

**Report on** *The quantitative Fatou property,  $\varepsilon$ -approximability and Carleson measures*  
**by Marcin Gryszówka**

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This is a report on a PhD thesis written by mr Marcin Gryszówka under the supervision of professor Tomasz Adamowicz. I received the required documents on June 24th 2025. According to them all formal requirements are met and thus I shall focus below solely on the content.

## **Content**

The thesis is based on three manuscripts, dubbed [Gr], [AdGr] and [AGG]. The first one is a published paper in *J. Geom. Anal.*, the second is co-authored with the advisor and is just published in *Math. Nachr.*, while the third is a joint work with the advisor and prof. González and is still in a form of a preprint.

The thesis consists of 5 parts - an Introduction containing an outline and historical background, Preliminaries with detailed definitions of the pertinent objects and three main chapters. Roughly speaking, Chapter 3 deals with the developed theory in the setting of Euclidean domains, Chapter 4 is devoted to analyzing the situation on Riemannian manifolds whereas the final chapter deals with analysis in the sub-Riemannian setting focusing on Heisenberg groups.

Before getting any further let me emphasize that the manuscript under review contains a massive amount of highly nontrivial mathematics coupled with a numerous explanations and motivating examples. I really appreciate the amount of work, both of the PhD Candidate and its advisor, put into the note.

The main theme considered in the work is that of *boundary values* of harmonic and close-to-harmonic functions. This is an extremely important though largely neglected part of modern analysis. As a matter of fact vast majority of students are completely clueless what could happen with a *nice* function when one approaches the boundary of its domain of definition! As the thesis itself recalls there is a very intricate interplay between the boundary geometry, function type and the (non)existence of *boundary values* in the sense of (generalized) radial limits. Starting from Fatou's discovery this amazing field is still a subject of intensive studies leading to deep and important results.

**Chapter 3** The first part of the thesis is devoted to studying boundary values of harmonic-like functions in a special subclass of domains in Euclidean space. The Lipschitz domains are defined as epigraphs of entire Lipschitz functions. For such domains and functions  $u$  that are harmonic (or close-to-harmonic) the *quantitative Fatou property* is considered: a counting function  $N$  (depending on several geometric parameters) is constructed that essentially controls the *oscillation* of  $u$  as a boundary point is approached in a non-tangential way. The quantitative Fatou property boils down to establishing integrability and Carleson-type estimates for the counting function at the boundary. This, in turn, is based on the  $\varepsilon$ -approximability: the possibility to globally approximate  $u$  by a  $BV_{loc}$  function  $\varphi$  (with any precision) such that the gradient of  $\varphi$  defines a Carleson measure.

The main result of the chapter is that for close-to-harmonic functions the aforementioned  $\varepsilon$ -approximability is always possible on Lipschitz-type domains and hence the quantitative Fatou property always holds.

This is by far the most technical part of the thesis and it utilizes a great amount of non-trivial techniques ranging from geometric properties of Lipschitz boundaries to Calderon-Zygmund type decompositions and other methods from harmonic analysis. A very interesting addendum to this chapter is a discussion which classes of close-to-harmonic functions can be treated with such methods.

**Chapter 4** Chapter 4 contains a study of boundary values of harmonic functions in a Lipschitz domains within Riemannian manifolds.

My general impression is that numerous authors neglect the subtleties of having a *manifold* as an ambient space and use *hand-waving* arguments for justification that facts from Euclidean settings can be generalized. Contrary to this trend the Author carefully names all the new phenomena and difficulties arising in the manifold setting. As it turns out some of the needed notions do not seem to have been properly defined beforehand and reference [Gr] (on which the chapter is based) carefully fills-in these holes.

With all the technical work done the main result is that  $\varepsilon$ -approximability and then quantitative Fatou property hold for harmonic functions living on a compact Lipschitz subdomains of Riemannian manifolds. This is an analogue of the findings in Chapter 3 but, as is well explained in the thesis, the argument has to be adjusted in several places.

**Chapter 5** The last part still considers boundary values of harmonic functions but now the attention switches to the sub-Riemannian setting focusing on Heisenberg groups. In this part various characterizations of Carleson measures are provided. While most of the result generalize more classical ones from the Euclidean setting, the note focuses on the technical differences and subtleties that arise. A very interesting result, for me, is Theorem 1.5.5. stating that for sufficiently regular domain in  $\mathbb{H}^n$  and a harmonic function  $u$  with  $L^p$  integrable horizontal gradient ( $1 < p \leq 2n + 2$ ) the nontangential limits exist not only off a zero measure set but actually off a set of  $p$ -Sobolev capacity zero.

Quite contrary to the previous chapters, which are essentially self-contained, Chapter 5 borrows generously from existent literature and only references are provided for some of the most important results. I guess this is due to size/time limitations. Due to my limited knowledge of the sub-Riemannian realm I had to take all these results for granted and my understanding of what is going on in this chapter is still far from complete.

## Comments

- (1) **Novelty/Originality:** Without any doubt the thesis contains novel and original results which I consider valuable additions to the existent literature on boundary behavior of harmonic functions.
- (2) **Independence:** The thesis is based on three manuscripts. Two of them are co-authored with the thesis advisor. While this always raises questions, the third paper is a sole work of mr Gryszówka and convinces me that the Candidate is mature to conduct independent research. Additionally, the Candidate has started collaborations with other researchers which convinces me that mr Gryszówka has been guided properly through his PhD studies and is on the right track in his research career.
- (3) **Layout and style:** The thesis is written in a logical way with a lot of explanations, examples and illuminating pictures. This is in stark contrast to modern trends of simply merging published work together. As a consequence the size of the thesis is rather large. As already mentioned some exceptions to the general layout can be found in Chapter 5, where various results are merely cited, but I fully understand the purpose of such a decision.

- (4) **English:** As a non-specialist it is hard for me to judge, but my feeling is that the thesis is written using a good level English. Typical polish errors, such as the usage of *a/an* and *the* articles do appear here and there but my overall impression is quite positive. The amount of typos is also insignificant given the size of the manuscript.

## Criticisms

My overall impression is that the author has had insufficient amount of time for carefully polishing all details of such a huge work. Therefore below I list some remarks. They range from suggestions to undefined notation to computational errors. In my opinion, **none** of these does a serious damage, yet the number of issues is larger than expected.

- (1) on page 4 formula (1.4) the definition of Lipschitz-graph domains requires such a domain  $\Omega$  to be contained in  $\mathbb{R}_+^{n+1}$  - see also formula (2.1) on page 10. The role of the  $+$  is never explained and the additional information that  $\phi$  is  $\mathbb{R}$  rather than  $\mathbb{R}_+$  valued adds to my confusion.
- (2) Definition 2.5.1 and the immediate Definition 2.5.2 both contain a function  $f : \Omega \mapsto [0, \infty]$  but for the first definition the notation  $f(z, y)$  is used to separate the variables whereas  $f(y)$  is used immediately afterwards. While not formally a mistake this is somewhat confusing.
- (3) In definition 2.6.1  $\alpha$ -Carleson measures are introduced but no  $\alpha$  ever appears!
- (4) In definition 2.8.1, as written, it seems that the whole space  $X$  rather than the subset  $E$  is Ahlfors-David regular despite the fact that  $E$  appears in the formula below.
- (5) In Definition 2.9.1 it seems to me that the for the exterior corkscrew condition one should use  $X \setminus \overline{\Omega}$  as  $X \setminus \Omega$  is not open.
- (6) In Definition 2.11.2 the function  $\phi$  is taken from  $BV(\Omega)$  rather than from  $BV_{loc}(\Omega)$ . I wonder what is the reason for this restriction.
- (7) On page 21 line -1 the number of the figure is not correct - it should be 3.1. The same problem occurs afterwards - see page 23 line 3.
- (8) On page 27 line 2 It seems to me that  $x^{\hat{Q}_j}$  should be  $x_{\hat{Q}_j}$ .
- (9) On page 32 line 7 a left  $|$  is missing.
- (10) On page 42 line -3 It is unclear what is the lower bound for  $\alpha$ .
- (11) In definition 4.1.5 I would suggest to write  $\nabla f$  instead of  $\nabla$  as  $\nabla$  is formally only a symbol and not a vector field.
- (12) Definition 4.1.9 I have two issues here: first of all it is unclear to me what  $\|A\|_\infty \leq C$  means if bounded coefficients have to be imposed afterwards. Secondly, it is unclear what  $\nabla A(X)$  really means here. If  $X$ , following the domain of the integral, is a point on the manifold then  $A(X)$  is a linear map from  $T_X M$  to  $T_X M$  and as such  $\nabla$  should be rather interpreted as an (undefined) covariant derivative. If, in turn,  $\nabla A$  is to be understood as the gradient of the coefficient matrix then there are issues with the dependence on coordinates.
- (13) Proposition 4.2.1 As written, there is no restriction to take  $e_3$  inside  $\Omega$  and then the dependence is unclear. I am guessing that  $\Omega$  should be a subset of  $\mathbb{S}^2 \setminus \{e_3\}$  or  $e_3$  is taken after a possible rotation. This is what the proof of Lemma 4.2.2 starts with.

- (14) The computation of the Jacobian matrix on page 51 is, unfortunately, incorrect. Wolfram alpha calculates the Jacobian matrix to be

$$a(x, y) \begin{pmatrix} \sqrt{1 - x^2 - y^2} - 1 + y^2 & -xy \\ -xy & \sqrt{1 - x^2 - y^2} - 1 + x^2 \end{pmatrix},$$

which makes the latter computations irrelevant, although, naturally, an equivalent outcome would follow with the correct entries.

- (15) On page 53 line 20 strictly speaking the conditions on page 48 are inequalities rather than equalities. Perhaps it would have been better to replace equations with *conditions* here.
- (16) On page 53 line 22 the *UR character* is a notion that seems not to be explained in the text.
- (17) On page 57 line 8 While one *can* arrange preimages of  $V_l$  to be Lipschitz subtler arguments are needed to justify this.
- (18) On Page 60 line 8 I don't quite understand the claim: the shape of  $A$  depends on the operator and not on the harmonicity of  $u$ . Also  $t^2$  should be  $|t|^2$  here and below.
- (19) On Page 60 line 10 I guess the rather than boundedness the distance of  $\Omega$  to  $e_3$  is crucial here.
- (20) On page 60 line -12 a minus sign is missing in the computation of the derivative.
- (21) On page 61 line 11 while the remaining computation is OK this first inequality is, as written, incorrect (the absolute value sign is misplaced).
- (22) On page 63 lines -10 to -8 I am not convinced that compactness and Lipschitzness of  $\partial\Omega$  is sufficient for the claim. To simplify, the claim in  $2D$  is that given a Lipschitz curve passing through the origin of a small Euclidean disc all the points within the disc have their closest point to the curve contained in the disc. While true for relatively compact subsets this cannot hold for all the disc (especially close to the boundary intersection) unless the curve crosses the boundary circle orthogonally.
- (23) On page 68 line 10 the choice of  $\phi - j$ 's for angular parameters is not optimal given the role  $\phi$  plays in the argument.
- (24) On page 75 line -2 I completely miss the argument why  $\tilde{h}$  should be Lipschitz.
- (25) On page 81 line 15 obviously the *harmonic* functions are discussed here.
- (26) Theorem 5.1.4 at this stage  $\omega^x$  is not yet defined.
- (27) On page 83 line -8 shouldn't it be  $\omega^x$  rather than  $\omega^z$ ?
- (28) the claim after 5.1.12 requires a reference or at least a comment what happens if the point  $z$  approaches the boundary.
- (29) page 85 line -2 Shouldn't it be  $\geq \frac{1}{C} \frac{d(x, \partial\Omega)}{r}$ ?
- (30) page 90 line 8 the letter  $\omega$  for a parameter is a bad choice given the role of the harmonic measure in this chapter.
- (31) On page 99 line -12 While I am unfamiliar with the reference [CG] the logic of the text suggests that  $C_j$  should be bounded by  $C2^{-\kappa j}$  rather than  $C2^{\kappa j}$ .
- (32) In (5.45) in order to apply John-Nirenberg it seems to me that  $\chi_{\Delta_1}$  rather than  $\chi_{c\Delta_1}$  has to be used (unless I misunderstand the role of  $c$  here).
- (33) On page 106 line -11 the integration should be against  $dy$ .

## Conclusion

Mr Grzyszkówka has written a deep thesis dealing with boundary values of harmonic and close-to-harmonic functions. The results require an enormous apparatus from geometric and harmonic analysis coupled with Riemannian and sub-Riemannian differential geometric tools. The findings are valuable and original. The spotted inaccuracies do not affect the correctness of the proofs. Given all this **I warmly recommend to accept the thesis** and proceed on with the necessary procedures granting mr Marcin Grzyszkówka a PhD in the field of Mathematics.

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