqcup}_{U(1)\times U(1)} D\times U(1)$$

$$U(1)=S^1$$

$$\Big\downarrow$$

$$S^2 = D \mathop{\sqcup}_{U(1)} D$$

<u>Remark D</u>: The principal bundle from Remark A is <u>not</u> even piecewise trivial. However, every locally trivial compact principal bundle is piecewise trivial by the partition of unity argument (existence of a closed refinement).

(2) Generalized from $F^{*n} * F \to F^{*n}_{fG}$ an arbitrary compact Hausdorff topological space X with a cont. free action of ~~a finite group~~ ~~to~~ any compact Hausdorff topological group G with torsion $(\exists g \neq e : g^n = e)$: $X * G \to X$

(Volovikov and, later on, Passer).

(3) Generalized by Passer from X to any unital C^*-algebra A.

(4) The Torsion Noncommutative Borsuk–Ulam Thm (Dąbrowski, H., Neshveyev).

Let A be a unital C^*-algebra with a free action $\delta : A \to A \underset{\min}{\otimes} H$ of a compact quantum group (H, Δ). Then, if H admits a character different from the counit whose finite convolution power is the counit, $\nexists$ an H-equiv. $*$-hom $A \to A \circledast^{\delta} H$.

Classical Borsuk-Ulam-type conjecture

(Baum, Dąbrowski, H.): Let X be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact group G. Then

$\nexists$ G-equiv. cont. $X * G \longrightarrow X$.

Corollary (weak Hilbert-Smith conjecture): If G is a locally compact group acting continuously, freely and properly (want effectively) on a topological manifold M, so that the orbit space M/G is f. dim. (want no assumption), then G is a Lie group. (Dąbrowski, Chirvasitu, Tobolski) Here it is key to prove BU for $G = \mathbb{Z}_p$.

The Brouwer fixed-point theorem for balls:

Every continuous map $B^n \to B^n$ has a fixed point.

This is <u>equivalent</u> to the non-contractibility of spheres:

$\nexists$ cont. $B^{n+1} \xrightarrow{A} S^n$: $S^n \hookrightarrow B^{n+1} \xrightarrow{f} S^n$

$= S^n \xrightarrow{id} S^n$.

The classical version Borsuk-Ulam theorem implies:

<u>Corollary</u>: If X is a compact Hausdorff space with a cont. free action of a non-trivial finite group, then X is <u>not</u> contractible.

<u>Corollary</u>: Contractible compact Hausdorff spaces do <u>not</u> admit a cont. free action of any non-trivial finite group.

Theorem (Dąbrowski, H., Meshveyer):

Let A be a unital C^*-algebra equipped with a free action $\delta: A \to A \underset{min}{\otimes} H$ of a compact quantum group (H, Δ). Then it follows from

$$\boxed{\nexists \text{ an } H\text{-equiv. } *\text{-homo } A \to A \overset{\delta}{\circledast} H}$$

that

$$\boxed{\nexists\ *\text{-homo } A \xrightarrow{\gamma} \mathcal{C}A \text{ s.t. } ev_1 \circ \gamma \text{ is } H\text{-equiv.}}$$

Furthermore, the opposite implication is true if (H, Δ) admits a counit.

Corollary: If A is a unital C^*-algebra with a free action of a non-trivial finite group, then A is not contractible:

$\nexists\ *$-homo $A \xrightarrow{\gamma} \mathcal{C}A$ s.t. $ev_1 \circ \gamma = id$.

Theorem (Dąbrowski, H., Neshveyev):

Let (H,Δ) be any compact quantum group. Then H is not contractible $\Longleftrightarrow$ $\nexists$ H-equiv. $*$-homo $H \to H \circledast^{\Delta} H$.

Proof: It suffices to show that, if $f: H \to H$ is an H-equiv $*$-homo, then H is an isomorphism. For any topological group G, if $F: G \to G$ is a continuous map s.t. $F(gh) = F(g)h$, then F is a homeomorphism. Indeed,

$$F^{-1}(g) := F(e)^{-1}g, \quad F(F^{-1}(g)) = F(eF^{-1}(g)) = F(e)F^{-1}(g) = F(e)F(e)^{-1}g = g, \quad F^{-1}(F(g)) = F(e)^{-1}F(g) = F(e)^{-1}F(eg) = F(e)^{-1}F(e)g = g.$$

For $\dim H < \infty$, we define $f^{-1}(h) := (\varepsilon \circ f \circ S)(h_{(1)})h_{(2)}$.

Then

$$f(f^{-1}(h)) = (\varepsilon \circ f \circ S)(h_{(1)}) f(h_{(2)}) = (\varepsilon \circ f \circ S)(h_{(1)})\, \varepsilon(f(h_{(2)})_{(1)})\, f(h_{(2)})_{(2)} = (\varepsilon \circ f \circ S)(h_{(1)})(\varepsilon \circ f)(h_{(2)})h_{(3)} = (\varepsilon \circ f)(S(h_{(1)})h_{(2)})h_{(3)} = h$$

and $f^{-1}(f(h)) = (\varepsilon \circ f \circ S)(f(h)_{(1)}) f(h)_{(2)}$

$= (\varepsilon \circ f \circ S)(\varepsilon(f(h)_{(1)}) \cdot f(h)_{(2)}) f(h)_{(3)} =$

$= (\varepsilon \circ f \circ S)(\varepsilon(f(h_{(1)})) h_{(2)}) h_{(3)} =$

$= (\varepsilon \circ f)(h_{(1)}) (\varepsilon \circ f) \cdot (S(h_{(2)})) h_{(3)} =$

$= (\varepsilon \circ f)(h_{(1)} S(h_{(2)})) h_{(3)} = h$.

For $\dim H = \infty$, we use Woronowicz's Peter-Weyl theory. $\square$

<u>Conjecture</u>: The above calculation can be generalized to locally compact quantum groups.

$$(\varepsilon \circ f \circ S)(f(h)_{(1)})\,\varepsilon(f(h_{(1)})_{(2)})\,h_{(2)}$$

$$(\varepsilon \circ f \circ S)(f(h_{(1)}))\,\varepsilon(h_{(2)})\,h_{(3)}$$

$$(\varepsilon \circ f \circ S)(f(h_{(1)}))\,h_{(2)}$$

$$(\varepsilon \circ f \circ S)\Big(\varepsilon\big(f(h_{(1)})_{(1)}\big)\big(f(h_{(1)})_{(2)}\big)\Big)\,h_{(2)}$$

$$(\varepsilon \circ f \circ S)\big(\varepsilon(f(h_{(1)}))\,h_{(2)}\big)\,h_{(3)}$$