CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS
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INSTITUTO DE CIENCIAS MATEMÁTICAS

Assessment of the Doctoral Thesis "The Quantitative Fatou Property, ε-Approximability and Carleson Measures" by Marcin Gryszówka

The dissertation entitled *"The Quantitative Fatou Property, ε-Approximability and Carleson Measures"*, authored by Marcin Gryszówka, explores the boundary behavior of generalized harmonic functions across several analytic and geometric frameworks. The research focuses on understanding the interplay between the Quantitative Fatou Property (QFP), ε-approximability, and Carleson measure estimates, and on how these concepts capture the interplay between analytic phenomena and geometric features in Euclidean domains, Riemannian manifolds, and Heisenberg groups. The topic lies at the crossroads of partial differential equations, harmonic analysis, and geometric measure theory. The novelty of the thesis lies in extending known results for harmonic functions both to broader classes of functions and to settings beyond the Euclidean framework, including Riemannian manifolds and Heisenberg groups.

The thesis is organized into five chapters. The opening two chapters provide an overview of the motivation and aims of the study, introduce the notation and key definitions, and outline the relevant background from analysis and geometry. The remaining three chapters constitute the original research contributions, each corresponding to one of the candidate's publications.

Chapter 3 investigates ε-approximability and the Quantitative Fatou Property for a broad class of functions that are not necessarily harmonic. The results extend the known theory —originally established for harmonic functions or for solutions to elliptic equations in divergence form— to functions satisfying a quantitative oscillation-type condition. The main result (Theorem 1.3.1) proves ε-approximability for bounded functions fulfilling this condition in the setting of Lipschitz domains, which, as a corollary, yields a quantitative version of the Fatou theorem. This work, coauthored with T. Adamowicz and M. J. González, represents a substantial generalization of earlier results and has already been posted on arXiv (arXiv:2412.13072).

Chapter 4 extends the analysis to Lipschitz domains in complete Riemannian manifolds. The author successfully adapts techniques from the Euclidean setting to the manifold context, addressing the challenges posed by curvature and the absence of global coordinates. The main results (Theorems 1.4.1 and 1.4.2) establish ε-approximability for bounded harmonic and, more generally, A-harmonic functions (more precisely, solutions to divergence-form elliptic operators whose coefficients satisfy a natural L^1 Kenig-Pipher condition). This, in turn, is shown to lead to the Quantitative Fatou Theorem in this non-Euclidean framework. This chapter is based on a paper authored by Gryszówka and published in the *Journal of Geometric Analysis.*

Chapter 5 focuses on Carleson measures and related boundary phenomena in Heisenberg groups, particularly in $\mathbb{H}^1$. The chapter presents new characterizations of Carleson measures for non-tangentially accessible (NTA) and admissible Dirichlet problem (ADP) domains (Theorems 1.5.1), together with L^2-boundedness estimates for the square function of subelliptic harmonic functions (Theorems 1.5.3) and Carleson-type estimates involving the Green function (Theorems 1.5.4). It concludes with a Fatou-type theorem on (ε, δ)-domains in $\mathbb{H}^n$, showing that non-tangential limits exist outside sets of vanishing p-Sobolev capacity (Theorems 1.5.5). This chapter corresponds to a paper coauthored with T. Adamowicz, accepted for publication in *Mathematische Nachrichten*.

The dissertation is written in a clear and readable style. The candidate handles mathematical language with accuracy, using notation consistently and explaining new concepts when needed. Although the material is highly technical, the presentation remains approachable. It is evident that the author made a real effort to organize the exposition carefully and to highlight the ideas behind the technical arguments.

The structure of the thesis makes it easy to read. Each section begins with a short explanation of its purpose, and the results appear in a natural order. Proofs are detailed enough to be self-contained but are not overloaded with routine computations. This gives the impression of someone who understands the subject deeply and knows how to communicate it effectively. The notation is coherent across chapters, which is not trivial in a work that combines Euclidean, Riemannian, and sub-Riemannian settings. The bibliography is complete and relevant, reflecting both a solid knowledge of the classical results and familiarity with the most recent literature in harmonic analysis, PDEs, and geometric measure theory.

Overall, the dissertation reads well and conveys the impression of a careful, mature, and thoughtful piece of research. It is technically demanding but presented in a way that makes it accessible to specialists from neighboring areas. The clarity of exposition and attention to detail contribute greatly to the overall quality of the work.

In **Chapter 3**, one of the most remarkable aspects is the fact that the author proves ε-approximability —and consequently the Quantitative Fatou Property— for a class of functions that are not necessarily solutions to any elliptic equation. This is a significant achievement, as the existing theory had been developed for harmonic or solutions to divergence-form elliptic operators. The proof relies on tools that are natural in this setting, such as dyadic decompositions and Whitney coverings, which are employed with considerable skill and elegance. The geometric constructions are illustrated with helpful diagrams that make the intricate structure of the argument easier to follow. This part of the thesis deals with topics that are technically very demanding, and the author demonstrates a clear mastery of the previous literature and techniques developed in this area. Extending the ε-approximability argument to functions that are not necessarily solutions is far from straightforward and requires a deep understanding of the delicate balance between geometry and analysis in Lipschitz domains.

Natural follow-up questions arise. First, it is natural to investigate to what extent these ideas can be pushed beyond Lipschitz domains (for instance, to NTA domains with Ahlfors-David regular boundaries or to domains with uniformly rectifiable boundaries) by leveraging the Hofmann-Martell-Mayboroda program, where ε-approximability is tightly connected to uniform rectifiability for suitable divergence-form operators. Second, one may ask whether the present non-solution viewpoint relates to other generalizations in the Hofmann-Martell-Mayboroda approach that treats broader classes of functions, and whether there exists a unifying set of hypotheses under which the Quantitative Fatou Property continues to hold.

In **Chapter 4**, one of the most notable achievements is the extension of ε-approximability and the Quantitative Fatou Property from the Euclidean setting to Lipschitz domains on complete Riemannian manifolds. The author successfully translates ideas from the Euclidean framework into the manifold setting, overcoming geometric difficulties such as curvature and the absence of global coordinates. The results establish ε-approximability for bounded harmonic and A-harmonic functions and, as a consequence, a Quantitative Fatou Theorem in this non-Euclidean context. This chapter, based on a paper published in the *Journal of Geometric Analysis*, shows the author's ability to merge analytical and geometric techniques with precision and clarity. Dealing with ε-approximability on manifolds requires substantial technical care, demonstrating a deep understanding of both PDE methods and geometric intuition.

This part of the work also opens several natural directions for further investigation. It points toward a deeper analysis of how the quantitative estimates depend on the underlying geometry of the manifold, and toward possible links between ε-approximability on manifolds and geometric notions that play a role similar to uniform rectifiability in the Euclidean setting —potentially building a bridge between harmonic analysis and geometric measure theory in non-Euclidean frameworks. Furthermore, as already noted in the context of **Chapter 3**, this chapter naturally leads to the consideration of whether these ideas could be extended to more irregular domains, for instance, NTA domains or complements of uniformly rectifiable sets, using ideas from Hofmann-Martell-Mayboroda.

In **Chapter 5**, the author moves beyond the Euclidean and Riemannian settings to the framework of Heisenberg groups, focusing primarily on $\mathbb{H}^1$. This chapter develops a characterization of Carleson measures for non-tangentially accessible (NTA) and admissible Dirichlet problem (ADP) domains and provides L^2 estimates for square functions of subelliptic harmonic functions. It also establishes Carleson-type estimates involving the Green function and concludes with a Fatou-type theorem for (ε, δ)-domains in $\mathbb{H}^n$, showing that nontangential limits exist outside sets of vanishing p-Sobolev capacity. This work, co-authored with T. Adamowicz and accepted for publication in *Mathematische Nachrichten*, highlights the author's ability to extend boundary-value techniques to non-Euclidean and sub-Riemannian contexts. The analytical and geometric challenges of the Heisenberg setting are substantial, since one must operate within a subelliptic framework where standard Euclidean tools are no longer directly applicable. The author manages these difficulties effectively, employing techniques from geometric measure theory, potential theory, and sub-Riemannian geometry in a coherent way.

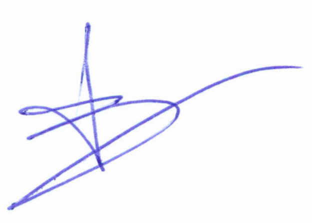
This chapter also opens several avenues for future exploration. It naturally invites a closer examination of the role of the ADP condition and whether this condition could be weakened using more geometric or analytic hypotheses without losing the main results. In the Euclidean context, it is known that in 1-sided chord-arc domains the solvability of the Dirichlet problem is equivalent to Carleson measure estimates for harmonic functions. Establishing whether similar equivalences hold in the Heisenberg framework would provide valuable insight into how Carleson-type conditions encode geometric information about the boundary and could serve as a bridge between geometric measure theory and subelliptic potential theory.

In summary, this dissertation constitutes a substantial and original contribution to the field of harmonic analysis and partial differential equations. It combines deep analytic techniques with geometric insight, extending fundamental results beyond the Euclidean setting into Riemannian and sub-Riemannian frameworks. The work is coherent and clearly presented, reflecting a high level of scientific maturity.

Although the author does not explicitly address the notion of uniform rectifiability in this thesis—given that it is not yet fully developed in non-Euclidean contexts—the results naturally point toward the possibility of defining and exploring analogous concepts in these settings. In the Euclidean framework, uniform rectifiability plays a central role in linking geometric and analytic properties of domains, being closely related to Carleson measure estimates and the solvability of boundary value problems. Extending this perspective to the manifold or Heisenberg contexts would be a highly promising direction for future research, potentially opening new connections between harmonic analysis, geometric measure theory, and sub-Riemannian geometry.

I would also like to acknowledge the valuable guidance provided by the dissertation advisor, as the coherence, depth, and focus of the work clearly reflect strong mentorship. It is worth noting that two of the main chapters have already been accepted for publication in leading international journals, and the third is available on arXiv and likely to appear soon. This level of dissemination attests to the scientific quality and relevance of the research.

Overall, this dissertation stands out for its originality, rigor, and scope. It demonstrates the candidate's ability to work independently at a high level and to contribute meaningfully to an active area of mathematics. I would characterize this work as outstanding and place it comfortably among the top 20% of dissertations in its area. I am therefore pleased to recommend that the candidate be awarded the doctoral degree.



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