INSTITUTE OF MATHEMATICS
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THE QUANTITATIVE FATOU PROPERTY, ε-APPROXIMABILITY AND CARLESON MEASURES

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I hereby declare that the dissertation is my own work.

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The dissertation is ready to be reviewed.

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Summary

This thesis is concerned with studying the Quantitative Fatou Property (QFP) and the ε-approximability, as well as notions such as the nontangential maximal function, the area function and Carleson measures. Let us briefly describe QFP and ε-approximability. Suppose that Ω is a domain in a space that we are interested in and suppose there is a function $u : \Omega \to \mathbb{R}$. The ε-approximability states that there is a function which is sufficiently regular such that it is ε close to u in L^∞ norm and such that the norm of its gradient gives rise to a Carleson measure. QFP usually follows from ε-approximability. QFP states that a function counting oscillations of u is in the space $L^1_{loc}(\partial\Omega)$. It is a property stronger than Fatou theorem which reads that for a harmonic function there exists a nontangential limit at almost every point of the boundary of Ω. The thesis focuses on extending the results known for harmonic functions in the Euclidean setting. The conducted research contains results pertaining to not necessarily harmonic function in the Euclidean setting and to harmonic functions in settings that are not Euclidean. To be precise, these non Euclidean settings are Riemanninan manifolds and Heisenberg groups. The thesis is based on three papers, [AGG], [Gr], [AdGr].

Firstly, in Chapter 3 based on [AGG], the case of not necessarily harmonic functions in the Euclidean setting is dealt with. We show that for Lipschitz-graph domains, i.e. superlevel sets of Lipschitz functions, a certain class of functions satisfies QFP. This class contains harmonic functions, but it is broader as nonnegative subharmonic functions are also elements of this class. We first show that for such functions ε-approximability holds and then how QFP follows from it.

Next chapter, that is Chapter 4 based on [Gr], handles the case of harmonic functions in Riemannian manifolds. We deal with Lipschitz domains. We prove ε-approximability of harmonic functions, and more generally A-harmonic functions. Then we proceed with the proof of QFP.

Finally, in the last Chapter 5 based on [AdGr], we work in the setting of Heisenberg groups with nontangentially accessible domains (NTA) and domains admissible for Dirichlet problem (ADP). We prove several theorems concerning the Carleson measures, the nontangential maximal functions and the area functions. We say that a measure μ defined on Ω is a Carleson measure if a measure of a ball intersected with Ω, i.e. $\mu(\Omega \cap B(x,r))$ for $x \in \partial\Omega$, is comparable with the measure of a boundary ball, i.e. $\sigma(\partial\Omega \cap B(x,r))$, where σ denotes a surface measure. The nontangential maximal function of function u is the supremum over a cone with vertex at the boundary of Ω of the absolute value of u. The area function of u at point $x \in \partial\Omega$ is the integral over a cone with vertex at x of the square of the norm of the gradient of u multiplied by the distance to $\partial\Omega$ raised to the appropriate power. First, we prove the characterization of Carleson measures in the first Heisenberg group $\mathbb{H}^1$ using the nontangential maximal function for regular enough domains. Then, we prove characterization of Carleson measures on balls. We then prove that for a harmonic function u on NTA domain Ω with boundary data f the L^2 norm of the area function is bounded by the L^2 norm of f. We also prove the Carleson type estimate, saying that the squared norm of the gradient of a harmonic function multiplied with the Green function defines a Carleson measure. Lastly, we prove a refined version of the Fatou theorem. The refinement lies in the fact that we prove that the set where the nontangential limit does not exist is of capacity zero, rather than of measure zero.
Keywords: Quantitative Fatou Property, ε-approximability, nontangential maximal function, area function, Carleson measure.
AMS Subject Classification 2024: 58J05, 35J05, 35R01, 35R03, 31B05, 31C05.

Streszczenie

Ta rozprawa dotyczy studiów nad Ilościową Własnością Fatou (IWF) oraz ε-aproksymowalnością, jak również pojęciami takimi jak niestyczna funckja maksymalna, funkcja area, czy miary Carlesona. Teraz krótko opiszemy IWF i ε-aproksymowalność. Niech Ω będzie dziedziną w interesującej nas przestrzeni, a $u : \Omega \to \mathbb{R}$ będzie funkcją. Własność ε-aproksymowalności mówi, że istnieje dostatecznie regularna funkcja, która jest odległa of u o nie więcej niż ε w normie L^∞ oraz norma jej gradientu zadaje miarę Carlesona. Zazwyczaj IWF wynika z ε-aproksymowalności. IWF mówi, że funkcja licząca oscylacje u jest w $L^1_{loc}(\partial\Omega)$. Ta własność jest silniejsza niż twierdzenie Fatou mówiące, że funkcja harmoniczna ma granice niestyczne w prawie każdym punkcie brzegu Ω. Rozprawa skupia się na rozszerzeniu wyników znanych dla funkcji harmonicznych w przestrzeniach euklidesowych. Przeprowadzone badania zawierają wyniki dotyczące funkcji z klasy szerszej niż tylko harmoniczne w przestrzeniach euklidesowych oraz dotyczące funkcji harmonicznych w przestrzeniach nieeuklidesowych. Precyzyjniej, w rozmaitościach riemannowskich i grupach Heisenberga. Rozprawa jest oparta na trzech artykułach [AGG], [Gr], [AdGr].

Wpierw, w rozdziale 3 opartym na [AGG], zajmujemy się przypadkiem funkcji niekoniecznie harmonicznych w przestrzeniach euklidesowych. Pokazujemy, że dla dziedzin, które są nadpoziomicami funkcji lipszycowskich, szczególna klasa funkcji spełnia IWF. Ta klasa zawiera funkcje harmoniczne, ale jest szersza, gdyż nieujemne funkcje subharmoniczne również są jej elementami. Najpierw pokazujemy ε-aproksymowalność, a potem jak wynika z niej IWF.

W rozdziale 4 opartym na [Gr] zajmujemy się przypadkiem funkcji harmonicznych na rozmaitościach riemannowskich. Pracujemy z dziedzinami lipszycowskimi. Dowodzimy ε-aproksymowalność funkcji harmonicznych, ogólniej A-harmonicznych. Następnie pokazujemy IWF.

W ostatnim rozdziale 5 opartym na [AdGr] pracujemy w grupach Heisenberga z dziedzinami osiągalnymi niestycznie (NTA) oraz dziedzinami dopuszczalnymi dla zagadnienia Dirichleta (ADP). Dowodzimy kilka twierdzeń dotyczących miar Carlesona, niestycznej funkcji maksymalnej i funkcji area. Mówimy, że miara μ zdefiniowana na Ω jest miarą Carlesona, jeśli miara przecięcia kuli z Ω, tj. $\mu(\Omega \cap B(x,r))$ dla $x \in \partial\Omega$, jest porównywalna z miarą kuli brzegowej, tj. $\sigma(\partial\Omega \cap B(x,r))$, gdzie σ oznacza miarę powierzchniową. Niestyczna funkcja maksymalna funkcji u jest supremum wartości bezwzględnej u po stożku o wierzchołku na brzegu Ω. Funkcja area funkcji u w punkcie $x \in \partial\Omega$ jest całką po stożku o wierzchołku w x z normy gradientu u w kwadracie pomnożonej przez odległość do brzegu podniesioną do odpowiedniej potęgi. Wpierw, dowodzimy charakteryzację miar Carlesona w pierwszej grupie Heisenberga $\mathbb{H}^1$ dla dostatecznie regularnych dziedzin, używając niestycznej funcji maksymalnej. Następnie, dowodzimy charakteryzację miar Carlesona dla kul. Potem pokazujemy, że dla funkcji harmonicznej u na dziedzinie Ω, która jest NTA, z wartością brzegową f, norma L^2 funkcji area dla funkcji u szacuje się przez normę L^2 funkcji f. Dowodzimy również oszacowanie typu Carlesona mówiące, że kwadrat normy gradientu funkcji harmonicznej pomnożony przez funkcję Greena definiuje miarę Carlesona. Na koniec, dowodzimy mocniejszą wersję twierdzenia Fatou. Zmiana polega na tym, że pokazujemy, że niestyczna granica nie istnieje na zbiorze o pojemności zero, a nie tylko o mierze zero.

Słowa kluczowe: Ilościowa Własność Fatou, ε-aproksymowalność, niestyczna funkcja maksymalna, funkcja area, miara Carlesona.

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Contents

Chapter 1

Introduction

1.1 Outline of the thesis

This thesis is divided into five chapters.

The first chapter is Introduction. It contains main theorems included in the thesis as well as motivation for investigating given topics.

The second chapter is Preliminaries. It consists of definitions and notions used throughout the thesis. Nevertheless, some definitions or notions may be repeated later when they are needed, so that it is more comfortable for the reader.

Chapters three, four and five constitute the main part of the thesis. Each of them is based on one paper and, in particular, includes proofs of theorems presented in the Introduction.

1.2 Quantitative Fatou Property and ε-approximability

The Quantitative Fatou Property and the ε-approximability are the key notions throughout the research that led to this thesis.

The main motivation of the author was to expand the knowledge concerning Quantitative Fatou Property (QFP) and ε-approximability which is an essential tool used to prove QFP.

The author's research interests grow from the studies of harmonic functions and their boundary behaviour. A special interest is devoted to various settings of metric spaces, such as the Riemannian manifolds, see Def. 4.1.1, and the Heisenberg groups, see Chapter 5.1.1, as well as important types of domains, including Lipschitz and nontangentially accessible domains (NTA domains), see Chapters 2.9, 5.1.2 for detailed discussion of such domains and Definitions 2.9.4, 5.1.1 for definition of such domains. Moreover, the Carleson measures, see Chapter 2.6 and Def. 5.1.8, play an important role in these studies, along with the area/square functions and nontangential maximal functions, see Chapters 2.5 and 5.1.4. The investigations involve tools from the geometric analysis and PDEs, harmonic analysis and geometric measure theory.

Let us now describe the motivation for our studies and the prior results.

In 1906 it was proved by Fatou that a harmonic function defined on a planar disc has a radial limit at almost every point of its boundary, i.e. a unit circle. Since then there was a huge advancement in this area of research. Namely, theorems concerning existence of not only radial limits but rather nontangential limits were proved in a wide array of settings, up to nontangentially accessible domains. Let

us list a few of the advancements that happened in the last 100 years. Stein proved the Fatou theorem for half-space in [St1, Chapter VII]. Jerison-Kenig proved it for NTA domains in [JK]. Carleson introduced (nowadays called) Carleson measures to deal with nontangential phenomena in [Car1], [Car2]. Fefferman-Stein dealt with estimates concerning nontangential maximal function and square function in [FS]. There are also negative results giving examples when the Fatou theorem does not hold. Wolff in [W] proved that there are p-harmonic functions for which nontangential limit exists only on the set of measure zero, Manfredi-Weitsman in [MW] proved there is a bound on the Hausdorff dimension of that set. See also Akman-Lewis-Vogel in [ALV] for more theory concerning such functions. The Fatou theorem was investigated not only in the Euclidean space, but also in the setting of Carnot-Carathéodory groups, see e.g. Capogna-Garofalo [CG].
Furthermore, there is a quantitative version of the Fatou theorem which gives a bound on the integral oscillations of a function. Let us be more precise.
From now on Ω will denote an open connected bounded set which is a subset of either Euclidean space or Riemannian manifold, unless stated otherwise.

We will denote by N a function that counts oscillations of a harmonic function u. That function for each point in the boundary $q \in \partial\Omega$ takes each sequence of points, inside a truncated cone of radius r with vertex at q, such that function u varies on consecutive points by at least ε, and these points converge sufficiently quickly (controlled by θ). Then it takes the supremum of lengths of such sequences. We write $N(r, \varepsilon, \theta)(q)$, so that it is obvious what are the parameters. For precise definitions of counting functions in different settings, see Chapter 2.4.

In different settings a counting function may be defined in different ways, see Bortz- Hofmann [BH, Section 1], Garnett [G, Chapter VIII, Section 6], Kenig-Koch-Pipher-Toro [KKPT, Section 2]. For Lipschitz domains it is natural to consider cones. Nevertheless, in every of mentioned cases the idea is the same. The goal is to somehow count oscillations.

One can reformulate a classical Fatou theorem, stating that nontangential limit exists at a.e. point of boundary, in terms of counting functions. It is equivalent to saying that $N(r, \varepsilon, \theta)$ is finite for almost every point $q \in \partial\Omega$ and every $\varepsilon > 0$. However, a stronger quantitative Fatou theorem (QFT) reads:

Let $u : \Omega \to \mathbb{R}$ be a bounded harmonic function with $\|u\|_\infty \leq 1$. Then for every point $p \in \partial\Omega$

$$\sup_{0<r<\operatorname{diam}(\Omega)} \frac{1}{r^{n-1}} \int_{\partial\Omega \cap B(p,r)} N(r, \varepsilon, \theta)(q) d\sigma(q) \leq C(\varepsilon, \alpha, \theta, n, \Omega),$$

where ε, θ are constants in the definition of the counting function and α denotes an aperture of a cone. In particular, constant C is independent of u.

From this statement the classical theorem follows, but it is much stronger. Garnett proved that for a harmonic function u defined on upper half-plane and satisfying $\|u\|_\infty \leq 1$ the counting function satisfies the estimate

$$\int_I N_\varepsilon(x) dx \leq C\varepsilon^{-7}$$

for every $\varepsilon > 0$ with constant depending on $\varepsilon, \alpha, \theta$, but independent of u, where I is any interval of length 1.

It was proved by Bortz and Hofmann, see [BH], that in the Euclidean space for nontangentially accessible (NTA) domains QFP is equivalent to uniform rectifiability of a boundary of a domain. NTA domains are a wide class of domains which include e.g. Lipschitz domains, Zygmund domains or quasispheres, see e.g. Jerison-Kenig [JK].

Theorem 1.1 in [BH]. Let $\Omega \subset \mathbb{R}^{n+1}$*,* $n \geq 2$*, be an open set satisfying an interior corkscrew condition, whose boundary is n-dimensional Ahlfors-David regular* (see Chapter 2.8). *Suppose that* $\mathcal{L} = -\operatorname{div} A\nabla$ *is a uniformly elliptic divergence form operator whose coefficients satisfy Eqs. 2.5 and 2.6. Then a quantitative Fatou theorem holds for bounded null solutions of* $\mathcal{L}$ *and its adjoint* $\mathcal{L}^*$*, if and only if* $\partial\Omega$ *is uniformly rectifiable.*

Let us notice how remarkable this result is. It gives a deep connection between a theory of PDEs and geometry of a domain. What is more, it states that a certain property of solutions of some PDEs is equivalent to some geometric properties of a domain. One can say that such a theorem builds a bridge between two branches of mathematics.

It is worth mentioning the importance of the notion of uniform rectifiability, see Def. 4.3.3. It was introduced by David and Semmes, see [DS1], [DS2], [DS3]. See Mattila [M] for a survey about uniform rectifiability. See also Bate-Hyde-Schul [BHS] for uniform rectifiability in metric spaces. It was developed to prove boundedness of certain singular integral operators. It turns out that uniform rectifiability is equivalent to that boundedness. It is a natural refinement of the notion of rectifiability, which in a sense provides the broadest class of sets worth considering in geometric measure theory. Making the notion of rectifiability uniform, allows to obtain a variety of quantitative results such as e.g. QFT.

A crucial step in proving QFT is the so-called ε-approximability. It was first established on the upper half-plane for some $1 > \varepsilon > 0$ by Varopoulos in [Va1], [Va2], and then proved for any $\varepsilon > 0$ by Garnett in [G, Chapter VIII, Section 6, Theorem 6.1].

Later the property was proved for harmonic functions on Lipschitz domains by Dahlberg, see [D1]. In [KKPT] the result is proven for A-harmonic functions, i.e. solutions to real divergence form equation $\operatorname{div} A\nabla u = 0$, in Lipschitz domains.

Fairly recently, it was shown that for real divergence form operator satisfying the Carleson measure condition and pointwise local Lipschitz bound, ε-approximability is equivalent to uniform rectifiability, see Hofmann-Martell-Mayboroda [HMM1], Azzam-Garnett-Mourgoglou [AGMT].

Definition 1.2.1 (ε-approximability)**.** Let $\varepsilon > 0$ and $\Omega \subset \mathbb{R}^{n+1}_+$ satisfy (1.2). We say that a function $u : \Omega \to \mathbb{R}$ is *ε-approximable*, if there exists a function $\varphi \in BV_{loc}(\Omega)$ such that

1. $\|u - \varphi\|_\infty < \varepsilon$,
2. $|\nabla\varphi|$ defines a Carleson measure on Ω, i.e. for every $x \in \partial\Omega$

$$\sup_{r\in(0,\operatorname{diam}\Omega)} \frac{1}{r^n} \int_{\Omega\cap B(x,r)} |\nabla\varphi(y)| \mathrm{d}\mathscr{L}^{n+1}(y) \leq C_\varepsilon, \tag{1.1}$$

   where $\mathscr{L}^{n+1}$ denotes $(n+1)$-dimensional Lebesgue measure.

We refer to Chapter 2.10 for a definition of BV functions used in the above definition.

Let us remark that condition (1.1) can be equivalently formulated in terms of the surface measure, since domain Ω is given by the Lipschitz graph, and thus the surface measure is n-Ahlfors regular on the boundary, implying that $\sigma(B(x,r)\cap\partial\Omega)\approx r^n$.

Here again, different authors have different definitions of ε-approximability, see [D1], [G], [BH]. The difference lies in the function space to which φ is supposed to belong to. It turns out that in the Euclidean setting all these definitions are equivalent. However, in a different setting it may not be the case, see e.g. [Gr].

Let us stress that the importance of this condition comes from an observation that a natural candidate for a Carleson measure of a harmonic function, namely $|\nabla u(x)|\mathrm{d}x$, may fail to be a Carleson measure, see e.g. [G, Section 6, Ch. VIII]. In order to bypass this problem, the notion of ε-approximability has been introduced and has turned out to be important in the studies of the BMO extension problems and Corona theorems ([G], Hofmann-Tapiola [HT]), the characterization of the uniform rectifiability (Hofmann-Le-Morris [HLM], [HMM1], Hofmann-Martell-Mayboroda-Toro-Zhao [HMMTZ]) and in the Quantitiative Fatou theorems ([G], [BH]).

Investigating QFP or ε-approximability is important because it gives a connection between properties of certain functions defined on Ω and its geometry. In Euclidean setting QFP is equivalent to ε-approximabilty, but it is also equivalent to uniform rectifiability of a boundary, see [BH], [HMM1]. Hence, in a sense geometry is determined by some properties of solutions of PDEs and vice versa.

In the hitherto work the author focused on investigating QFP and other notions related with harmonic analysis on Euclidean spaces and beyond them. Firstly, in [AGG], we have been studying ε-approximability in Euclidean space for a certain class of functions beyond harmonic ones. Then, in [Gr], we proved ε-approximability and QFT for Lipschitz domains in Riemannian manifolds. Lastly, in [AdGr], we investigated properties of nontangential maximal functions, square functions and Carleson measures in the Heisenberg groups.

1.3 Introduction to the third chapter

The third chapter is based on the manuscript [AGG]. There, we consider the Lipschitz-type domains in the form

$$\Omega = \{(x,y)\in\mathbb{R}^{n+1}_+ \,:\, y>\phi(x)\}, \tag{1.2}$$

where $\phi:\mathbb{R}^n\to\mathbb{R}$ is an L-Lipschitz function. On such domains, we study functions $u\in C^2(\Omega)$ which satisfy the following condition on any ball $B_r\subset\Omega$ such that $2B_r\subset\Omega$:

$$\operatorname{osc}_{B_r}(u)\le C\left(r^{1-n}\int_{(1+\eta)B_r}\left(|\nabla u|^2+|u\Delta u|\right)\mathrm{d}\mathscr{L}^{n+1}\right)^{\frac{1}{2}} \tag{$*$}$$

for some $\eta\in[0,1)$ and $C>0$. Such a class has been considered by González-Koskela-Llorente-Nicolau in [GKLN], when studying the relationships between the nontangential maximal function and convenient versions of the area function of general (nonharmonic) functions. A priori, it might not be clear how wide is this family of functions. However, Proposition 5.1 in [GKLN] shows that ($*$) follows from the following pointwise condition:

$$|u\Delta u|\le\theta|\nabla u|^2 \text{ in } \Omega \tag{\#}$$

for some $\theta > 0$. However, further restriction on θ is necessary in order to control the area function by the nontangential maximal function of u. Namely, we need to assume that $0 < \theta < 1$. From now on we will say that a function u satisfies condition (#) if $0 < \theta < 1$.

The class of functions (#) clearly encloses harmonic ones, but also others, see Proposition 3.3.3 in Chapter 3.3. However, what is perhaps more important from our point of view is that, the oscillation condition ($*$) holds for several non-harmonic examples, for instance for non-negative C^2 subharmonic ones, see Proposition 3.3.1 or non-negative C^2 functions u with subharmonic $|\nabla u|^\alpha$, for $\alpha \in (0, 2]$, see Proposition 3.3.2. Estimate ($*$) together with (#) imply that

$$(\operatorname{osc}_{B_r}(u))^2 \lesssim_{n,\theta} r^{1-n} \int_{(1+\eta)B_r} |\nabla u|^2 \, \mathrm{d}\mathscr{L}^{n+1}, \tag{1.3}$$

which can be understood as the Morrey-type estimate for u.

The main goal of the third chapter is to prove the following result.

Theorem 1.3.1. *Let $\Omega \subset \mathbb{R}^{n+1}_+$ be the Lipschitz-graph domain as in (1.2) and let further $u : \Omega \to \mathbb{R}$ be bounded and satisfy condition (#). Then for every $\varepsilon > 0$ function u is ε-approximable in Ω.*

The result generalizes the existing ones, as it is to best of our knowledge, first ε-approximability result for functions that need not be solutions of PDEs in the divergence form. Moreover, we would like to emphasize that condition (#) can be obsolete for some classes of functions and ($*$) instead suffices, as illustrated by nonnegative subharmonic functions, see Proposition 3.3.1 in Chapter 3.3. This observation follows from a brief analysis of the proofs of Theorem 1.1 and Lemmas 4.3 and 4.5 in [GKLN].

The key consequence of Theorem 1.3.1 is the following Quantitative Fatou Theorem in Corollary 1.3.2 (see Definition 2.4.1 of the counting function).

Corollary 1.3.2 (Quantitative Fatou Theorem)**.** *Let $\Omega \subset \mathbb{R}^{n+1}_+$ be the Lipschitz-graph domain as in (1.2) and let further $u : \Omega \to \mathbb{R}$ satisfy condition (#) and be bounded with $\|u\|_\infty \leq 1$. Then for every point $\omega \in \partial\Omega$*

$$\sup_{0<r<r_0} \frac{1}{r^n} \int_{\partial\Omega \cap B(\omega,r)} N(r, \varepsilon, \beta)(z) d\sigma(z) \leq C(\varepsilon, \alpha, \beta, n, \Omega),$$

where $\varepsilon, \alpha, \beta$ are constants in the definition of the counting function N. In particular, constant C is a independent of u.

The proof of Theorem 1.3.1 utilizes methods used in [G, Chapter VIII, Section 6] and in [HMM1]. We use dyadic decomposition of the boundary and a Whitney-type covering of a domain chosen in such a way that a union of appropriate elements of it forms a "nice" set. However, our approach mixes constructions from [G] and [HMM1] in a new way. What is more, we introduced some new elements to the constructions employed in the proof. Let us mention Proposition 3.2.6. In the aforementioned works it is an essential part of the proof of ε-approximability. However, in our setting it was impossible to use the same methods as the mentioned authors. Therefore, we had to come up with a new idea of a proof of that result.

1.4 Introduction to the fourth chapter

The fourth chapter is based on the paper [Gr].

Our goal is to extend Quantitative Fatou Property to the setting of Riemannian manifolds. They are much broader than Euclidean space, however in a sense they are a first step in generalizing any results from $\mathbb{R}^n$ to more general metric measure spaces, see Mitrea-Taylor [MT]. We believe that Riemannian manifolds are an interesting class of spaces as they arise naturally in a variety of problems and hence better understanding of boundary behaviour of harmonic functions may be useful. Moreover, understanding QFP in Riemannian setting gives as an insight into the possibility of investigating it in other non-Euclidean spaces. A priori one does not know whether QFP should hold in a setting different than the Euclidean one. Knowing that QFP is true in Riemannian setting gives hope that it may hold in even more general settings. Furthermore, the fact that QFP holds in different settings would suggest that it is a notion deeply intertwined with the notion of harmonicity and independent from the underlying space. The main difference between the Euclidean space and a Riemannian manifold is the fact that there are no global coordinates in a Riemannian manifold. Therefore, we need to deal with charts and then glue them. What is more, the fact that the space is curved makes a geometry different and finding "nice" paths joining points requires more effort.

Since some notions are not even yet defined outside Euclidean space or fully established, e.g. uniform rectifiability, see [BHS], [M, Chapters 6 and 9], we deal with the case of Lipschitz domains. We prove the following:

Theorem 1.4.1. *Let M be a complete Riemannian manifold and let further $\Omega \subset M^n$ be a Lipschitz domain. Furthermore, let $u : \Omega \to \mathbb{R}$ be a harmonic bounded function with $\|u\|_\infty \leq 1$. Then for every point $p \in \partial\Omega$*

$$\sup_{0<r<\mathrm{r}_{\mathrm{inj}}} \frac{1}{r^{n-1}} \int_{\partial\Omega \cap B(p,r)} N(r,\varepsilon,\theta)(q)\, d\sigma(q) \leq C(\varepsilon,\alpha,\theta,n,\Omega),$$

where $\varepsilon, \alpha, \theta$ are constants in the definition of the counting function. In particular, constant C is independent of u.

One of the key auxiliary results to prove Theorem 1.4.1 is the following ε-approximability property:

Theorem 1.4.2. *Let M be an n-dimensional complete Riemannian manifold and $\Omega \subset M$ be an open bounded connected Lipschitz set. Let u be a harmonic bounded function in Ω. Then u is ε-approximable for every $\varepsilon > 0$.*

Main difficulty is the fact that we do not have one map available on whole of Ω. Therefore we deal with pieces of Ω where there are maps. However, we cannot take any open sets as our charts. It is essential that we choose them to be Lipschitz. What is more, we need to make sure that the number of sets covering our domain is bounded. On these pieces we take local ε-approximants, which exists due to [HMM1]. We need to show that we can choose all maps in a uniform way and that we can later glue everything together to obtain ε-approximant φ on Ω. Then we have to show that the integrals of the norm of its derivative over cones, i.e. $\int_\Gamma |\nabla\varphi|$ gives a Carleson measure.

To obtain Theorem 1.4.1 we need to come up with a bit different approach than in the Euclidean case. In the Euclidean case there is usually a certain way to integrate counting function to obtain desired estimate. We deal with it by taking an appropriately constructed curve and its neighbourhood contained in a nontangential cone, see Lemma 4.4.4. The curve joins consecutive points from admissible sequence, see definition of a counting function in Chapter 2.4. We need it to have bounded length and derivative. Then, using coarea formula enables us to arrive at our estimate, see Lemma 4.4.7. In our proof we follow the idea of [KKPT], but adjust it to the setting of Riemannian manifolds.

1.5 Introduction to the fifth chapter

The fifth chapter is based on the paper [AdGr]. The Carleson measures play an important role in geometric mapping theory and, especially in recent years, also in the studies of relations between geometry, analysis and the measure theory. The importance of such measures has been growing in the last decade via the results on PDEs on rough domains, for instance, the studies of the solvability of the Dirichlet problems for elliptic equations, in analysis of the boundary behaviour of harmonic functions, also in relations to the square functions on NTA domains or uniformly rectifiable sets, see e.g. [HMM1], [HLM], [HMMTZ]. From our point of view the two main motivations come from the investigations of the uniform rectifiability and the ε-approximation, see e.g. [BH], [GMT], [HT] and from the Hardy spaces of quasiconformal mappings, see [AK], [AF]. Moreover, it turns out that the Carleson measures are closely related to the geometry of functions and mappings also in the settings beyond the Euclidean one, for example on homogeneous spaces Hofmann-Mitrea-Mitrea-Morris [HMMM] and on Riemannian manifolds, see Mitrea-Mitrea-Mitrea-Schmutzler [MMMS], [Gr] and in the Heisenberg group $\mathbb{H}^1$ see [AF]. Even though, the need for further studies of Carleson measures in the non-Euclidean setting arises, this topic in the subriemannian setting has not yet been explored as much, as in the Euclidean spaces. Therefore, one of the goals of the fifth chapter is to pursue this direction of investigations. In particular, we focus our attention on the Heisenberg groups $\mathbb{H}^n$, especially on the first Heisenberg group $\mathbb{H}^1$ and on the subelliptic harmonic functions, see Chapter 5.1.3 for definition of such functions, on bounded nontangentially accessible domains (NTA domains) and on bounded domains admissible for the Dirichlet problem (ADP domains), see Chapter 5.1.2 for details about NTA and ADP domains. The fundamental results in the Euclidean setting that have inspired us are discussed in Chapters I and VI of the book [G] and in [JK], while the main tools in the potential theory in the Heisenberg groups employed in this work are proven in [CG], Capogna-Garofalo-Nhieu [CGN].

Let us present and briefly discuss our main results. We show the following characterization of the Carleson measures on ADP domain in $\mathbb{H}^1$ in terms of the level sets of the harmonic functions. The lemma is well known in the setting of the upper-half plane, see Lemma 5.5, Chapter I in [G]. We refer to Chapter 2.6 for the discussion of the Carleson measures and their properties.

Theorem 1.5.1. *Let $\Omega \subset \mathbb{H}^1$ be a smooth ADP domain with 3-regular boundary and μ be a positive measure on Ω. Then μ is a Carleson measure on Ω if and only if there exists a constant $C = C(\alpha)$ such that for every harmonic function u on Ω and every $\lambda > 0$ it holds that*

$$\mu(\{x \in \Omega : |u(x)| > \lambda\}) \leq C\sigma(\{\omega \in \partial\Omega : N_\alpha u(\omega) > \lambda\}), \tag{1.4}$$

where $N_\alpha u$ stands for the nontangential maximal function of u (see Definition 5.1.9) and σ is the surface measure on $\partial\Omega$, i.e. $\sigma = H^2\lfloor\partial\Omega$. Moreover, if C is the least constant such that (1.4) holds, then the Carleson constant of μ satisfies $\gamma_\mu \approx_\alpha C$.

While the proof of the sufficiency part of the theorem follows by applying fairly general approach based on the Whitney-type decomposition, the proof of the necessity part relies on the potential-theoretic properties of harmonic functions, including the boundary Harnack estimate in Garofalo-Phuc [GP] and the results proven in [CGN].

Our next result generalizes a characterization of Carleson measures on the unit disc in the Euclidean plane, cf. Lemma 3.3 in Chapter VI.3 in [G]. One of the key features that give the result in the plane is the rich family of Möbius self-transformations of a disc, a property which is no longer true in the subriemannian setting due to the rigidity of Carnot groups. However, recently in [AF, Section 4.1] a counterpart of Möbius self-maps of a ball in $\mathbb{R}^n$ has been introduced on the Korányi-Reimann unit ball $B(0,1) \subset \mathbb{H}^1$ by the price of giving up that the target domain remains a ball, see the definition of maps $T := T_{x,a,\rho}$ in (5.16) and their property (5.18). The following result characterizes the Carleson measures on $B(0,1)$ in terms of the boundary growth of 1-quasiconformal mappings T.

Theorem 1.5.2. *A measure μ on the unit gauge ball $B := B(0,1) \subset \mathbb{H}^1$ is a Carleson measure if and only if*

$$\int_B \left(\frac{d(T_{x,a,\rho}(y), \partial T_{x,a,\rho}(B))}{d(y, \partial B)} \right)^3 \mathrm{d}\mu(y) = M < \infty, \tag{1.5}$$

for all $x \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$, and $\rho > 0$ such that $\rho \lesssim \min\{d(x,\partial B), d(a,\partial B)\}$ and $\rho \approx d(a,x)$.

In Remark 5.3.1 we also point to the generalization of the above theorem to the setting of higher order Heisenberg groups $\mathbb{H}^n$ for $n \geq 2$.

One of the main results of this chapter is the L^2-estimate for the square function of a subelliptic harmonic function on a bounded NTA domain in $\mathbb{H}^n$ with respect to the L^2 boundary data and the harmonic measure ω. The result generalizes Theorem 9.1 in [JK] proved for bounded NTA domains in $\mathbb{R}^n$. We refer to Chapters 2.5 and 5.1.4 for the definition and further discussion of the square functions.

Theorem 1.5.3 (L^2-boundedness of the square function)**.** *Let $\Omega \subset \mathbb{H}^n$ be a bounded NTA domain. Let further $f \in L^2(\mathrm{d}\omega)$ and $u(x) := \int_{\partial\Omega} f(y)\mathrm{d}\omega^x(y)$. Then, the following estimate holds for the square function S_α of a subelliptic harmonic function u in Ω*

$$\|S_\alpha u\|_{L^2(\mathrm{d}\omega)} \leq C\|f\|_{L^2(\mathrm{d}\omega)},$$

where the constant C depends on n, M, constant from Harnack inequality, α, Ω.

Our second main result is the subriemannian counterpart of the Euclidean result, i.e. Theorem 9.6 in [JK]. Moreover, it also generalizes Theorem 3.4 in [G, Chapter VI.3] for the unit disc in the plane, see Remark 5.4.3. We further refer to Example 5.4.4 for the case of the unit gauge ball in $\mathbb{H}^n$, where the Green function G in the assertion of Theorem 1.5.4 can be explicitly estimated from below in terms of the distance function, thus providing more classical and handy estimate (5.26). In order to obtain this estimate we prove Proposition 5.5.5 in the Appendix.

Theorem 1.5.4 (Carleson measure estimate)**.** *Let $\Omega \subset \mathbb{H}^n$ be a bounded NTA domain and u be subelliptic harmonic in Ω such that $u(x) = \int_{\partial\Omega} f(y)\mathrm{d}\omega^x(y)$ for some $f \in BMO(\partial\Omega)$. Further, let $D > 1$. Then for any choice of D there exists a constant $C = C(D) > 0$ such that for any ball $B(x_0, r)$ centered at $x_0 \in \partial\Omega \setminus \Sigma_\Omega$ with any $0 < r < r_0 \leq \min\{1, \frac{d(x_0,\Sigma_\Omega)}{M}\}$ it holds that*

$$\int_{B(x_0,r)\cap\Omega} |\nabla_H u|^2 G(x, A_r(x_0))\mathrm{d}x \leq C\omega(B(x_0,r) \cap \partial\Omega),$$

where constant C depends on n, M, r_0 and $\|f\|_{BMO(\partial\Omega)}$.

Among corollaries of Theorem 1.5.4 we show the corresponding Carleson estimates on an ADP domain (Corollary 5.4.1) and on the (Euclidean) $C^{1,1}$-domain (Corollary 5.4.2).

The proof of Theorem 1.5.4 consists of several steps and auxiliary observations which largely follow the steps of the corresponding proof of Theorem 9.6 in [JK]. However, we expand several arguments in [JK] and clarify steps which in the new setting of Heisenberg groups require using the subriemannian tools. In particular, we frequently use a variety of properties of subelliptic harmonic functions such as e.g. Harnack inequality. What is more, we heavily rely on the theory of Green functions in subriemannian setting as well as harmonic measures in that setting. We also employ results from the theory of metric measure spaces concerning existence of Whitney-type decompositions. Moreover, our proof requires John-Nirenberg theorem in the subriemannian setting.

Our last result, is a counterpart of the classical Fatou theorem for harmonic functions on (ε, δ)-domains in $\mathbb{H}^n$, under the condition of the L^p-integrability of the gradient of the function. The (ε, δ)-domains in $\mathbb{H}^n$ can be thought of as the quantified version of the uniform domains and contain large family of NTA domains.

Theorem 1.5.5. *Let $\Omega \subset \mathbb{H}^n$ be a bounded (ε, δ)-domain and let further u be harmonic in Ω. If $\int_\Omega |\nabla_H u|^p < \infty$ for some $1 < p \leq 2n + 2$, then u has nontangential limits on $\partial\Omega$ along horizontal curves in Ω outside the set of p-Sobolev capacity zero.*

This result extends previous observations in the Heisenberg setting in two ways:

1. the considered domains are slightly more general than in a Fatou theorem on NTA domains in $\mathbb{H}^n$ [CG, Theorem 4] and in $\mathbb{R}^n$ [JK, Theorem 6.4];

2. the assertion gives the existence of nontangential limits not only up to the measure zero set as e.g. in [CG], but outside the set of p-Sobolev capacity zero, which is a refined measure.

Chapter 2

Preliminaries

The goal of this chapter is to introduce and discuss some necessary definitions and theorems used throughout this thesis. The chapter is divided into subchapters in such a way that each subchapter corresponds to one of the notion used in the thesis.

2.1 Notation

Let us begin by introducing some notation used throughout this work.

- $a \approx b$ means that there exists a constant $C > 0$ such that

$$\frac{1}{C} a \leq b \leq C a,$$

  if the symbol $\approx$ has something in the lower index e.g. $\approx_n$ it means that the constant depends on n,

- $a \lesssim b$ means that there exists a constant $C > 0$ such that

$$a \leq C b,$$

- $\mathcal{H}^n$ denotes the n-Hausdorff measure of a set,

- $d(x, E) := \operatorname{dist}(x, E)$ for a point $x \in X$ and $E \subset X$, where (X, d) is a metric space,

2.2 Lipschitz-graph domain

In Chapter 3, we consider the Lipschitz-graph domains in the form

$$\Omega = \{(x, y) \in \mathbb{R}^{n+1}_{+} : y > \phi(x)\}, \tag{2.1}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is an L-Lipschitz function.

Unless specified otherwise, in Chapter 3, by Ω we always denote a Lipschitz-type domain as in (2.1).

Such domains were studied primarily in [GKLN]. The authors proved inequalities between norms of the area function and the nontangential maximal function of functions from a certain class which was broader than harmonic functions. Notice that a half-space is a special case of such domains. The case of a half-space was studied in [G], where the author proved ε-approximability of harmonic functions on a half-plane. The case of half-spaces was also researched in [FS], where the authors proved inequalities similar to those in [GKLN].

Notice that Lipschitz-graph domains are also a special case of Lipschitz domains. The notions of ε-approximability and inequalities between the area and nontangential maximal functions for Lipschitz domains were studied in e.g. [D1] and [KKPT].

2.3 Cones

In Chapter 3 we use the following definitions of a cone, see also Figures 2.1 and 2.2 below.

Definition 2.3.1. For $\alpha > 0$ a cone with a vertex at point $(x, \phi(x)) \in \partial\Omega$ and aperture α is defined as follows

$$\Gamma_\alpha(x) := \{(z, y) \in \mathbb{R}^{n+1}_+ : |z - x| < \alpha(y - \phi(x))\}.$$

Notice that for every $x \in \mathbb{R}^n$ a cone $\Gamma_\alpha(x)$ is congruent to a cone $\{(x, y) \in \mathbb{R}^{n+1}_+ : |x| < \alpha y\}$. However, such cones need not be contained in domain Ω. Therefore, we introduce the *truncated cone*:

$$\Gamma_{\alpha,s,t}(x) := \Gamma_\alpha(x) \cap \{(z, y) : \phi(z) + s < y < \phi(z) + t\},$$

where $0 \le s \le t \le \infty$. In that notation $\Gamma_\alpha(x) = \Gamma_{\alpha,0,\infty}(x)$. Since function ϕ is L-Lipschitz, it holds that $\Gamma_{\alpha,0,t}(x) \subset \Omega$ only for $\alpha < \frac{1}{L}$ (and hence, from now on we only consider $\alpha < \frac{1}{L}$).

In Chapter 3 we also use a different notion of truncated cones in Corollary 1.3.2, in particular in the definition of the counting function, see. Def. 2.4.1.

Definition 2.3.2 (Cones truncated with hypersurfaces)**.** For $0 < \alpha < \frac{1}{L}$, $0 \le s < t \le \infty$ and a point $(x, \phi(x)) \in \partial\Omega$ we set

$$\widetilde{\Gamma}_{\alpha,s,t}(x) = \Gamma_\alpha(x) \cap \left(H^x_{\phi,s} \setminus \overline{H^x_{\phi,t}}\right),$$

where $H^x_{\phi,r} = \{(z, y) \in \mathbb{R}^{n+1} : y > \phi(x) + r\}$.

However, in Chapter 4 we use a different definition of a cone, more appropriate for the Riemannian setting. We define these different cones in Chapter 4.1 so that it is more convenient to the reader, see Def. 4.1.13, 4.1.14.

The two notions of a cone are different from each other. The cones used in Chapter 3 are congruent to Euclidean cones, whereas the cones used in Chapter 4 are curvilinear and their shape depends on the distance to the boundary of a domain.

2.4 Counting function

First, we present the definition used in Chapter 3.

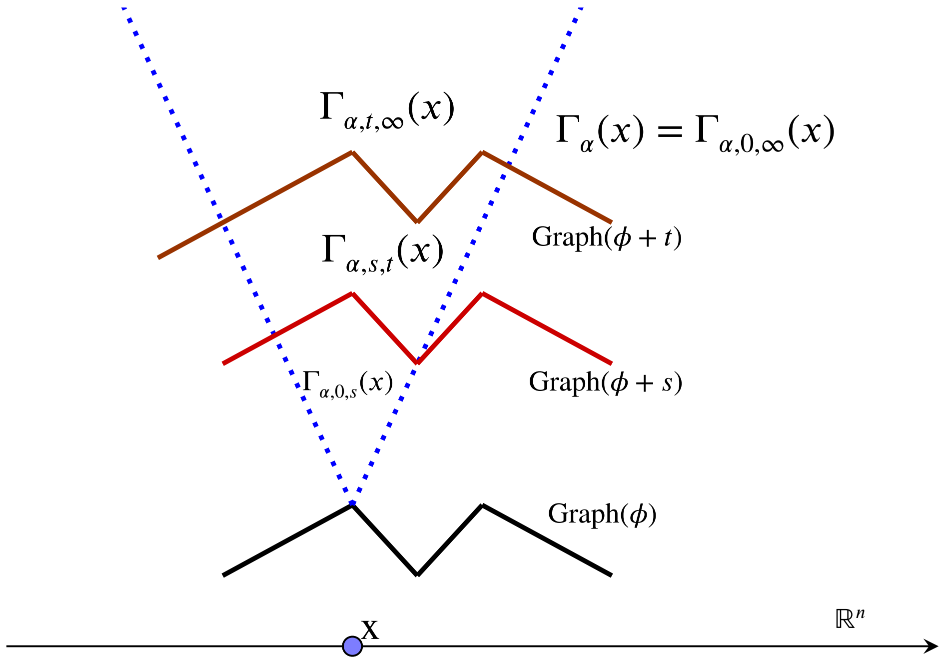

Figure 2.1: This figure depicts cones used in Chapter 3 from Def. 2.3.1.

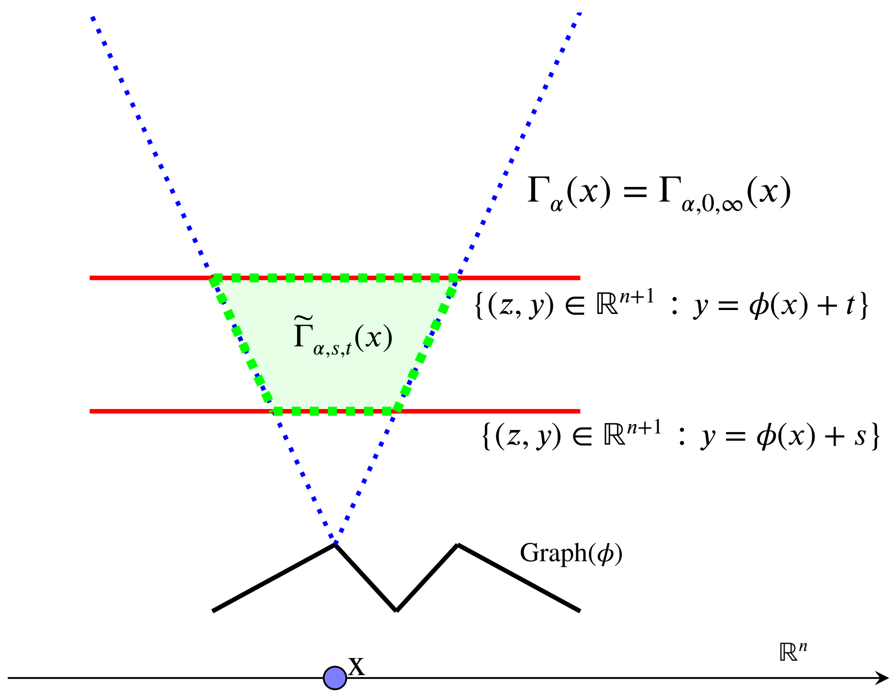

Figure 2.2: This figure depicts cones used in Chapter 3 from Def. 2.3.2.

Definition 2.4.1 (Counting function)**.** Let $\widetilde{\Gamma}_{\alpha,0,r}(x)$ be a truncated cone with the vertex at a point $(x,\phi(x)) \in \partial\Omega$. Let u be a continuous function defined on Ω. Fix $\varepsilon > 0$, $0 < \beta < 1$ and $0 < r < 1$. We say that a sequence of points (x_n, t_n) such that $(x_n, t_n) \in \widetilde{\Gamma}_{\alpha,0,r}(x)$ is $(r, \varepsilon, \beta, x)$-admissible for u if

$$|u(x_n,t_n) - u(x_{n-1},t_{n-1})| \geq \varepsilon \quad \text{and} \quad t_n - \phi(x) < \beta(t_{n-1} - \phi(x)).$$

Set

$$N(r,\varepsilon,\beta)(x) := \sup\{k : \text{there exists an } (r,\varepsilon,\beta,x)\text{-admissible sequence of length } k\}.$$

We will call N a *counting function.*

In Chapter 4 we use a different definition of the counting function N, more appropriate for the Riemannian setting. We give that definition in Chapter 4.1, see Definition 4.1.15.

The main difference is the type of cones used. The first definition is utilized in Chapter 3 and the definition with Riemannian manifolds is employed in Chapter 4. Another discrepancy lies in the distances. The first definition makes use of the Euclidean distance, while the second uses the distance on a manifold.

2.5 Area and nontangential maximal functions

One may find area function (sometimes called square function) and nontangential maximal functions e.g. in works [D1], [FS], [JK], [KKPT] and many more. The area function and the nontangential maximal function have been studied by many authors. Let us mention the work of Dahlberg [D2] and Dahlberg-Jerison-Kenig [DJK], see also Stein [St2] for an interesting account on the history of the notion of the area/square function. In these papers the authors prove the comparability of L^p norms of the area function and the nontangential maximal function under certain conditions. The nontangential maximal function plays a role similar to the role of Hardy-Littlewood maximal function in classical analysis. The area/square function has been investigated by many authors, see e.g. [St1], [FS]. It arises naturally in a lot of estimates regarding e.g. harmonic functions and that is why doing research pertaining to it is useful in harmonic analysis. These notions provide useful tools in harmonic analysis and related fields. It actually turns out that the comparability of their norms is equivalent to a uniform rectifiability of a boundary of the domain, see e.g. [HMM1], [HMM2]. Hence, it gives a connection between analysis and geometry.

Definition 2.5.1 (Area function)**.** Let $f : \Omega \to [0,\infty]$ be a measurable function. The *area function* associated to the density f is defined by

$$(A_\alpha f)(x) = \left(\int_{\Gamma_\alpha(x)} f(z,y)(y-\phi(x))^{1-n} \mathrm{d}z\mathrm{d}y\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

Similarly, we define the truncated version of the area function $A_{\alpha,s,t}f$ with respect to cones $\Gamma_{\alpha,s,t}$.

In what follows we are mostly interested in the case $f = |\nabla u|^2$ for a function $u \in C^2(\Omega)$. Then we write

$$(A_{\alpha,s,t}u)(x) := (A_{\alpha,s,t}|\nabla u|^2)(x) = \left(\int_{\Gamma_{\alpha,s,t}(x)} |\nabla u(z,y)|^2 (y-\phi(x))^{1-n} \mathrm{d}z\mathrm{d}y\right)^{\frac{1}{2}}.$$

Definition 2.5.2 (Nontangential maximal function)**.** Let $f : \Omega \to [0, \infty]$ be a continuous function. The *nontangential maximal function function of u* is defined as follows

$$(N_\alpha f)(x) = \sup_{\Gamma_\alpha(x)} |f(y)|, \quad x \in \mathbb{R}^n.$$

As above, the truncated nontangential maximal function of u, denoted by $N_{\alpha,s,t}u$ is defined analogously with respect to cones $\Gamma_{\alpha,s,t}$.

2.6 Carleson measures

The following notion will be employed in Chapters 3.2, 4.3.

Definition 2.6.1 (Carleson measure in $\mathbb{R}^{n+1}$)**.** Let Ω be an open set in $\mathbb{R}^{n+1}$. We say that a (positive) Borel measure μ on Ω is an *α-Carleson measure on* Ω, if there exists a constant $C > 0$ such that

$$\mu(\Omega \cap B(x, r)) \leq Cr^n, \quad \text{for all } x \in \partial\Omega \text{ and } r > 0.$$

The *Carleson measure constant* of μ is defined as the infimum of constants C above.

Carleson measures were introduced by Carleson in [Car1] to deal with interpolating by bounded analytic functions and the famous corona problem. Carleson measures are also used in the definition of ε-approximability. Namely, the gradient of an approximation gives rise to a Carleson measure. As already mentioned, it may happen that a gradient of even a harmonic function is not a Carleson measure. Therefore, it is necessary to have an object such as a Carleson measure to obtain Quantitative Fatou Property. Since Carleson defined these measures, they have been used by many authors. Garnett used this notion in [G] to prove ε-approximability of harmonic functions in upper half-plane. They were used by Dahlberg in [D1], to prove ε-approximability in Lipschitz domains. It was used in other works concerning harmonic analysis such as [KKPT], [HMM1], [BH] and many more.

It is a useful tool that allows to obtain estimates of certain integrals by the measure of the boundary of a set.

For the definition of the Carleson measures in Heisenberg groups, see Definition 5.1.8 in Chapter 5.1.4.

2.7 Cavalieri's principle

The following well-known representation of an integral of the superlevel sets will frequently be used in Chapter 3. Let $\Omega \subset \mathbb{R}^n$, μ be a measure on Ω and $f : \Omega \to \mathbb{R}_+$ be measurable. then for every monotone $\Phi \in C^1$, $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\Phi(0) = 0$, we have

$$\int_\Omega \Phi(f(x))\mathrm{d}\mu = \int_0^\infty \Phi'(\lambda)\mu(\{x \in \Omega : f(x) > \lambda\})\mathrm{d}\lambda.$$

2.8 Ahlfors-David regularity

The following definitions will be employed in Chapters 4.3, 4.4, 5.2.

Definition 2.8.1. Let (X, d, μ) be a metric measure space. We say that X is *n-Ahlfors-David regular* (n-ADR), if there exists a constant $C < 0$ such that

$$\frac{1}{C} r^{n-1} \leq \mu(B(p, r) \cap E) \leq C r^{n-1}, \qquad \text{for all } p \in E \text{ and } 0 < r < \operatorname{diam}(E).$$

In particular, if M is an n-dimensional Riemannian manifold and $(n-1)$-dimensional $E \subset M$ we can take $(E, d_M, \mathcal{H}^{n-1})$ as a metric measure space and obtain:

Definition 2.8.2. We say that set $E \subset M$ is *Ahlfors-David regular*, of Hausdorff dimension $n-1$, if it is closed and there exists a constant $C < 0$ such that

$$\frac{1}{C} r^{n-1} \leq \sigma(B(p, r) \cap E) \leq C r^{n-1}, \qquad \text{for all } p \in E \text{ and } 0 < r < \operatorname{diam}(E),$$

where $\sigma = \mathcal{H}^{n-1}|_E$ denotes a surface measure on E.

Among examples of $(n-1)$-Ahlfors-David regular spaces let us mention boundaries of smooth, C^k or Lipschitz domains in $\mathbb{R}^n$ or Riemannian manifolds of dimension n. Furthermore, examples of Ahlfors-David regular sets with n being noninteger enclose a boundary of Koch snowflake or Cantor set.

If $X = \mathbb{H}^n$ and $\Omega \subset \mathbb{H}^n$ is a non-empty open connected set, we get the following definition.

Definition 2.8.3. We say that Ω has *s-regular boundary* for some $s > 0$, if there exists a constant $C \geq 1$ such that

$$\frac{1}{C} r^s \leq \mathcal{H}^s(B(x, r) \cap \partial\Omega) \leq C r^s, \qquad \text{for all } x \in \partial\Omega \text{ and } 0 < r < \operatorname{diam}(\partial\Omega).$$

2.9 Nontangentially accessible domains

We now recall one of the fundamental notions of the dissertation, used in Chapters 4.3, 5.2, 5.4, 5.5.

Let (X, d) be a metric space.

Definition 2.9.1 (Interior corkscrew condition)**.** Let $\Omega \subset X$ be an open set. We say that it satisfies interior corkscrew condition if there exists a constant c such that for every set $B(p, r) \cap \partial\Omega$ with $p \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$ there is a ball $B(\widetilde{x}, cr) \subset B(p, r) \cap \Omega$. We say that Ω satisfies exterior corkscrew condition if $X \setminus \Omega$ satisfies interior corkscrew condition.

Definition 2.9.2 (Exterior corkscrew condition)**.** Let $\Omega \subset X$ be an open set. We say that it satisfies exterior corkscrew condition if its exterior $\Omega_{ext} = X \setminus \overline{\Omega}$ satisfies interior corkscrew condition.

See Figures 2.3 and 2.4 for the illustration of the above definitions. Let us add that the corkscrew conditions play a role in studies of the solvability of the Dirichlet problem, see Chapter 15.4 in [GT].

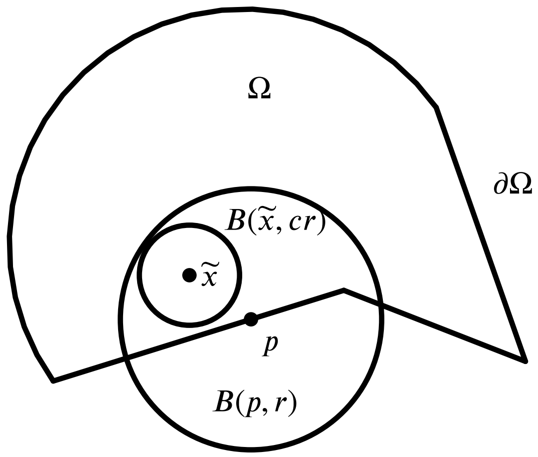

Figure 2.3: This figure shows interior corkscrew condition. Namely, for every point $p \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, there exist a constant $c > 0$ and a point $\widetilde{x} \in \Omega \cap B(p, r)$ such that $B(\widetilde{x}, cr) \subset \Omega \cap B(p, r)$.

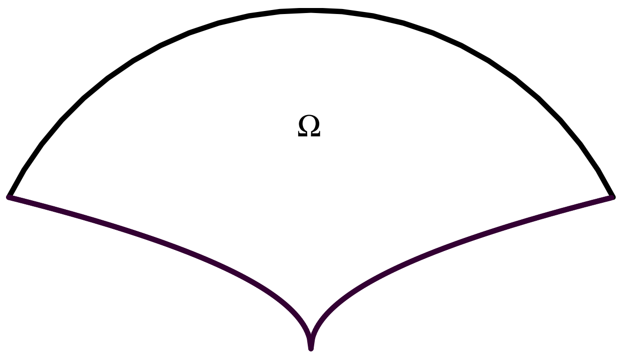

Figure 2.4: This figure depicts an example of a domain which does not satisfy interior corkscrew condition. The cusp is the reason why the condition is not met.

Definition 2.9.3 (Harnack chain condition)**.** Let $\Omega \subset X$ be an open set. We say that Ω satisfies Harnack chain condition, if for every $\varepsilon > 0$ and $x, y \in \Omega$ such that $d(x, \partial\Omega) > \varepsilon$, $d(y, \partial\Omega) > \varepsilon$ and $d(x, y) < C\varepsilon$ there exists a sequence of balls $B_1, \ldots, B_p$ with the following properties:

1. $x \in B_1$ and $y \in B_p$,
2. $\frac{r}{M} < d(B_i(x, r), \partial\Omega) < Mr$ for every $i = 1, \ldots, p$,
3. $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, p - 1$,
4. length of the chain p depends on C but not on ε.

The Harnack chain condition appears in a variety of problems in geometric analysis, for instance in working with John domains or uniform domains, see e.g. [TT].

Definition 2.9.4 (Nontangentially accessible domains (NTA))**.** Let $\Omega \subset X$ be an open set. We say that it is a nontangentially accessible domain (NTA) if it satisfies both interior and exterior corkscrew conditions and Harnack chain condition.

Examples of NTA domains enclose smooth, C^k or Lipschitz domains in $\mathbb{R}^n$ or in a Riemannian manifold. Furthermore, Zygmund domains are NTA as well as quasispheres are NTA, see [JK]. As an example of one-sided NTA, i.e. without exterior corkscrew condition, one can consider a ball in $\mathbb{R}^2$ with a corner 1-dimensional Cantor set removed.

In Heisenberg group, NTA sets are, for example, Karányi-Reimann balls, but not balls in Carnot-Carathéodory distance, or upper half space, $\mathbb{H}^n_+ = \{(z_1, \dots, z_n, t) \in \mathbb{H}^n : t > 0\}$. Let us mention that $\mathbb{H}^n_+$ is not even a Lipschitz set in $\mathbb{H}^n$, see [CG]. Therefore, in the setting of Carnot-Carathéodory groups the class of NTA domains in a sense is the smallest class for which it is possible to do a reasonable analysis. It is the case, because even sets that would seem to be "nice", such as half-space, are not regular in any traditional sense.

2.10 BV functions

The following definition will largely be used in Chapters 3, 4.3, 5.4, see also the next definition of ε-approximability.

Definition 2.10.1 (Local BV functions)**.** Let Ω be an open set in $\mathbb{R}^{n+1}$. We say that an L^1_{loc}-function f has *locally bounded variation in* Ω, and denote it by $f \in BV_{loc}(\Omega)$, if for any open set $\Omega' \Subset \Omega$ the total variation of f over Ω' is finite:

$$\sup_{\Psi \in C^1_0(\Omega', \mathbb{R}^{n+1}),\, \|\Psi\|_{L^\infty} \leq 1} \int_{\Omega'} f(x)\, \mathrm{div}\Psi(x)\, \mathrm{d}x < \infty.$$

BV functions are a natural class of functions to use for our research. We require that a norm of a gradient of ε-approximation gives rise to a Carleson measure (see the definition below). Gradients of BV functions are Radon measures. Hence, the class of BV functions is exactly what is needed.

According to our best knowledge BV functions were first introduced by Jordan in [J], to deal with convergence of Fourier series. After him the notion was vastly developed and used for various applications in e.g. geometric measure theory, calculus of variations or partial differential equations.

Let us now generalize the definition of BV functions to the setting of Riemannian manifolds.

Definition 2.10.2. Let $\Omega \subset M$ be an open set and $u \in L^1(\Omega)$. We say that u has *bounded variation* in Ω and denote it by $u \in BV(\Omega)$ if

$$\sup \left\{ \int_\Omega u\, \mathrm{div}(\phi X) \,:\, X \in \Gamma(\Omega), \phi \in C^\infty_c(\Omega, \mathbb{R}), |\phi| \leq 1 \right\} < \infty,$$

where

$\Gamma(\Omega)$ is a family is of smooth vector fields on Ω such that $g(X(x), X(x)) \leq 1$ for every $x \in \Omega$, where g denotes the metric on M. The above supremum is called a *variation* of u.

If $\Omega \subset M = \mathbb{R}^n$, then we retrieve the definition of functions of bounded variation in Ω in Definition 2.10.1.

2.11 ε-Approximability

Definition 2.11.1 (ε-approximability)**.** Let $\varepsilon > 0$ and $\Omega \subset \mathbb{R}^{n+1}_+$ satisfy (1.2). We say that a function $u : \Omega \to \mathbb{R}$ is *ε-approximable*, if there exists a function $\varphi \in BV_{loc}(\Omega)$ such that

1. $\|u - \varphi\|_{L^\infty(\Omega)} < \varepsilon$,
2. $|\nabla\varphi|$ defines a Carleson measure on Ω, i.e. for every $x \in \partial\Omega$

$$\sup_{r\in(0,\operatorname{diam}\Omega)} \frac{1}{r^n} \int_{\Omega\cap B(x,r)} |\nabla\varphi(y)| \mathrm{d}\mathscr{L}^{n+1}(y) \leq C_\varepsilon. \tag{2.2}$$

However, to our best knowledge so far, it has only been used in Euclidean setting. Therefore, we give the definition in the setting of Riemannian manifolds.

Definition 2.11.2. Let $\Omega \subset M$ be a Lipschitz domain on a Riemannian manifold M. Let $u : \Omega \to \mathbb{R}$ be a harmonic function with $\|u\|_\infty \leq 1$. We will say that function u is *ε-approximable* for some $\varepsilon > 0$ if there exists a function $\phi \in BV(\Omega)$ such that

1. $\|u - \phi\|_{L^\infty(\Omega)} < \varepsilon$,
2. $|\nabla\phi|$ defines a Carleson measure on Ω, i.e. for every $x \in \partial\Omega$

$$\sup_{r\in(0,\operatorname{diam}\Omega)} \frac{1}{r^{n-1}} \int_{B(p,r)\cap\Omega} |\nabla\phi| \mathrm{d}X \leq C_\varepsilon.$$

Notice that these definitions are basically the same. The only difference is the measure with respect to which we integrate the norm of the gradient of ϕ.

It is worth noting that the notion of ε-approximability is essential in proving the results that we are interested in. Let us repeat that it turns out, as indicated by Exc. 9 Ch. VI in [G], that for a harmonic function u it may happen that $|\nabla u|$ does not give rise to a Carleson measure. What is more, one cannot assume that ϕ may be taken as another harmonic function as indicated by Exc. 12 Ch. VIII in [G].

Chapter 3

ε-Approximability and Quantitative Fatou Property for non-harmonic functions

This chapter is based on the manuscript [AGG] written jointly with Tomasz Adamowicz and Maria J. González. Recall that a Lipschitz-graph domain is in the form of

$$\Omega = \{(x, y) \in \mathbb{R}^{n+1}_{+} \,:\, y > \phi(x)\}, \tag{3.1}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is an L-Lipschitz function.

The main goal of this chapter is to prove the following result:

Theorem. 1.3.1 *Let $\Omega \subset \mathbb{R}^{n+1}_{+}$ be the Lipschitz-graph domain as in (1.2) and let further $u : \Omega \to \mathbb{R}$ be bounded and satisfy condition (#). Then for every $\varepsilon > 0$ function u is ε-approximable in Ω.*

The result generalizes the existing ones (see e.g. [G], [D1], [KKPT], [HMM1]), as it is to best of our knowledge, first ε-approximability result for functions that need not be solutions of PDEs of divergence form. Moreover, we would like to emphasize that condition (#) can be obsolete for some classes of functions and ($*$) instead suffices, as illustrated by nonnegative subharmonic functions, see Proposition 3.3.1 in Chapter 3.3. This observation follows from a brief analysis of the proofs of Theorem 1.1 and Lemmas 4.3 and 4.5 in [GKLN], which we present in Chapter 3.3.

The key consequence of Theorem 1.3.1 is the following Quantitative Fatou Theorem (see Definition 2.4.1 of the counting function).

Corollary. 1.3.2 (Quantitative Fatou Theorem) *Let $\Omega \subset \mathbb{R}^{n+1}_{+}$ be the Lipschitz-graph domain as in (3.1) and let further $u : \Omega \to \mathbb{R}$ satisfy condition (#) and be bounded with $\|u\|_{L^\infty(\Omega)} \leq 1$. Then for every point $\omega \in \partial\Omega$*

$$\sup_{0<r<r_0} \frac{1}{r^n} \int_{\partial\Omega \cap B(\omega,r)} N(r, \varepsilon, \beta)(z) d\sigma(z) \leq C(\varepsilon, \alpha, \beta, n, \Omega),$$

where $\varepsilon, \alpha, \beta$ are constants in the definition of the counting function N. In particular, constant C is a independent of u.

The proof of the corollary is a direct repetition of the proof of Lemma 2.9 in [KKPT] and, therefore, we only briefly sketch it at the end of Chapter 3.2.

Lemma (Lemma 2.9 in [KKPT]). *Suppose u is $\frac{\varepsilon}{4}$-approximable in $\Omega \subset \mathbb{R}^n$. Then*

$$\int_{\partial\Omega\cap B_r(Q)} N(r,\varepsilon,\beta)(z)\mathrm{d}\sigma(z) \leq Cr^{n-1},$$

where C depends on $\varepsilon, \alpha, \beta, n$ and the Lipschitz constant of Ω.

Let us recall, that the notion of the counting function is known in the literature, see for instance [G, KKPT, BH]. However, the definition varies depending on the authors. Nevertheless, the essential purpose of introducing the counting function always remains the same. It provides a way to estimate how much a function oscillates while approaching the boundary.

3.1 Preliminaries and notation

In this chapter we use $|\cdot|$ to denote the norm of a vector or an n-Hausdorff measure of a set, depending on the context. Symbol $\mathscr{L}^{n+1}$ denotes the $(n+1)$-Lebesgue measure.

In what follows we will use the notions of cones, see Definition 2.3.1 in Chapter 2.3, also the counting function (Def. 2.4.1), area function (Def. 2.5.1), nontangential maximal function (Def. 2.5.2). Moreover, recall the definition of a Carleson measure (Def. 2.6.1).

Next, we introduce some geometric constructions used in the proof of our main result.

Curved cubes and associated centers. Fix $\varepsilon > 0$ and denote by Q_0 the unit cube in $\mathbb{R}^n$. We denote by $\{Q^m_{j_1,\dots,j_n}\}$ the family of dyadic cubes in the dyadic decomposition of Q_0:

$$Q^m_{j_1,\dots,j_n} = \{(x_1,\dots,x_n) \in \mathbb{R}^n \,:\, j_i 2^{-m} \leq x_i \leq (j_i+1)2^{-m}\},\ \text{for } m \in \mathbb{N} \text{ and } j_1,\dots,j_n \in \{0,\dots,2^m-1\}.$$

In the case parameters m and $j_1,\dots,j_n$ are fixed or their exact values are not important for the discussion, we will write Q to denote a cube in the m-th generation for some m. For the sake of notation, in what follows we will usually denote the side length of Q by $l(Q)$ rather than 2^{-m}.

Let further

$$\hat{Q}_0 = \{(x,y) \in \mathbb{R}^{n+1} \,:\, x \in Q_0, \phi(x) \leq y \leq 1+\phi(x)\}$$

be an associated curved unit cube in $\mathbb{R}^{n+1}$, where $\phi : Q_0 \to \mathbb{R}$ is a Lipschitz function. Similarly, for a given cube Q, we define the *curved cube*

$$\hat{Q} = \{(x,y) \in \mathbb{R}^{n+1} \,:\, x \in Q, \phi(x) \leq y \leq \phi(x) + l(Q)\}.$$

In what follows we will often omit the word *curved* when discussing sets $\hat{Q}$ and instead simply write cube.

Let $x_{\hat{Q}}$ denote a center of a (curved) cube $\hat{Q}$, i.e. $x_{\hat{Q}} := (x_Q, \phi(x_Q) + 2^{-m-1})$, where x_Q is a center of Q. Note that since by (1.2) it holds that ϕ is L-Lipschitz, we have the following inclusions:

$$B\left(x_{\hat{Q}}, \frac{1}{\sqrt{1+L^2}}\frac{l(Q)}{2}\right) \subset \hat{Q} \subset \overline{B\big(x_{\hat{Q}}, C(L)l(Q)\big)}, \quad C(L) := \frac{1}{2}\sqrt{L^2+2L+2}. \tag{3.2}$$

Let us prove the above inclusions.

Proof of (3.2). We begin with the left inclusion. Without loss of generality, we may assume that $x_Q = 0$ and $\phi(0) = 0$. The bottom face of $\hat{Q}$ is given by the graph of ϕ and the top one by the graph of $\phi + l(Q)$. Since ϕ is a Lipschitz function, it holds that its graph lies in a region bounded by a cone such that its generatrix is given by a line with the slant L. It follows that we can restrict ourselves to the case when dimension is equal to 1, as we are only interested in the distance of a point $x_{\hat{Q}}$ to the surface of that cone. Therefore, we can project $(n+1)$-dimensional space to its 2-dimensional subspace, which corresponds with the case $n = 1$. And so the constant $C(L)$ does not depend on n.

To prove the inclusion, we need to know that the distance of $x_{\hat{Q}}$ to the boundary of $\hat{Q}$ is no bigger than $\frac{1}{\sqrt{1+L^2}}\frac{l(Q)}{2}$.

Obviously, the distance to the vertical sides of $\hat{Q}$ is equal to $\frac{l(Q)}{2}$. It remains to calculate the distances to the top and bottom sides.

To bound the distance to the bottom side, it suffices to calculate the distance to the cone which bounds the graph of ϕ. In the case of $n = 1$ the cone is given by two lines with equations: $L_1 : y = Lx$ and $L_2 : y = -Lx$. We directly calculate that:

$$d(x_{\hat{Q}}, L_1) = d(x_{\hat{Q}}, L_2) = d\left(\left(0, \frac{l(Q)}{2}\right), \left(\frac{L}{1+L^2}\frac{l(Q)}{2}, \frac{L^2}{1+L^2}\frac{l(Q)}{2}\right)\right) = \frac{1}{\sqrt{1+L^2}}\frac{l(Q)}{2}.$$

One can similarly calculate the distance to the top side of $\hat{Q}$. Therefore, we get $B\big(x_{\hat{Q}}, \frac{1}{\sqrt{1+L^2}}\frac{l(Q)}{2}\big) \subset \hat{Q}$.

Let us now prove the second inclusion. In order to obtain it we need to estimate the distance of $x_{\hat{Q}}$ to the boundary of $\hat{Q}$, but this time we need to estimate from above. Let us estimate it for any x such that $|x| \le \frac{l(Q)}{2}$:

$$\begin{aligned} d^2(x_{\hat{Q}}, (x, \phi(x))) &= |x|^2 + \left(\frac{l(Q)}{2} - \phi(x)\right)^2 \\ &\le \frac{l(Q)^2}{4} + \frac{l(Q)^2}{4} + L\frac{l(Q)^2}{2} + L^2\frac{l(Q)^2}{4} \\ &= \frac{l(Q)^2}{4}\left(L^2 + 2L + 2\right), \end{aligned}$$

where the inequality is a consequence of ϕ being L-Lipschitz. Thus, the proof of the second inclusion is concluded. □

Next, we define the *associated center of $\hat{Q}$* as follows:

$$x^l_{\hat{Q}} = x_{\hat{Q}} + \overline{e_{n+1}}l(Q) = (x_Q, \phi(x_Q) + 2^{-m-1} + l(Q)) = \left(x_Q, \phi(x_Q) + \frac{3}{2}2^{-m}\right). \tag{3.3}$$

The name of this point is justified by the fact that $x^l_{\hat{Q}}$ does not lie inside $\hat{Q}$, and is the center of the curved cube lying directly above cube $\hat{Q}$ and obtained by shifting up $\hat{Q}$ in $l(Q)$, see Figure 1.

We will now describe the stopping time procedure, which is one of the key underlying technique used in the proof of Theorem 1.3.1. This kind of technique is commonly used in Harmonic Analysis, see e.g. [G], [D1], [HMM1], [BH].

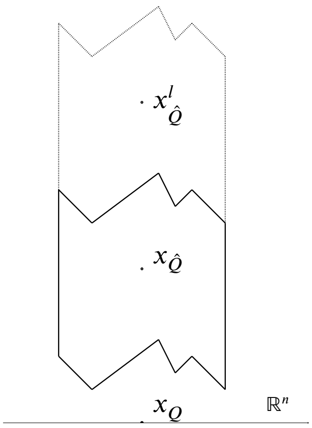

Figure 3.1: The center $x_{\hat{Q}}$ of a cube $\hat{Q}$ and its associated center $x^l_{\hat{Q}}$.

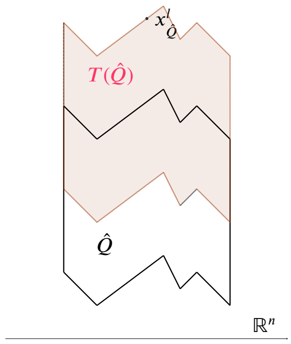

Figure 3.2: The set $T(\hat{Q})$ (brown set) with respect to the set $\hat{Q}$ (set bounded by black line).

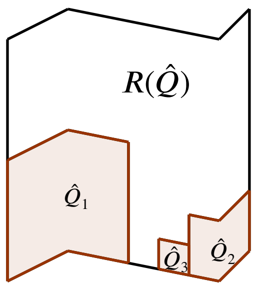

Figure 3.3: An example of how a set $R(\hat{Q})$ may look like. Cubes $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$ are removed from cube $\hat{Q}$ to obtain $R(\hat{Q})$. In general there may be infinitely many sets that are removed from $\hat{Q}$.

Stopping time conditions.
Set

$$G_0 := \{\hat{Q}_0\} \text{ and denote by}$$
$$G_1 := \text{ a family of maximal curved cubes } \hat{Q} \subset \hat{Q}_0 \text{ such that } |u(x_{\hat{Q}_0}) - u(x^l_{\hat{Q}})| > \varepsilon.$$

Next, define

$$G_2 = \bigcup_{\hat{Q} \in G_1} G_1(\hat{Q}),$$

where $G_1(\hat{Q})$ is defined the same way as G_1 with $\hat{Q}_0$ replaced with $\hat{Q}$. Then define inductively families of sets G_k, for $k = 2, \ldots$. Denote by

$$G = \bigcup_{k=0} G_k. \tag{3.4}$$

Domains $R(\hat{Q})$. Let us introduce a domain which roughly can be understood as follows: given any curved cube $\hat{Q} \in G$ consider its subset constructed by removing those maximal curved cubes $\hat{Q}_i$, where the jump of the values of u at associated centers is big: $|u(x_{\hat{Q}}) - u(x^l_{\hat{Q}_i})| > \varepsilon$, i.e. we define

$$R(\hat{Q}) := \hat{Q} \setminus \bigcup_{\hat{Q}_i \in G_1(\hat{Q})} \hat{Q}_i, \quad \text{for any } \hat{Q} \in G.$$

Thus, set $R(\hat{Q})$ consists of all curved subcubes in $\hat{Q}$ with small oscillations of u, see Figure 3. We remark that this construction is similar to the one in Garnett's book, see the proof of Theorem 6.1 in [G, Section 6 in Ch. VIII].

Notice that given two different sets $\hat{Q}, \widehat{W} \in G$, the corresponding domains $R(\hat{Q})$ and $R(\widehat{W})$ can only intersect piecewise along boundaries, but their interiors are pairwise disjoint.

Blue and red sets. Finally, we define blue and red sets, which are essential in our construction. Denote by $T(\hat{Q})$ the set $\hat{Q}$ translated vertically by $\frac{1}{2}l(Q)$:

$$T(\hat{Q}) = \{(x, y) \in \mathbb{R}^{n+1} : x \in Q, \phi(x) + \frac{1}{2}l(Q) \leq y \leq \phi(x) + \frac{3}{2}l(Q)\}. \tag{3.5}$$

The key feature of sets $T(\hat{Q})$, to which we appeal several times below, is that they are separated from the graph of the Lipschitz function ϕ, i.e. from the boundary of Ω. Moreover, an important feature of sets $T(\hat{Q})$ is that the associated center of $\hat{Q}$ is the center of an upper side of $T(\hat{Q})$, see Figure 2.

Sets $T(\hat{Q})$ are not disjoint. However, for a given set $\hat{Q}$ a set $T(\hat{Q})$ intersects only finitely many other sets of form $T(\hat{Q}_j)$. Moreover, the cardinality of a family of sets $\#\{j : T(\hat{Q}) \cap T(\hat{Q}_j) \neq \emptyset\}$ is uniformly bounded for all choices of $\hat{Q}$. When dealing with set $\hat{Q}_0$, we set $T(\hat{Q}_0) := \{(x, y) \in \mathbb{R}^{n+1} : x \in Q_0, \phi(x) + \frac{1}{2}l(Q_0) \leq y \leq \phi(x) + l(Q_0)\}$, i.e. its upper half.

Let $k > 0$. We say that $T(\hat{Q})$ is *blue*, if

$$\mathrm{osc}_{T(\hat{Q})} u \leq k\varepsilon.$$

Otherwise, we say that $T(\hat{Q})$ is *red*.

3.2 Proof of Theorem 1.3.1

Let us briefly describe our approach to the proof of the main result. First, we construct function φ_1, the first approximation of φ, see (3.6) and show in Proposition 3.2.1 that φ_1 gives rise to the Carleson measure. The proof of Proposition 3.2.1 relies on two auxiliary observations, namely Lemmas 3.2.2 and 3.2.3. The first one gives a lower bound estimate for area function and is applied in the proof of Lemma 3.2.3 to control the sum of volumes of cubes obtained by the stopping procedure. Then, we construct the function φ, see (3.18) and show that it ε-approximates function u in the L^∞-norm. In order to show condition (2.2) in Definition 2.11.1, we study the decomposition of the gradient of φ, see (3.19), and show that each of its terms leads to the Carleson condition, see estimates (Car1) and (Car2). An important auxiliary result, perhaps of the independent interest, is presented in Proposition 3.2.6 and proved in the Appendix. It gives the L^2 bounds for the area function on cubes. The above approach has been inspired by the discussion in [G, Section 6, Ch. VIII]) and also by [HMM1].

Proof of Theorem 1.3.1. First, we define an auxiliary function $\varphi_1 : \bigcup_{\hat{Q}_k \in G} R(\hat{Q}_k) \to \mathbb{R}$, which later on will be used to define the ε-approximation of u, cf. (3.18)

$$\varphi_1(z) := \sum_{j=1}^{\infty} \sum_{\hat{Q}_k \in G_j} u(x_{\hat{Q}_k}) \chi_{R(\hat{Q}_k)}(z). \tag{3.6}$$

Notice that, φ_1 is in fact defined for all $z \in \hat{Q}$ and, moreover, for any $\hat{Q} \in G$ it holds that

$$\int_{\hat{Q}} |\nabla \varphi_1| \, \mathrm{d}\mathscr{L}^{n+1} \leq \sum_{\hat{Q}_j \in G} |\hat{Q} \cap \partial R(\hat{Q}_j)|, \tag{3.7}$$

where $|\cdot|$ denotes the n-Hausdorff measure. Here, the expression $|\nabla \varphi_1|$ is understood only in the distributional sense and the component functions of $\nabla \varphi_1$ are the signed measures supported on the appropriate faces in $\partial R(\hat{Q}_j)$, see the discussion for the upper-half space in $\mathbb{R}^2$ on pg. 345 in [G, Section 6, Ch. VIII]. Therefore, $|\chi_{R(\hat{Q}_k)}|$ in (3.6) are the n-Hausdorff measures of $\hat{Q} \cap \partial R(\hat{Q}_j)$ and the above estimate is justified.

Our first step is to prove the following observation, which applied at (3.7) shows that $|\nabla \varphi_1| \, \mathrm{d}\mathscr{L}^{n+1}$ is a Carleson measure.

Proposition 3.2.1. *For any $\hat{Q}$ it holds that $\sum_{\hat{Q}_j \in G} |\hat{Q} \cap \partial R(\hat{Q}_j)| \leq C\varepsilon^{-2} l(Q)^n$.*

Proof. We may assume, without loss of the generality, that $\hat{Q} \in G$. For otherwise, we consider a family $M(\hat{Q})$ of cubes such that $\hat{Q}_1 \in M(\hat{Q})$ if $\hat{Q}_1 \subset \hat{Q}$, $\hat{Q}_1 \in G$ and $\hat{Q}_1$ is maximal. Then it suffices to prove the assertion for each of the cubes in $M(\hat{Q})$. Hence, from now on we assume that $\hat{Q} \in G$. In order to show the assertion of Proposition 3.2.1 we consider two cases depending whether $\hat{Q}_j$ is contained in $\hat{Q}$ or not and then prove two auxiliary observations in Lemmas 3.2.2 and 3.2.3.

Case 1: $\hat{Q}_j$ is such that $\hat{Q} \cap \partial R(\hat{Q}_j) \neq \emptyset$ and $\hat{Q}_j \not\subset \hat{Q}$.

(1.1) Let $l(Q_j) \leq l(Q)$. Then, it holds that $\mathrm{int}\hat{Q} \cap \mathrm{int}\hat{Q}_j = \emptyset$, but the boundaries of curved cubes $\hat{Q}$ and $\hat{Q}_j$ still intersect.

It holds that $\hat{Q} \cap \partial R(\hat{Q}_j)$ is a subset of the vertical faces of $\hat{Q}$ (throughout this chapter, by vertical faces we mean those different from the bottom and the top deck of a cube/curved cube). It is the case, since: (1) $\hat{Q}_j$ has to touch $\hat{Q}$, as otherwise $\hat{Q} \cap \partial R(\hat{Q}_j) = \emptyset$ and such a curved cube does not contribute to the sum $\sum_{\hat{Q}_j \in G} |\hat{Q} \cap \partial R(\hat{Q}_j)|$; (2) since $l(Q_j) \leq l(Q)$, only vertical sides can touch.

For different curved cubes $\hat{Q}_j$ satisfying $l(Q_j) \leq l(Q)$, the corresponding sets $\hat{Q} \cap \partial R(\hat{Q}_j)$ can intersect along a set of positive $(n-1)$-Hausdorff measure only, due to the definition of G and $R(\hat{Q}_j)$. Indeed, let $\hat{Q}_l \neq \hat{Q}_k$ be such cubes. Then we have three cases:

(a) cubes $\hat{Q}_l$ and $\hat{Q}_k$ have no common face and $\mathrm{int}\hat{Q}_l \cap \mathrm{int}\hat{Q}_k = \emptyset$ in which case the corresponding sets $\hat{Q} \cap \partial R(\hat{Q}_k)$ and $\hat{Q} \cap \partial R(\hat{Q}_l)$ can intersect along a set of positive $(n-1)$-Hausdorff measure only. See Figure 4.

(b) cubes $\hat{Q}_l$ and $\hat{Q}_k$ have a common face and $\mathrm{int}\hat{Q}_l \cap \mathrm{int}\hat{Q}_k = \emptyset$. Then sets $\hat{Q} \cap \partial R(\hat{Q}_k)$ and $\hat{Q} \cap \partial R(\hat{Q}_l)$ are subsets of a common face of $\hat{Q}$, which can only intersect along an $(n-1)$ dimensional set $\partial \hat{Q} \cap \partial \hat{Q}_k \cap \partial \hat{Q}_l$.

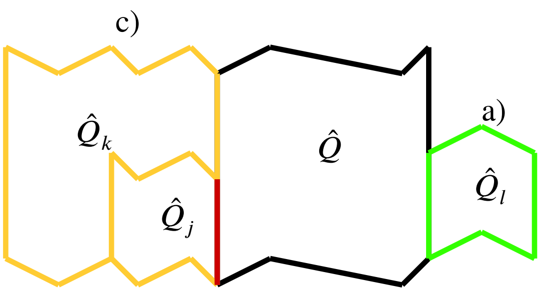

Figure 3.4: This figure illustrates a) and c) in Case 1.1 in Proposition 3.2.1. Since b) may only be observed if the dimension is greater than two, it is not shown as a figure. Red line is a set $\partial\hat{Q} \cap \partial\hat{Q}_j$. A set $\partial R(\hat{Q}_j) \cap \hat{Q}$ is a subset of a red set, whereas a set $\partial R(\hat{Q}_k) \cap \hat{Q}$ is contained in a yellow line above a red one. Therefore these sets may only intersect along a set of dimension $n-1$.

(c) interiors of cubes $\hat{Q}_j$ and $\hat{Q}_k$ intersect, but this means that one of the cubes contains another, for instance let $\hat{Q}_j \subset \hat{Q}_k$. However, then $\hat{Q}_j \cap R(\hat{Q}_k) = \emptyset$ and so the conclusion is as in case (a) above.

Therefore, all such $\hat{Q}_j$ amount to at most $C(n)l(Q)^n$ in $\sum_{\hat{Q}_j \in G} |\hat{Q} \cap \partial R(\hat{Q}_j)|$, as they cover at most all vertical faces of $\hat{Q}$.

(1.2) Let $l(Q_j) > l(Q)$.

Then, there are at most $C(n)$ of such cubes $\hat{Q}_j$. In order to see that this holds, let us consider two cases. If $\hat{Q} \not\subset \hat{Q}_j$, then there cannot be more of such $\hat{Q}_j$ than faces of $\hat{Q}$. This is a consequence of the following observations: (1) $\hat{Q} \cap \partial R(\hat{Q}_j) \neq \emptyset$ by assumptions, and so $\hat{Q}$ and $\hat{Q}_j$ have to touch; (2) since $\hat{Q}_j \in G$ and $l(Q_j) > l(Q)$, then for each face F of $\hat{Q}$ there is at most one curved cube in G such that it touches F with the face of side length bigger than $l(Q)$ and, moreover, $\hat{Q} \cap \partial R(\hat{Q}_j) \neq \emptyset$ (see also Figure 5).

Let now $\hat{Q} \subset \hat{Q}_j$, then there exists exactly one cube in family G such that $\hat{Q} \cap \partial R(\hat{Q}_j) \neq \emptyset$. To prove it, note that for any bigger cube $\hat{Q}_k \in G$ with $\hat{Q} \subset \hat{Q}_j \subset \hat{Q}_k$ it holds that $\hat{Q} \cap \partial R(\hat{Q}_k) = \emptyset$, as for such $\hat{Q}_k$, the cube $\hat{Q}_j$ is not contained in $R(\hat{Q}_k)$, as it had to be removed in the construction of $R(\hat{Q}_k)$. Therefore, there is only one cube such that $\hat{Q} \subset \hat{Q}_j$ and $\hat{Q} \cap \partial R(\hat{Q}_j) \neq \emptyset$.

Thus, similarly to case (1.1), such cubes contribute at most $C(n)l(Q)^n$ to the sum $\sum_{\hat{Q}_j \in G} |\hat{Q} \cap \partial R(\hat{Q}_j)|$.

In summary, the discussion in cases (1.1) and (1.2) gives that

$$\sum_{\hat{Q}_j \in G, \hat{Q}_j \not\subset \hat{Q}} |\hat{Q} \cap \partial R(\hat{Q}_j)| \leq C(n)l(Q)^n. \tag{3.8}$$

CASE 2: $\hat{Q}_j$ is such that $\hat{Q} \cap \partial R(\hat{Q}_j) \neq \emptyset$ and $\hat{Q}_j \subset \hat{Q}$.
Then, trivially we have that

$$\sum_{\hat{Q}_j \in G, \hat{Q}_j \subset \hat{Q}} |\hat{Q} \cap \partial R(\hat{Q}_j)| \leq C(n) \sum_{\hat{Q}_j \in G, \hat{Q}_j \subset \hat{Q}} l(Q_j)^n. \tag{3.9}$$

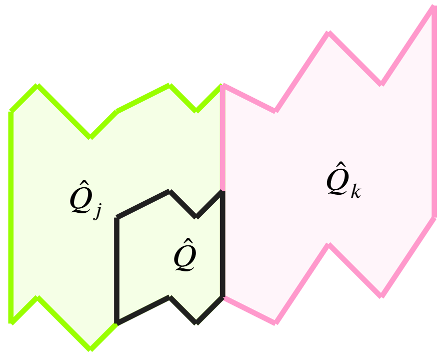

Figure 3.5: This figure shows Case 1.2 in Proposition 3.2.1. The purple cube refers to the case $\hat{Q} \not\subset \hat{Q}_j$ and the green one refers to the case $\hat{Q} \subset \hat{Q}_j$.

To continue the proof of Proposition 3.2.1 let us prove the following observation.

Lemma 3.2.2. *Let $\hat{Q} \in G$, where G is as in (3.4). It holds that*

$$\sum_{\hat{Q}_j \in G_1(\hat{Q})} l(Q_j)^n \leq C\varepsilon^{-2} \int_{\widetilde{R(\hat{Q})}} |\nabla u(x,y)|^2 (y - \phi(x))\,\mathrm{d}x\mathrm{d}y,$$

where $C = C(n, L, \theta, \eta)$ and the set $\widetilde{R(\hat{Q})}$ is defined as follows:

$$\widetilde{R(\hat{Q})} := \bigcup_{\hat{Q}_j \in G_1(\hat{Q})} \widetilde{\hat{Q}_j} \quad \text{where} \quad \widetilde{\hat{Q}_j} := \begin{cases} T(\hat{Q}_j), & \text{if } T(\hat{Q}_j) \text{ is red} \\ \bigcup_{X \in U\hat{Q}_j} \Gamma_{\alpha,0,\frac{1}{2}l(Q_j)}(X), & \text{if } T(\hat{Q}_j) \text{ is blue.} \end{cases} \tag{3.10}$$

By $U\hat{Q}_j$, we denote the upper deck of $\hat{Q}_j$. (We refer to the discussion in the proof below, see (3.15), where the set $\widetilde{R(\hat{Q})}$ is constructed and its meaning explained).

Proof. Let $\hat{Q}_j \in G_1(\hat{Q})$.

Case 1: The translated curved cube $T(\hat{Q}_j)$ is red (cf. (3.5) for the definition of $T(\hat{Q}_j)$). Then, it follows by (1.3) and (3.2) that

$$k^2\varepsilon^2 \leq (\operatorname{osc}_{T(\hat{Q}_j)} u)^2 \lesssim_{n,L,\theta,\eta} l(Q_j)^{1-n} \int_{T(\hat{Q}_j)} |\nabla u|^2,$$

for some $k > 0$ whose exact value will be determined later in this proof. Hence, since $T(\hat{Q}_j) \cap \partial\Omega = \emptyset$ we have that $y - \phi(x) \approx_{n,L} l(Q_j)$ for all $(x,y) \in T(\hat{Q}_j)$. Thus, we get

$$k^2 l(Q_j)^n \lesssim_{n,L,\theta,\eta} \varepsilon^{-2} \int_{T(\hat{Q}_j)} |\nabla u(x,y)|^2 (y - \phi(x))\,\mathrm{d}x\mathrm{d}y. \tag{3.11}$$

Case 2: Set $T(\hat{Q}_j)$ is blue.

Since $\hat{Q}_j \in G_1(\hat{Q})$, we know that $|u(x^l_{\hat{Q}_j}) - u(x_{\hat{Q}})| > \varepsilon$. Next, let us define the point

$$x^{\frac{1}{2}l}_{\hat{Q}_j} := x^{\hat{Q}_j} + \frac{1}{2}l(Q_j)\overline{e_{n+1}},$$

which has the same x coordinate as the center of the curved cube $x_{\hat{Q}_j}$ but its y coordinate equals $\phi(x)+l(Q_j)$. Thus, one can think that such point is a vertical projection of the center of the cube $\hat{Q}_j$ on the upper deck of $\hat{Q}_j$, denoted by $U\hat{Q}_j$. However, notice that $x^{\frac{1}{2}l}_{\hat{Q}_j}$ does not lie in the boundary $\partial\Omega$ while we would like to consider a cone with the vertex at that point. Therefore, we let $\Omega_j = \Omega + \overline{e_{n+1}}l(Q_j)$ be a subdomain of Ω obtained by shifting Ω vertically up by $l(Q_j)$. Now $x^{\frac{1}{2}l}_{\hat{Q}_j} \in \partial\Omega_j$.

Therefore, we have

$$N_{\alpha,0,\frac{1}{2}l(Q_j)}(u - u(x_{\hat{Q}}))(x^{\frac{1}{2}l}_{\hat{Q}_j}) > \varepsilon, \tag{3.12}$$

where the (truncated) nontangential maximal function N is considered with respect to domain Ω_j.

We now show that estimate (3.12) holds not only at $x^{\frac{1}{2}l}_{\hat{Q}_j}$, the center of the upper deck of $\hat{Q}_j$, but in fact at its all points X, i.e.

$$N_{\alpha,0,\frac{1}{2}l(Q_j)}(u - u(P_{\hat{Q}}))(X) \gtrsim \varepsilon.$$

Let us consider vertical shifts of points $X \in U\hat{Q}_j$ so that they belong to $T(\hat{Q}_j)\backslash\hat{Q}_j$, e.g. $X+\frac{1}{4}\overline{e_{n+1}}l(Q_j)$ and notice that they satisfy

$$X + \tfrac{1}{4}\overline{e_{n+1}}l(Q_j) \in \Gamma_\alpha(X) \quad \text{and} \quad X + \tfrac{1}{4}\overline{e_{n+1}}l(Q_j) \in T(\hat{Q}_j).$$

As a consequence we get, by the triangle inequality and since $T(\hat{Q}_j)$ is blue, that

$$\begin{aligned}\varepsilon < |u(x^l_{\hat{Q}}) - u(x_{\hat{Q}})| &\le |u(x^l_{\hat{Q}}) - u(X + \tfrac{1}{4}\overline{e_{n+1}}l(Q_j))| + |u(X + \tfrac{1}{4}\overline{e_{n+1}}l(Q_j)) - u(x_{\hat{Q}})| \\ &\le k\varepsilon + |u(X + \tfrac{1}{4}\overline{e_{n+1}}l(Q_j)) - u(x_{\hat{Q}})|\end{aligned}$$

and hence

$$|u(X + \tfrac{1}{4}\overline{e_{n+1}}l(Q_j)) - u(x_{\hat{Q}})| > (1-k)\varepsilon. \tag{3.13}$$

Therefore, for every $X \in U\hat{Q}_j$, we obtain the following estimate

$$(1-k)\varepsilon \le N_{\alpha,0,\frac{1}{2}l(Q_j)}(u - u(x_{\hat{Q}}))(X).$$

Hence, for any $X \in U\hat{Q}_j$

$$\begin{aligned}(1-k)^2\varepsilon^2 l(Q_j)^n &\lesssim_{n,L} (N_{\alpha,0,\frac{1}{2}l(Q_j)}(u - u(x_{\hat{Q}})))^2(X) \int_{U\hat{Q}_j} \mathrm{d}\mathscr{H}^n \\ &\le \int_{U\hat{Q}_j} (N_{\alpha,0,\frac{1}{2}l(Q_j)}(u - u(x_{\hat{Q}})))^2(X)\,\mathrm{d}\mathscr{H}^n \\ &\lesssim \int_{U\hat{Q}_j}\int_{\Gamma_{\alpha,0,1/2l(Q_j)}(X)} |\nabla u|^2(y - \phi(x) - l(Q_j))^{1-n}\,\mathrm{d}x\mathrm{d}y && (N \lesssim A)\end{aligned}$$

$$\lesssim \int_{\bigcup_{X\in U\hat{Q}_j} \Gamma_{\alpha,0,1/2l(Q_j)}(X)} |\nabla u|^2(y-\phi(x)-l(Q_j))\,\mathrm{d}x\mathrm{d}y \qquad \text{(Fubini's Theorem)}$$

$$\leq \int_{\bigcup_{X\in U\hat{Q}_j} \Gamma_{\alpha,0,1/2l(Q_j)}(X)} |\nabla u|^2(y-\phi(x))\,\mathrm{d}x\mathrm{d}y, \tag{3.14}$$

where the third ($N \lesssim A$) inequality follows by the fact that $U\hat{Q}_j \subset \partial\Omega_j$ and by applying the local version of Theorem 1.1 (b) for $p=2$ in [GKLN] allowing us the consider the truncated versions of the N_α and the A_α functions, see the comment following the statement of Theorem 1.2 in [GKLN].

The inequalities ($N \lesssim A$) and ($A \lesssim N$) refer to inequalities between L^p norms of nontangential maximal function and area function. They were pioneered by Fefferman and Stein in [FS] for half-space. Dahlberg proved similar results in [D2] for Lipschitz domains. One can find such results also in [KKPT] for the L-harmonic functions. The result that we use comes from [GKLN] and we now recall it, specialized to our setting of Condition ($*$) and $\varphi(t) := ct$, cf. page 194 in [GKLN].

Theorem (1.1 [GKLN])**.** *Let u be a C^2 function which satisfies ($*$) in $\mathbb{R}^{n+1}_+$. Fix $0<\alpha<\beta$ and assume there exists $x_0 \in \mathbb{R}^n$ such that $(N_{\alpha,0,\infty}u)(x_0) < \infty$. Then,*

(a) For a.e. $x \in \{x \in \mathbb{R}^n : (S_\beta u)(x) < \infty\}$, the function $u(w,y)$ has finite limit when $(w,y) \in \Gamma_\alpha(x)$ tends to x.

(b) Assume that $\lim u(x,y) = 0$ as $\|(x,y)\| \to \infty$. For $0<p<\infty$, there exists a constant C depending on p,α,β,n such that

$$\|N_\alpha u\|_{L^p(\mathbb{R}^n)} \leq C\|S_\beta u\|_{L^p(\mathbb{R}^n)}.$$

We continue the proof of Case (2) in Lemma 3.2.2 and notice that the set $\bigcup_{X\in U\hat{Q}_j} \Gamma_{\alpha,0,\frac{1}{2}l(Q_j)}(X)$ consists of the upper-half of $T(\hat{Q}_j)$ and additional parts belonging to neighbouring curved cubes.

However, those parts may only be contained in cubes in the same generation (in the dyadic decomposition), say generation m, as $T(\hat{Q}_j)$ or in a previous generation $m-1$ and intersect only finitely many of such cubes whose number is estimated by a constant $C(n,\alpha)$, see Figure 3.8. To be more specific, notice that the distance of a point in $\Gamma_{\alpha,0,\frac{1}{2}l(Q_j)}(X)$ to the axis of the cone can be at most $\frac{1}{2}\alpha l(Q_j)$. Hence, for cubes in the same generation as $T(\hat{Q}_j)$, in each direction such a cone can only intersect at most $\lceil\frac{\alpha}{2}\rceil$ other cubes. For cubes in the previous generation we have the same estimate as there are fewer cubes in the previous generation. Therefore, for every direction there are at most $2\lceil\frac{\alpha}{2}\rceil$ cubes that a cone can intersect. Moreover, as faces of $\hat{Q}_j$ are n-dimensional, a cone can overlap with up to $\omega_n(2\lceil\frac{\alpha}{2}\rceil)^n$ other cubes, where ω_n stands for the measure of n-dimensional unit ball. Therefore, upon adding up in (3.14) over all cubes $\hat{Q}_j \in G_1(\hat{Q})$, we increase the constant on the right-hand side only by a factor of $C(n,\alpha)+1$. Thus, also the discussion of case 2 is completed.

In order to estimate the sum in the assertion of the lemma we now combine cases 1 and 2. For this, we also need to analyze how a red set $T(\hat{Q}_j)$ may intersect other red sets. Notice that the case of cubes in the same generation as a red $T(\hat{Q}_j)$ is already taken care of above. However, it may happen that $T(\hat{Q}_j)$ intersects with sets that belong to one generation below the one of $T(\hat{Q}_j)$ or one above, i.e. to

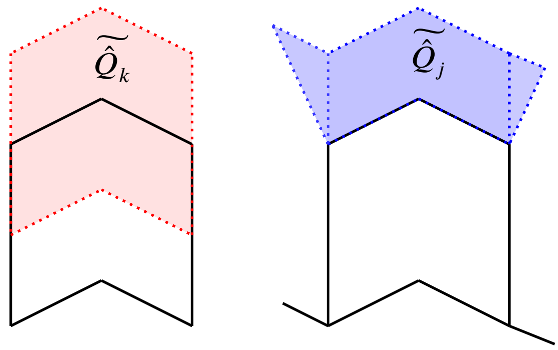

Figure 3.6: This figure shows how sets $\widetilde{\hat{Q}_k}$ and $\widetilde{\hat{Q}_j}$ look like for $T(\hat{Q}_k)$ red and $T(\hat{Q}_j)$ blue, respectively. Notice that for a blue $T(\hat{Q}_j)$ we drew a bit more of a graph of ϕ as a blue set is a union of truncated cones and the way in which the cone is truncated depends on ϕ.

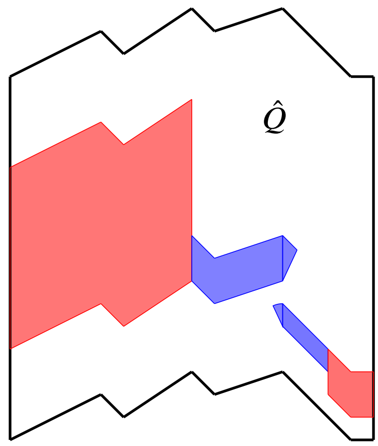

Figure 3.7: This figure shows how a domain $\widetilde{R(\hat{Q})}$ is constructed. It is a union of red and blue sets of the form $\widetilde{\hat{Q}_k}$.

$(m+1)$-th or $(m-1)$-th generation for $T(\hat{Q}_j)$ belonging to the m-th generation, for some m, see Figure 3.8. However, since the number of such cubes is finite, $T(\hat{Q}_j)$ can only intersect $C(n)$ of such cubes.

Finally, we combine estimates (3.11) and (3.14) to arrive at the assertion of Lemma 3.2.2:

$$\sum_{\hat{Q}_j \in G_1(\hat{Q})} l(Q_j)^n \leq C\varepsilon^{-2} \int_{\widetilde{R(\hat{Q})}} |\nabla u(y)|^2 (y - \phi(x)),$$

where $\widetilde{R(\hat{Q})} := \bigcup_{\hat{Q}_j \in G_1(\hat{Q})} \widetilde{\hat{Q}_j}$ with

$$\widetilde{\hat{Q}_j} = \begin{cases} T(\hat{Q}_j), & \text{if } T(\hat{Q}_j) \text{ is red} \\ \bigcup_{X \in U\hat{Q}_j} \Gamma_{\alpha,0,\frac{1}{2}l(Q_j)}(X), & \text{if } T(\hat{Q}_j) \text{ is blue.} \end{cases} \tag{3.15}$$

See Figures 3.6 and 3.7 illustrating the construction of the set $\widetilde{R(\hat{Q})}$.

Notice, that by (3.11) and (3.14), the assertion of the lemma holds with C depending on $\max\{k^{-2}, (1-k)^{-2}\}$ and, thus taking into account also (3.13), any $0 < k < 1$ is suitable. $\square$

Lemma 3.2.2 implies the following observation.

Lemma 3.2.3. *Let $\hat{Q} \in G$. Then, $\sum_{\hat{Q}_j \in G, \hat{Q}_j \subset \hat{Q}} l(Q_j)^n \leq C\varepsilon^{-2} l(Q)^n$.*

Before we prove the lemma, let us recall the following notion of shadow of a point and show the claim needed to complete the proof of Lemma 3.2.3.

Definition 3.2.4. Let $\omega \in \mathbb{R}^n$ and $z \in \Omega$. The *shadow of* z, denoted by $S(z) := S_{\alpha,s,t}(z)$, is a subset of $\partial\Omega$, defined in the following way:

$$(\omega, \phi(\omega)) \in S_{\alpha,s,t}(z) \Leftrightarrow z \in \Gamma_{\alpha,s,t}(\omega).$$

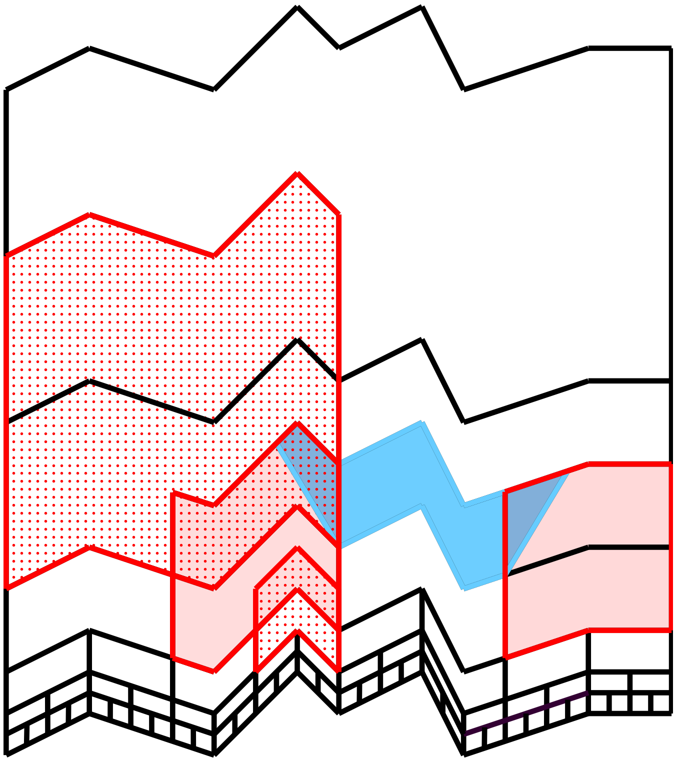

Figure 3.8: This figure depicts the blue set $\bigcup_{X \in U\hat{Q}_j} \Gamma_{\alpha,0,\frac{1}{2}l(Q_j)}(X)$ and how it can intersect sets of the form $T(\hat{Q}_j)$ which are drawn with red color. Additionally, on the left side of the figure dotted red sets indicate how different red sets can intersect each other.

Claim 3.2.5. *Let $z = (x, y) \in C(n, \alpha)\hat{Q}$. Then*

$$B\Big(\big(x, \phi(x)\big), \frac{\alpha}{1 + L\alpha}\big(y - \phi(x)\big)\Big) \cap \partial\Omega \subset S_{\alpha,0,C(n,\alpha)l(Q)}(z).$$

Proof. First, we may assume that $z = (0, t)$ and $\phi(0) = 0$. As we did in the proof of (3.2), we can restrict ourselves to the case when $n = 1$. First, we find $\eta \in \mathbb{R}^n$ such that $(\eta, \phi(\eta)) \in S_{\alpha,0,C(n,\alpha)l(Q)}(z)$. One of the sides of the cone $\Gamma_{\alpha,0,C(n,\alpha)l(Q)}(0)$ is given by the equation $y = \frac{1}{\alpha}x$. Suppose that $\eta > 0$. Then, one of the sides of the cone $\Gamma_{\alpha,o,C(n,\alpha)l(Q)}(\eta)$ is given by $y = -\frac{1}{\alpha}x + \frac{1}{\alpha}\eta + \phi(\eta)$.

By Definition 3.2.4, point $(\eta, \phi(\eta)) \in S_{\alpha,0,C(n,\alpha)l(Q)}(z)$ if $z \in \Gamma_{\alpha,o,C(n,\alpha)l(Q)}(\eta)$, which happens if $t > \frac{1}{\alpha}\eta + \phi(\eta)$. Therefore, we have

$$\frac{1}{\alpha}\eta + \phi(\eta) < t < C(n, \alpha)l(Q), \text{ and so} \quad \eta < -\alpha\phi(\eta) + \alpha C(n, \alpha)l(Q).$$

By the Lipschitzness of ϕ and since $\phi(0) = 0$, we get

$$-L\alpha\eta < -\alpha\phi(\eta) < L\alpha\eta.$$

Hence, we obtain

$$\eta < -L\alpha\eta + \alpha C(n,\alpha)l(Q), \text{ and hence } \quad \eta < \frac{\alpha}{1+L\alpha}C(n,\alpha)l(Q).$$

Therefore for points $(\eta, \phi(\eta))$ with $|\eta| < \frac{\alpha}{1+L\alpha}C(n,\alpha)l(Q)$ it holds that $(\eta, \phi(\eta)) \in S_{\alpha,0,C(n,\alpha)l(Q)}(z)$.

Notice that by the definition of a cone we have $t < C(n,\alpha)l(Q)$.

It follows that for $\eta < \frac{\alpha}{1+L\alpha}t$, it holds that $\eta \in S_{\alpha,0,C(n,\alpha)l(Q)}(z)$.

Therefore,

$$B\left((0,0), \frac{\alpha}{1+L\alpha}t\right) \cap \partial\Omega \subset S_{\alpha,0,C(n,\alpha)l(Q)}(z).$$

If we now let $z = (x, y)$ we obtain

$$B\left((x,\phi(x)), \frac{\alpha}{1+L\alpha}(y-\phi(x))\right) \cap \partial\Omega \subset S_{\alpha,0,C(n,\alpha)l(Q)}(z),$$

which concludes the proof of the claim. □

Proof of Lemma 3.2.3. It holds that

$$\begin{aligned}
\sum_{\hat{Q}_j \in G, \hat{Q}_j \subset \hat{Q}} l(Q_j)^n &= \sum_{k\geq 0} \sum_{\hat{Q}_j \in G_k(\hat{Q})} l(Q_j)^n \\
&= l(Q)^n + \sum_{k\geq 1} \sum_{\hat{Q}_j \in G_k(\hat{Q})} l(Q_j)^n \\
&= l(Q)^n + \sum_{k\geq 1} \sum_{\hat{Q}' \in G_{k-1}(\hat{Q})} \sum_{\hat{Q}_j \in G_1(\hat{Q}')} l(Q_j)^n \\
&\lesssim_{n,L,\theta,\eta} l(Q)^n + \varepsilon^{-2} \sum_{k\geq 1} \sum_{\hat{Q}' \in G_{k-1}(\hat{Q})} \int_{\widetilde{R(\hat{Q}')}} |\nabla u(x,y)|^2 (y-\phi(x))\,\mathrm{d}x\mathrm{d}y \\
&\qquad \text{(Lemma 3.2.2)} \\
&\lesssim_{n,L,\theta,\eta} l(Q)^n + \varepsilon^{-2} \int_{C(n,\alpha)\hat{Q}} |\nabla u(x,y)|^2 (y-\phi(x))\,\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where the second inequality follows, by the discussion similar to the one at the end of the proof of Lemma 3.2.2, from the fact that any cube may be counted at most finitely many times with the uniform constant depending on n and α. However, since sets $\widetilde{R(\hat{Q}')}$ may contain also unions of cones, we may need to consider a cube bigger than $\hat{Q}$ so that $\bigcup \widetilde{R(\hat{Q}')} \subset C(n,\alpha)\hat{Q}$. We enlarge the cube because the union of cones need not be contained in $\hat{Q}$. The proof of Lemma 3.2.3 will be completed once we show that

$$\int_{C(n,\alpha)\hat{Q}} |\nabla u(x,y)|^2 (y-\phi(x))\,\mathrm{d}x\mathrm{d}y \lesssim_{n,L,\theta,\eta} l(Q)^n. \tag{3.16}$$

In order to prove this estimate, notice that for $z = (x,y) \in C(n,\alpha)\hat{Q}$ it holds that $y - \phi(x) \lesssim_{n,\alpha} d(z, \partial\Omega)$.

Then Claim 3.2.5 together with the Fubini theorem allow us to obtain the following estimate

$$\begin{aligned}
&\int_{C(n,\alpha)\hat{Q}} |\nabla u(x,y)|^2 (y-\phi(x))\,\mathrm{d}x\mathrm{d}y \\
&\approx \int_{C(n,\alpha)\hat{Q}} |\nabla u(x,y)|^2 (y-\phi(x))^{1-n} (y-\phi(x))^n\,\mathrm{d}x\mathrm{d}y \qquad (3.17)\\
&\approx_{n,L,\alpha} \int_{C(n,\alpha)\hat{Q}} |\nabla u(x,y)|^2 (y-\phi(x))^{1-n} \left(\int_{\partial\Omega} \chi_{B((x,\phi(x)),\frac{\alpha}{1+L\alpha}(y-\phi(x))\cap\partial\Omega}\mathrm{d}\sigma \right) \mathrm{d}x\mathrm{d}y \\
&\lesssim_{n,L,\alpha} \int_{C(n,\alpha)\hat{Q}} |\nabla u(x,y)|^2 (y-\phi(x))^{1-n} \left(\int_{\partial\Omega} \chi_{S_{\alpha,0,C(n,\alpha)l(Q)}(z)}\mathrm{d}\sigma \right) \mathrm{d}x\mathrm{d}y \qquad \text{(by Claim 3.2.5)}\\
&= \int_{C(n,\alpha)\hat{Q}} \int_{\partial\Omega} |\nabla u(x,y)|^2 (y-\phi(x))^{1-n} \chi_{S_{\alpha,0,C(n,\alpha)l(Q)}(z)}\mathrm{d}\sigma\mathrm{d}x\mathrm{d}y \\
&= \int_{\partial\Omega} \left(\int_{C(n,\alpha)\hat{Q}} \nabla u(x,y)|^2 (y-\phi(x))^{1-n} \chi_{\Gamma_{\alpha,0,C(n,\alpha)l(Q)}(x)}\mathrm{d}x\mathrm{d}y \right) \mathrm{d}\sigma \qquad \text{(Fubini's theorem)}\\
&\approx_{n,L,\alpha} \int_Q \left(A_{\alpha,0,C(n,\alpha)l(Q)} u \right)^2 (x)\,\mathrm{d}x \lesssim (C(n,\alpha)l(Q))^n.
\end{aligned}$$

The last inequality follows from the following observation, whose proof we present in the appendix.

Proposition 3.2.6. *Let $\Omega \subset \mathbb{R}^{n+1}_+$ be the Lipschitz-graph domain as in (1.2) and let further $u : \Omega \to \mathbb{R}$ be bounded and satisfy condition (#). Then for any dyadic cube $Q \subset \mathbb{R}^n$ it holds that*

$$\int_Q \left(A_{\alpha,0,l(Q)} u \right)^2 (x)\,\mathrm{d}x < c(l(Q))^n,$$

where the constant c depends only on α, θ as in (#), n and the Lipschitz constant L of ϕ.

Therefore, the inequality (3.16) is proven and, hence the proof of Lemma 3.2.3 is completed. □

Upon combining the discussion in (3.8) and (3.9) together with Lemma 3.2.2 we complete the proof of Proposition 3.2.1. □

CONTINUATION OF THE PROOF OF THEOREM 1.3.1:

Recall, that as already mentioned in the discussion following (3.6) and (3.7), Proposition 3.2.1 shows that $|\nabla\varphi_1|\mathrm{d}\mathscr{L}^{n+1}$ is a Carleson measure in Ω. Let us now define the following function $\varphi : \Omega \to \mathbb{R}$:

$$\varphi(z) = \begin{cases} u(z) & \text{if } z \text{ belongs to any red } T(\hat{Q}), \\ \varphi_1(z) & \text{otherwise.} \end{cases} \qquad (3.18)$$

Our goal is to prove that φ is an ε-approximation of u as in Definition 2.11.1. Denote by

RED the union of all red sets $T(\hat{Q})$ and by BLUE the union of all blue sets $T(\hat{Q})$, for $\hat{Q} \subset \hat{Q}_0$.

If $z \in$ RED, then $u(z) - \varphi(z) = 0$, whereas if $z \in$ BLUE, then $z \in R(\hat{Q})$ for some $\hat{Q} \in G$. Suppose that $z \in T(\hat{Q}_1)$. Since, by the definition (3.5), the set $T(\hat{Q}_1)$ is a vertical translation of cube $\hat{Q}_1$, its upper half may be a subset of one R-set (i.e. $R(\hat{Q})$ for some $\hat{Q} \in G$), while its lower half may lie in another R-set. Moreover, it can also happen that $T(\hat{Q}_1)$ is entirely contained in one R-set. This discussion leads to the following two cases.

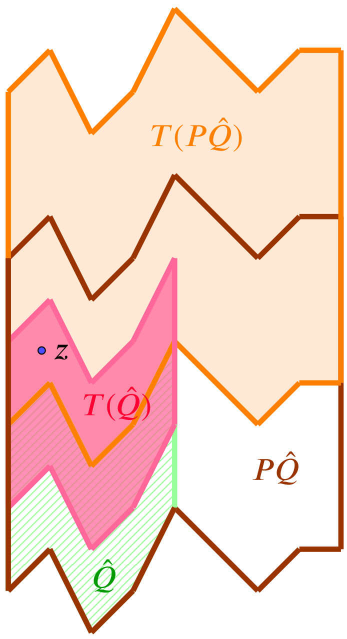

Figure 3.9: This figure shows the situation when z belongs to the upper half of $T(\hat{Q}_1)$. A green cube is a cube $\hat{Q}_1$, the cube bounded by a brown line is a parent of $\hat{Q}_1$, i.e. $P\hat{Q}_1$.

If $T(\hat{Q}_1) \subset R(\hat{Q})$, then

$$\begin{aligned}|u(z) - \varphi(z)| &= |u(z) - \varphi_1(z)| \\ &= |u(z) - u(x_{\hat{Q}_1})\chi_{R(\hat{Q})}(z)| \leq |u(z) - u(x^l_{\hat{Q}_1})| + |u(x^l_{\hat{Q}_1}) - u(x_{\hat{Q}})| \leq k\varepsilon + \varepsilon,\end{aligned}$$

where the first $k\varepsilon$ comes from the fact that $T(\hat{Q}_1)$ is blue and the second ε is obtained because $\hat{Q}_1$ is not entirely removed from $R(\hat{Q})$.

If $T(\hat{Q}_1) \not\subset R(\hat{Q})$, then suppose first that z belongs to the lower half of $T(\hat{Q}_1)$, i.e. $z \in T(\hat{Q}_1) \cap \hat{Q}_1$. Then $z \in R(\hat{Q}_1)$ and since $T(\hat{Q}_1)$ is blue, we have that $|u(z) - \varphi(z)| \leq k\varepsilon$.

If z belongs to the upper half of $T(\hat{Q}_1)$, i.e. $z \in T(\hat{Q}_1) \setminus \hat{Q}_1$, then z lies in $T(P\hat{Q}_1)$, where $P\hat{Q}_1$ denotes the *parent* of $\hat{Q}_1$, meaning the smallest cube $\hat{Q}$ containing $\hat{Q}_1$. Moreover, $z \in T(P\hat{Q}_1) \cap P\hat{Q}_1$. If $T(P\hat{Q}_1)$ is blue, then $|u(z) - \varphi(z)| \leq k\varepsilon$ in a similar manner as before. Otherwise, if $T(P\hat{Q}_1)$ is red, then this case has already been taken care of above (see also Figure 3.9).

This discussion shows that $\|u - \varphi\|_\infty \leq (k+1)\varepsilon$.

Notice that

$$\nabla\varphi = (\nabla\varphi_1)\chi_{\hat{Q}_0 \setminus RED} + (\nabla u)\chi_{RED} + J, \tag{3.19}$$

where J denotes jumps along boundaries of RED. We already proved that $|\nabla\varphi_1| d\mathscr{L}^{n+1}$ is a Carleson measure. Therefore, by the definition of φ in (3.18), it remains to prove that on the set RED, the measure $|\nabla\varphi| d\mathscr{L}^{n+1} = |\nabla u| d\mathscr{L}^{n+1}$ is a Carleson measure and that also J gives a Carleson measure. Since $\|u\|_\infty \leq 1$ and $\|u - \varphi\|_\infty \lesssim_k \varepsilon$ it follows that

$$|J| \lesssim (1+\varepsilon) \sum_{\text{red } T(\hat{Q}_j)} l(Q_j)^n.$$

From those it will follow that $|\nabla\varphi|$ defines a Carleson measure.

Our goal amounts to proving the two inequalities (Car1) and (Car2). The first one allows us to handle the ∇u term in (3.19), while (Car2) takes care of the J part.

$$\sum_{T(\hat{Q}_j)\text{ red},\,\hat{Q}_j\subset\hat{Q}} \int_{T(\hat{Q}_j)} |\nabla u| \lesssim_{n,L,\eta} \frac{1}{\varepsilon} l(Q)^n, \tag{Car1}$$

$$\sum_{T(\hat{Q}_j)\text{ red},\,\hat{Q}_j\subset\hat{Q}} l(Q_j)^n \lesssim_{n,L,\eta} \frac{1}{\varepsilon^2} l(Q)^n. \tag{Car2}$$

Let us begin with proving (Car2). Recall that we say that a ball B is a hyperbolic ball if its radius is comparable to its distance to the boundary $\partial\Omega$ and $B\subset\Omega$. Let us choose a finite cover by hyperbolic balls centered at points of any given cube $T(\hat{Q}_j)$. Since we are interested now in red cubes, we have by (1.3) that for any such ball B_r of radius r from the covering of $T(\hat{Q}_j)$ it holds

$$\varepsilon^2 \le (\operatorname{osc}_{B_r} u)^2 \lesssim r^{1-n} \int_{(1+\eta)B_r} |\nabla u|^2, \quad \text{for some fixed } \eta\in[0,1). \tag{3.20}$$

Notice that for any point $z\in T(\hat{Q}_j)$ it holds that the distance $\delta_j(z)$ of z to the bottom face of $\hat{Q}_j$, satisfies $\frac{1}{2}l(Q_j)\le\delta_j(z)\le l(Q_j)$. Moreover, for any hyperbolic ball B_r containing z it holds, by the definition of a hyperbolic ball, that $r\le\delta_j(z)\le Cr$, for some fixed numerical constant $C>0$. Thus, we have that

$$r\approx\delta_j(z)\approx l(Q_j).$$

Let us fix one ball B_r from the covering of $T(\hat{Q}_j)$, centered at the point $x_{\hat{Q}_j}+\overline{e_{n+1}}\frac{1}{2}l(Q_j)$ a center of $T(\hat{Q}_j)$ and such that $(1+\eta)B_r \Subset T(\hat{Q}_j)$. Such a ball can be obtained by similar reasoning as in (3.2) and hence, there is a constant $C(n,L)>0$ such that $r:=\frac{l(Q_j)}{(1+\eta)C(n,L)}$ suffices.

Therefore, upon multiplying inequality (3.20) by $l(Q_j)^n$, we get

$$l(Q_j)^n \le \frac{1}{\varepsilon^2}\int_{T(\hat{Q}_j)} |\nabla u|^2\delta_j(z), \tag{3.21}$$

as the above constructed ball B_r satisfies $(1+\eta)B_r\subset T(\hat{Q}_j)$. From this and the Hölder inequality together with (3.2) we infer the following estimate

$$\left(\int_{T(\hat{Q}_j)}|\nabla u|\right)^2 \lesssim_{n,L} \left(\int_{T(\hat{Q}_j)}|\nabla u|^2\right) l(Q_j)^{n+1} \approx \left(\int_{T(\hat{Q}_j)}|\nabla u|^2\delta_j\right) l(Q_j)^n \lesssim \frac{1}{\varepsilon^2}\left(\int_{T(\hat{Q}_j)}|\nabla u|^2\delta_j\right)^2,$$

which gives

$$\int_{T(\hat{Q}_j)}|\nabla u| \lesssim_{n,L} \frac{1}{\varepsilon}\int_{T(\hat{Q}_j)}|\nabla u|^2\delta_j.$$

We now proceed with the first inequality (Car1), as it turns out that proving it, will also complete the proof of (Car2).

$$\sum_{T(\hat{Q}_j)\text{ red},\,\hat{Q}_j\subset\hat{Q}} \int_{T(\hat{Q}_j)} |\nabla u(z)| \lesssim \sum_{T(\hat{Q}_j)\text{ red},\,\hat{Q}_j\subset\hat{Q}} \frac{1}{\varepsilon}\int_{T(\hat{Q}_j)} |\nabla u(z)|^2\delta_j(z) \lesssim_{n,L} \frac{1}{\varepsilon}\int_{\hat{Q}} |\nabla u(z)|^2\delta(z), \tag{3.22}$$

where in the last inequality, by $\delta(z)$ we denote the distance of point z to the bottom face of $\hat{Q}$. Moreover, the last inequality holds true, due to observation that although sets $T(\hat{Q}_j)$ may, in general intersect, for different $\hat{Q}_j$, each point in $\hat{Q}$ belongs to at most two sets $T(\hat{Q}_j)$. Thus, the integral on the right-hand side of the last estimate in (3.22) may increase at most twice. Finally, similarly to the discussion of estimate (3.16) in the proof of Lemma 3.2.3, we observe that for any point $\omega \in \hat{Q} \cap \partial\Omega$, i.e. in the bottom face of cube $\hat{Q}$, it holds that

$$z = (x, y) \in \Gamma_{\alpha,0,l(Q)}(\omega) \Leftrightarrow (\omega, \phi(\omega)) \in S_{\alpha,0,l(Q)}(z) \supset B\Big((x, \phi(x)), \frac{\alpha}{1 + L\alpha}(y - \phi(x))\Big) \cap \partial\Omega.$$

Moreover, notice that for $z = (x, y) \in \hat{Q}$ it holds that $\delta(z) \leq l(Q) \approx_{n,L} y - \phi(x)$. These observations, together with the analogous computations as in (3.17) and Proposition 3.2.6, imply that

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\hat{Q}} |\nabla u(z)|^2 \delta(z)\, \mathrm{d}z &\approx_{n,L} \frac{1}{\varepsilon}\int_{\hat{Q}} |\nabla u(x, y)|^2 (y - \phi(x))^{1-n}(y - \phi(x))^n \,\mathrm{d}x\mathrm{d}y \\ &\approx_{n,L,\alpha} \frac{1}{\varepsilon}\int_{Q} \left(A_{\alpha,0,l(Q)}u\right)^2(x)\,\mathrm{d}x \lesssim \frac{1}{\varepsilon}(l(Q))^n. \end{aligned} \tag{3.23}$$

This completes the argument for inequality (Car1) and the proof of (Car2) follows as well, upon combining (3.21) with (3.22) and (3.23).

Hence, $|\nabla\varphi|\mathrm{d}\mathscr{L}^{n+1}$ is a Carleson measure and, therefore, the proof of the ε-approximability of u in $\hat{Q}_0$ is completed.

Notice that in the proof it is not important that we consider a unit cube. Hence, our reasoning gives ε-approximation for any cube $\hat{Q}$ regardless of its side length. To obtain an ε-approximation in set Ω, we follow the approach in the end of the proof of Theorem 1.3 in [HMM1, Section 5]. Namely, let us choose a point x_0 in $\mathbb{R}^n$. Let Q_k be a family of cubes in $\mathbb{R}^n$ such that x_0 is a center of each of those cubes and $l(Q_k) = 2^k$. Denote by φ_k an ε-approximation on set $\hat{Q}_k$. Define

$$\varphi := \varphi_0 + \sum_{k=0}^{\infty} \varphi_k \chi_{\hat{Q}_{k+1}\setminus\hat{Q}_k}.$$

Let us verify that φ is an ε-approximation in Ω. Obviously it holds that $\|u - \varphi\|_{L^\infty(\Omega)} < \varepsilon$. It remains to check that $|\nabla\varphi|\mathrm{d}\mathscr{L}^{n+1}$ gives rise to the Carleson measure. Take $r > 0$ and $x \in \hat{Q}_l \setminus \hat{Q}_{l-1}$ if $l \geq 1$ or $x \in \hat{Q}_0$ otherwise (then assume $l = 0$) and consider a set $S = B(x, r) \cap \Omega$. We need to obtain

$$\int_{B(x,r)\cap\Omega} |\nabla\varphi|\mathrm{d}\mathscr{L}^{n+1} \leq C_\varepsilon r^n.$$

Let $k \in \mathbb{N}$ be such that $2^k < r \leq 2^{k+1}$ if $r > 1$. Then the integral of $|\nabla\varphi|$ consists of the jump terms J over the faces of $\hat{Q}_k \cap \Big(B(x, r) \cap \Omega\Big)$ and also the term $\int_{B(x,r)\cap\Omega\cap\hat{Q}_k} |\nabla\varphi|\mathrm{d}\mathscr{L}^{n+1}$.

Let us begin with estimating the jump terms. If $r \leq 1$ then S only lies in no more than three of the sets $\hat{Q}_k$. If $r > 1$ then S lies in no more than $k+2$ of sets $\hat{Q}_k$. In the latter case, we have

$$\begin{aligned}
|J| &\lesssim C_\varepsilon((1)^n + (2^1)^n + \cdots + (2^{k+2})^n) \\
&= C_\varepsilon\left(\frac{2^{nk+2n}-1}{2^n-1}\right) \\
&\lesssim_{k,n} C_\varepsilon\left(\frac{2^{2n}r^n-1}{2^n-1}\right) \\
&\lesssim_{k,n} C_\varepsilon\left(2^n r^n - \frac{1}{2^n-1}\right) \\
&\lesssim_{k,n} C_{\varepsilon,n} r^n.
\end{aligned} \tag{3.24}$$

In case $r \le 1$ we have

$$|J| \le C_\varepsilon 2r^n.$$

Hence, the jump terms are taken care of. We now proceed to deal with the remaining terms. First suppose that $r > 1$, then

$$\int_{S\cap\hat{Q}_0} |\nabla\varphi| d\mathscr{L}^{n+1} + \sum_{i=0}^{i=k+1} \int_{S\cap(\hat{Q}_{i+1}\setminus\hat{Q}_i)} |\nabla\varphi| d\mathscr{L}^{n+1} \le C_\varepsilon\left(1^n + \sum_{i=0}^{i=k+1}(2^{i+1})^n\right) \le C_{\varepsilon,n} r^n,$$

by the estimate in (3.24).

If $r \le 1$, then we have

$$\int_{S\cap\hat{Q}_0} |\nabla\varphi| d\mathscr{L}^{n+1} + \sum_{i=0}^{i=k+1} \int_{S\cap(\hat{Q}_{i+1}\setminus\hat{Q}_i)} |\nabla\varphi| d\mathscr{L}^{n+1} \lesssim C_\varepsilon 3r^n.$$

Therefore, $|\nabla\varphi| d\mathscr{L}^{n+1}$ gives rise to a Carleson measure. Thus, the proof of ε-approximability of u in Ω is completed. □

Remark 3.2.7. As observed in several works (e.g. [HMM1, G, HT]), the regularity of the ε-approximation φ obtained in the proof above, can be improved to C^∞. Indeed, this follows by Lemmas 3.2 (i) 3.6 and 3.8 in [HT] and by the standard mollification procedure, see e.g. [EG, Section 4.2].

Let us now prove Corollary 1.3.2. The proof is the repetition of the proof of Lemma 2.9 in [KKPT]. Recall that for a definition of a counting function we use cones from Def. 2.3.2, not from Def. 2.3.1. The reason is that for dealing with QFP it is more convenient with cones truncated with hypersurfaces, than cones truncated with graphs. Whereas for proving ε-approximability it is more convenient to use the latter.

Sketch of a proof of Corollary 1.3.2. Consider a family of truncated cones $\widetilde{\Gamma}_{\beta,0,s}(x)$ with $x \in \mathbb{R}^n$ such that $\Gamma_\alpha \subset \Gamma_\beta$. Put

$$A_s(\varphi)(x) = \int_{\widetilde{\Gamma}_{\beta,0,s}(x)} |\nabla\varphi| \frac{dz}{|z-(x,\phi(x))|^n}.$$

Let us first state the claim and show how it implies assertion of Corollary 1.3.2.

Claim 3.2.8. *Suppose that $N(\alpha, s, \varepsilon, \theta)(x) \geq k$ and φ is the $\frac{\varepsilon}{4}$-approximation of u. Then*

$$A_s(\varphi)(x) \geq kC_{\varepsilon,\theta}.$$

Now notice that the Claim implies that

$$\int_{\partial\Omega\cap B(x,r)} A_r(\varphi)(q)\mathrm{d}\sigma(q) \leq C \int_{\Omega\cap B(x,r)} |\nabla\varphi(z)|\mathrm{d}z,$$

and hence

$$\int_{\partial\Omega\cap B(x,r)} N(\alpha, s, \varepsilon, \theta)(q)\mathrm{d}\sigma(q) \leq \int_{\partial\Omega\cap B(x,r)} A_r(\varphi)(q)\mathrm{d}\sigma(q) \leq C \int_{\Omega\cap B(x,r)} |\nabla\varphi(z)|\mathrm{d}z \lesssim r^n.$$

The last estimate follows by Theorem 1.3.1, as φ is the ε-approximation and defines the Carleson measure on Ω. □

Proof of the Claim. We follow closely the steps of the proof of Lemma 2.9 in [KKPT]. Assume that $s = 1$, $x = 0$, $\phi(x) = 0$. Since the counting function $N(\alpha, 1, \varepsilon, \theta)(0) \geq k$, there is a sequence of points $Z_j = (x_j, t_j)$ with $j = 1, \ldots, k$ such that $|x_j| < \alpha t_j$ and $0 \leq t_k \leq \cdots \leq t_1 \leq 1$ satisfying $|u(Z_j) - u(Z_{j-1})| \geq \varepsilon$. Since u is C^2, and so, in particular, Hölder continuous and $\|u\|_\infty \leq 1$, there is δ depending on ε such that $|u(Z) - u(Z_j)| < \frac{\varepsilon}{8}$ for $Z \in L_j = \{Z = (x, t) : |x - x_j| < \delta t_j, t = t_j\}$. We get similar result for $j - 1$. Hence, we get for any $Z \in L_j$ and $Y \in L_{j-1}$ that $|u(Z) - u(Y)| \geq \frac{3\varepsilon}{4}$. By taking δ small enough, we make sure that any L_j is contained in the cone $\widetilde{\Gamma}_{\beta,0,1}$.

Let φ be a smooth $\frac{\varepsilon}{4}$-approximation of u given by Theorem 1.3.1. For $Z \in L_j$ and $Y \in L_{j-1}$ we get $|\varphi(Z) - \varphi(Y)| \geq \frac{\varepsilon}{4}$. For $Z = (x, t_j) \in L_j$ and $1 \leq y \leq y_j = \frac{t_{j-1}}{t_j}$, set

$$X_y = \left((x - x_j)y + \left(1 - \frac{y-1}{y_j - 1}\right)x_j + \frac{y-1}{y_j-1}x_{j-1}, yt_j\right).$$

Then $X_y \in \Gamma_\beta$, at $y = y_j$ we have $X_{y_j} \in L_{j-1}$ and $X_1 \in L_j$. Hence, $|\int_1^{y_j} \frac{\partial}{\partial y}\varphi(X_y)\mathrm{d}y| \geq \frac{\varepsilon}{4}$. Moreover,

$$\begin{aligned}\frac{\partial}{\partial y}X_y &= \left((x - x_j) - \frac{1}{y_j - 1}x_j + \frac{1}{y_j-1}x_{j-1}, t_j\right)\\ &= \left((x - x_j) + \frac{x_{j-1} - x_j}{y_j - 1}, t_j\right),\end{aligned}$$

and therefore, $|\frac{\partial}{\partial y}X_y| \leq \delta t_j + \frac{2\alpha t_{j-1}}{y_j - 1} + y_j \leq Ct_j$, because $t_{j-1} - t_j \geq (1-\theta)t_{j-1}$.

Let us now consider the change of variables $(x, y) \mapsto X_y = (z, s)$, where $|x - x_j| \leq \delta t_j$ and $1 \leq y \leq y_j$. This map is one to one and $\mathrm{d}x\mathrm{d}y = \frac{t_j^{n-1}}{s^n}\mathrm{d}z\mathrm{d}s$ as the Jacobian is given by the inverse of

$$\det\begin{bmatrix} y & & & (*) \\ 0 & \ddots & & \\ \vdots & & y & \\ 0 & \ldots & 0 & t_j \end{bmatrix} = y^n t_j = (yt_j)^n t_j^{1-n} = s^n t_j^{1-n},$$

where $(*)$ stands for some coefficients that do not affect the determinant. Hence, for $\Gamma_{\beta,j} = \Gamma_\beta \cap \{(x,t) : t_j \le t \le t_{j-1}\}$,

$$\int_{\Gamma_{\beta,j}} |\nabla\varphi| \frac{\mathrm{d}z\mathrm{d}s}{s^n} \ge C_\delta \left(\frac{1}{\delta t_j^n} \int_{|x-x_j|\le\delta t_j} \int_1^{y_j} |\frac{\partial\varphi}{\partial y}(X_y)|\mathrm{d}y\mathrm{d}x \right) \ge C_\delta \frac{\varepsilon}{4}.$$

Now we sum over j with $j = 2, \ldots, k$ to obtain

$$A_s(\varphi)(x) = \int_{\widetilde{\Gamma}_{\beta,0,s}} |\nabla\varphi| \frac{\mathrm{d}z\mathrm{d}s}{s^n} \ge (k-1)C_\delta \frac{\varepsilon}{4}.$$

□

3.3 Examples of functions satisfying Theorem 1.3.1 and the related PDEs

In this chapter we provide an example of the class of functions satisfying condition $(*)$ and, moreover, that are also ε-approximable. Furthermore, we also discuss some sufficient conditions giving $(*)$ and (#). The importance of the latter condition comes from the fact that it implies $(*)$, see Chapter 1.3. Our examples illustrate that for some classes of functions, a condition $(*)$ more general than (#), suffices for the ε-approximability to hold.

Let us recall Proposition 5.2 in [GKLN]. It asserts that if u is a C^2-regular function defined in a ball $B \subset \mathbb{R}^{n+1}$ such that u^{2k} is subharmonic in B for some positive integer $k > 0$, then u satisfies condition $(*)$ in B.

Let us recall lemmas 4.5 and 4.6 from [GKLN].

***Lemma 4.5** in [GKLN]. Suppose $u \in C^2(\Omega)$ satisfies (#) for some $0 < \theta < 1$. Then, for each $(x,y) \in \Omega$ and any $0 < \varepsilon < \frac{1}{4\sqrt{1+L^2}}$, one has*

$$\int_B |\nabla u|^2 \le \frac{C}{r^2} \int_{2B} u^2,$$

where $r = \varepsilon(y - \phi(x))$, $B = B((x,y),r)$ and C depends only on n and θ.

***Lemma 4.6** in [GKLN]. Suppose $u \in C^2(\Omega)$ satisfies (#) for some $0 < \theta < 1$ and $|u| \le 1$ in Ω. Then for any cube $Q \subset \mathbb{R}^n$ of side length l,*

$$\int_Q (A^2_{\alpha,0,l}u)(x)\mathrm{d}\omega^*(x) \le C,$$

where C depends on L, n, α and θ, the constant in (#).

These observations give us the following wide class of functions satisfying Theorem 1.3.1.

Proposition 3.3.1. *Let $u \in C^2$ be nonnegative and subharmonic in an open set $\Omega \subset \mathbb{R}^{n+1}$, i.e. $\Delta u \ge 0$. Then u satisfies $(*)$. Moreover, u is ε-approximable in domains Ω as in (1.2).*

The proof of the first assertion follows by direct computations showing that $\Delta u^{2k} \geq 0$ for any positive integer $k > 0$. Indeed, we have that:

$$
\begin{aligned}
\Delta u^{2k} &= \sum_{i=1}^{i=n+1} \frac{\partial^2}{\partial x_i^2} u^{2k} = \\
&= \sum_{i=1}^{i=n+1} \frac{\partial}{\partial x_i}\left(2k u^{2k-1} \frac{\partial}{\partial x_i} u\right) \\
&= 2k \sum_{i=1}^{i=n+1} (2k-1) u^{2k-2} \left(\frac{\partial}{\partial x_i} u\right)^2 + u^{2k-1} \frac{\partial^2}{\partial x_i^2} u \\
&= 2k u^{2k-2}((2k-1)|\nabla u|^2 + u\Delta u) \geq 0.
\end{aligned}
$$

Hence, u^{2k} is subharmonic and by Proposition 5.2 in [GKLN] it satisfies ($*$).

The second assertion follows immediately from Theorem 1.3.1, upon noticing that proofs of Lemmas 4.5 and 4.6 in [GKLN] hold as well for functions satisfying assumptions of the proposition. Indeed, the scrutiny of the proofs of Lemmas 4.5 and 4.6 reveals that under assumptions of the proposition, it holds that $\Delta u^2 = 2(|\nabla u|^2 + u\Delta u) \geq 2|\nabla u|^2$ and Lemma 4.6 in [GKLN] follows, if u satisfies ($*$).

Proposition 3.3.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and $u \in C^2(\Omega)$ be nonnegative and such that*

$$\Delta |\nabla u|^\alpha \geq 0$$

for any $0 < \alpha \leq 2$. Then, conditions ($*$) holds.

Proof. Since $|\nabla u|^\alpha$ is C^2-regular subharmonic, it satisfies the submean value property on Euclidean balls $\overline{B} \subset \Omega$. Hence for any $x, y \in B$ and some point $z \in B$, lying on a line segment joining x and y, we have

$$
\begin{aligned}
|u(x) - u(y)|^\alpha &\leq |\nabla u(z)|^\alpha |x-y|^\alpha \\
&\leq C(n,\eta) \left(\fint_{(1+\eta)B} |\nabla u|^\alpha \right) r^\alpha \\
&\leq C(n,\eta) \left(\frac{1}{r^{n+1-2}} \int_{(1+\eta)B} |\nabla u|^2 \right)^{\frac{\alpha}{2}} \\
&\leq C(n,\eta) \left[\left(\frac{1}{r^{n-1}} \int_{(1+\eta)B} |\nabla u|^2 + |u\Delta u| \right)^{\frac{1}{2}} \right]^\alpha .
\end{aligned}
$$

Therefore, we proved that ($*$) holds with $\phi(t) = Ct$ with C depending on n, η and the diameter of domain Ω. $\square$

Recall that a C^2-function satisfies (#), if $|u\Delta u| \leq \theta |\nabla u|^2$ for some $0 < \theta < 1$.

Proposition 3.3.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and u be a C^2-function. If $u\Delta u \geq 0$ in Ω, then each of the following conditions implies ($*$): $\Delta \ln u \leq 0$, $\Delta u^{-1} \geq 0$. Moreover, if $\Delta u^\alpha \leq 0$ for some $0 < \alpha < 1$, then condition (#) holds with $\theta := 1 - \alpha$, and hence, also ($*$) holds.*

Proof. The proof of the first assertion is based on the same type of computations and therefore, we will show only argument for the first of the two conditions. We have that, at points in Ω where $u \neq 0$, it holds that

$$0 \geq \Delta \ln u = \operatorname{div}\left(\frac{1}{u}\nabla u\right) = \frac{u\Delta u - |\nabla u|^2}{u^2},$$

and so $|u\Delta u| \leq |\nabla u|^2$ holds in Ω (as, if $u = 0$ at some point in Ω, then this inequality holds trivially). Thus, by the comment following definition of (#) in Introduction, condition ($*$) follows from Proposition 5.1 in [GKLN], even though $\theta = 1$. By analogy, the following direct calculations give us condition (#) and show the second assertion:

$$0 \geq \Delta u^\alpha = \operatorname{div}\left(\frac{\alpha}{u^{1-\alpha}}\nabla u\right) = \frac{\alpha}{u^{2-\alpha}}(u\Delta u - (1-\alpha)|\nabla u|^2).$$

□

Appendix: the proof of Proposition 3.2.6

The reasoning relies on the presentation in [GKLN, Section 4] and on a variant of the observation stated on page 261 in Garnett's book [G, Exc. 4], see also [S]. Let us state Exc. 4:

Let $f(x)$ be measurable on $\mathbb{R}$. Suppose there exist $\alpha < \frac{1}{2}$ and $\lambda > 0$ such that for each interval I there is some constant a_I such that

$$|\{x \in I : |f(x) - a_I| > \lambda\}| \leq \alpha|I|.$$

Then $f \in BMO$. Let us recall the definition of BMO functions:

Definition 3.3.4. Let $f \in L^1_{loc}(\Omega)$. For a ball $B \subset\subset \Omega$ define a mean value of f over B:

$$f_B = \frac{1}{|B|}\int_B f(x)\mathrm{d}x.$$

We say that $f \in BMO(\Omega)$ if

$$\sup_B \frac{1}{|B|}\int_B |f(x) - f_B|\mathrm{d}x < \infty,$$

where sup is taken over all balls compactly contained in Ω.

It is worth noting that instead of balls in the above definition we could take cubes Q.

According to our best knowledge there is no simple proof of this result for $n > 1$ in the literature. Therefore, we state our version of it and provide its proof.

First, we state the claim and show how it implies the assertion of Proposition 3.2.6. Then, we prove the claim.

CLAIM. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function and let $c \in (0, \frac{1}{2})$ and $\lambda > 0$. If for any cube $Q \subset \mathbb{R}^n$ there exists a constant a_Q such that*

$$\left|\left\{x \in Q : |f(x) - a_Q| > \lambda\right\}\right| < c|Q|,$$

then it holds that

$$\left|\left\{x \in Q : |f(x) - a_Q| > t\right\}\right| < e^{-c_2 t}|Q|,$$

where $t = 3n\lambda$, while $c_2 = (3\lambda)^{-1}\ln(4/3)$.

Suppose that the claim is proven. Then Lemma 4.2 (i) in [GKLN] asserts that $A_{\alpha,0,l}(f) < A_{\alpha',0,l'}(f_{B_\varepsilon})$, for some $\alpha' > \alpha$, $l' > l$, all $0 < \varepsilon < \varepsilon_0(\alpha, L)$ and a nonnegative measurable function f. Here, f_{B_ε} stands for the mean value integral of f over a hyperbolic ball $B_\varepsilon = B((z, y), \varepsilon(y - \phi(z)))$. Namely:

$$f_{B_\varepsilon} := f_{B_\varepsilon(z,y)} = \fint_{B((z,y),\varepsilon(y-\phi(z)))} f, \quad (z, y) \in \Omega.$$

Thus, Proposition 3.2.6 will be proven provided that we show that

$$\int_Q \left(A_{\alpha',0,l'}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\,\mathrm{d}x < c|Q| = c(l(Q))^n.$$

However, we find that

$$\begin{aligned}
&\int_Q \left(A_{\alpha',0,l'}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\,\mathrm{d}x\\
&= \int_Q \left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\,\mathrm{d}x\\
&\le \int_Q \left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x_Q)\,\mathrm{d}x \qquad (3.25)\\
&+ \int_Q \left|\left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x_Q) - \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\right|\,\mathrm{d}x.
\end{aligned}$$

The second integral on the right-hand side is bounded above by $c(\alpha, n, L)|Q|$ in a consequence of applying Lemma 4.3 in [GKLN] with $f(z) := \fint_{B_\varepsilon(z,y)} |\nabla u|^2$, provided that we know that

$$\fint_{B_\varepsilon(z,y)} |\nabla u|^2 \le \frac{c}{(y - \phi(z))^2}, \quad \text{for } (z, y) \in \Omega.$$

However, this condition immediately follows from the Caccioppoli inequality, see Lemma 4.5 in [GKLN], with the constant $c = c(n, \theta)\varepsilon^{-2}\|u\|^2_{L^\infty}$.

***Lemma 4.3** in [GKLN]. Assume that $f \ge 0$ is measurable in Ω and satisfies the uniform estimate*

$$f(z, y) \le \frac{A}{(y - \phi(z))^2}$$

for some $A > 0$ and any $(z, y) \in \Omega$. Then if $Q \subset \mathbb{R}^n$ is a cube of side length l, $) < \alpha < \frac{1}{M}$ we have

$$|A^2_{\alpha,l,\infty}(f)(x_1) - A^2_{\alpha,l,\infty}(f)(x_2)| < C$$

for any $x_1, x_2 \in Q$, where C is a constant depending only on L, α, A, n.

As for the integral (3.25) we estimate it by the Cavalieri's formula, see Chapter 2.7, for $\Phi(x) = x$ as follows:

$$\int_Q \left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x_Q)\,\mathrm{d}x$$
$$= \int_0^\infty \left|\left\{x \in Q : \left|\left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - a_Q\right| > t\right\}\right| \mathrm{d}t,$$

where $a_Q := \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x_Q)$.
Let us employ reasoning analogous to the proof of Theorem 4.7 in [GKLN] and Corollary 4.8 in [GKLN].

Corollary 4.6 in [GKLN] states that

$$\omega^*\left(\{x \in Q : \left(A_{\alpha',0,l'}|\nabla u|^2_{B_\varepsilon}\right)^2(x) > t)\}\right) \le C\frac{\omega^*(Q)}{t}.$$

Using the fact that harmonic measure and surface measure are mutually absolutely continuous by Theorem 2.1 in [GKLN], we obtain:

$$\left|\{x \in Q : \left(A_{\alpha',0,l'}|\nabla u|^2_{B_\varepsilon}\right)^2(x) > t)\}\right| \le C\frac{|Q|}{t^b}$$

for some $b > 0$. Now, we apply Lemma 4.3 from [GKLN] to get:

$$\left|\left(A_{\alpha',0,l'}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - \left(A_{\alpha',0,l'}|\nabla u|^2_{B_\varepsilon}\right)^2(y)\right| \le C_1$$

for any $x, y \in Q$. Hence, we have:

$$\begin{aligned}
t &< \left|\left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x_Q)\right| \\
&\le \left|\left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x_Q) - \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\right| \\
&+ \left|\left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\right| \\
&= \left|\left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x_Q) - \left(A_{\alpha',l',\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\right| \\
&+ \left|\left(A_{\alpha',0,l'}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\right| \\
&\le C_1 + \left|\left(A_{\alpha',0,l'}|\nabla u|^2_{B_\varepsilon}\right)^2(x)\right|.
\end{aligned}$$

Thus, it follows that

$$\left|\{x \in Q : \left|\left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - a_Q\right| > t\}\right| \le \frac{C|Q|}{(t - C_1)^b}.$$

We know that there exists t_0 such that for all $t > t_0$ it holds that

$$\left|\left\{x \in Q : \left|\left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2(x) - a_Q\right| > t\right\}\right| \le \frac{1}{4}|Q|,$$

as by following the notation of [GKLN], see page 217, we may choose t_0 such that $\frac{C}{(t_0-C_1)^b} = \frac{1}{4}$. By the claim applied to $f := \left(A_{\alpha',0,\infty}|\nabla u|^2_{B_\varepsilon}\right)^2$, the latter estimate implies a corresponding one with $e^{-ct}|Q|$, which in turn gives us the assertion of Proposition 3.2.6.

It remains to show the above Claim. Without the loss of generality let $c = \frac{1}{4}$ for the constant c as in the assumptions of the claim.

First, let $Q_j \subset Q$ denote any cube in the dyadic decomposition of cube Q and set $G_1 := \bigcup Q_j$, the union of all maximal cubes Q_j satisfying the following stopping time condition:

$$|\{x \in Q_j : |f(x) - a_Q| > \lambda\}| \geq \frac{1}{3}|Q|.$$

The family of cubes G_1 has the following properties:

(i) $Q \notin G_1$, as $c = \frac{1}{4}$.

(ii) If $Q_j \in G_1$ and $\hat{Q}_j$ denotes a parent of Q_j, i.e. $\hat{Q}_j$ is the minimal cube containing Q_j, then as $\hat{Q}_j \notin G_1$, we have that

$$\frac{1}{3}|Q_j| \leq |\{x \in Q_j : |f(x) - a_Q| > \lambda\}| \leq |\{x \in \hat{Q}_j : |f(x) - a_Q| > \lambda\}| < \frac{1}{3}|\hat{Q}_j| = \frac{2}{3}|Q_j|.$$

(iii) If $x \notin G_1$, then $|f(x) - a_Q| \leq \lambda$ a.e. in Q. Indeed, if $x \in Q_k$ for a cube not satisfying the stopping condition, then for a set $E := \{x \in Q : |f(x) - a_Q| > \lambda\}$, we have that

$$\fint_E 1_E = \frac{|E \cap Q_k|}{|Q_k|} < \frac{1}{3}, \text{ and hence } 1_E(x) = 0 \text{ and } x \notin E.$$

The Lebesgue Differentiation Theorem applied to 1_E, a characteristic function of set E, gives the property (iii) to hold at a.e. point of Q.

(iv) $\sum_{Q_j \in G_1} |Q_j| \leq \frac{3}{4}|Q|$. Indeed, since by the stopping condition $|Q_j| \leq |Q| \leq 3|\{x \in Q_j : |f(x) - a_Q| > \lambda\}|$, we get

$$\sum_{Q_j \in G_1} |Q_j| \leq \sum_{Q_j \in G_1} 3|\{x \in Q_j : |f(x) - a_Q| > \lambda\}| \leq 3|\{x \in Q : |f(x) - a_Q| > \lambda\}| < \frac{3}{4}|Q|.$$

Next, we construct a family of cubes $G_2 := \bigcup Q_k$, consisting of maximal subcubes of cubes in G_1 satisfying the following stopping time condition:

$$|\{x \in Q_k : |f(x) - a_{Q_j}| > \lambda\}| \geq \frac{1}{3}|Q_j|, \text{ for some } Q_j \in G_1.$$

By property (iv) we get that

$$\sum_{Q_k \in G_2} |Q_k| \leq \sum_{Q_j \in G_1} \Big(\sum_{Q_k \subset Q_j, Q_k \in G_2} |Q_k| \Big) \leq \sum_{Q_j \in G_1} \frac{3}{4}|Q_j| \leq \left(\frac{3}{4}\right)^2 |Q|. \tag{3.26}$$

Furthermore, for a.e. point $x \notin G_2$ it holds that

$$|f(x) - a_Q| \leq 3\lambda. \tag{3.27}$$

In order to show (3.27), note that if $x \notin G_2$, then we consider two cases: either (1) $x \notin G_1$, or (2) $x \in Q_j$ for some $Q_j \in G_1$. In the first case, we have $|f(x) - a_Q| \leq \lambda$ by property (ii). In the second case, an argument similar to the one giving property (iii) shows that $|f(x) - a_{Q_j}| < \lambda$ for a.e $x \notin G_2$.

Next, we show that $|a_Q - a_{Q_j}| < 2\lambda$. Define sets

$$E_1 := \{y \in Q_j : |f(y) - a_Q| > \lambda\}, \quad E_2 := \{y \in Q_j : |f(y) - a_{Q_j}| > \lambda\}.$$

Then, by property (ii) it holds that $|E_1| < \frac{2}{3}|Q_j|$ and, moreover, by the hypotheses of the claim (recall that we fixed $c = \frac{1}{4}$) we have $|E_2| < \frac{1}{4}|Q_j|$. Furthermore, $(Q_j \setminus E_1) \cap (Q_j \setminus E_2) \neq \emptyset$, as otherwise

$$|Q_j| \geq |Q_j \setminus E_1| + |Q_j \setminus E_2| > (1 - \frac{2}{3})|Q_j| + (1 - \frac{1}{4})|Q_j| > |Q_j|.$$

Therefore, there exists $y \in Q_j$ such that $|f(y) - a_Q| \leq \lambda$ and $|f(y) - a_{Q_j}| \leq \lambda$. This immediately results in the desired estimate

$$|a_Q - a_{Q_j}| \leq |f(y) - a_Q| + |f(y) - a_{Q_j}| \leq 2\lambda.$$

Hence, (3.27) follows, as

$$|f(x) - a_Q| \leq |f(x) - a_{Q_j}| + |a_Q - a_{Q_j}| \leq 3\lambda.$$

We iterate the above stopping time procedure and after n steps obtain the family of cubes G_n with the following properties, cf. property (iv) and (3.26), (3.27) and :

$$(1) \sum_{Q_l \in G_n} |Q_l| \leq \left(\frac{3}{4}\right)^n |Q|, \quad (2)\ |f(x) - a_Q| < 3n\lambda \text{ for a.e. } x \notin G_n.$$

In a consequence, we get that $\left|\left\{x \in Q : |f(x) - a_Q| > 3n\lambda\right\}\right| \leq \left(\frac{3}{4}\right)^n |Q|$. The latter implies, upon setting $t := 3n\lambda$, the assertion of Claim, as $(3/4)^n = e^{-(\ln 4/3)(3\lambda)^{-1}t}$. This completes the proof of Claim and the proof of Proposition 3.2.6 is completed as well.

Chapter 4

ε-Approximability and Quantitative Fatou Theorem on Riemannian manifolds

This chapter is based on the paper [Gr].

We deal with ε-approximability and Quantitative Fatou Property on Riemannian manifolds. We first prove ε-approximability. In order to obtain that goal we construct the covering of the domain with desired properties. Then we take ε-approximant on each of the sets from that covering. To this end we apply Theorem 1.3 from [HMM1], which states that there is an ε-approximant in the Euclidean space. Finally, we glue these approximations together to get ε-approximation on a domain in a Riemannian manifold.

Having ε-approximation in our hands we may proceed to proving Quantitative Fatou Property. We follow the approach from Lemma 2.9 in [KKPT]. Since we are in the Riemannian setting, not the Euclidean one, we have to prove all the necessary claims in this broader setting and also develop some of them.

The following are the key results of this chapter:

Theorem. 1.4.2 *Let M be an n-dimensional complete Riemannian manifold and $\Omega \subset M$ be an open bounded connected Lipschitz set. Let u be a harmonic bounded function in Ω. Then u is ε-approximable for every $\varepsilon > 0$.*

Theorem. 1.4.1 *Let M be a complete Riemannian manifold and let further $\Omega \subset M^n$ be a Lipschitz domain. Furthermore, let $u : \Omega \to \mathbb{R}$ be a harmonic bounded function with $\|u\|_\infty \leq 1$. Then for every point $p \in \partial\Omega$*

$$\sup_{0<r<r_{\mathrm{inj}}} \frac{1}{r^{n-1}} \int_{\partial\Omega \cap B(p,r)} N(r, \varepsilon, \theta)(q) d\sigma(q) \leq C(\varepsilon, \alpha, \theta, n, \Omega),$$

where $\varepsilon, \alpha, \theta$ are constants in the definition of the counting function. In particular, constant C is independent of u.

Even though we state our theorems for harmonic functions, we are actually able to prove these results for A-harmonic functions on Riemannian manifolds, see Definition 4.1.9.

4.1 Preliminaries

In what follows we will investigate the setting of Riemannian manifolds and, therefore, we recall some of the basic properties and definitions.

Definition 4.1.1. Let M^n be a smooth n-dimensional manifold. We will say that M^n is a Riemannian manifold if it is equipped with a scalar product $g_x : T_xM^n \times T_xM^n \to \mathbb{R}$ depending smoothly on the point $x \in M$. Usually g is represented by a matrix which is also denoted by g. Its determinant is denoted by $\det g$ and its coefficients are written with lower indices as g_{ij} for $i, j = 1, \dots, n$. The inverse matrix, which exists at every point since g is invertible as scalar product, is denoted by g^{-1} and its coefficients are written with upper indices g^{ij} for $i, j = 1, \dots, n$. In what follows we will write $M := M^n$ to denote the Riemannian manifold when the dimension n is fixed.

Recall that on a Riemannian manifold there exists a canonical measure defined by a Riemannian metric given by a volume form. Throughout this chapter we simply write $\mathrm{d}X$ when dealing with any integral. But it always means integrating with respect to this canonical measure.

When working with manifold one often uses local coordinates. In our case a certain choice of coordinates is convenient.

Definition 4.1.2. Let M be a Riemannian manifold. For any point $p \in M$ and any neighbourhood of p we introduce *normal coordinates* in a following way. Let $\exp_p : T_pM \to M$ be a map such that $\exp_p(v) = \gamma_p(1)$, where γ_p is a unique geodesic satisfying $\gamma_p(0) = p$ and $\dot{\gamma}_p(0) = v$. It is known that one can find such a neighbourhood of point p that $\exp_p$ is a diffeomorphism, call it U_p. Since the tangent space T_pM can be identified with space $\mathbb{R}^n$, say by isomorphism $T : T_pM \to \mathbb{R}^n$, the map $T \circ \exp_p^{-1} : M \supset U_p \to \mathbb{R}^n$ defines a local coordinate chart which we will call *normal coordinates*. In the sequel with an abuse of notation we will omit the isomorphism between a tangent space and Euclidean space and use only $\exp_p$ or $\exp_p^{-1}$.

Notice that if one takes $V \subset\subset U_p$, then $\exp_p$ is also a Lipschitz map.

Definition 4.1.3 (Injectivity radius)**.** Let M be a Riemannian manifold. For a point $x \in M$ we define injectivity radius $\mathrm{r_{inj}}(x)$ at x as the supremum of the set of all numbers $r > 0$ such that on a ball $B(0, r) \subset T_xM$ the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism. Then, the injectivity radius of a set $\Omega \subset M$ is defined by:

$$\mathrm{r_{inj}}_\Omega = \inf_{x \in \Omega} \mathrm{r_{inj}}(x).$$

Usually, we omit the lower case Ω if the set is fixed.

For more information on geometry of manifolds we refer e.g. to [H], [Li].

We will mostly deal with Lipschitz domains. Therefore, it is necessary to recall what we mean by a Lipschitz set on a manifold M.

Definition 4.1.4. Let Ω be an open connected subset of a manifold M and let $p \in \partial\Omega$. We say that Ω is *locally Lipschitz* at p if there exist a neighbourhood U of p in M and a local chart $f : U \to f(U) \subset \mathbb{R}^n$ such that $f(U \cap \Omega)$ is a Lipschitz set in $\mathbb{R}^n$. We say that Ω is *a Lipschitz set* if it is locally Lipschitz at every point of its boundary $\partial\Omega$.

Definition 4.1.5 (Gradient)**.** Let M be a Riemannian manifold with metric g. In particular at point $x \in M$ we have a scalar product g_x defined on the tangent space T_xM at x. Let f be a scalar function on M. The gradient ∇ is a vector field defined by the following property:

$$g_x(\nabla f(x), v_x) = df(x)(v_x),$$

where $v_x \in T_xM$ and d denotes differential.

In the local coordinates one obtains:

$$(\nabla f)^i = \partial^i f = \sum_j g^{ij}\partial_j f,$$

where g^{ij} are coefficients of the inverse of g.

Definition 4.1.6 (Divergence)**.** Let M be an n-dimensional Riemannian manifold with metric g. Let X be a vector field on M. The divergence of X denoted by $\mathrm{div}X$ is defined by:

$$\mathrm{dix}X \,\mathrm{vol}_n = L_X \,\mathrm{vol}_n,$$

where in local coordinates $\mathrm{vol}_n = \sqrt{|\det g|}dx^1 \wedge \cdots \wedge dx^n$ denotes the volume form on M and L_X is a Lie derivative along X.

In local coordinates we obtain:

$$\mathrm{div}X = \frac{1}{\sqrt{|\det g|}} \sum_i \partial_i \left(\sqrt{|\det g|}X^i \right).$$

Definition 4.1.7 (Laplacian)**.** Let M be a Riemannian manifold. The Laplace (or the Laplace-Beltrami) operator is defined on functions u defined on M as follows

$$\Delta u = \mathrm{div}(\nabla u).$$

In local coordinates we get:

$$\Delta u = \frac{1}{\sqrt{|\det g|}} \sum_i \partial_i \left(\sqrt{|\det g|} \sum_j g^{ij}\partial_j f \right).$$

Definition 4.1.8. We say that a Sobolev function u on M is harmonic if $\Delta u = 0$.

The following generalizes Definition 2.4 in [BH] to the setting of Riemannian manifolds. We retrieve that definition for $M = \mathbb{R}^n$.

We call a class of functions satisfying the definition below A-harmonic. Some authors use the name L-harmonic. This class of function is a natural generalization of harmonic functions. In [N] and [DG] one can find the proof of continuity of L-harmonic functions. They have been studied for many years. Moser proved Harnack inequality for such functions in [Mos], in [CFK] one can find discussion about L-harmonic measures. In [CFMS] pertains to existence on nontangential limits of A-harmonic functions at the boundary of the domain. The authors of [DJK] prove the comparability of p-norms of nontangential maximal function and square/area function for A-harmonic functions. In [HMM1] one can find the proof of ε-approximability of such functions and in [BH] there is a proof of Quantitative Fatou Property of such functions.

For an overview of some properties of A-harmonic functions on Riemannian manifold, see [CM].

Definition 4.1.9. We will say that a Sobolev function $u : \Omega \to \mathbb{R}$ is *A-harmonic* if it satisfies the following equation

$$\operatorname{div}(A\nabla u) = 0, \tag{4.1}$$

understood in the weak sense, where $A : TM \to TM$ is such that for each point $x \in \Omega$ we have $A(x) : T_xM \to T_xM$ is a linear map and for every fixed vector field X the map $A(\cdot, X)$ is measurable. Furthermore, we assume that there is a constant $C > 1$ such that

$$C^{-1}|\xi|^2 \leq g_x(A(x)\xi, \xi), \quad \|A\|_\infty \leq C,$$

for all $x \in \Omega$ and $\xi \in T_xM$. Moreover, we impose that operator A has bounded coefficients and satisfies the following conditions (&) and (&&) in [BH]

- the Carleson measure condition

$$\sup_{x\in\partial\Omega, 0<r<\operatorname{diam}(\partial\Omega)} \frac{1}{\mathcal{H}^{n-1}(B(x,r)\cap\partial\Omega)} \int_{B(x,r)\cap\Omega} |\nabla A(X)| \mathrm{d}X \leq C < \infty, \tag{\&}$$

- the pointwise gradient estimate

$$|\nabla A(X)| \leq \frac{C}{\operatorname{dist}(X, \partial\Omega)}, \quad \text{for all } X \in \Omega. \tag{\&\&}$$

Condition (&) means that the norm of the gradient of A gives rise to a Carleson measure. Since we ultimately want to prove ε-approximability, which requires that $|\nabla\varphi|\mathrm{d}X$ is a Carleson measure, it is a natural condition. Notice that for $A = Id$ this condition is satisfied. Condition (&&) means that A cannot change too rapidly as we approach the boundary. Again $A = Id$ satisfies this condition. A case $A = Id$ is important, because it recovers the Laplace-Beltrami operator. Examples of operators which satisfy conditions (&) and (&&) are for instance operators with constant coefficients or operators with linear coefficients on bounded domains.

Remark 4.1.10. If in the above definition operator A is such that for every $x \in \Omega$ we have $A(x) : T_xM \to T_xM$ is an identity transformation, then we obtain a Laplace-Beltrami operator. For more information about harmonic functions on manifolds, see e.g. [Li].

Recall that in the Preliminaries we define the functions with bounded variation in the Euclidean setting as well as in the setting of Riemannian manifolds. Below, we recall the generalized Definition 2.10.2 in the setting of Riemannian manifolds.

Definition 4.1.11. Let $\Omega \subset M$ be an open set and $u \in L^1(\Omega)$. We say that u has *bounded variation* in Ω and denote it by $u \in BV(\Omega)$ if

$$\sup\left\{ \int_\Omega u \operatorname{div}(\phi X) : X \in \Gamma(\Omega), \phi \in C_c^\infty(\Omega, \mathbb{R}), |\phi| \leq 1 \right\} < \infty,$$

where

$\Gamma(\Omega)$ is a family is of smooth vector fields on Ω such that $g(X(x), X(x)) \leq 1$ for every $x \in \Omega$, where g denotes the metric on M. The above supremum is called a *variation* of u.

If $\Omega \subset M = \mathbb{R}^n$, then we retrieve the definition of functions of bounded variation in Ω in Definition 2.10.1.

The main advantage of BV functions is the fact that it is a class wider that $W^{1,1}$ functions and in particular, it allows discontinuities along hypersurfaces. Therefore, it is well-suited for studying geometric variation problems. Let us mention that the BV functions on manifolds have been studied e.g. in [GuP], [MPPP], [CaM].

Let us now recall the definition of ε-approximability. The discussion regarding that notion is contained in the Preliminaries. It has often appeared in the literature for many years. However, to our best knowledge so far, it has only been used in Euclidean setting. Therefore, we give the definition in the setting of Riemannian manifolds.

Definition 4.1.12. Let $\Omega \subset M$ be a Lipschitz domain on a Riemannian manifold M. Let $u : \Omega \to \mathbb{R}$ be a harmonic function with $\|u\|_\infty \leq 1$. We will say that function u is *ε-approximable* for some $\varepsilon > 0$ if there exists a function $\phi \in BV(\Omega)$ such that

1. $\|u - \phi\|_{L^\infty(\Omega)} < \varepsilon$,
2. $|\nabla\phi|$ defines a Carleson measure on Ω, i.e. for every $x \in \partial\Omega$

$$\sup_{r\in(0,\operatorname{diam}\Omega)} \frac{1}{r^{n-1}} \int_{B(p,r)\cap\Omega} |\nabla\phi| \mathrm{d}X \leq C_\varepsilon.$$

Now we will define a generalized cone, a notion that in the Euclidean setting corresponds to the notion of the cone.

Definition 4.1.13. Let Ω be a connected, open subset of a Riemannian manifold M and let $p \in \partial\Omega$. Let $0 < \alpha < \infty$. The set

$$\Gamma(p) := \Gamma_\alpha(p) = \{q \in \Omega : d(p,q) \leq (1+\alpha)d(q,\partial\Omega)\},$$

is called a *generalized cone*. Moreover, we also define a *truncated generalized cone* as follows

$$\Gamma^r(p) := \Gamma(p) \cap B(p,r).$$

For the sake of simplicity we will use a name "cone" instead of generalized cone. In literature a name nontangential approach region is also used, see [KKPT]. We call it a generalized cone because in the case when $M = \mathbb{R}^n$ and $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$ we obtain for $p \in \partial\Omega$ as $\Gamma(p)$ the usual Euclidean cone. Compare this Definition to Definition 2.3.1 in Preliminaries.

Similarly to Definition 4.1.14 we introduce the following notion.

Definition 4.1.14. Let $\Omega \subset M$ be open, bounded, connected and $p \in \partial\Omega$. Let further $\Gamma(p)$ denote a cone at p. We define a *doubly truncated cone* $\Gamma_{r_1,r_2}(p)$ as follows

$$\Gamma_{r_1,r_2}(p) := (\Gamma(p) \cap B(p,r_1)) \setminus (\Gamma(p) \cap B(p,r_2)),$$

where $0 < r_1, r_2 < \infty$.

Let us state one of the key definitions used in this chapter.

Definition 4.1.15 (Counting function)**.** Let $\Gamma^r(p)$ be a cone for some point p in the boundary of an open set $\Omega \subset M^n$. Let u be a harmonic function defined on Ω. Denote by d a Riemannian distance in manifold M. Fix $\varepsilon > 0$, $0 < \theta < 1$ and $0 < r < 1$. We say that a sequence of points (Q_n) such that $Q_n \in \Gamma^r(p)$ is $(r, \varepsilon, \theta, p)$-*admissible* for u if

$$|u(Q_n) - u(Q_{n-1})| \geq \varepsilon \quad \text{and} \quad d(Q_n, p) < \theta d(Q_{n-1}, p).$$

Set

$$N(r, \varepsilon, \theta)(p) = \sup\{k : \text{there exists an } (r, \varepsilon, \theta, p)\text{-admissible sequence of length } k\}.$$

We call N a *counting function*.

4.2 A special case of ε-approximability

Before presenting the general case of n-dimensional complete manifold, we would like to give the taste of the ε-approximability in the special, but important, case of $M = \mathbb{S}^2$- the unit sphere in $\mathbb{R}^3$. Since then the maps can be found explicitly, the ε-approximability can be proven directly.

Proposition 4.2.1. *Let $\Omega \subset \mathbb{S}^2$ be a Lipschitz domain in the 2-dimensional sphere such that the surface measure σ of the set $\mathbb{S}^2 \setminus \Omega$ is positive. Let u be a Laplace–Beltrami harmonic function in Ω such that there exists a point $p^* \in \Omega$ with $u(p^*) = 0$. Then for every $\varepsilon > 0$ there is a BV function ϕ such that $|u(x) - \phi(x)| < \varepsilon$ for every $x \in \Omega$ and $|\nabla\phi|$ defines a Carleson measure i.e.*

$$\int_{B_y(r)\cap\Omega} |\nabla\phi| dA \leq Cr$$

for every point $y \in \partial\Omega$, $0 < r < \operatorname{diam}(\Omega)$ and for some positive constant C depending on Ω. Namely, C depends on $\operatorname{dist}(\Omega, e_3)$, where e_3 denotes unit vector $(0, 0, 1)$ in $\mathbb{R}^3$.

In the proof of the above proposition, we employ the following auxiliary observation.

Lemma 4.2.2. *The stereographic projection $f : \mathbb{R}^3 \supset \mathbb{S}^2 \to \mathbb{R}^2$ given by the formula*

$$f(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right)$$

is a Lipschitz map on any open set $\Omega \subset \mathbb{S}^2$ such that $\sigma(\mathbb{S}^2 \setminus \Omega) > 0$. Moreover, there exists constant $C = C(\operatorname{dist}(\Omega, e_3))$ such that the Jacobian of f satisfies

$$|Jf(x, y, z)| \approx C.$$

Here and in what follows by $\{e_i\}_{i=1}^n$ we denote the standard orthonormal vector basis in $\mathbb{R}^n$.

Proof of Lemma 4.2.2. Let us first notice that, upon rotating the sphere, we may assume that the north pole $e_3 \in \mathbb{S}^2 \setminus \Omega$ and the distance on a sphere of e_3 to Ω is positive. Since rotation is an isometry, it does not affect the harmonicity of function u. Moreover, as the Jacobian of rotation equals 1, the Carleson measure estimates are not affected. Finally, recall that f is a conformal diffeomorphism.

Let us cover $\mathbb{S}^2$ with six sets open in the topology of the sphere U_x^+, U_x^-, U_y^+, U_y^-, U_z^+, U_z^-, where

$$U_x^+ = \left\{(x,y,z) \in \mathbb{S}^2 \, : \, x > \frac{1}{10}\right\}, \quad U_x^- = \left\{(x,y,z) \in \mathbb{S}^2 \, : \, x < -\frac{1}{10}\right\}$$

and similarly for the remaining of those sets. We choose $\frac{1}{10}$, because we want to be separated from a circle $\{x = 0\}$. Analogously for sets with subindices y or z we want to be separated from circles $\{y = 0\}$, $\{z = 0\}$ respectively. We could choose any constant such that aforementioned sets cover the sphere. Here we take $\frac{1}{10}$ for convenience. Then on each of sets U, the remaining two coordinates form local coordinates on sphere, e.g. on U_x^+ coordinates are given by (y,z). Next, we express f on any of aforementioned sets with these new coordinates. In particular, on U_z^+ we have $f(x,y) = \left(\frac{x}{1-\sqrt{1-x^2-y^2}}, \frac{y}{1-\sqrt{1-x^2-y^2}}\right)$, and so by the direct computations we find that

$$Df(x,y) = a(x,y)\begin{pmatrix} \sqrt{1-x^2-y^2}-1+2x^2+y^2 & xy \\ xy & \sqrt{1-x^2-y^2}-1+x^2+2y^2 \end{pmatrix},$$

where $a(x,y) = \left[\sqrt{1-x^2-y^2}\,(1-\sqrt{1-x^2-y^2})^2\right]^{-1}$. Since $(x,y,z) \in U_z^+$, we know that $\frac{1}{\sqrt{1-x^2-y^2}} = \frac{1}{z^2} < 100$. Moreover, upon setting $d := d(\Omega, e_3) > 0$, we infer that

$$\frac{1}{(1-z)^2} = \frac{1}{(1-\sqrt{1-x^2-y^2})^2} < \frac{1}{(1-\cos d)^2}.$$

An elementary estimate gives us that $|\sqrt{1-x^2-y^2} - 1 + 2x^2 + y^2| \leq 5$ for all $(x,y,z) \in U_z^+$ and, hence on $\Omega \cap U_z^+$ we have

$$\|Df\|_{L^\infty} \leq \frac{500}{(1-\cos d)^2}.$$

The case of U_z^- is handled by analogous computations. The cases of sets with indices x and y are different, because in the formula for the stereographic projection we divide by $1 - z$ and on these sets z is one of coordinates. However, again by similar computations we see that whether it is set U_x^+, U_x^-, U_y^+ or U_y^-, the result is the same. Moreover, the formula for stereographic projection allows us to handle sets U_y^+ and U_y^- in a similar fashion. On U_x^+ we have $f(y,z) = \left(\frac{\sqrt{1-y^2-z^2}}{1-z}, \frac{y}{1-z}\right)$ and hence

$$Df(y,z) = b(y,z)\begin{pmatrix} -y(1-z) & 1-y^2-z \\ (1-z)\sqrt{1-y^2-z^2} & y\sqrt{1-y^2-z^2} \end{pmatrix},$$

where $b(y,z) = \left[\sqrt{1-y^2-z^2}(1-z)^2\right]^{-1}$. The computations analogous to the above allow us to estimate L^∞-norm again by $\frac{500}{(1-\cos d)^2}$. The remaining three sets U_x^-, U_y^+, U_y^- can be handled in the same way, and therefore, $\|Df\|_{L^\infty(\Omega)}$ is bounded. Hence f is Lipschitz on Ω with Lipschitz constant $L = \frac{500}{(1-\cos d)^2}$.

Finally, let us show the bounds for the Jacobian of f. First, we compute directly that on U_z^+ it holds

$$|Jf(x,y)| = \frac{1}{z^2(1-z)^4}|z||1-z|^2| - 2z - 1| = \frac{2z+1}{z(1-z)^2},$$

which combined with the fact that $\frac{1}{10} < z < \cos d$ gives us the following inequalities:

$$\frac{12}{10}\left(\frac{10}{9}\right)^2 \frac{1}{\cos d} < |Jf(x,y)| < 10\,\frac{2\cos d + 1}{(1-\cos d)^3}.$$

Similarly on U_x^+ we have

$$|Jf(y,z)| = \frac{1}{x(1-z)^2}$$

which, in a consequence, results in the following bounds for $|Jf|$:

$$1 \cdot \frac{1}{(1+\frac{3\sqrt{11}}{10})^2} < |Jf(y,z)| < 10\,\max\left\{\frac{1}{(1-\cos d)^2}, \frac{1}{(1-\frac{3\sqrt{11}}{10})^2}\right\}.$$

By analogous computations we find bounds for Jf on other sets U. $\square$

We are now in a position to show the proof of ε-approximability on Lipschitz domains on $\mathbb{S}^2$.

Proof of Proposition 4.2.1. Let f be the stereographic projection as in Lemma 4.2.2. Since u is harmonic, we get that $u \circ f^{-1}$ is also harmonic on $f(\Omega)$ due to conformal invariance of harmonicity in dimension 2. Furthermore, by Lemma 4.2.2 it holds that $f(\Omega) \subset \mathbb{R}^2$ is a Lipschitz domain. Therefore, by Dahlberg's result [D1, Theorem 1] for any $\varepsilon > 0$, we find a function $\widetilde{\phi}$ which ε-approximates $u \circ f^{-1}$. Upon setting $\phi := \widetilde{\phi} \circ f$ we immediately obtain that ϕ is a desired function in the first part of the assertion of the theorem, that is ϕ ε-approximates u. Indeed,

$$|u(x) - \phi(x)| = |u \circ f^{-1}(f(x)) - \widetilde{\phi}(f(x))| < \varepsilon$$

as $\widetilde{\phi}$ approximates $u \circ f^{-1}$ and f is a bijection.

The proof that $|\nabla \phi|$ gives a Carleson measure requires slightly longer argument. Note that by definition of ϕ, Lipschitz bound on $|Df|$ and by the standard identity $|Jf(x)|\,|Jf^{-1}(f(x))| = 1$, we have

$$\begin{aligned} &\int_{B_y(r)\cap\Omega} |\nabla \phi(x)| \mathrm{d}\sigma(x) \\ &\le \int_{B_y(r)\cap\Omega} |\nabla \widetilde{\phi}(f(x))|\,|Df(x)| \mathrm{d}\sigma(x) \\ &\le L \int_{B_y(r)\cap\Omega} |\nabla \widetilde{\phi}(f(x))|\,|Jf(x)|\,|Jf^{-1}(f(x))| \mathrm{d}\sigma(x), \end{aligned} \tag{4.2}$$

where L is a Lipschitz constant, which depends also on $d = \operatorname{dist}(\Omega, e_3)$, see the proof of Lemma 4.2.2 for details. Since $f(\Omega)$ is bounded and f is a diffeomorphism, it holds by the second part of assertion in Lemma 4.2.2 that $|Jf^{-1}(f(x))|$ is bounded as well by $C = C(d)$. Therefore,

$$\begin{aligned} &\int_{B_y(r)\cap\Omega} |\nabla \widetilde{\phi}(f(x))|\,|Jf(x)|\,|Jf^{-1}(f(x))| \mathrm{d}\sigma(x) \\ &\qquad \le C \int_{B_y(r)\cap\Omega} |\nabla \widetilde{\phi}(f(x))|\,|Jf(x)| \mathrm{d}\sigma(x) \\ &\qquad = C \int_{f(B_y(r)\cap\Omega)} |\nabla \widetilde{\phi}(z)| \mathrm{d}z, \end{aligned} \tag{4.3}$$

where in the last step we use the change of variables formula. The set $f(B_y(r) \cap \Omega)$ need not be equal to $f(\Omega) \cap B_{f(y)}(r')$ for some r', but there is such $\tilde{r} = \tilde{r}(y)$ that $f(B_y(r) \cap \Omega) \subset f(\Omega) \cap B_{f(y)}(\tilde{r})$.

Since, by Dahlberg's result, $|\nabla\widetilde{\phi}|$ defines a Carleson measure on $\mathbb{R}^2$, we trivially have that

$$\int_{f(\Omega)\cap B_{f(y)}(\tilde{r})} |\nabla\widetilde{\phi}(z)|\mathrm{d}z \leq \tilde{C}\tilde{r} \leq \tilde{C}Kr. \tag{4.4}$$

The last inequality is a consequence of a fact that we can find a global constant K such that for each $y \in \partial\Omega$ it holds that $\tilde{r}(y) < Kr$. For example, take $K = L + 1$.

By combining estimates (4.2)-(4.4) we obtain that $|\nabla\phi|$ defines a Carleson measure on $\Omega \subset \mathbb{S}^2$. $\square$

4.3 Harmonic and A-harmonic ε-approximability

The crucial part of the proof of Quantitative Fatou Theorem is the ε-approximability. Definition 4.1.12 shows that the ε-approximation function ϕ is close to harmonic function u, but has a property that its gradient gives the Carleson measure, which may not necessarily by true for any harmonic function, see the discussion in Chapter 1.2. This is the essential part of estimates needed to obtain Quantitative Fatou Theorem.

Theorem. 1.4.2 *Let M be an n-dimensional complete Riemannian manifold and $\Omega \subset M$ be an open bounded connected Lipschitz set. Let u be a bounded harmonic function in Ω. Then u is ε-approximable for every $\varepsilon > 0$.*

In the proof of Theorem 1.4.2 we use the following deep result, see [BH, Theorem 2.15]. For the reader's convenience let us recall this theorem:

Theorem. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying the (interior) corkscrew condition such that $\partial\Omega$ is uniformly rectifiable and $\mathcal{L}$ is an A-harmonic elliptic operator with coefficients satisfying equations (&) and (&&). Then all bounded solutions to $\mathcal{L}u = 0$ in Ω are ε-approximable for all $\varepsilon \in (0, 1)$ with constant C_ε depending on equations (&), (&&), ε, n, and the UR character of $\partial\Omega$.*

Therefore, in the case when M is the Euclidean space and Ω is a Lipschitz domain and operator $\mathcal{L}$ is given by Laplace-Beltrami operator in local coordinates, all the assertions of this theorem are satisfied and we are allowed to use it.

Our strategy is as follows: we cover set Ω with finitely many open sets such that on each we are able to use normal coordinates. The properties of these coordinates and the fact that Ω is Lipschitz enable us to prove existence of ε-approximation on each of the sets in the covering. Finally we glue these approximations and show that it is an ε-approximation on Ω. We need normal coordinates on a manifold in order to express the Laplace equation on the manifold in these coordinates as an A-harmonic equation. We prove that a Lipschitz set satisfies interior corkscrew condition, which is an essential property of a set Ω. Due to Lipschitzness, the boundary $\partial\Omega$ satisfies the Ahlfors-David regularity condition, which is necessary for our theorem to hold.

Before proving Theorem 1.4.2 let us discuss two auxiliary results. First we establish a relation between BV functions on manifolds and in $\mathbb{R}^n$. Lemma 4.3.1 enables us to produce a BV function on a manifold M if we have a BV function defined on a subset of $\mathbb{R}^n$. Lemma 4.3.2 allows us says that bounded Lipschitz sets in $\mathbb{R}^n$ satisfy interior corkscrew condition. This property is necessary in the proof of Theorem 1.4.2.

Lemma 4.3.1. *Let M be a Riemannian manifold. Furthermore, let $\phi : U \to \mathbb{R}$ be a BV function defined on a set $U \subset V \subset \mathbb{R}^n$, where V is such a set that $\exp_p$ is a diffeomorphism on V into M for some point $p \in V$. Then, the composition $\phi \circ \exp_p^{-1}$ is a BV function on $\exp_p(U) \subset M$.*

Proof. By the definition, a function $\phi \in BV(U)$ if

$$\sup \int_U \phi \operatorname{div} v < \infty,$$

where the supremum is taken over the set of all $v \in C_c^\infty(U, \mathbb{R}^n)$ with $|v| \leq 1$. We would like to find uniform estimate for the integral $\int_{\exp_p(U)} (\phi \circ \exp_p^{-1}) \operatorname{div} v$, where $v \in C_c^\infty(\exp_p(U), \mathbb{R}^n)$. By the change of variables formula we obtain

$$\int_{\exp_p(U)} (\phi \circ \exp_p^{-1}) \operatorname{div} v = \int_U \left((\phi \circ \exp_p^{-1}) \circ \exp_p \right) \operatorname{div}(v \circ \exp_p) |J \exp_p| \lesssim \int_U \phi \operatorname{div}(v \circ \exp_p).$$

The approximate inequality is the consequence of the fact that Jacobian of $\exp_p$ is bounded on U by a constant depending on U and M, since set $U \subset V$ and so, by assumptions, $\exp_p$ is a diffeomorphism on U. In the last integral, instead of v, we now have $v \circ \exp_p$, which may a priori change the set of functions over which supremum is taken. However, it turns out that any function $w \in C_c^\infty(U, \mathbb{R}^n)$, $|w| \leq 1$ can be written as $v \circ \exp_p$ for some function $v \in C_c^\infty(\exp_p(U), \mathbb{R}^n)$, $|v| \leq 1$. Indeed, this follows from writing $w = (w \circ \exp_p^{-1}) \circ \exp_p$ and setting $v = w \circ \exp_p^{-1}$. This completes the proof of the lemma. □

We now state as a lemma the well-known result. Its proof is contained in the Appendix to this chapter. It seems to us that this fact is the mathematical folklore in geometric analysis. However, we were not able to find any written proof of it and hence we decided to include ours.

Lemma 4.3.2. *A bounded Lipschitz set in the Euclidean space satisfies interior corkscrew condition.*

We are now in a position to prove one of the main results of this chapter (the second being Quantitative Fatou Property).

Proof of Theorem 1.4.2. Before we start the proof, let us outline the main steps.

Step 1. Construct a covering of Ω. Each set in the covering is diffeomorphic to a subset of $\mathbb{R}^n$ and partial derivatives of diffeomorphisms are uniformly bounded and each set is Lipschitz.
Step 2. Use Theorem 1.3 in [HMM1] to obtain ε-approximation on each of the sets of the covering.
Step 3. Glue together all ε-approximations to get an ε-approximation on Ω.

Step 1.
We begin with covering domain Ω with appropriately constructed Lipschitz sets, such that each of these sets is diffeomorphic to a subset of $\mathbb{R}^n$ and all the partial derivatives of such diffeomorphisms are uniformly bounded in Ω. Let us describe how it can be achieved.

Let us cover $\bar{\Omega}$ with sets V_p for $p \in \Omega$, such that each V_p is a neighbourhood of p with the property that $\exp_p$ is a diffeomorphism on $\exp_p^{-1}(V_p)$. Since $\bar{\Omega}$ is closed and bounded, it holds that $\bar{\Omega}$ is compact by completeness of M. Therefore, there exists a finite cover of $\bar{\Omega}$ by sets denoted by V_l with

$l \in \{1, 2, \dots, m\}$, where m is a number of sets in the cover. Moreover, by p_l we denote points p corresponding to sets V_l.

Claim: Sets V_l can be chosen to be Lipschitz.

In order to prove the claim, take any ball $B(p_l, R_l) \subset M$, where $R_l = \mathrm{r_{inj}}(p_l)$ is injectivity radius at $p_l \in \Omega$, for $l = 1, \dots, m$. Due to the fact that $\bar{\Omega}$ is compact, it holds that

$$\inf_{p\in\Omega} \mathrm{r_{inj}}(p) = \min_{p\in\bar{\Omega}} \mathrm{r_{inj}(p)} := \mathrm{r_{inj}} > 0.$$

Next, consider balls $B(p_l, R_l - \delta)$ with $\delta > 0$ small enough, say $\delta = \frac{1}{10}\mathrm{r_{inj}}$. If it turns out that such smaller balls do not cover Ω, then increase m so that the new larger family of balls covers Ω. Since $B(p_l, R_l - \delta) \subset\subset B(p_l, R_l)$ and the map $\exp_{p_l}^{-1}$ is a diffeomorphism on $B(p_l, R_l)$, we have that the derivative of $\exp_{p_l}$ is bounded on each of $B(p_l, R_l - \delta)$. Moreover, balls $B(p_l, R_l - \delta)$ are Lipschitz because they are images of balls $B(0, R_l - \delta)$ in Euclidean space under $\exp_{p_l}$. This holds since the latter balls are Lipschitz sets in $\mathbb{R}^n$ and $\exp_p$ is a Lipschitz map.

Since in our setting function u is defined only on Ω, we need to intersect balls $B(p_l, R_l - \delta)$ with Ω. Unfortunately, sets $V_l' := B(p_l, R_l - \delta) \cap \Omega$ need not be Lipschitz nor satisfy interior corkscrew condition. Thus, it is necessary to augment our covering.

Notice, that as Ω is Lipschitz, sets V_l' have finitely many connected components. Since $l \leq m$, the set of all connected components of these sets is finite as well. Hence we may further assume that sets V_l' are connected.

Each set V_l' is locally Lipschitz at almost every point of the boundary $\partial V_l'$. In general, the intersection $b_l := \partial\Omega \cap \partial B(p_l, R_l - \delta)$ may consist of several connected components and, moreover, different components may have different dimensions. Some components may have dimension $n-1$ when a piece of $\partial B(p_l, R_l - \delta)$ is also a piece of $\partial\Omega$, i.e. when locally Ω and $B(p_l, R_l - \delta)$ share a boundary. However, these connected components of b_l are Lipschitz because Ω is Lipschitz. Therefore, only lower dimensional components of b_l may form a set of points b_l' such that $\partial V_l'$ is not locally Lipschitz at points from b_l'. Hence, indeed V_l' is locally Lipschitz at almost every point of its boundary.

Take such n-dimensional neighbourhoods $A_l \supset b_l'$ that sets $V_l' \setminus A_l$ are Lipschitz. We can also choose A_l so that $d(V_l' \setminus A_l, b_l') < \frac{\mathrm{r_{inj}}}{2}$. It can be done because each set V_l' is "almost Lipschitz". Let us be more precise and clear what we mean by "almost Lipschitz". The boundary of V_l' consists of parts that are subsets of $\partial\Omega$ and subsets of $\partial B(p_l, R_l - \delta)$. Both of these sets are Lipschitz. Hence, the subset of the boundary of V_l' which may be an obstacle to V_l' being Lipschitz has measure zero. Therefore, we only have to take A_l small enough so that the part of its boundary inside V_l' is Lipschitz and does not form cusps with $\partial V_l'$. The latter means that $V_l' \setminus A_l$ satisfies interior and exterior corkscrew conditions. Hence, $(\partial A_l \cap V_l') \cup (\partial V_l' \setminus A_l)$ is locally Lipschitz. Since there are finitely many sets V_l' the Lipschitz constants of all sets are uniformly bounded. Put $\mathcal{B}_l := A_l \cap V_l'$ and set

$$V_l'' := \begin{cases} V_l' & \text{if } V_l' \text{ is Lipschitz} \\ V_l' \setminus \mathcal{B}_l & \text{if } V_l' \text{ is not Lipschitz.} \end{cases}$$

Sets V_l'' are Lipschitz but they need not cover whole set Ω. Thus we need to deal with sets $\mathcal{B}_l$. Divide sets $\mathcal{B}_l$ into subsets $\mathcal{B}_{l,i}$ such that:

- $\#\{i \in \mathbb{N} : \mathcal{B}_{l,i}\} \lesssim \frac{\mathrm{Vol}(\mathcal{B}_l)}{\mathrm{r_{inj}}^n} \lesssim \frac{\mathrm{Vol}(\Omega)}{\mathrm{r_{inj}}^n}$ for every l.

- $\operatorname{diam}(\mathcal{B}_{l,i}) \leq \mathrm{r_{inj}}$ for every i and every l,
- $\bigcup_i \mathcal{B}_{l,i} = \mathcal{B}_l$ for every l.

To construct such a partition, it is sufficient to cover set $\overline{\mathcal{B}_l}$ with sets $B(p, \frac{\mathrm{r_{inj}}}{2}) \cap \overline{\mathcal{B}_l}$ for points $p \in \mathcal{B}_l$ and take a finite subcover. Let us denote the sets from this finite subcover by $\mathcal{B}_{l,i} := B(p_{l,i}, \frac{\mathrm{r_{inj}}}{2}) \cap \mathcal{B}_l$. In particular, we refer to the centers of the balls by $p_{l,i}$. Let us mention that sets $\mathcal{B}_{l,i}$ need not be connected. However, again, they may have only finitely many connected components and hence we will assume that $\mathcal{B}_{l,i}$ are connected.

Define sets $C_{l,i} := B(p_{l,i}, \mathrm{r_{inj}} - \delta) \cap \Omega$. Let us again assume that $C_{l,i}$ are connected. Notice that

$$\mathcal{B}_{l,i} \subset C_{l,i}, \quad \mathcal{B}_{l,i} \subset\subset B(p_{l,i}, \mathrm{r_{inj}} - \delta), \quad d(\mathcal{B}_{l,i}, \partial B(p_{l,i}, \mathrm{r_{inj}} - \delta)) > \frac{\mathrm{r_{inj}}}{4}.$$

Moreover, sets $C_{l,i}$ need not be Lipschitz.

Now we improve a family of sets $C_{l,i}$ to a family of sets $C'_{l,i}$ which are constructed as Lipschitz and $\mathcal{B}_{i,l} \subset C'_{i,l}$. That such $C'_{l,i}$ can be chosen in such a way follows from

$$d\left(\mathcal{B}_{l,i}, \partial B(p_{l,i}, \mathrm{r_{inj}} - \delta)\right) > \frac{\mathrm{r_{inj}}}{4}.$$

Notice that this means that sets $\mathcal{B}_{l,i}$ and $\partial B(p_{l,i}, \mathrm{r_{inj}} - \delta)$ are separated from each other. Moreover, sets $C_{l,i}$ are already "almost Lipschitz" as their boundaries consist of pieces of $\partial\Omega$ and $\partial B(p_{l,i}, \mathrm{r_{inj}})$. Therefore, one can find a set $C'_{l,i}$ such that it is Lipschitz and $\mathcal{B}_{l,i} \subset C'_{l,i}$. Again, because there are finitely many sets $C_{l,i}, C'_{l,i}$ all their Lipschitz constants are uniformly bounded. Hence we obtained that all $\mathcal{B}_l \subset \bigcup_i C'_{l,i}$. Therefore, we covered our "bad" sets $\mathcal{B}_l$ with Lipschitz sets.

Altogether sets V''_l and $C'_{l,i}$ provide the covering of Ω with Lipschitz sets. Let us rename and renumber the constructed collection of sets to obtain the covering $\{V_l\}$ of Ω with Lipschitz sets.

Therefore, we can assume that sets V_l are Lipschitz and the claim is proven.

The estimate for m, the number of sets V_l in the constructed covering.

Observe that, since the image of a ball centered at the origin under exp is a ball with the same radius and in the proof of the Claim we set $\delta = \frac{1}{10}\mathrm{r_{inj}}$, we have that $\operatorname{diam}(V''_l) \approx \mathrm{r_{inj}}$. Next, notice that since the radius of any geodesic ball in Ω does not exceed $\mathrm{r_{inj}}$, each such a ball has volume comparable to $\left(\frac{9}{10}\mathrm{r_{inj}}\right)^n$ and the same holds for sets $C'_{i,l}$. Therefore, the number m of sets V_l required to cover Ω is

$$m \lesssim_\Omega \left(\frac{\operatorname{Vol}(\Omega)}{\mathrm{r_{inj}}^n}\right)^2.$$

The square in the above estimate is the result of the fact that the number of sets V'_l is comparable to $\frac{\operatorname{Vol}(\Omega)}{\mathrm{r_{inj}}^n}$ and for each set V''_l the number of sets $C'_{l,i}$ is, again, comparable to $\frac{\operatorname{Vol}(\Omega)}{\mathrm{r_{inj}}^n}$. The implicit constant depends only on the geometry of Ω, as the volume of a ball (with a small enough radius) can be bounded by the n-th power of radius multiplied by a function depending on dimension and curvature.

Due to our construction, on each set V_l we have normal coordinates. This allows us to write the Laplace-Beltrami equation $\Delta_M u = 0$ in local coordinates as follows:

$$\frac{1}{\sqrt{\det g}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{n} \sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x^j} \right) = 0, \tag{4.5}$$

where g denotes a metric tensor on M.

From this representation we infer that on every $V_l \subset M$ the equation $\Delta_M u = 0$ corresponds to $\operatorname{div}(A\nabla u) = 0$ on $\exp_{p_l}^{-1}(V_l) \subset \mathbb{R}^n$, where the square matrix $A \in M^{n\times n}$ depends on a point on M and

$$A = \sqrt{\det g}(g^{ij}). \tag{4.6}$$

Let us notice that any set $\exp^{-1}(V_l)$ is Lipschitz, because the derivative $|D\exp_{p_l}|$ is bounded on each V_l. Moreover, since there are finitely many sets V_l, the derivatives $D\exp_{p_l}$ are uniformly bounded on $\bar{\Omega}$. Therefore, in maps our harmonic equation locally reduces to an A-harmonic one on Lipschitz domains in $\mathbb{R}^n$.

Step 2.

At this point we would like to apply [BH, Theorem 2.15], see also [HMM1, Theorem 1.3], which provides conditions on an underlying domain and a bounded A-harmonic function implying its ε-approximability. Namely, the domain has to satisfy the interior corkscrew condition and its boundary must be uniformly rectifiable. Here, we study domains

$$W_l := \exp_{p_l}^{-1}(V_l) \subset B(0, R_l - \delta) \subset T_{p_l}M = \mathbb{R}^n, \quad l = 1, \dots, m. \tag{4.7}$$

Notice that, by Lemma 4.3.2, V_l, and hence also W_l satisfy the interior corkscrew condition. Furthermore, since all V_l, and hence also all W_l by the above discussion, are Lipschitz, the uniform rectifiability condition holds because every Lipschitz set is in particular uniformly rectifiable, by direct verification of Definition 2.7 in [BH] with θ and M_0 depending on Lipschitz constant of Ω and curvature of Ω. Indeed,for the reader's convenience, let us recall that definition.

Definition 4.3.3 (2.7 [BH])**.** An n-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ is *uniformly rectifiable* if and only if it contains "Big Pieces of Lipschitz Images" of $\mathbb{R}^n$, see e.g. [BH], [M, Section 6]. This means that there are positive constants θ and M_0, such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than M_0, such that

$$\mathcal{H}^n\Big(E \cap B(x,r) \cap \rho(\{z \in \mathbb{R}^n : |z < r|\})\Big) \geq \theta r^n.$$

As for the assumptions on the A-harmonic function u, the matrix A in [BH, Theorem 2.15] has bounded coefficients and defines an elliptic operator (cf. Definition 2.4 in [BH]). This is the case of matrix in (4.6), since W_l are bounded, metric g is smooth and positive definite. Moreover, as in [BH] coefficients of an A-harmonic operator are locally Lipschitz, again by the smoothness of g. What remains to be checked are conditions (&) and (&&) in Definition 4.1.9.

In order to check (&&) we compute the gradient of A, cf. (4.6). For $i, j, k = 1, \dots, n$ it holds that

$$\begin{aligned} \frac{\partial a_{ij}}{\partial x^k} &= \frac{\partial}{\partial x^k}\left(\sqrt{\det g}\, g^{ij}\right) \\ &= \frac{1}{2\sqrt{\det g}} \frac{\partial \det g}{\partial x^k} g^{ij} + \sqrt{\det g}\frac{\partial}{x^k} g^{ij} \\ &= \frac{1}{2\sqrt{\det g}} \det g \operatorname{tr}\left(g^{-1}\frac{\partial}{\partial x^k} g\right) g^{ij} + \sqrt{\det g}\frac{\partial}{\partial x^k} g^{ij} \\ &= \sqrt{\det g}\left[\frac{1}{2}\left(\sum_{a,b} g^{ab}\frac{\partial}{\partial x^k} g_{ba}\right) g^{ij} + \frac{\partial}{\partial x^k} g^{ij}\right], \end{aligned} \tag{4.8}$$

where in (4.8) we use the following Jacobi's formula for derivatives of a matrix determinant.

$$\frac{d}{dt}\det A(t) = \operatorname{tr}\left(\operatorname{adj}\big(A(t)\big)\frac{dA(t)}{dt}\right) = \big(\det A(t)\big)\cdot \operatorname{tr}\left(A(t)^{-1}\cdot\frac{dA(t)}{dt}\right).$$

By compactness of $\bar{W}_l$ and continuity of g, g_{ij}, g^{ij} and their derivatives we have that

$$C_l := \sup_{x\in\bar{W}_l}\left\{|g(x)|, |g_{ij}(x)|, |g^{ij}(x)|, \left|\frac{\partial}{\partial x^k} g_{ij}(x)\right|, \left|\frac{\partial}{\partial x^k} g^{ij}(x)\right|\right\} < \infty. \tag{4.9}$$

Let further

$$d_l := \sup_{x\in\bar{W}_l} \operatorname{dist}(x, \partial W_l). \tag{4.10}$$

Then for all $i, j, k = 1, \dots, n$ it holds that

$$\left|\frac{\partial a_{ij}}{\partial x^k}(x)\right| \le \sqrt{n! C_l^n}\left(\frac{1}{2} n^2 C_l^3 + C_l\right) \le \frac{C(n, C_l)}{d_l} \le \frac{\widetilde{C}}{\operatorname{dist}(x, \partial W_l)}. \tag{4.11}$$

Therefore, (&&) holds and it remains to prove (&). First, since W_l are Lipschitz, then ∂W_l satisfy the Ahlfors–David regularity condition (see Definition 2.8.1). This observation, together with the above estimates of partial derivatives of a_{ij} imply the following inequality for $x \in \partial W_l$ and $0 < r < \operatorname{diam}(\partial W_l)$

$$\begin{aligned} \frac{1}{\mathcal{H}^{n-1}(B(x,r)\cap\partial W_l)}\int_{B(x,r)\cap W_l} |\nabla A(X)|\mathrm{d}X &\lesssim_{n,C_l} \frac{1}{r^{n-1}}\int_{B(x,r)\cap W_l}\frac{1}{d_l}\mathrm{d}X \\ &\lesssim_{n,C_l} \frac{1}{r^{n-1}}\int_{B(x,r)\cap W_l}\frac{1}{r}\mathrm{d}X \\ &= \frac{1}{r^n}\int_{B(x,r)\cap W_l}\mathrm{d}X \le \frac{\mathcal{H}^n(B(x,r))}{r^n} \le C(n). \end{aligned}$$

Upon taking the supremum over $x \in \partial W_l$ we arrive at (&&). In consequence, Theorem 2.15 in [BH] gives us the ε-approximation of u by BV functions on sets $\exp_{p_l}^{-1}(V_l)$ for $l = 1, \dots, m$.

Step 3.

In the last step of the proof, we glue together the BV functions constructed above, to obtain one BV function approximating u on Ω.

For each V_l, for $l = 1, \dots, m$, denote by ϕ_l a BV function approximating u on V_l. Let σ be a permutation of indices $l = 1, \dots, m$ satisfying following conditions: The value of permutation $\sigma(1)$, is any number from 1 to m. The value of $\sigma(2)$ is any index l such that $V_{\sigma(1)} \cap V_{\sigma(2)} \neq \emptyset$. Then $\sigma(3)$ is such an index that $V_{\sigma(3)} \cap (V_{\sigma(1)} \cup V_{\sigma(2)}) \neq \emptyset$ and so on. Let us denote by $\phi_{\sigma(j)}$ a BV function approximating u on $V_{\sigma(j)}$. On a set $V_{\sigma(2)} \setminus V_{\sigma(1)}$ take a function $\phi_{\sigma(2)}\Big|_{V_{\sigma(2)} \setminus V_{\sigma(1)}}$. Let function $\phi^1 : V_{\sigma(1)} \cup V_{\sigma(2)} \to \mathbb{R}$ be defined as follows

$$\phi^1 = \begin{cases} \phi_{\sigma(1)} & \text{on } V_{\sigma(1)} \\ \phi_{\sigma(2)} & \text{on } V_{\sigma(2)} \setminus V_{\sigma(1)}. \end{cases}$$

Next choose any set $V_{\sigma(3)}$ that has a nonempty intersection with $V_{\sigma(1)} \cup V_{\sigma(2)}$ and a function $\phi_{\sigma(3)}\Big|_{V_{\sigma(3)} \setminus (V_{\sigma(1)} \cup V_{\sigma(2)})}$. Similarly as above, we define function ϕ^2 as follows:

$$\phi^2 = \begin{cases} \phi_{\sigma(1)} & \text{on } V_{\sigma(1)} \\ \phi_{\sigma(2)} & \text{on } V_{\sigma(2)} \setminus V_{\sigma(1)} \\ \phi_{\sigma(3)} & \text{on } V_{\sigma(3)} \setminus (V_{\sigma(1)} \cup V_{\sigma(2)}). \end{cases}$$

After finitely many steps we construct a function $\phi := \phi^{m-1}$ defined on Ω. Such a function has bounded variation as a consequence of the following reasoning. Firstly, each ϕ_l has bounded variation. Secondly, on sets $\partial V_l \cap V_k$ where the function ϕ may have additional jumps, the norm of the derivative of ϕ is bounded by $(1 + \varepsilon)$ times surface measure of these boundaries. Since all sets V_i are bounded Lipschitz and there are finitely many of them, the variation of ϕ remains finite. Finally, the fact that $|\nabla \phi|$ gives rise to the Carleson measure follows from the fact that $|\nabla \phi_i|$ all give rise to Carleson measures and the surface measures generated by jumps mentioned above are already Carleson measures. Indeed, $|\nabla \phi_i| \mathrm{d}X$ are Carleson measures because they were obtain by Theorem 1.3 in [HMM1]. The surface measures are Carleson measures because the measure of any ball is equal to the measure of its intersection with the boundary which is $(n-1)$-dimensional. The proof of Theorem 1.4.2 is therefore completed. □

Let us illustrate the above theorem with the example of the Laplace-Beltrami harmonic equation on the n-dimensional sphere $\mathbb{S}^n$.

Example 4.3.4. Let us consider an n-dimensional sphere $\mathbb{S}^n$ equipped with the coordinates $t = (t_1, \dots, t_n)$ and $|t|^2 = \sum t_i^2$ which come from stereographic projection. Take a stereographic projection and denote by $(x_1, \dots, x_{n+1})$ the standard coordinates in $\mathbb{R}^{n+1}$. Then we have:

$$x_i = \frac{2t_i}{|t|^2 + 1} \qquad \text{for } i = 1, \dots, n,$$
$$x_{n+1} = \frac{|t|^2 - 1}{|t|^2 + 1}.$$

Therefore, the metric induced on a sphere has a form

$$g = \frac{4}{(|t|^2+1)^2} I,$$

where I is an $n \times n$ identity matrix. Hence, we get $\det g = \left(\frac{4}{(|t|^2+1)^2}\right)^n$. Notice also that

$$g^{-1} = \frac{(|t|^2+1)^2}{4} I.$$

Let us now use Definition 4.1.7. Then the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n} u$ reads

$$\Delta_{\mathbb{S}^n} u(t_1, \dots, t_n) = \left(\frac{|t|^2+1}{2}\right)^n \sum_{i=1}^n \frac{\partial}{\partial t_i}\left(\left(\frac{2}{|t|^2+1}\right)^{n-2} \frac{\partial u}{\partial t_i}\right)$$
$$= \begin{cases} -\frac{(|t|^2+1)(n-2)}{2} \sum_{i=1}^n t_i \frac{\partial u}{\partial t_i} + \frac{(|t|^2+1)^2}{4} \sum_{i=1}^n \frac{\partial^2 u}{\partial t_i^2} & \text{for } n \geq 3, \\ \frac{(|t|^2+1)^2}{4} \sum_{i=1}^n \frac{\partial^2 u}{\partial t_i^2} & \text{for } n = 2. \\ \end{cases}$$

Hence, if u is harmonic i.e. $\Delta_{\mathbb{S}^n} u = 0$, then the associated matrix A from Definition 4.1.9 is diagonal and takes form $A = \operatorname{diag}\left(\left(\frac{2}{t^2+1}\right)^{n-2}\right)$. Let $\Omega \subset \mathbb{S}^n$ be a Lipschitz domain as in Theorem 1.4.2. Set $d_\Omega := \sup_{x\in\Omega} \operatorname{dist}_{\mathbb{S}^n}(x, \partial\Omega) \leq \operatorname{diam}(\partial\Omega)$ and since Ω is bounded we have $|t_i| \leq K_\Omega$ for some constant $K_\Omega > 0$. By direct computations, we obtain that:

$$\frac{\partial a_{ii}}{\partial t_k} = (n-2)\left(\frac{2}{1+t^2}\right)^{n-1} t_k, \qquad i, k = 1, \dots, n$$
$$|\nabla A| = (n-2)\sqrt{n}\left(\frac{2}{t^2+1}\right)^{n-1} t \leq (n-2)\sqrt{n}\left(2^{n-1}\sqrt{n} K_\Omega\right)$$
$$= \frac{2^{n-1} n(n-2) K_\Omega d_\Omega}{d_\Omega} = \frac{C_\Omega}{d_\Omega} \leq \frac{C_\Omega}{\operatorname{dist}(x, \partial\Omega)},$$

where $C_\Omega = 2^{n-1} n(n-2) K_\Omega$. Therefore, condition (&&) holds for matrix A.

Moreover, for $x \in \partial\Omega$ also condition (&) in Definition 4.1.9 can be verified directly:

$$\frac{1}{r^{n-1}} \int_{B(x,r)\cap\Omega} |\nabla A| \mathrm{d}X \leq \frac{1}{r^{n-1}} \int_{B(x,r)\cap\Omega} \frac{C_\Omega}{d_\Omega} \mathrm{d}X \lesssim_{C_\Omega} \frac{1}{r^{n-1}} \int_{B(x,r)\cap\Omega} \frac{1}{r} \mathrm{d}X \approx \frac{1}{r^n} r^n = 1.$$

Remark 4.3.5. Notice that everything that was proved so far in this chapter applies as much to A-harmonic functions on an open set $\Omega \subset M$. Indeed, in local coordinates equation (4.1) takes the form

$$\frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x^i}\left(\sum_l \sum_t \sqrt{\det g}\, a_{il}(x) g^{lt} \frac{\partial u}{\partial x^t}\right) = 0.$$

Equivalently, it can be written as

$$\frac{1}{\sqrt{\det g}} \operatorname{div}(B(x)\nabla u) = 0, \tag{4.12}$$

where for any $x \in \Omega$, the matrix $B(x)$ has coefficients $b_{ij}(x) = \sum_{l=1}^{n} \sqrt{\det g(x)}(a_{il}(x)g^{lj}(x))$, $j = 1, \dots, n$.

Hence, Theorem 1.4.2 holds for A-harmonic functions on Lipschitz domains on manifolds. In order to show this, it is enough to prove that $B(x)$ satisfies (&) and (&&). In order to prove (&&) we directly compute that for all $i, j, k = 1, \dots, n$ it holds that

$$\begin{aligned}
\frac{\partial b_{ij}}{\partial x^k} &= \frac{\partial}{\partial x^k}\left(\sum_{s=1}^{n} \sqrt{\det g}\, a_{is}(x) g^{sj}\right) \\
&= \sum_{s=1}^{n}\left(\left(\frac{\partial}{\partial x^k}\sqrt{\det g}\right) a_{is}(x) g^{sj} + \sqrt{\det g}\frac{\partial a_{is}(x)}{\partial x^k} g^{sj} + \sqrt{\det g}\, a_{is}(x)\frac{\partial g^{sj}}{\partial x^k}\right) \\
&= \sum_{s=1}^{n}\left(\frac{1}{2\sqrt{\det g}} \det g \operatorname{tr}\left(g^{-1}\frac{\partial g}{\partial x^k}\right) a_{is}(x) g^{sj} + \sqrt{\det g}\frac{\partial a_{is}(x)}{\partial x^k} g^{sj} + \sqrt{\det g}\, a_{is}(x)\frac{\partial g^{sj}}{x^k}\right) \\
&= \sqrt{\det g}\sum_{s=1}^{n}\left[\frac{1}{2}\left(\sum_{a,b=1,\dots,n} g^{ab}\frac{\partial g_{ba}}{\partial x^k}\right) a_{is}(x) g^{sj} + \frac{\partial a_{is}(x)}{\partial x^k} g^{sj} + a_{is}(x)\frac{\partial g^{sj}}{x^k}\right].
\end{aligned}$$

Denote by $M := \|A\|_{L^\infty}(\Omega)$ and let constants C_l be as in (4.9). Then we have

$$\begin{aligned}
\left|\frac{\partial b_{ij}}{\partial x^k}\right| &\le \sqrt{n! C_l^n}\sum_{s=1}^{n}\left|\frac{1}{2} n^2 C_l^3 a_{is}(x) + \frac{\partial a_{is}(x)}{\partial x^k} C_l + a_{is}(x) C_l\right| \\
&\le \sqrt{n! C_l^n}\, n\left(\frac{1}{2} n^2 C_l^3 M + |\nabla A| C_l + M C_l\right) \\
&\le \sqrt{n! C_l^n}\, n\left(\frac{1}{2} n^2 C_l^3 M + M C_l\right) + \sqrt{n! C_l^n}\, n C_l |\nabla A| \\
&\lesssim_{n,C_l} \frac{M}{d_l} + \frac{1}{\operatorname{dist}(x, \partial W_l)} \qquad \text{by (4.7), (4.10) and (4.11)} \qquad (4.13) \\
&\lesssim \frac{1}{\operatorname{dist}(x, \partial W_l)}.
\end{aligned}$$

Let us now proceed to proving (&). We have for each $l = 1, \dots, n$

$$\begin{aligned}
&\frac{1}{\mathcal{H}^{n-1}(B(x,r)\cap\partial W_l)}\int_{B(x,r)\cap W_l} |\nabla B(X)|\, dX \\
&\qquad \lesssim_{M,C_l} \frac{1}{r^{n-1}}\int_{B(x,r)\cap W_l} \frac{1}{\operatorname{diam} W_l} + \sqrt{n! C_l^n}\, n C_l |\nabla A|\, dX \qquad (4.14) \\
&\qquad \lesssim \frac{1}{r^{n-1}}\int_{B(x,r)\cap W_l} \frac{1}{r} + \sqrt{n! C_l^n}\, n C_l |\nabla A|\, dX \qquad (4.15) \\
&\qquad \lesssim_{n,M} C, \qquad (4.16)
\end{aligned}$$

where in the inequality (4.14) we use the fact that $d_l \approx \operatorname{diam} W_l$ and inequality (4.13). In the inequality (4.15) we notice that $r < \operatorname{diam} W_l$. Lastly, in the inequality (4.16) we use property (&) for matrix A.

Therefore, upon repeating the gluing argument (**Step 3**) above, Theorem 1.4.2 extends to the setting of A-harmonic functions with $A = A(x)$ as in Definition 4.1.9.

4.4 Quantitative Fatou Property

The goal of this chapter is to apply the ε-approximability to prove the Quantitative Fatou Property on Lipschitz domains in Riemannian manifolds.

Since the choice of good maps and their properties will be important in what follows, let us briefly recall some necessary facts.

If M is a Riemannian manifold, then we always have a chart preserving the Lipschitzness of a set, namely $\exp_p^{-1}$ taken on such set $U \subset M$ that $\exp_p^{-1}$ is a diffeomorphism,see the proof of Theorem 1.4.2 above.

However, the stronger result is available. Namely, we can take charts that preserve bounded Lipschitz sets. Indeed, Lemma 4.4.2 shows that any chart would do as long as Lipschitz domains are 1-connected at the boundary and the image is bounded. Therefore, we do not need to necessarily use exponential maps. Nevertheless, we use them because they are convenient and handy to work with, but any chart with similar properties would be sufficient. By similar properties we mean that:

- we can take such sets U_i as charts, that each U_i contains a ball with radius uniformly bounded from below,
- the Lipschitz constants of maps are uniformly bounded from above.

The only difference between the above choice of Lipschitz maps and the exponential map is that now different charts will have different Lipschitz constants. However, it only affects the constants in the estimates, which yields that all results are still true.

In the next definition we recall topological notion that plays a crucial role in the studies of the extension of mappings, including the continuous and homeomorphic extensions, see e.g. theorem in [V, Chapter 2, Section 17]. Moreover, see [A], where the notion of prime ends is used to determine the existence of extension.

Definition 4.4.1. Let X be a metric space. We will say that a set $U \subset X$ is 1-connected at the boundary if for every point $x \in \partial U$ there exists its arbitrarily small neighbourhood U_x such that $U \cap U_x$ is connected.

An example of a set that is not 1-connected is a slit disc. It is a disc $B(p, r) \subset \mathbb{R}^2$ with a removed line segment joining the center p with a boundary, say at point x. Then at point x any small enough neighbourhood U_x has a property that $B(p, r) \cap U_x$ has two connected components.

Lemma 4.4.2. *Let $U \subset (X, d)$ be an open, connected, precompact and 1-connected at the boundary set. Let further $h : U \to h(U)$ be a homeomorphism such that $h(U)$ is bounded in $(Y, \tilde{d})$. Then for any bounded Lipschitz subset $U' \subset U$ it holds that $h(U')$ is also bounded Lipschitz in $h(U)$.*

The proof of the lemma is in the Appendix of this chapter.

Remark 4.4.3. One can approach defining the counting function N either independently of maps or in maps. In what follows we take the first approach as it is more natural in the manifold setting. Nevertheless we would like to briefly comment on the approach via maps. Namely, at every boundary point we can choose the local coordinates and in those coordinates define locally the N function. Then we cover $\partial\Omega$ with balls of radius $R << \mathrm{r_{inj}}$, e.g. $R = \frac{1}{20}\mathrm{r_{inj}}$ centered at some points $p_i \in$

$\partial\Omega$. Define counting function in the set $B(p_1, R) \cap \partial\Omega$ by using a chart that preserves Lipschitzness on that set, e.g. $\exp_{p_1}^{-1}$. Then proceed inductively. Namely, define a counting function on the set $(B(p_2, R) \setminus B(p_1, R)) \cap \partial\Omega$ by using a chart on that set, and continue until all boundary is covered. For different choices of charts we obtain different counting functions, but the Quantitative Fatou Property holds for all of them with different constants. Now, since we already know that a harmonic function defined on Ω is ε-approximable for every ε (Chapter 4.3) we may apply Lemma 2.9 in [KKPT] and get the Quantitative Fatou Property for Lipschitz domains in complete Riemannian manifolds on balls with radii $r < c < \mathrm{r_{inj}}$.

Recall Definition 4.1.15 and observe that the counting function N defined in such a way does not depend on a chosen chart. We would like to prove the Quantitative Fatou Property for such N, since it would be desirable that the QFP is independent of charts and relies only on the Riemannian structure of the manifold.

Recall Definition 4.1.13 of the (generalized) cone. In what follows, we apply this definition to r_1 and r_2 the distances of p to the consecutive points in an $(r, \varepsilon, \theta, p)$-admissible sequence, cf. Def. 4.1.15. Moreover, if point p is fixed , then we skip writing it and denote $\Gamma_{r_1,r_2} := \Gamma_{r_1,r_2}(p)$.

Lemma 4.4.4. *Let $\Omega \subset M$ be a Lipschitz domain and Γ_{r_1,r_2}, the doubly truncated cone with the aperture α, be connected. Let further $x_1 \in \Gamma_{r_1,r_2} \cap S(p, r_1)$, $x_2 \in \Gamma_{r_1,r_2} \cap S(p, r_2)$ be elements of the $(r, \varepsilon, \theta, p)$-admissible sequence corresponding to r_1 and r_2, respectively. Denote by $\widetilde{\Gamma}_{r_1,r_2}(p)$ a doubly truncated cone with the aperture $\widetilde{\alpha} > \alpha$, so that $\Gamma_{r_1,r_2} \subset \widetilde{\Gamma}_{r_1,r_2}$. Then there exists a curve $\gamma : [r_2, r_1] \to \widetilde{\Gamma}_{r_1,r_2}$ with $\gamma(r_2) = x_2$ and $\gamma(r_1) = x_1$ and such that $\gamma(r) \in \widetilde{\Gamma}_{r_1,r_2}$ for all $r \in [r_2, r_1]$, with the following properties:*

- *the length $l(\gamma)$ satisfies $l(\gamma) \leq Kd(x_1, x_2)$ for some $K > 0$,*
- *$\left|\frac{\partial\gamma}{\partial r}\right| \leq C_{\alpha,\theta}$ for some constant $C_{\alpha,\theta}$ depending only on α and θ,*
- *γ intersects every sphere $S(p, r)$ for $r_1 < r < r_2$ exactly once.*

The set Γ_{r_1,r_2} may fail to be Lipschitz, because at the points of intersection of a cone Γ_r and a sphere $S(p, r_2)$ the regularity of the boundary of Γ_{r_1,r_2} may worsen, for instance be only Hölder as some cusps may occur. That is why we take a bigger set $\widetilde{\Gamma}_{r_1,r_2}$.

Proof. First, by using the exponential map we can reduce the discussion to the ambient space $\mathbb{R}^n$. Notice also that the Lipschitzness of Ω implies that by taking Γ_R with R small enough, we may ensure that Γ_{r_1,r_2} is connected for every pair r_1, r_2. It is enough to consider such R that for every point $x \in B(p, R)$ the distance $d(x, \partial\Omega)$ is achieved at some point $y \in \partial\Omega \cap B(p, R)$. Such R exists because $\partial\Omega$ is compact and it is Lipschitz. Moreover, such R depends on Lipschitz constant of $\partial\Omega$ and Ω.

Note that for small enough R, the boundary $\partial\Gamma_R$ does not "turn". By turn we mean the following property. Take a tangent space to $\partial\Omega$ at p, denoted by $T_p(\partial\Omega)$. Such a space exists at almost every point $p \in \partial\Omega$, because $\partial\Omega$ is a Lipschitz set. Moreover, that space is an $(n-1)$-dimensional subspace of n-dimensional tangent space $T_p(M)$. Then any line perpendicular to $T_p(\partial\Omega)$ intersects $\partial\Gamma_R$ as long as that line is close enough to point p. Furthermore, such a line intersects $\partial\Gamma_R$ at least twice: once when it intersects the surface, where $d(q, p) = (1 + \alpha)d(q, \partial\Omega)$ and the second time when it intersects sphere $S(p, R)$. However, it can occur that the surface defined by the equation $d(q, p) = (1 + \alpha)d(q, \partial\Omega)$ is intersected more than once. If it happens, then we say that that $\partial\Gamma_R$ turns, whereas if there are only two points of intersection we will say that $\partial\Gamma_R$ does not turn. Again, due to compactness and Lipschitz property of $\partial\Omega$ we can ensure that for R small enough $\partial\Gamma_R$ does not turn. Moreover, compactness of $\overline{\Omega}$ allows us to choose R small enough satisfying all the aforementioned properties at every point of boundary $\partial\Omega$ i.e. sets $\Gamma_{r_1,r_2}(p)$ are connected for all $p \in \partial\Omega$ and all $0 < r_1 < r_2 \leq R$ and $\Gamma_R(p)$ does not turn.

Next, let us prove the following observation.

Claim 4.4.5. *There exists a constant $\delta_\alpha > 0$ such that for every point $x \in \partial\Gamma_{r_1,r_2}$ a ball $B(x, \delta_\alpha d(x,p)) \subset \widetilde{\Gamma}_{(1+\delta_\alpha)r_1,(1-\delta_\alpha)r_2}$.*

Proof. Let $\widetilde{x} \in B(x, \delta_\alpha d(x,p))$. Denote by q and $\widetilde{q}$ points on $\partial\Omega$ such that $d(x,\partial\Omega)$ and $d(\widetilde{x},\partial\Omega)$ are attained, i.e. $d(q,x) = d(x,\partial\Omega)$ and $d(\widetilde{q},\widetilde{x}) = d(\widetilde{q},\partial\Omega)$, respectively. Then

$$\begin{aligned} d(x,\partial\Omega) &\leq d(x,\widetilde{q}) \leq d(\widetilde{x},\partial\Omega) + d(x,\widetilde{x}) \\ &\leq d(\widetilde{x},\partial\Omega) + \delta_\alpha d(x,p) \\ &= d(\widetilde{x},\partial\Omega) + \delta_\alpha(1+\alpha)d(x,\partial\Omega), \end{aligned}$$

where the equality is the consequence of $x \in \partial\Gamma_{r_1,r_2}$ and so, in particular x satisfies the equation of the boundary of $\Gamma(p)$, cf. Definition 4.1.13.

Hence

$$(1-\delta_\alpha(1+\alpha))d(x,\partial\Omega) \leq d(\widetilde{x},\partial\Omega). \tag{4.17}$$

Note, that for this inequality to make sense, $\delta_\alpha < \frac{1}{1+\alpha}$. Now we can estimate the distance of $\widetilde{x}$ to vertex p:

$$\begin{aligned} d(\widetilde{x},p) &\leq d(x,p) + d(\widetilde{x},x) \leq (1+\delta_\alpha)d(x,p) \\ &= (1+\delta_\alpha)(1+\alpha)d(x,\partial\Omega) \\ &\leq \frac{(1+\delta_\alpha)(1+\alpha)}{1-\delta_\alpha(1+\alpha)} d(\widetilde{x},\partial\Omega) \qquad\qquad \text{(by (4.17))} \\ &= \left(1 + \frac{\alpha + 2\delta_\alpha(1+\alpha)}{1-\delta_\alpha(1+\alpha)}\right) d(\widetilde{x},\partial\Omega). \end{aligned}$$

Therefore, in order to make sure that ball $B(x,\delta_\alpha r) \subset \widetilde{\Gamma}_{r_1,r_2}$ we need to find δ_α such that

$$\frac{\alpha + 2\delta_\alpha(1+\alpha)}{1-\delta_\alpha(1+\alpha)} \leq \widetilde{\alpha}$$

which gives

$$\delta_\alpha \leq \frac{\widetilde{\alpha} - \alpha}{(1+\alpha)(2+\widetilde{\alpha})} < \frac{1}{1+\alpha}$$

and completes the proof of the claim. □

We now show that there exist certain two-dimensional quasirectangles contained in $\widetilde{\Gamma}_{r_1,r_2}$ which enable to choose curve γ in such a way that $\left|\frac{\partial\gamma}{\partial r}\right|$ is uniformly bounded in $\widetilde{\Gamma}_{r_1,r_2}$. Recall that by assumption x_1 and x_2 are given points such that $x_1 \in \Gamma_{r_1,r_2} \cap S(p,r_1)$, $x_2 \in \Gamma_{r_1,r_2} \cap S(p,r_2)$. Let l_1 and l_2 denote line segments beginning at p and crossing x_1 and x_2 respectively. Let further L_{12} denote a two-dimensional cone spanned between l_1 and l_2. Set $\hat{r} = (1-\delta_\alpha)r_1$. We would like to show now that a quasirectangle $K_{r_1,\hat{r}} := L_{12} \cap (B(p,r_1) \setminus B(p,\hat{r})) \subset \widetilde{\Gamma}_{r_1,r_2}$.

First we need to know that $\hat{r} > r_2$. Since $r_2 \le \theta r_1$ it is enough to take $\delta_\alpha < 1-\theta$. We slightly abuse the notation and let

$$\delta_{\alpha,\theta} := \min\{\delta_\alpha, 1-\theta\}. \tag{4.18}$$

Then we change $\hat{r}$, if necessary, and let $\hat{r} := (1-\delta_{\alpha,\theta})r_1$. For every point in $K_{r_1,\hat{r}}$ there exists a line segment $l_{\widetilde{x}}$ joining point p with some point $\widetilde{x} \in \partial\Gamma_{\hat{r}}$ such that this point lies on $l_{\widetilde{x}}$. Denote by $\widetilde{r} := d(\widetilde{x},p)$. Since by the previous step of the proof we know that a ball $B(\widetilde{x},\delta_{\alpha,\theta}\widetilde{r}) \subset \widetilde{\Gamma}_{r_1,r_2}$ it suffices to observe that $l_{\widetilde{x}} \subset B(\widetilde{x},\delta_{\alpha,\theta}\widetilde{r})$. Indeed, since $d(p,K_{r_1,\hat{r}}) > \hat{r}$, it is therefore enough to show that

$$\widetilde{r} - \hat{r} = \widetilde{r} - (1-\delta_{\alpha,\theta})r_1 < \delta_{\alpha,\theta}\widetilde{r}.$$

Moreover, the latter is trivially equivalent to $\widetilde{r} < r_1$ which is always true.

Let us now construct a curve γ as in the assertion of the lemma. It consists of two subcurves. First one, denoted by γ_1, is contained in a line segment starting at p and containing x_2 and the second one, denoted by γ_2, is contained in a quasirectangle $K_{r_1,(1-\delta_{\alpha,\theta})r_1}$ between r_1 and $(1-\delta_{\alpha,\theta})r_1$. Moreover, we can choose γ_2 in such a way that its derivative is bounded. Indeed, on γ_1 it holds that $|\frac{\partial\gamma_1}{\partial r}| = 1$, while on γ_2 we can estimate that

$$\left|\frac{\partial\gamma_2}{\partial r}\right| \le 1 + \frac{\alpha}{\delta_{\alpha,\theta}}.$$

To see the above estimate take as γ_2 a quasidiagonal of quasirectangle. By this, we mean a curve that in polar coordinates in the plane L_{12} with point p corresponding to 0 is given by

$$\gamma(r) = \left(r, \phi_2 + \frac{\phi_1-\phi_2}{\delta_{\alpha,\theta}r_1}\big(r-(1-\delta_{\alpha,\theta})r_1\big)\right)$$

with $r \in [(1-\delta_{\alpha,\theta})r_1, r_1]$, where $x_1 = (r_1,\phi_1)$ and $x_2 = (r_2,\phi_2)$. This curve starts at the endpoint of γ_1 and ends at x_1. One gets that $\frac{\partial\gamma_2}{\partial r} = \left(1, \frac{\phi_1-\phi_2}{\delta_{\alpha,\theta}r_1}\right)$. Hence,

$$\left|\frac{\partial\gamma_2}{\partial r}\right| = \sqrt{1 + r^2\left(\frac{\phi_1-\phi_2}{\delta_{\alpha,\theta}r_1}\right)^2} \le \sqrt{1+\frac{\alpha^2}{\delta_{\alpha,\theta}^2}} \le 1 + \frac{\alpha}{\delta_{\alpha,\theta}}.$$

Thus, the derivative with respect to r is bounded on both curves. Furthermore, we can choose γ_2 such that its length with respect to Euclidean distance $l(\gamma_2) < 2\pi r_1$, since any two points on different concentric spheres can be connected by a curve of length smaller than perimeter of a bigger of those two spheres and quasidiagonal is such a curve. Quasidiagonal also intersects every sphere centered at p with radius between r_2 and r_1 exactly once.

Finally we estimate the Euclidean length of γ:

$$
\begin{aligned}
l(\gamma) = l(\gamma_1) + l(\gamma_2) &\le ((1-\delta_{\alpha,\theta})r_1 - r_2) + 2\pi r_1 \\
&\le (r_1 - r_2) + 2\pi r_1 \le (r_1 - r_2) + \frac{2\pi}{1-\theta}(r_1 - r_2) \\
&\le \left(1 + \frac{2\pi}{1-\theta}\right) d(x_1, x_2),
\end{aligned}
\tag{4.19}
$$

where in (4.19) we use that x_1 and x_2 belong to the $(r, \varepsilon, \theta, p)$-admissible sequence and so $r_2 < \theta r_1$. Let us notice that constants in all estimates depend only on α, $\widetilde{\alpha}$, θ and the Lipschitz constant of the exponential map, but there is no dependence on r_1 and r_2. □

Remark 4.4.6. In order to apply Lemma 4.4.4 we need radius r to be sufficiently small. Fortunately for the Quantitative Fatou Property it is necessary to know only the behaviour of harmonic function u close to boundary $\partial\Omega$. Therefore, it is not a problem that we need to restrict possible r, as long as we can find uniformly some radius r for all boundary points such that every Γ_r satisfies all our assumptions at every point p of $\partial\Omega$. Since Ω is compact, it can be achieved.

If r is small enough, i.e. $r < r_{\text{inj}}$, then $\Gamma_r(p)$ is contained in a ball centered at p such that there are local coordinates due to the exponential map which is a bounded diffeomorphism on that ball. Therefore, for sufficiently small r we can always assume that the ambient space is Euclidean.

The following lemma is the key auxiliary observation needed in the proof of the Quantitative Fatou Property, see Theorem 4.4.8 below. Moreover, the lemma is a Riemannian counterpart of the main claim in the proof of Lemma 2.9 in [KKPT].

Lemma 4.4.7. *Let $\Omega \subset M$ be a Lipschitz domain and $u : \Omega \to \mathbb{R}$ be a bounded harmonic function with $\|u\|_\infty \le 1$. Suppose that $\varepsilon > 0$ and ϕ is an $\frac{\varepsilon}{4}$-approximation of u. If the counting function $N(r, \varepsilon, \theta)(p) \ge k$ for some $k \in \mathbb{N}$, then the following holds*

$$
\int_{\Gamma_r(p)} \frac{|\nabla\phi(x)|}{d(x,p)^{n-1}} \mathrm{d}x \ge k C_{n,\varepsilon,\theta,\alpha}, \tag{4.20}
$$

where $C_{n,\varepsilon,\theta,\alpha} > 0$ is a constant depending only on $n, \varepsilon, \theta, \alpha$.

Proof. Without loss of generality we may assume that $\Omega \subset \mathbb{R}^n$, see Remark 4.4.6. Let us also assume that $p = 0$. Since, by assumptions $N(r, \varepsilon, \theta)(p) \ge k$, there is a finite sequence of points $x_1, \ldots, x_k \in \Gamma_r(0)$ such that

$$
0 < |x_k| < \cdots < |x_1| < r, \qquad |x_{j+1}| \le \theta |x_j| \qquad \text{for } j = 1, \ldots, k-1
$$

and

$$
|u(x_j) - u(x_{j+1})| \ge \varepsilon.
$$

Since u after composing with exponential map is Lipschitz, it is in particular Hölder continuous and bounded. Thus, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$
|u(x) - u(x_j)| < \frac{\varepsilon}{8} \quad \text{for } x \in l_j := \left\{ y \in \Gamma_r(0) \cap S(0, |x_j|) : d_{S_j}(y, x_j) < \delta |x_j| \right\}, \tag{4.21}
$$

where d_{S_j} denotes a distance on a sphere $S(0,|x_j|)$. Therefore, for all $x \in l_j$ and $y \in l_{j+1}$ it holds that: $|u(x)-u(y)| \geq \frac{3\varepsilon}{4}$. Moreover, we have $|\phi(x)-\phi(y)| \geq \frac{\varepsilon}{4}$.

Let U_j be a doubly truncated Euclidean cone such that its angle is 2δ, its vertex is 0 and for every $z \in U_j$ the following holds: $r_j \leq |z| \leq r_{j-1}$. Let $\gamma_j \subset U_j$ be a curve given by the assertion of Lemma 4.4.4. Consider the transformation $F_j : U_j \to \mathbb{R}^n$ with the following properties:

1. the image of a symmetry axis of U_j, denoted by l_{U_j} is γ_j, i.e. $F_j(l_{U_j}) = \gamma_j$, $F_j(x) = \gamma_j(|x|)$ for every $x \in l_{U_j}$.
2. for every r it holds that $F_j|_{U_j \cap S(0,r)}$ is a rotation such that a point on a symmetry axis is transformed into $\gamma_j(r)$.

Such F_j is piecewise smooth, because γ_j is piecewise smooth. Furthermore, F_j does not change the volume of a set U_j, and hence the absolute value of its Jacobi determinant equals 1. To see this claim let $U \subset U_j$ be measurable and compute that

$$\mathrm{Vol}(F_j(U)) := \int_{F_j(U)} 1 \mathrm{d}x = \int_{r_j}^{r_{j-1}} \int_{F_j(U)\cap S(0,r)} 1 \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}r.$$

Here we apply the coarea formula with function $f(x) = |x|$, see [EG, Chapter 3.4]. Moreover, the Jacobian of f equals 1, see [EG, Chapter 3.2] for the definition of the Jacobian of a real-valued function. Since the $(n-1)$-Hausdorff measure on a sphere is rotation invariant we get

$$\int_{r_j}^{r_{j-1}} \int_{F_j(U)\cap S(0,r)} 1 \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}r = \int_{r_j}^{r_{j-1}} \int_{U\cap S(0,r)} 1 \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}r = \int_U 1 \mathrm{d}x,$$

where the latter equality follows again from the coarea formula. Hence for every measurable set $U \subset U_j$ we have

$$\mathrm{Vol}(F_j(U)) = \mathrm{Vol}(U)$$

and the claim is proven.

Notice that $F_j(U_j \cap S(0,r_j)) = l_j$ and $F_j(U_j \cap S(0,r_{j+1})) = l_{j+1}$. It follows that

$$\left| \int_{r_j}^{r_{j-1}} \frac{\partial}{\partial r} \phi(F_j) \mathrm{d}r \right| \geq \frac{\varepsilon}{4}. \tag{4.22}$$

Since F_j is given by a rotation, its partial derivative with respect to r is solely determined by $\frac{\partial \gamma}{\partial r}$. However, due to Lemma 4.4.4 we know that $\left|\frac{\partial \gamma}{\partial r}\right| \leq 1 + \frac{\alpha}{\delta_{\alpha,\theta}}$ and hence $\left|\frac{\partial}{\partial r} F_j\right| \leq 1 + \frac{\alpha}{\delta_{\alpha,\theta}}$. We are now in a position to show assertion (4.20). It holds that:

$$\int_{\Gamma_{r_j,r_{j+1}}} \frac{|\nabla \phi(x)|}{|x|^{n-1}} \mathrm{d}x \geq \int_{F_j(U_j)} \frac{|\nabla \phi(x)|}{|x|^{n-1}} \mathrm{d}x = \int_{U_j} \frac{|\nabla \phi(F_j(\widetilde{x}))|}{|\widetilde{x}|^{n-1}} \mathrm{d}\widetilde{x},$$

as $F_j(U_j) \subset \Gamma_{r_j,r_{j+1}}$ and by the change of variables formula.

Let us notice that by the chain rule we have

$$\nabla(\phi \circ F_j)(\widetilde{x}) = (\mathrm{ad} DF_j)(\widetilde{x}) \nabla \phi(F_j(\widetilde{x})),$$

where by $(\mathrm{ad} DF_j)(\widetilde{x})$ we mean the adjoint operator of $DF_j(\widetilde{x})$ defined via the scalar product given by the Riemannian metric g:

$$g(DF_j(\widetilde{x})X, Y) = g(X, (\mathrm{ad} DF_j)(\widetilde{x})Y) \tag{4.23}$$

for all $X, Y \in T_{\widetilde{x}}M^n$. Therefore, in the spherical coordinates $(r, \phi_1, \dots, \phi_{n-1})$ on Ω it holds that

$$\frac{\partial}{\partial r}(\phi \circ F_j)(\widetilde{x}) = \langle r_1((\mathrm{ad} DF_j)(\widetilde{x})), \nabla \phi(F_j(\widetilde{x}))\rangle,$$

where $r_1((\mathrm{ad} DF_j)(\widetilde{x}))$ stands for the first row of matrix $(\mathrm{ad} DF_j)(\widetilde{x})$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. Hence we get

$$|\nabla \phi(F_j(\widetilde{x}))| \geq \frac{\left|\frac{\partial}{\partial r}(\phi \circ F_j)(\widetilde{x})\right|}{|r_1((\mathrm{ad} DF_j)(\widetilde{x}))|}. \tag{4.24}$$

Let us now explain how to obtain metric g in spherical coordinates. Set f to be a parametrization:

$$\begin{aligned} x_1 &= r\cos(\phi_1), \\ x_s &= r\cos(\phi_s)\prod_{m=1}^{s-1}\sin(\phi_m) \quad \text{for } s = 2, \dots, n-1, \\ x_n &= r\prod_{m=1}^{n-1}\sin(\phi_m). \end{aligned}$$

In the standard coordinates x_i the metric has a form $g_{ij} = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. The pullback metric $g'_{ab} = (f^*g)_{ab}$. We get the following equation

$$g'_{ab} = \sum_{i,j=1}^{n} g_{ij}\frac{\partial x_i}{\partial_a}\frac{\partial x_j}{\partial_b} = \sum_{i=1}^{n}\frac{\partial x_i}{\partial_a}\frac{\partial x_i}{\partial_b}, \tag{4.25}$$

where ∂_a, ∂_b denote derivatives with respect to either r or one of ϕ_m and we have a convention that $a = 1$ corresponds to coordinate r and $a > 1$ corresponds to ϕ_{a-1}.

Notice that if $a \neq b$ the sum (4.25) becomes zero. When $a = b$ we get

$$\begin{aligned} g'_{11} &= 1, \\ g'_{22} &= r^2, \\ g'_{aa} &= r^2\prod_{m=1}^{a-2}\sin^2(\phi_m) \qquad \text{for } a = 3, \dots, n. \end{aligned}$$

Therefore, metric g' in spherical coordinates is given by the following matrix (we change the name g' back to g):

$$g = \begin{bmatrix} 1 & & & & \\ & r^2 & & & \\ & & r^2\sin^2(\phi_1) & & \\ & & & \ddots & \\ & & & & r^2\sin^2(\phi_1)\cdot\ldots\cdot\sin^2(\phi_{n-2}) \end{bmatrix},$$

where r stands for $|\widetilde{x}|$.

We are now in the position to compute $r_1((\mathrm{ad}DF_j)(\widetilde{x}))$. Let us denote tangent vectors $X, Y \in T_{\widetilde{x}}M$ by $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n)$ and recall that they are arbitrary. Let us also denote by $(z_1, z_2, \dots, z_n) := (r, \phi_1, \dots, \phi_{n-1})$. For brevity we skip writing $\widetilde{x}$. By using (4.23), we get:

$$
\begin{aligned}
&Y_1 \sum \frac{\partial F_{j,1}}{\partial z_i} X_i + Y_2 r^2 \sum \frac{\partial F_{j,2}}{\partial z_i} X_i + \dots + Y_n r^2 \sin^2(\phi_1) \cdot \dots \cdot \sin^2(\phi_{n-2}) \sum \frac{\partial F_{j,n}}{\partial z_i} X_i \\
&= X_1 \sum (\mathrm{ad}DF_j)_{1,i} Y_i + X_2 r^2 \sum (\mathrm{ad}DF_j)_{2,i} Y_i + \dots + X_n r^2 \sin^2(\phi_1) \cdot \dots \cdot \sin^2(\phi_{n-2}) \sum (\mathrm{ad}DF_j)_{n,i} Y_i .
\end{aligned}
$$

We now continue the computations to obtain that

$$
\begin{aligned}
&X_1 \sum (\mathrm{ad}DF_j)_{1,i} Y_i + X_2 r^2 \sum (\mathrm{ad}DF_j)_{2,i} Y_i + \dots + X_n r^2 \sin^2(\phi_1) \cdot \dots \cdot \sin^2(\phi_{n-2}) \sum (\mathrm{ad}DF_j)_{n,i} Y_i \\
&= Y_1 (X_1 (\mathrm{ad}DF_j)_{1,1} + X_2 r^2 (\mathrm{ad}DF_j)_{2,1} + \dots + X_n r^2 \sin^2(\phi_1) \cdot \dots \cdot \sin^2(\phi_{n-2}) (\mathrm{ad}DF_j)_{n,1}) \\
&+ Y_2 (X_1 (\mathrm{ad}DF_j)_{1,2} + X_2 r^2 (\mathrm{ad}DF_j)_{2,2} + \dots + X_n r^2 \sin^2(\phi_1) \cdot \dots \cdot \sin^2(\phi_{n-2}) (\mathrm{ad}DF_j)_{n,2}) + \dots \\
&+ Y_n (X_1 (\mathrm{ad}DF_j)_{1,n} + X_2 r^2 (\mathrm{ad}DF_j)_{2,n} + \dots + X_n r^2 \sin^2(\phi_1) \cdot \dots \cdot \sin^2(\phi_{n-2}) (\mathrm{ad}DF_j)_{n,n}).
\end{aligned}
$$

Since X, Y are arbitrary we may think about above expressions as polynomials in $X_1, \dots, X_n, Y_1, \dots, Y_n$. Moreover, since we are interested in the first row of the adjoint operator $r_1((\mathrm{ad}DF_j)(\widetilde{x}))$ we need to obtain $((\mathrm{ad}DF_j)(\widetilde{x}))_{1,i}$ for $i = 1, \dots, n$. Hence, we need to compare the appropriate coefficients in these polynomials.

We get

$$
\begin{aligned}
&(\mathrm{ad}DF_j)_{1,1} = \frac{\partial F_{j,1}}{\partial z_1} = \frac{\partial F_{j,1}}{\partial r} \quad \text{for } Y_1 X_1, \\
&(\mathrm{ad}DF_j)_{1,2} = r^2 \frac{\partial F_{j,2}}{\partial z_1} = r^2 \frac{\partial F_{j,2}}{\partial r} \quad \text{for } Y_2 X_1, \\
&\qquad \vdots \\
&(\mathrm{ad}DF_j)_{1,n} = r^2 \sin^2(\phi_1) \cdot \dots \cdot \sin^2(\phi_{n-2}) \frac{\partial F_{j,n}}{\partial z_1} = r^2 \sin^2(\phi_1) \cdot \dots \cdot \sin^2(\phi_{n-2}) \frac{\partial F_{j,n}}{\partial r} \quad \text{for } Y_n X_1.
\end{aligned}
$$

Finally, we have

$$
r_1((\mathrm{ad}DF_j)(\widetilde{x})) = \left(\frac{\partial}{\partial r} F_{j,1}, r^2 \frac{\partial}{\partial r} F_{j,2}, r^2 \sin^2(\phi_1) \frac{\partial}{\partial r} F_{j,3}, \dots, r^2 \sin^2(\phi_1) \cdot \ldots \cdot \sin^2(\phi_{n-2}) \frac{\partial}{\partial r} F_{j,n} \right).
$$

Furthermore, we can now find the following estimate needed to complete (4.24):

$$
\begin{aligned}
&|r_1((\mathrm{ad}DF_j)(\widetilde{x}))|^2 \\
&= \left(\frac{\partial}{\partial r} F_{j,1} \right)^2 + r^2 \left(r^2 \frac{\partial}{\partial r} F_{j,2} \right)^2 + \dots \\
&+ r^2 \sin^2(\phi_1) \cdot \ldots \cdot \sin^2(\phi_{n-2}) \left(r^2 \sin^2(\phi_1) \cdot \ldots \cdot \sin^2(\phi_{n-2}) \frac{\partial}{\partial r} F_{j,n} \right)^2 \\
&\leq \left(\frac{\partial}{\partial r} F_{j,1} \right)^2 + r^4 \left(r^2 \left(\frac{\partial}{\partial r} F_{j,2} \right)^2 + \dots + r^2 \sin^2(\phi_1) \cdot \ldots \cdot \sin^2(\phi_{n-2}) \left(\frac{\partial}{\partial r} F_{j,n} \right)^2 \right)
\end{aligned}
$$

$$\leq \left(\left(\frac{\partial}{\partial r} F_{j,1} \right)^2 + r^2 \left(\frac{\partial}{\partial r} F_{j,2} \right)^2 + \cdots + r^2 \sin^2(\phi_1) \cdot \ldots \cdot \sin^2(\phi_{n-2}) \left(\frac{\partial}{\partial r} F_{j,n} \right)^2 \right) \cdot \left(1 + (n-1)^2 r^4 \right)$$

$$= \left| \frac{\partial}{\partial r} F_j(\widetilde{x}) \right|^2 \left(1 + (n-1)^2 r^4 \right) \lesssim 2(n-1)^2 \left| \frac{\partial}{\partial r} F_j(\widetilde{x}) \right|^2,$$

where $|\cdot|$ stands for the length of a vector with respect to the scalar product g also in the first inequality we use that sin is bounded by 1. We employ the above estimate in(4.24) and get the following inequality:

$$\int_{U_j} \frac{|\nabla \phi(F_j(\widetilde{x}))|}{|\widetilde{x}|^{n-1}} \mathrm{d}\widetilde{x} \geq \int_{U_j} \frac{1}{|r_1((\mathrm{ad}DF_j)(\widetilde{x}))|} \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(\widetilde{x}) \right|}{|\widetilde{x}|^{n-1}} \mathrm{d}\widetilde{x}$$

$$\gtrsim \frac{1}{\sqrt{2}(n-1)} \int_{U_j} \frac{1}{\left| \frac{\partial}{\partial r} DF_j(\widetilde{x}) \right|} \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(\widetilde{x}) \right|}{|\widetilde{x}|^{n-1}} \mathrm{d}\widetilde{x}. \tag{4.26}$$

Hence, due to Lemma 4.4.4 we obtain

$$\int_{U_j} \frac{1}{\left| \frac{\partial}{\partial r} F_j(\widetilde{x}) \right|} \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(\widetilde{x}) \right|}{|\widetilde{x}|^{n-1}} \mathrm{d}\widetilde{x} \gtrsim \frac{\delta_{\alpha,\theta}}{\delta_{\alpha,\theta} + \alpha} \int_{U_j} \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(\widetilde{x}) \right|}{|\widetilde{x}|^{n-1}} \mathrm{d}\widetilde{x}, \tag{4.27}$$

where $\delta_{\alpha,\theta}$ stands for the constant in the proof of Lemma 4.4.4, see (4.18).

Since U_j is measurable and the integral on the right-hand side of (4.27) exists, we may apply the coarea formula with the Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\widetilde{x}) = |\widetilde{x}| = t$, see [EG, Chapter 3]. Therefore,

$$\int_{U_j} \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(\widetilde{x}) \right|}{|\widetilde{x}|^{n-1}} \mathrm{d}\widetilde{x} = \int_{r_j}^{r_{j-1}} \int_{U_j \cap S(0,r)} \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(\omega_r) \right|}{r^{n-1}} \mathrm{d}\mathcal{H}^{n-1}(\omega_r) \mathrm{d}r,$$

where ω_r stands for a point in set $U_j \cap S(0,r)$. By the change of variables $\omega_r \mapsto \frac{\omega_r}{r} = \omega$, we scale every sphere to the unit sphere $S(0,1)$ and obtain

$$\int_{r_j}^{r_{j-1}} \int_{U_j \cap S(0,r)} \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(\omega_r) \right|}{r^{n-1}} \mathrm{d}\mathcal{H}^{n-1}(\omega_r) \mathrm{d}r = \int_{r_j}^{r_{j-1}} \int_A \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(r\omega) \right|}{r^{n-1}} r^{n-1} \mathrm{d}\mathcal{H}^{n-1}(\omega) \mathrm{d}r,$$

where $A = \{x \in S(0,1) : d_{S(0,1)}(y,x) < \delta\}$ for some $y \in S(0,1)$, also see (4.21) to recall how we define $\delta = \delta(\varepsilon)$. In order to understand the geometry of set A, recall that U_j is a doubly truncated cone. Since $\mathcal{H}^{n-1}$-measure of A is independent of choice of y, it holds that A is just a radial projection of U_j on a sphere with radius 1 while point y only denotes the projection of the axis of U_j onto that sphere. Now we can use the Fubini theorem to change the order of integration and get

$$\int_{r_j}^{r_{j-1}} \int_A \frac{\left| \frac{\partial}{\partial r} (\phi \circ F_j)(r\omega) \right|}{r^{n-1}} r^{n-1} \mathrm{d}\mathcal{H}^{n-1}(\omega) \mathrm{d}r = \int_A \int_{r_j}^{r_{j-1}} \left| \frac{\partial}{\partial r} (\phi \circ F_j)(r\omega) \right| \mathrm{d}r \, \mathrm{d}\mathcal{H}^{n-1}(\omega).$$

This together with (4.26), (4.27) and (4.22) imply the following

$$\begin{aligned}\int_{U_j} \frac{|\nabla\phi(F_j(\widetilde{x}))|}{|\widetilde{x}|^{n-1}} \mathrm{d}\widetilde{x} &\geq \frac{\delta_{\alpha,\theta}}{\delta_{\alpha,\theta}+\alpha} \int_A \int_{r_j}^{r_{j-1}} \left|\frac{\partial}{\partial r}(\phi\circ F_j)(t\omega)\right| \mathrm{d}t\, \mathrm{d}\mathcal{H}^{n-1}(\omega) \\ &\geq \frac{\delta_{\alpha,\theta}}{\delta_{\alpha,\theta}+\alpha} \int_A \frac{\varepsilon}{4} \mathrm{d}\mathcal{H}^{n-1} \approx C(n,\varepsilon,\alpha,\theta).\end{aligned}$$

Finally, recall, by the discussion at the beginning of this proof that in fact $\delta = \delta(\varepsilon)$. Thus, it is enough to sum over $j = 2, \dots, k$ to get the assertion of a Lemma. □

Recall that $\mathrm{r}_{\mathrm{inj}}(\Omega)$ denotes the infimum of injectivity radii taken over set Ω. When Ω is fixed, we will write $\mathrm{r}_{\mathrm{inj}} := \mathrm{r}_{\mathrm{inj}}(\Omega)$ for the sake of simplicity of the notation. We are now ready to prove one of the key results of our work, namely the Quantitative Fatou Theorem for harmonic functions on Riemannian manifolds.

Theorem 4.4.8. *Let M be a complete Riemannian manifold and let further $\Omega \subset M^n$ be a Lipschitz domain. Furthermore, let $u : \Omega \to \mathbb{R}$ be a bounded harmonic function with $\|u\|_\infty \leq 1$. Then, for every point $p \in \partial\Omega$*

$$\sup_{0<r<r_{inj}} \frac{1}{r^{n-1}} \int_{\partial\Omega\cap B(p,r)} N(r,\varepsilon,\theta)(q)d\sigma(q) \leq C(\varepsilon,\alpha,\theta,n,\Omega),$$

where $\varepsilon, \alpha, \theta$ are constants in the definition of the counting function. In particular, constant C is independent of u.

Proof. Recall that a shadow of a point $x \in \Omega$, denoted by $S(x)$, is defined as follows, cf. Definition 3.2.4

$$S(x) := \{q \in \partial\Omega : x \in \widetilde{\Gamma}(q)\}.$$

Furthermore, the shadow and the cone are related as follows:

$$x \in \widetilde{\Gamma}(q) \Leftrightarrow q \in S(x). \tag{4.28}$$

Let us first estimate the following integral

$$\begin{aligned}&\int_{\partial\Omega\cap B(p,r)} \int_{\widetilde{\Gamma}^r(q)} |\nabla\phi(x)| d(x,q)^{1-n} \mathrm{d}x\mathrm{d}\sigma(q) \\ &\qquad = \int_{\partial\Omega\cap B(p,r)} \int_{\Omega\cap B(p,2r)} |\nabla\phi(x)| d(x,q)^{1-n} \chi_{\widetilde{\Gamma}^r(q)}(x)\mathrm{d}x\mathrm{d}\sigma(q).\end{aligned}$$

Here, the integration over $\widetilde{\Gamma}^r(q)$ can be replaced with the integration over $\Omega \cap B(p, 2r)$ with characteristic function of $\widetilde{\Gamma}^r(q)$, as every truncated cone $\widetilde{\Gamma}_r$ is contained in a ball with radius $2r$.

Now we use the Fubini theorem to change the order of integration:

$$\begin{aligned}&\int_{\partial\Omega\cap B(p,r)} \int_{\Omega\cap B(p,2r)} |\nabla\phi(x)| d(x,q)^{1-n} \chi_{\widetilde{\Gamma}^r(q)}(x)\mathrm{d}x\mathrm{d}\sigma(q) \\ &\qquad = \int_{\Omega\cap B(p,2r)} |\nabla\phi(x)| \int_{\partial\Omega\cap B(p,r)} d(x,q)^{1-n} \chi_{\widetilde{\Gamma}^r(q)}(x)\mathrm{d}\sigma(q)\mathrm{d}x \\ &\qquad \leq \int_{\Omega\cap B(p,2r)} |\nabla\phi(x)| \int_{\partial\Omega\cap B(p,r)} d(x,q)^{1-n} \chi_{S(x)}(q)\mathrm{d}\sigma(q)\mathrm{d}x,\end{aligned}$$

where in the last inequality we use (4.28) and the fact that $\widetilde{\Gamma}_r(q) \subset \widetilde{\Gamma}(q)$. Furthermore, since $d(x,q) \geq d(x,\partial\Omega)$ we have the following estimate:

$$\int_{\Omega\cap B(p,2r)} |\nabla\phi(x)| \int_{\partial\Omega\cap B(p,r)} d(x,q)^{1-n}\chi_{S(x)}(q)\mathrm{d}\sigma(q)\mathrm{d}x \\ \leq \int_{\Omega\cap B(p,2r)} |\nabla\phi(x)|d(x,\partial\Omega)^{1-n} \int_{\partial\Omega\cap B(p,r)} \chi_{S(x)}(q)\mathrm{d}\sigma(q)\mathrm{d}x.$$

Next, we need the following observation. For every $x \in \Omega$, it holds that

$$S(x) \subset \partial\Omega \cap B(q_x, (2+\widetilde{\alpha})d(x,\partial\Omega)),$$

where q_x denotes a point in Ω where $d(x,\partial\Omega)$ is attained. Indeed, let $y \in S(x)$. Then

$$d(y,q_x) \leq d(x,q_x) + d(x,y) = d(x,\partial\Omega) + d(x,y) \leq (2+\widetilde{\alpha})d(x,\partial\Omega).$$

Therefore,

$$\begin{aligned} &\int_{\Omega\cap B(p,2r)} |\nabla\phi(x)|d(x,\partial\Omega)^{1-n} \int_{\partial\Omega\cap B(p,r)} \chi_{S(x)}(q)\mathrm{d}\sigma(q)\mathrm{d}x \\ &\quad\leq \int_{\Omega\cap B(p,2r)} |\nabla\phi(x)|d(x,\partial\Omega)^{1-n} \int_{\partial\Omega\cap B(p,r)} \chi_{\partial\Omega\cap B(q_x,(2+\widetilde{\alpha})d(x,\partial\Omega))}(q)\mathrm{d}\sigma(q)\mathrm{d}x \\ &\quad\lesssim_{\widetilde{\alpha}} \int_{\Omega\cap B(p,2r)} |\nabla\phi(x)|d(x,\partial\Omega)^{1-n}d(x,\partial\Omega)^{n-1}\mathrm{d}x \\ &\quad= \int_{\Omega\cap B(p,2r)} |\nabla\phi(x)|\mathrm{d}x \leq C(\Omega)(2r)^{n-1}, \end{aligned} \tag{4.29}$$

where in the second inequality we appeal to the Ahlfors-David regularity of $\partial\Omega$ which results in the estimate

$$\sigma\Big(\partial\Omega \cap B\big(q_x,(2+\widetilde{\alpha})d(x,\partial\Omega)\big)\Big) \lesssim (2+\widetilde{\alpha})^{n-1}(d(x,\partial\Omega))^{n-1},$$

while in the last inequality in (4.29) we use the fact that ϕ is ε-approximation of u. Finally, by Lemma 4.4.7 and (4.29) we get the assertion of the theorem

$$C_{\varepsilon,\theta}\int_{\partial\Omega\cap B(p,r)} N(r,\varepsilon,\theta)(q)\mathrm{d}\sigma(q) \leq \int_{\partial\Omega\cap B(p,r)}\int_{\widetilde{\Gamma}^r(q)} |\nabla\phi(x)|d(x,q)^{1-n}\mathrm{d}x\mathrm{d}\sigma(q) \\ \leq C(\Omega)(2r)^{n-1} \lesssim C(\varepsilon,\alpha,\theta,n,\Omega)r^{n-1},$$

which proves the theorem.

□

Appendix

The following result is a mathematical folklore. However, since our argument is elementary and direct, we decided to include it in the dissertation.

Lipschitz sets in $\mathbb{R}^n$ satisfy the interior corkscrew condition

Proof of Lemma 4.3.2. Let Ω be any bounded Lipschitz set in $\mathbb{R}^n$. By definition of the Lipschitz set, for each $z \in \partial\Omega$ there are a hyperplane H such that $z \in H$ and numbers $\tilde{r}$, h with a cylinder $C = \{x + y\mathbf{n} : x \in B(z, \tilde{r}) \cap H, -h < y < h\}$ and a Lipschitz function $g : H \to \mathbb{R}$ such that

1. $\Omega \cap C = \{x + y\mathbf{n} : x \in B(z, \tilde{r}) \cap H, -h < y < g(x)\}$,
2. $\partial\Omega \cap C = \{x + y\mathbf{n} : x \in B(z, \tilde{r}) \cap H, y = g(x)\}$,

where $\mathbf{n}$ is a unit vector normal to H that is outer with respect to Ω. If at point $z \in \partial\Omega$ the boundary is of class C^1, then we take as a hyperplane H a tangent plane at z. Otherwise, we take any hyperplane that satisfies the aforementioned above conditions. In other words, there is a cone contained in Ω with vertex at z, an angle α such that $\tan\alpha = -\frac{2L}{1-L^2}$, where L denotes the Lipschitz constant of g, and height h. Since $\partial\Omega$ is compact there exist minimal h, denoted by $\tilde{H}$, minimal $\tilde{r}$, denoted by $\tilde{R}$, and maximal Lipschitz constant, denoted by $\tilde{L}$, such that any cone with vertex in $\partial\Omega$ and parameters given by $\tilde{H}$, $\tilde{R}$ and $\tilde{L}$ is contained in Ω. Let us denote a cone with such parameters and vertex at z by $K(z)$. We would like to show that the interior corkscrew condition holds, i.e. that there exists a constant $c > 0$ such that for each $z \in \partial\Omega$ and each $0 < r < \operatorname{diam}(\Omega)$ there exists a point $\tilde{z} \in \Omega \cap B(z, r)$ with the property that $B(\tilde{z}, cr) \subset \Omega \cap B(z, r)$. For a point $z \in \partial\Omega$ set $\tilde{z} = z - \frac{1}{2}\min(\tilde{H}, r)\mathbf{n}$. We notice that the distance from $\tilde{z}$ to the lateral surface of a cone $K(z)$ is given by $\frac{\min(\tilde{H}, r)}{2\sqrt{1+\tilde{L}^2}}$ and the distance between $\tilde{z}$ and the base of a cone $K(z)$ is given by $\tilde{H} - \frac{1}{2}\min(\tilde{H}, r)$. We need to find a constant F such that the ball $B(\tilde{z}, Fr)$ is contained in both ball $B(z, r)$ and cone $K(z)$. To ensure that a ball with radius Fr is contained in a cone $K(z)$ the following inequalities have to be satisfied:

$$Fr < d = \frac{\min(r, \tilde{H})}{2\sqrt{1 + \tilde{L}^2}} \quad \text{and} \quad Fr < \tilde{H} - \frac{1}{2}\min(r, \tilde{H}). \tag{4.30}$$

The condition needed for ball with radius Fr to be contained in $B(z, r)$ reads:

$$Fr + \frac{1}{2}\min(r, \tilde{H}) < r. \tag{4.31}$$

Upon choosing

$$F := \frac{1}{2}\frac{1}{2\sqrt{1 + \tilde{L}^2}}\frac{\tilde{H}}{\operatorname{diam}(\Omega)}$$

we now directly check that such F satisfies all the necessary conditions.

After inserting F, the first inequality in (4.30) becomes

$$\frac{\tilde{H}r}{2\operatorname{diam}(\Omega))} < \min(r, \tilde{H}).$$

If $\tilde{H} \leq r$ we get $\frac{r}{2\operatorname{diam}(\Omega)} < 1$, which holds, as $r < \operatorname{diam}(\Omega)$. Otherwise, $r < \tilde{H}$ and we have $\frac{\tilde{H}}{2\operatorname{diam}(\Omega)} < 1$, which holds as $\tilde{H} < \operatorname{diam}(\Omega)$.

Similarly, the second inequality in (4.30) becomes

$$\frac{1}{2}\frac{1}{2\sqrt{1 + \tilde{L}^2}}\frac{\tilde{H}r}{\operatorname{diam}(\Omega)} < \tilde{H} - \frac{1}{2}\min(r, \tilde{H}).$$

Again, there are two cases to be considered. If $\tilde{H} \leq r$ we get

$$\frac{1}{2}\frac{1}{2\sqrt{1+\tilde{L}^2}}\frac{r}{\operatorname{diam}(\Omega)} < \frac{1}{2},$$

which holds as $r < \operatorname{diam}(\Omega)$. Otherwise, $r < \tilde{H}$ and we get

$$\frac{1}{2}\frac{1}{2\sqrt{1+\tilde{L}^2}}\frac{\tilde{H}r}{\operatorname{diam}(\Omega)} < \tilde{H} - \frac{1}{2}r.$$

Notice that $\tilde{H} - \frac{1}{2}r > \frac{1}{2}\tilde{H}$, and hence it is enough to prove

$$\frac{1}{2}\frac{1}{2\sqrt{1+\tilde{L}^2}}\frac{\tilde{H}r}{\operatorname{diam}(\Omega)} < \frac{1}{2}\tilde{H},$$

which reduces the discussion to the first case.

Finally, the inequality in (4.31) becomes

$$\frac{1}{2\sqrt{1+\tilde{L}^2}}\frac{\tilde{H}}{\operatorname{diam}(\Omega)} + \frac{\min(r,\tilde{H})}{r} < 2,$$

which holds as $\frac{\tilde{H}}{\operatorname{diam}(\Omega)} < 1$ and $\frac{\min(r,\tilde{H})}{r} \leq 1$.

Thus, it holds that a ball $B(\tilde{z}, Fr)$ is contained both in a cone $K(z)$ and in a ball $B(z,r)$. Hence $B(\tilde{z}, Fr) \subset \Omega \cap B(z,r)$. □

Proof of Lemma 4.4.2

Proof. Recall that $U \subset (X,d)$ is an open, connected, precompact and 1-connected at the boundary set and $h : U \to h(U)$ is a homeomorphism such that $h(U)$ is bounded in $(Y,\tilde{d})$.

First we prove that h can be extended to $\overline{U}$. Let $x \in \partial U$ and let further (x_n) be any sequence of points $x_n \in U$ such that (x_n) converges to x. We define $\widetilde{h} : \overline{U} \to \overline{h(U)}$ by the following formula:

$$\widetilde{h}(x) := \begin{cases} h(x), x \in U, \\ \lim_{n\to\infty} h(x_n), x \in \partial U. \end{cases}$$

We have to check whether $h(x_n)$ converges. Since $h(U)$ is bounded, we may take a convergent subsequence $\big(h(x_{n_k})\big)$ and denote its limit as y. Notice that $y \in \overline{h(U)}$. Suppose that there is another convergent subsequence $\big(h(x_{n_l})\big)$ and it has a different limit $\tilde{y}$. Since y and $\tilde{y}$ are distinct we can find their disjoint neighbourhoods V and $\tilde{V}$ such that almost all of $h(x_{n_k})$ and $h(x_{n_l})$ are in V and $\tilde{V}$, respectively. Now we consider $V \cap h(U)$ and $\tilde{V} \cap h(U)$ and notice that their preimages under h are also disjoint subsets in U. However, these preimages contain subsequences (x_{n_k}) and (x_{n_l}), respectively. There are two cases to be considered. In the first case, we find subsequences of (x_n) which converge to different limits, which leads to contradiction with convergence of (x_n). In the second case, both (x_{n_k}) and (x_{n_l}) converge to x and $h^{-1}(V)$ and $h^{-1}(\tilde{V})$ are disjoint while x belongs to both of their boundaries.

However, since there is a connected neighbourhood of x in U, denoted by $U \cap U_x$, and by convergence almost all x_n are in that neighbourhood, it holds that $h(U \cap U_x)$ is connected and almost all $h(x_n)$ belong to this image. On the other hand, we may choose U_x with arbitrarily small diameter. Therefore, $\big(h(x_{n_k})\big)$ and $\big(h(x_{n_l})\big)$ have to converge to the same limit, and hence $y = \tilde{y}$.

For our extension mapping $\widetilde{h}$ to be properly defined, we need to prove that if (z_n) is a different sequence converging to x, then $\big(h(z_n)\big)$ also converges to y. If we assume that $\big(h(z_n)\big)$ converges to $z \neq y$, we can take disjoint neighbourhoods V_z, V_y of z and y, respectively, and intersect them with $h(U)$ to obtain sets W_z, W_y. Both of these disjoint sets contain almost all points $h(z_n)$ and $h(x_n)$, respectively, and hence their preimages under h, i.e. $h^{-1}(W_z), h^{-1}(W_y)$, contain almost all points (z_n) and (x_n), respectively. However, (x_n) and (z_n) both converge to x, so x belongs to both of their boundaries $\partial(h^{-1}(W_z))$ and $\partial(h^{-1}(W_y))$. Again, by using existence of a connected neighbourhood of x, we can prove that $z = y$. To summarize, we have proved that the extension $\widetilde{h}$ is well defined and due to the construction, it is continuous.

Next, we prove that $\tilde{h}$ is actually a homeomorphism. Suppose that there is $y \in \partial\overline{h(U)}$ that is not the image of a point from ∂U. Then $\tilde{h}(\overline{U}) \neq \overline{h(U)}$. However, as U is precompact, then $\overline{U}$ is compact and therefore $\tilde{h}(\overline{U})$ is compact. Moreover, $\tilde{h}(\overline{U})$ contains $h(U)$ and by the definition, $\overline{h(U)}$ is the smallest closed set containing $h(U)$. Hence, $\overline{h(U)} \subset \tilde{h}(\overline{U})$. The inverse inclusion is assured because of the definition of $\tilde{h}$. Therefore, we proved that $\tilde{h}$ is onto.

We would like to know that a point $x \in \partial U$ is mapped to a point in $\partial h(U)$. Suppose that there is $y \in h(U)$ such that $x \in \tilde{h}^{-1}(y)$. We also know that there is some $\tilde{x} \in U$ that is also a preimage of y. We can find disjoint neighbourhoods of x and $\tilde{x}$. However, by assumptions h is the homeomorphism on U, so the images of these neighbourhoods would also have to be disjoint, but we assumed they have a common point y. This contradiction gives us that a point in ∂U is mapped to a point in $\partial h(U)$.

Take $x, \tilde{x} \in \partial U$ and suppose $\tilde{h}(x) = \tilde{h}(\tilde{x})$. Let sequences (x_n) and $(\widetilde{x_n})$ be converging to x and $\tilde{x}$, respectively. We may find disjoint neighbourhoods V of x and $\tilde{V}$ of $\tilde{x}$ such that almost all x_n and $\widetilde{x_n}$ are elements of V and $\tilde{V}$, respectively. Since h is a homeomorphism the images of intersections $V \cap U$ and $\tilde{V} \cap U$ are also disjoint, which means that $\tilde{h}(x) \neq \tilde{h}(\tilde{x})$. Hence, $\tilde{h}$ is one-to-one.

We know that $\tilde{h}$ is a continuous bijection, but $\overline{U}$ is compact and hence $\tilde{h}$ is a homeomorphism. Since $\overline{U}$ is compact, $\tilde{h}$ is Lipschitz. Similarly $\tilde{h}^{-1}$ is Lipschitz. Therefore, $\tilde{h}$ is bi-Lipschitz and preserves bounded Lipschitz sets, and thus the same applies to h. $\square$

Chapter 5

Carleson measures on domains in Heisenberg groups

This chapter is based on the manuscript [AdGr] written jointly with Tomasz Adamowicz. Main results of this chapter include Theorems 1.5.1, 1.5.2, 1.5.3, 1.5.4, 1.5.5 presented in Chapter 1.5.

5.1 Preliminaries

In this chapter we recall key definitions employed in the chapter. Our presentation includes the Heisenberg group, various types of domains and their geometry, basic information on subelliptic harmonic functions and Green functions in the sub-Riemannian setting, the Carleson measures, the nontangential maximal function and the BMO spaces.

5.1.1 Heisenberg groups

There are number of approaches to define the Heisenberg groups and, more general, Carnot-Carathéodory groups, see [BLU], [CDPT], [Gro], [Mon]. One approach is based on introducing the Lie algebra which defines a connected and simply connected Lie group. However, in what follows, we choose a different, although equivalent, approach to define the Heisenberg groups.

The n-th Heisenberg group $\mathbb{H}^n$ as a set is $\mathbb{R}^{2n}\times\mathbb{R}\simeq\mathbb{C}^n\times\mathbb{R}$ with the group law given by

$$(z_1,\ldots,z_n,t)\cdot(z_1^{'},\ldots,z_n^{'},t^{'})=\left(z_1+z_1^{'},\ldots,z_n+z_n^{'},t+t^{'}+2\mathrm{Im}\left(\sum_{i=1}^n z_i\overline{z_i^{'}}\right)\right),$$

where $(z_1,\ldots,z_n,t)=(x_1,y_1,\ldots,x_n,y_n,t)$. Furthermore, we define the following left-invariant vector fields

$$X_i(p)=\frac{\partial}{\partial x_i}+2y_i\frac{\partial}{\partial t},\qquad Y_i(p)=\frac{\partial}{\partial y_i}-2x_i\frac{\partial}{\partial t},\qquad T=\frac{\partial}{\partial t}$$

for which the only nontrivial brackets are

$$[X_i,Y_i]:=X_iY_i-Y_iX_i=-4T\quad i=1,\ldots,n.$$

It is worth noting that the above commutator relations are a reason for the name of the Heisenberg groups. They work exactly the same as position and momentum operator in quantum physics. Namely, if $\hat{x}$ denotes a position operator, $\hat{p}$ denotes momentum operator and I is identity operator, we get:

$$[\hat{x}, \hat{p}] = i\hbar I, \quad [\hat{x}, i\hbar I] = 0, \quad [\hat{p}, i\hbar I] = 0.$$

Due to the similarity with quantum mechanics, the Heisenberg group got it name alluding to one of the founders of that branch of physics. Let us mention that Heisenberg was working using matrices and the Heisenberg group may also be defined as a group of matrices.

The horizontal space at $p \in \mathbb{H}^n$ is given pointwise by

$$\mathrm{H}_p\mathbb{H}^n = \operatorname{span}\{X_1(p), Y_1(p), \ldots, X_n(p), Y_n(p)\}.$$

The horizontal space plays the role of the tangent space. In Riemannian geometry all curves on a manifold have their derivatives in a tangent space. However, the Heisenberg group is a subriemannian manifold. Hence, we only allow curves that have their derivatives in a subspace of a tangent space, called horizontal space. Spaces which are subriemannian manifolds occur naturally whenever one has to deal with a situation which a certain direction of movement is excluded. For example in the first Heisenberg group $\mathbb{H}^1$ at point $0 := (0, 0, 0)$, the horizontal space $\mathrm{H}_0\mathbb{H}^n$ is a 2-dimensional space spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and hence we are only allowed to use curves whose derivatives lie in that space. In particular, the direction along the z axis is excluded. It follows that the shortest path from a point $(0, 0, z)$ to 0 is achieved along a spiral.

Let us also notice that the Heisenberg group is exceptional among subriemannian manifolds, because it is also a group. Therefore, it is a Carnot-Carathéodory group. A Carnot-Carathéodory group of step k is a connected, simply connected, finitely-dimensional Lie group such that its Lie algebra $\mathfrak{g}$ admits a step-k stratification, i.e.

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k, \quad [V_1, V_i] = V_{i+1} \quad \text{for } i = 1, \ldots, k-1 \text{ and } [V_1, V_k] = 0.$$

For more information about Carnot-Carathéodory groups, see e.g. [BLU].

The fact that the Heisenberg groups have so many different structures such as being a subriemannian manifold, a Carnot-Carathéodory group and also a contact manifold, makes it a great object to study. The multitude of structures makes it a good starting point whenever one wants to work with subriemannian manifolds or Carnot-Carathéodory groups. It is often the first object studied in these settings as usually understanding any phenomenon in the Heisenberg groups gives an insight to what may be happening in more general situations.

Let $\gamma : [0, S] \to \mathbb{R}^{2n+1}$ be an absolutely continuous curve. We will say that γ is horizontal if $\dot{\gamma}(s) \in \mathrm{H}_{\gamma(s)}\mathbb{H}^n$ for almost every s. Now, we equip $\mathrm{H}_p\mathbb{H}^n$ with left invariant Riemannian metric such that fields X_i, Y_i are orthonormal and so if $v = \sum_{i=1}^n a_i X_i(p) + b_i Y_i(p)$, then the following expression defines a norm $|v|_H = \sqrt{\sum_{i=1}^n a_i^2 + b_i^2}$. In a consequence, we define the Carnot-Carathéodory distance in $\mathbb{H}^n$ as follows:

$$d_{CC}(p, q) = \inf_{\Gamma_{p,q}} \int_a^b |\dot{\gamma}(s)|_H \mathrm{d}s,$$

where $\Gamma_{p,q}$ denotes a set of horizontal curves joining p and q, such that γ joins points p and q: $\gamma(a) = p$ and $\gamma(b) = q$.

Equipped with the above structure, the Heisenberg group $\mathbb{H}^n$ becomes a subriemannian manifold and a Carnot-Carathéodory group, in addition to being a metric space. Nevertheless, Carnot-Carathéodory distance may be troublesome and, hence, we introduce the so-called Korányi-Reimann distance, defined as follows:

$$d_{\mathbb{H}^n}(p,q) = \|q^{-1} \cdot p\|,$$

where the pseudonorm is given by

$$\|p\| := \|(z,t)\| = \left(|z|^4 + t^2\right)^{\frac{1}{4}}.$$

The Korányi-Reimann distance is equivalent (comparable) to d_{CC} and hence both distances generate the same topology, see e.g. [Be]. However, $d_{\mathbb{H}^n}$ is easier in computations and therefore, throughout this work we use Korányi-Reimann distance $d_{\mathbb{H}^n}$. In particular, all balls are defined using that distance, i.e. $B(x,r) = \{y \in \mathbb{H}^n : d_{\mathbb{H}^n}(x,y) < r\}$.

Finally, we recall that the left-invariant Haar measure on $\mathbb{H}^n$ is simply the $(2n+1)$-dimensional Lebesgue measure on $\mathbb{H}^n$ and it follows that $\mathbb{H}^n$ is Q-Ahlfors regular, with $Q = 2n+2$, i.e. there exists a positive constant c such that for all balls B with radius $r > 0$ we have

$$\frac{1}{c} r^Q \leq \mathcal{H}^Q(B) \leq c r^Q,$$

where $\mathcal{H}^Q$ stands for the Q-dimensional Hausdorff measure induced by $d_{\mathbb{H}^n}$.

5.1.2 Geometry of domains

One of the fundamental types of domains studied in this chapter are the NTA domains and the ADP domains, whose definitions and basic properties we now recall, cf. Chapter 2.9.

Below, we partially repeat the presentation from Chapter 2.9, but we here we also discuss the NTA domains in more details, as they will play a key role in the results of this chapter.

Definition 5.1.1 (NTA domain, cf. Definition 5.11 in [CGN]). We say that $\Omega \subset \mathbb{H}^n$ is a nontangentially accessible domain (NTA, for short) if there exist constants $M, r_0 > 0$ such that:

(1) (Interior corkscrew condition). For any $x \in \partial\Omega$ and $r \leq r_0$ there exists $A_r(x) \in \Omega$ such that

$$\frac{r}{M} < d(A_r(x), x) \leq r \quad \text{and} \quad d(A_r(x), \partial\Omega) > \frac{r}{M}.$$

(2) (Exterior corkscrew condition). The complement $\Omega^c := \mathbb{H}^n \setminus \Omega$ satisfies interior corkscrew condition.

(3) (Harnack chain condition). For every $\varepsilon > 0$ and $x, y \in \Omega$ such that $d(x,\partial\Omega) > \varepsilon$, $d(y,\partial\Omega) > \varepsilon$ and $d(x,y) < C\varepsilon$ there exists a sequence of balls $B_1, \ldots, B_p$ with the following properties:

   (a) $x \in B_1$ and $y \in B_p$,

   (b) $\frac{r}{M} < d(B_i(x,r), \partial\Omega) < Mr$ for every $i = 1, \ldots, p$,

   (c) $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, p-1$,

(d) length of the chain p depends on C but not on ε.

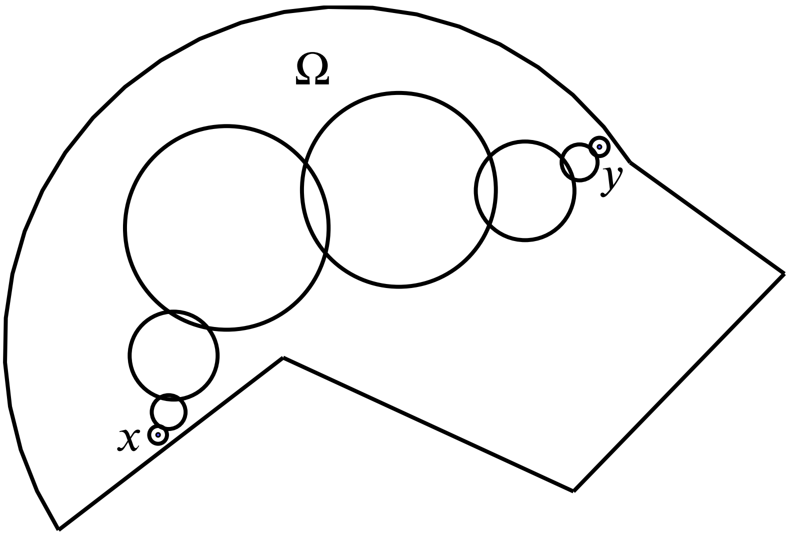

Figure 5.1: This figure depicts a Harnack chain of balls between points x and y.

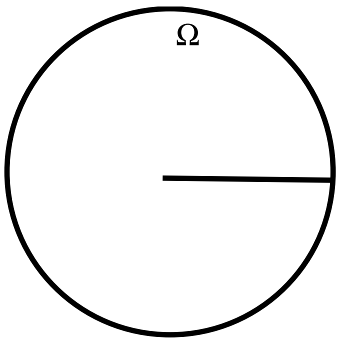

Figure 5.2: This figure shows an example of a domain which does not satisfy Harnack chain condition. Due to the presence of a slit one can take pairs of points on the opposite sides of a slit which are arbitrarily close to each other. However, then there is no bound on the length of a chain of balls joining them.

In the corresponding Definition 1 in [CG] the analogous notion of the X-NTA domains is considered. There, one lets $X = \{X_1, \ldots, X_m\}$ be a family of smooth vector fields satisfying the Hörmander rank condition, and so d_{CC} denotes the Carnot-Carathéodory distance related to X. For example, in the Heisenberg group $\mathbb{H}^n$ our family of vector fields is $X := \{X_1, Y_1, \ldots, X_n, Y_n\}$, see Chapter 5.1.1.

The notion of the NTA domain originates from a work of Jerison-Kenig, see [JK, Section 3]. Notice that the above definition makes sense also in the setting of metric space, in which case, the distance need not be induced by a family of vector fields.

Examples of NTA domains in $\mathbb{R}^n$ encompass:

- Lipschitz domains, see Proposition 3.6 in [JK],

- Zygmund domains, see Proposition 3.6 in [JK],

- quasispheres (snow-flake domains), see Section 2 in [GO], [Jon].

Another example of NTA domain is domain Ω, defined as a complement of a planar Cantor set in a large enough ball in $\mathbb{R}^n$. It turns out that Ω satisfies our definition, even though such a Cantor set is not rectifiable as a part of the 1-dimensional boundary of Ω. Intuitively speaking, one can think that conditions (1) and (2) exclude both interior and exterior cusps, while condition (3) eliminates a possibility of slits within a domain or narrowings that are infinitely thin. Examples of NTA domains in $\mathbb{H}^n$, or in more general Carnot groups, include:

- bounded $C^{1,1}$ sets with cylindrical symmetry (Theorem 5 in [CG]),

- level sets of fundamental solutions of the real part of the sub-Laplacian (Corollary 2 in [CG]),

- balls in the metric $d_{\mathbb{H}^n}$, see Corollary 4 and Proposition 1 in [CG],

- an image of an NTA domain $\mathbb{H}^n$ under the global quasiconformal map $f : \mathbb{H}^n \to \mathbb{H}^n$ is an NTA domain, see [CT].

We refer to Section 5 in [CG] for further examples of NTA domains. However, it turns out that balls in d_{CC} are not NTA domains. This partially motivates that from the point of view of our studies, the $d_{\mathbb{H}^n}$ distance has an advantage over Carnot-Carathéodory distance.

From now on, unless specified differently, let us denote by $d := d_{\mathbb{H}^n}$.

Basing on the notion of the NTA domains we now recall one of the second fundamental types of domains considered in this chapter, namely the so-called domains *admissible for the Dirichlet problem*, ADP in short, see [CGN]. Such a class is defined by combining the above notion of NTA domains with the existence of a uniform outer ball. As observed in [CGN], it can be viewed as the closest nonabelian counterpart of the class of $C^{1,1}$ domains from Euclidean analysis.

Definition 5.1.2 (cf. Definition 2.1 in [CGN])**.** We say that a bounded domain $\Omega \subset \mathbb{H}^1$ is *admissible for the Dirichlet problem*, denoted by ADP, if Ω is NTA and satisfies the uniform outer ball condition with respect to the metric d.

Examples of the ADP domains include:

- Korányi-Reimann ball, see result of Theorem 2.13 in [CGN],

- level sets of some entire solutions to Yamabe type equations, see Section 1 in [CGN],

- $C^{1,1}$ domains which are convex (at the level of Lie algebra) and which have partial symmetry near their characteristic sets, see Theorem 2.13 in [CGN],

- C^2 convex domains with strongly isolated characteristic points, see Theorem 2.16 in [CGN].

5.1.3 Subelliptic harmonic functions and Green functions

Below we collect some of the basic definitions and potential theoretic results for the theory of subelliptic functions in Heisenberg groups $\mathbb{H}^n$.

Let $\Omega \subset \mathbb{H}^n$ be an open set in the Heisenberg group $\mathbb{H}^n$. We say that a function $u : \Omega \to \mathbb{R}$ belongs to the *horizontal Sobolev space* $HW^{1,2}(\Omega)$, if $u \in L^2(\Omega)$ and the horizontal derivatives $X_i u, Y_i u$ for $i = 1, \ldots, n$ exist in the distributional sense and belong to $L^2(\Omega)$. Similarly, we define the local horizontal Sobolev space $HW^{1,2}_{loc}(\Omega)$.

The *horizontal gradient* $\nabla_H u$ is given by the following equation:

$$\nabla_H u = \sum_{i=1}^{n} (X_i u)X_i + (Y_i u)Y_i.$$

Next, we define the sub-Laplace operator of u:

$$\Delta_H u = \sum_{i=1}^{n} (X_i)^2 u + (Y_i)^2 u$$

and say that $u \in HW^{1,2}_{loc}(\Omega)$ is *subelliptic harmonic* in Ω, if $\Delta_H u = 0$ in the weak sense. In what follows, for the sake of simplicity, we will omit the word subelliptic and write *harmonic functions*, instead.

Recall that functions in $\mathbb{H}^n$ are smooth (in fact analytic) and satisfy the weak maximum principle and the Harnack inequality, see Chapters 8 and 5 in [BLU], respectively.

Let $G(x, y) = G(y, x) = G_\Omega(x, y)$ denote the Green function for the sub-Laplacian and for the domain $\Omega \subset \mathbb{H}^n$. We refer to [CG] and to Chapter 9 in [BLU] for definitions and basic properties of Green functions.

For the reader's convenience, we recall a definition of a Green function.

Definition 5.1.3 (cf. Definition 9.2.1 in [BLU])**.** Let $\Omega \subset \mathbb{H}^n$ be an open set and $x \in \Omega$. The function $y \mapsto \Gamma(x^{-1} \circ y)$ is superharmonic and nonnegative in Ω, where Γ denotes a fundamental solution for Δ_H. Then it has the greatest harmonic minorant in Ω. Let us denote it by h_x. The function

$$\Omega \times \Omega \ni (x, y) \mapsto G_\Omega(x, y) := \Gamma(x^{-1} \circ y) - h_x(y) \in (0, \infty)$$

is the Green function for Ω.

We write G instead of G_Ω whenever it is clear what Ω is. Let us remark that $G(x, \cdot)$ is harmonic in $\Omega \setminus \{x\}$. By symmetry of G also $G(\cdot, y)$ is a harmonic function in $\Omega \setminus \{y\}$.

Moreover, in the Appendix we provide a proof of one of the standard properties of Green functions needed in Example 5.4.4. The result is likely a mathematical folklore in $\mathbb{H}^n$, but since we did not find it explicitly in the literature for Carnot groups, we provide the full argument.

The following observations from [CG] will frequently be used, especially in Chapter 5.4 devoted to proofs of Theorem 1.5.3 and Theorem 1.5.4. Here, we formulate them for the gauge balls rather then for the metric balls. This is justified by the equivalence of both metrics in $\mathbb{H}^n$.

Let

$$\Delta(x, r) := B(x, r) \cap \partial\Omega \tag{5.1}$$

denote the surface ball at $x \in \partial\Omega$ with radius $r > 0$.

Theorem 5.1.4 (Dahlberg-type estimate, cf. Theorem 1 [CG]). *Let $\Omega \subset \mathbb{H}^n$ be an NTA domain with parameters $M, r_0 > 0$ and let further $x_0 \in \partial\Omega$ and $r < \frac{r_0}{2}$. Then, there exist $a > 1$ and $C > 0$, depending on Δ_H, M and r_0, such that for every $x \in \Omega \setminus B(x_0, ar)$*

$$C\frac{|B(x,r)|}{r^2}G(x, A_r(x_0)) \le \omega^x(\Delta(x_0, r)) \le C^{-1}\frac{|B(x,r)|}{r^2}G(x, A_r(x_0)),$$

where G denotes a Green function of Ω.

Theorem 5.1.5 (The Carleson-type estimate, Lemma 1 in [CG]). *Let $x_0 \in \partial\Omega$ and $r \le r_0$. There exist constants $C, \beta > 0$, depending on Δ_H, M and r_0 such that for any nonnegative subelliptic harmonic function u on $\Omega \cap B(x_0, 2r)$, vanishing continuously on $\Delta(x_0, 2r)$, one has*

$$u(x) \le C\left(\frac{d(x, x_0)}{r}\right)^{\beta} \sup_{\Omega\cap\partial B(x_0,r)} u$$

for any $x \in \Omega \cap B(x_0, r)$.

Theorem 5.1.6 (Local comparison theorem, cf. Theorem 3 [CG]). *Let $\Omega \subset \mathbb{H}^n$ be an NTA domain with parameters $M, r_0 > 0$ and let further $x_0 \in \partial\Omega$ and $0 < r < \frac{r_0}{M}$. If u, v are harmonic functions in Ω, that continuously vanish on $\Delta(x_0, Mr)$, then for any $x \in B(x_0, \frac{r}{2M}) \cap \Omega$ one has*

$$\frac{u(x)}{v(x)} \le C\frac{u(A_r(x_0))}{v(A_r(x)_0))},$$

for some constant $C > 0$ which depends only on Δ_H, M and r_0.

Theorem 5.1.7 (Theorem 9 in [CG]). *Let $x_0 \in \partial\Omega$, $r \le r_0$. There exists a positive constant C depending on Δ_H, M and r_0 such that for any nonnegative subelliptic harmonic function u in $\Omega \cap B(x_0, 2r)$, which vanishes continuously on $\Delta(x_0, 2r)$, one has*

$$u(x) \le Cu(A_r(x_0))$$

for any $x \in \Omega \cap B(x_0, r)$.

5.1.4 Carleson measures and related notions in Harmonic analysis

Recall that the definition of regularity of a set, also known as Ahlfors-David regularity for metric spaces was formulated in Definition 2.8.1.

Definition 5.1.8 (Carleson measure in $\mathbb{H}^n$). Let $1 \le \alpha < \infty$ and $s > 0$. We say that a positive Borel measure μ on an open connected set $\Omega \subset \mathbb{H}^n$ with non-empty s-regular boundary is an *α-Carleson measure on* Ω, if there exists a constant $C > 0$ such that

$$\mu(\Omega \cap B(x, r)) \le Cr^{\alpha s}, \quad \text{for all } x \in \partial\Omega \text{ and } r > 0. \tag{5.2}$$

The *α-Carleson measure constant* of μ is defined by

$$\gamma_\alpha(\mu) := \inf\{C > 0 \text{ such that (5.2) holds for all } x \in \partial\Omega \text{ and } r > 0\}$$

We also call 1-Carleson measures simply *Carleson measures*.

We define two objects that are essential for analyzing the behavior of harmonic functions. Both of these objects were studied in the setting of Euclidean spaces, but now we give the definitions for Heisenberg groups, cf. Chapter 2.5.

Definition 5.1.9. Let $u : \Omega \to \mathbb{R}$ be a continuous function. We define the *nontangential maximal function* $N_\alpha u : \partial\Omega \to \mathbb{R}$ as follows:

$$(N_\alpha u)(x) = \sup\{|u(y)| : y \in \Gamma_\alpha(x)\},$$

where $\Gamma_\alpha(x) = \{y \in \Omega : d(y,x) < (1+\alpha)d(y,\partial\Omega)\}$ is a cone with vertex $x \in \partial\Omega$ and aperture given by α.

In the next definition we assume that the function is C^1, but the Sobolev regularity $HW^{1,2}_{loc}$ would suffice as well, cf. [GMT] for the Euclidean setting. Since the definition below is applied only to subelliptic harmonic functions on $\mathbb{H}^n$, which are analytic, our regularity assumption is enough.

Definition 5.1.10. Let $u : \Omega \to \mathbb{R}$ be a $C^1(\Omega)$ function. We define the *square function* $(S_\alpha u)^2 : \partial\Omega \to \mathbb{R}$ as follows:

$$(S_\alpha u)^2(x) = \int_{\Gamma_\alpha(x)} |\nabla_H u(y)|^2 d(y,\partial\Omega)^{2-Q}\mathrm{d}y,$$

where $Q = 2n+2$ is a homogeneous dimension of $\mathbb{H}^n$.

Another important notion of this thesis is one of the harmonic measure.

Definition 5.1.11. Let $\Omega \subset \mathbb{H}^n$ be a domain. For a continuous function $f \in C(\partial\Omega)$ there exists a unique solution to the following boundary value problem:

$$\begin{cases} \Delta_H u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Hence, for each $x \in \Omega$, owing to linearity of Δ_H, we may define a linear functional

$$f \mapsto u_f(x),$$

where u_f is the unique subelliptic harmonic function with boundary data f. Hence, by the Riesz representation theorem, we get

$$u_f(x) = \int_{\partial\Omega} f(z)\mathrm{d}\omega^z$$

for some Borel regular measure ω^z, which we call the *harmonic measure*.

For a given domain $\Omega \subset \mathbb{H}^n$ choose a point $y \in \Omega$ and consider the harmonic measures ω^y on Ω. Then for a given $x \in \partial\Omega$ and $r > 0$ we let $\Delta(x,r) := B(x,r) \cap \partial\Omega$ and recall the mean-value of a function $f : \partial\Omega \to \mathbb{R}$ on $\Delta(x,r)$:

$$f_{\Delta(x,r)} := \fint_{\Delta(x,r)} f(z)\,\mathrm{d}\omega^y(z).$$

In Chapter 3.3 we recalled the definition of the BMO space for domains in $\mathbb{R}^n$, see Definition 3.3.4. Below, we recall the definition of the boundary BMO space in $\mathbb{H}^n$, see Definition 8.4 in [JK].

Definition 5.1.12 (Boundary BMO space)**.** Let $\Omega \subset \mathbb{H}^n$ be a domain. We say that a function $f : \partial\Omega \to \mathbb{R}$ belongs to the space $BMO(\partial\Omega, \mathrm{d}\omega)$ with respect to the harmonic measure ω in Ω, if

$$\sup_{\Delta(x,r)} \frac{1}{\omega(\Delta(x,r))} \int_{\Delta(x,r)} |f(y) - f_{\Delta(x,r)}| \mathrm{d}\omega < \infty.$$

When discussing the NTA domains in $\mathbb{H}^n$ we may omit the reference point in the harmonic measure $\mathrm{d}\omega^z$ and write ω for simplicity.

5.2 Characterizations of Carleson measures on ADP-domains

The purpose of this chapter is to show Theorem 1.5.1, which can be understood as the nonabelian counterpart of the following well-known characterization of the Carleson measures in the upper-half plane $\mathbb{R}^2_+$.

Theorem (Lemma 5.5 in [G] Section 5, Ch. I)**.** *Let σ be a positive measure on $H = \mathbb{R} \times \mathbb{R}_+$, and let $\alpha > 0$. Then σ is a Carleson measure with constant γ_σ if and only if there exists $A = A(\alpha)$ such that*

$$\sigma(\{|u(z)| > \lambda\}) \leq A|\{t : N_\alpha u(t) > \lambda\}|, \qquad \lambda > 0 \tag{5.3}$$

for every harmonic function u on H. If A is the least constant such that (5.3) holds, then $\gamma_\sigma \approx A$.

However, here we prove Theorem 1.5.1 only for bounded domains in $\mathbb{H}^1$.

Theorem. 1.5.1 *Let $\Omega \subset \mathbb{H}^1$ be a smooth ADP domain with 3-regular boundary and μ be a positive measure on Ω. Then μ is a Carleson measure on Ω if and only if there exists a constant $C = C(\alpha)$ such that for every harmonic function u on Ω and every $\lambda > 0$ it holds that*

$$\mu(\{x \in \Omega : |u(x)| > \lambda\}) \leq C\sigma(\{\omega \in \partial\Omega : N_\alpha u(\omega) > \lambda\}), \tag{5.4}$$

where σ is the surface measure on $\partial\Omega$, i.e. $\sigma = H^2 \lfloor \partial\Omega$. Moreover, if C is the least constant such that (5.4) holds, then the Carleson constant of μ satisfies $\gamma_\mu \approx_\alpha C$.

Remark 5.2.1. Upon the necessary modifications, Theorem 1.5.1 can be as well formulated for the smooth ADP domains in $\mathbb{H}^n$ for $n \geq 1$. However, for the sake of the simplicity of the presentation and in order to emphasize the similarity to the corresponding result in [G], we restrict our discussion to $\mathbb{H}^1$ only.

The proof of the sufficiency part relies on the corresponding one for Proposition 6.3 in [AF] and in fact holds for Borel regular functions in general metric spaces.

In order to show the necessity part of the assertion we adapt the idea of the proof of Lemma 5.5 in [G, Section 5, Ch. I] for the Carleson measures on the upper half plane $\mathbb{R}^2_+$ and the Euclidean harmonic functions. There, by choosing the constant boundary data 4λ with support contained in the interval $I \subset \mathbb{R}$ and by defining the harmonic function u as the convolution of the Poisson kernel in the upper half plane $\mathbb{R}^2_+$ with the function $4\lambda\chi_I$, one shows that the superlevel set $\{x \in \mathbb{R}^2_+ : u(x) > \lambda\}$ contains the square Q with base I and so its measure satisfies: $\mu(Q) \leq \mu(\{x \in \mathbb{R}^2_+ : u(x) > \lambda\})$. This combined with the weak-L^1 estimate for the Hardy–Littlewood maximal function gives the assertion of the theorem.

Our strategy of the proof relies on the following facts: first, existence of the Poisson kernel on ADP domains allows us to construct the appropriate harmonic function u. Then we invoke the harmonic measure representation of u together with the mutual absolute continuity of the harmonic measure with respect to the surface measure. Finally, the subelliptic counterparts of the weak-L^1 estimates and the estimates for the nontangential maximal function allow us to conclude the necessity part of the proof.

Definition 5.2.2. Let $\Omega \subset \mathbb{H}^n$ be a domain. We say that a point $x \in \partial\Omega$ is characteristic if the tangent space to $\partial\Omega$ at x is horizontal. The set of all such points in $\partial\Omega$ is denoted by Σ_Ω.

For the reader's convenience we will now recall results from [CGN] and [GP] that are essential for the proof of the Theorem 1.5.1:

(A) (Theorem 1.1 [CGN]). Let $\Omega \subset \mathbb{H}^n$ be a smooth ADP domain. Then, for any $x \in \Omega$ the (subelliptic) harmonic measure $\mathrm{d}\omega^x$ and the surface measure $\mathrm{d}\sigma$ are mutually absolutely continuous. Moreover, for every $p > 1$ it holds that $L^p(\partial\Omega, \mathrm{d}\sigma) \subset L^1(\partial\Omega, \mathrm{d}\omega^x)$.

(B) (Theorem 5.5 [CGN]). Let $\Omega \subset \mathbb{H}^n$ be an NTA domain. Fix $x_0 \in \Omega$ and for a given $\phi \in L^1(\partial\Omega, \mathrm{d}\omega^{x_0})$ define the following function

$$u(x) := \int_{\partial\Omega} \phi(y) \mathrm{d}\omega^x(y), \qquad x \in \partial\Omega.$$

Then u is subelliptic harmonic in Ω and the following estimate holds for the nontangential maximum function of u and the Hardy-Littlewood maximal operator:

$$(N_\alpha(u))(x) \leq C M_\omega(\phi)(x), \qquad x \in \partial\Omega.$$

Here,

$$M_\omega(\phi)(x) := \sup_{0<r<\operatorname{diam}\Omega} \frac{1}{\omega(\partial\Omega \cap B(x,r))} \int_{\partial\Omega \cap B(x,r)} |\phi(z)| \mathrm{d}\omega(z), \quad x \in \partial\Omega. \tag{5.5}$$

(C) (Theorem 4.9 in [CGN]). Let $\Omega \subset \mathbb{H}^n$ be a smooth domain, then $\sigma(\Sigma_\Omega) = 0$.

(D) (Theorem 1.1 in [GP]) Let $\Omega \subset \mathbb{H}^n$ be an ADP domain and let $x_0 \in \partial\Omega$, $0 < r < \frac{R_0}{6}$ where $R_0 > 0$ depends only on the ADP character of Ω. If u is a nonnegative p-harmonic function in $\Omega \cap B(x_0, 6r)$ which vanishes continuously on $\partial\Omega \cap B(x_0, 6r)$, then there exists $C = C(n, \Omega, p) > 0$ such that for every $x \in \Omega \cap B(x_0, r)$ one has

$$\frac{u(x)}{u(A_r(x_0))} \leq C \frac{d(x, \partial\Omega)}{r}.$$

(E) (Theorem 1.2 in [GP]) Let u be a nonnegative p-harmonic function in a bounded (Euclidean) $C^{1,1}$ domain $\Omega \subset \mathbb{H}^n$. then, there exists $M > 1$ depending only on Ω such that for every $x_0 \in \partial\Omega \setminus \Sigma_\Omega$ and every $0 < r < \frac{d(x_0, \Sigma_\Omega)}{M}$ one has for some constant $C = C(n, \Omega, p) > 0$

$$\frac{u(x)}{u(A_r(x_0))} \geq \frac{d(x, \partial\Omega)}{r}$$

for every $x \in \Omega \cap B(x_0, r)$.

Proof of Theorem 1.5.1. The sufficiency part of the proof follows from the discussion analogous to the one in the proof of Proposition 6.3 in [AF]. In particular, formula (6.5) in [AF] for $\alpha = 1$ and $s = 2$ gives assertion (5.4). For the sake of completeness of the presentation we now provide some key steps of the reasoning in [AF]. Moreover, for the sufficiency part it is enough that function u in (5.4) is a continuous function.

Let μ be a Carleson measure on Ω. We define the following superlevel sets

$$E(\lambda) := \{x \in \Omega : |u(x)| > \lambda\} \quad \text{and} \quad U(\lambda) := \{\omega \in \partial\Omega : N_\alpha u(\omega) > \lambda\}, \quad \lambda > 0.$$

In this notation, assertion (5.4) reads

$$\mu(E(\lambda)) \leq C\mathcal{H}^2(U(\lambda)) \quad \text{for all } \lambda > 0. \tag{5.6}$$

As in [AF] we employ the Whitney-type decomposition of $U(\lambda)$ based on the general result [HKST, Proposition 4.1.15] applied to the metric space $(\partial\Omega, d|_{\partial\Omega})$ and the open set $U(\lambda)$. It allows us to find a countable collection $\mathcal{W}_\lambda = \{B(\omega_i, r_i) : i = 1, 2, \dots\}$ of balls with centers $\omega_i \in U(\lambda)$ such that

$$U(\lambda) = \bigcup_{i=1,2,\dots} B(\omega_i, r_i) \cap \partial\Omega, \tag{5.7}$$

$$\sum_i \chi_{B(\omega_i, 2r_i)\cap\partial\Omega} \leq 2N^5, \tag{5.8}$$

where $r_i = (1/8)d(\omega_i, \partial\Omega \setminus U(\lambda))$ and N depends only on the 3-regularity constant of $\partial\Omega$. Let x be an arbitrary point in $E(\lambda)$, then

$$N_\alpha u(\omega) > \lambda, \quad \text{for all } \omega \in S(x) = B\left(x, (1+\alpha)d(x, \partial\Omega)\right) \cap \partial\Omega,$$

where $S(x)$ stands for shadow of x, see Definition 3.2.4. Hence,

$$S(x) \subset U(\lambda) \overset{(5.7)}{=} \bigcup_i B(\omega_i, r_i) \cap \partial\Omega, \quad \text{for all } x \in E(\lambda).$$

Next, for $x \in E(\lambda)$, let $\omega_x \in \partial B$ be such that $d(x, \omega_x) = d(x, \partial\Omega)$, due to compactness of $\partial\Omega$. Thus $\omega_x \in S(x)$ and, since $S(x) \subset U(\lambda)$, we moreover know that $d(\omega_x, \partial\Omega \setminus U(\lambda)) \geq d(\omega_x, \partial\Omega \setminus S(x))$.

Furthermore, there exists $i_x \in \{1, 2, \dots\}$ such that $\omega_x \in B(\omega_{i_x}, r_{i_x})$. By repeating the reasoning in [AF] we find that

$$x \in B\left(\omega_{i_x}, \left(\tfrac{9}{\alpha} + 1\right) r_{i_x}\right), \quad r_{i_x} = \frac{1}{8} d(\omega_{i_x}, \partial\Omega \setminus U(\lambda)).$$

Since x was chosen arbitrarily from $E(\lambda)$, we have thus shown that $E(\lambda)$ is covered by the countable family of balls $B(\omega_i, Cr_i)$, $i = 1, 2, \dots$, where $C = C(\alpha) = \frac{9}{\alpha} + 1$. By using the assumption that μ is a Carleson measure, the fact that $\mathcal{H}^2|_{\partial\Omega}$ is 2-regular, and that the multiplicity of Whitney balls is controlled by (5.8), we deduce that

$$\mu(E(\lambda)) \leq \sum_i \mu(B(\omega_i, Cr_i) \cap \Omega) \leq \gamma_\mu \left(\tfrac{9}{\alpha} + 1\right)^2 \sum_i r_i^2 \lesssim \sum_i \mathcal{H}^2(B(\omega_i, r_i) \cap \partial\Omega) \overset{(5.8)}{\lesssim} \mathcal{H}^2(U(\lambda)),$$

as desired. This concludes the proof of the sufficiency part of Theorem 1.5.1.

Next, let us prove the necessity part of the assertion. First, we need the solvability of the subelliptic harmonic Dirichlet problem for continuous boundary data. Such a result holds for bounded open sets in $\mathbb{H}^1$ satisfying the uniform outer ball condition, see Remark 3.4 in [LU] and references therein. Therefore, since every gauge ball B satisfies the uniform outer ball condition and, by assumptions, so is Ω, it holds that also $\Omega \cap B(x_0, 3r)$ satisfies the condition, see Remark 3.5 in [LU] for convex open sets. However, the set $\Omega \cap B(x_0, 3r)$ does not have to be convex, but a ball is convex and hence satisfies the uniform outer ball condition. Therefore, the intersection of two sets satisfying uniform outer ball condition also satisfies it, as it suffices to take a smaller radius of those defining outer balls for Ω and $B(x_0, 3r)$.

By the discussion at (4.1) in [LU] we define the Poisson kernel $P = P(x, \omega)$ related to Ω, for $x \in \Omega$ and $\omega \in \partial\Omega \setminus \Sigma_\Omega$. Recall, that $\sigma(\Sigma_\Omega) = 0$ by (C) in our presentation following Definition 5.2.2.

Let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function such that $\phi \equiv 4\lambda$ on the set $\partial\Omega \cap B(x_0, 6r)$, $\phi \equiv 0$ outside the set $\partial\Omega \cap B(x_0, 7r)$ and $0 \leq \phi \leq 4\lambda$. Then a function $u(y) = \int_{\partial D} P(y, \omega)\phi(\omega)\mathrm{d}\sigma(\omega)$ is the unique harmonic solution to the Dirichlet Problem in Ω for the Poisson kernel P of domain Ω with boundary data given by ϕ. Moreover, by the weak maximum principle in Theorem 8.2.19 (ii) in [BLU] we obtain the weak minimum principle for u, so that $0 \leq u \leq 4\lambda$ in Ω. Let us consider a function $w := 4\lambda - u$. Such a function is harmonic in Ω, satisfies $0 \leq w \leq 4\lambda$ and it has zero boundary values on $\partial\Omega \cap B(x_0, 6r)$. therefore, we can use Theorem 1.1 from [GP], see (D), to obtain

$$\frac{w(x)}{w(A_r(x_0))} = \frac{4\lambda - u(x)}{4\lambda - u(A_r(x_0))} \leq c\frac{d(x, \partial\Omega)}{r}.$$

Then it follows that

$$u(x) \geq 4\lambda \left(1 - c\frac{d(x, \partial\Omega)}{r}\right) + cu(A_r(x_0))\frac{d(x, \partial\Omega)}{r}$$

for $x \in \Omega \cap B(x_0, r)$ and $c = c(n, \Omega)$.

If $x \in \Omega \cap B(x_0, \widetilde{r})$ with $\widetilde{r} < \frac{3}{4c}r$, then $1 - c\frac{d(x,\partial\Omega)}{r} > \frac{1}{4}$. In the consequence, we get

$$u(x) > \lambda + cu(A_r(x_0))\frac{d(x, \partial\Omega)}{r} > \lambda.$$

In particular, $u > \lambda$ on $\Omega \cap B(x, \frac{1}{2c}r)$ and so it holds for any ball $B(x_0, r)$ that

$$\mu(B(x_0, r) \cap \Omega) \lesssim \mu(B(x_0, \frac{1}{2c}r) \cap \Omega) \leq \mu(\{x \in \Omega : u(x) > \lambda\}) \overset{(5.4)}{\leq} C\sigma(\{\omega \in \partial\Omega : N_\alpha u(\omega) > \lambda\}). \tag{5.9}$$

Since $\phi \in C(\partial\Omega)$, we have that $\phi \in L^p(\partial\Omega, \mathrm{d}\sigma)$ for any $1 \leq p \leq \infty$. This holds, as $\sigma(\partial\Omega) < \infty$, due to the 3-regularity of $\partial\Omega$ and the boundedness of the diameter of Ω. Hence, Theorem 1.1 in [CGN], see (A), implies that $\phi \in L^p(\partial\Omega, \mathrm{d}\omega^y)$ for any given $y \in D$. Moreover, it holds that $\mathrm{d}\omega^y = P(y, \cdot)\mathrm{d}\sigma$, for a Poisson kernel. Therefore, Theorem 5.5 (i) in [CGN], see (B), implies that

$$\{\omega \in \partial\Omega : N_\alpha u(\omega) > \lambda\} \subset \left\{\omega \in \partial\Omega : M_{\omega^y}(\phi)(\omega) > \frac{\lambda}{C}\right\}, \tag{5.10}$$

where $M_{\omega^y}(\phi)$ stands for the Hardy–Littlewood maximal operator of function $\phi \in L^1(\partial\Omega, \mathrm{d}\omega^y)$, see (5.5)

$$M_{\omega^y}(\phi)(\omega) := \sup_{0<r<\operatorname{diam}\Omega} \frac{1}{\omega^y(\partial\Omega \cap B(\omega, r))} \int_{\partial\Omega\cap B(\omega,r)} |\phi(z)|\mathrm{d}\omega^y(z), \quad \omega \in \partial\Omega.$$

Next, we appeal to the following relation between the maximal operator considered with respect to the harmonic measure ω^y and the surface measure σ, see (6.7) in [CGN]:

$$M_{\omega^y}(\phi)(\omega) \leq C\left(M_\sigma|\phi|^\beta\right)^{\frac{1}{\beta}}(\omega), \quad \omega \in \partial\Omega, \quad \text{any fixed } y \in \Omega. \tag{5.11}$$

The estimate holds for any $1 < p \leq \infty$ and $1 < \beta < p$. Since $\phi \in L^\infty(\partial\Omega, \mathrm{d}\omega^y)$ we may choose $p = \infty$.

By [CG], pg. 414 it holds that $(\partial\Omega, \mathrm{d}\omega^x, d_{\partial\Omega})$ is the homogeneous space. Recall that a metric measure space (X, d, μ) is the space of homogeneous type if (X, d) is a quasimetric space and μ is a doubling measure. Thus, by collecting estimates in (5.9)-(5.11) and by applying the weak-L^1 estimate for doubling spaces in Theorem 3.5.6 in [HKST] and by the definition of ϕ we obtain the following estimate

$$\begin{aligned}
\mu(B(x_0, r) \cap \Omega) &\leq C\sigma\left(\left\{\omega \in \partial\Omega \,:\, M_{\omega^y}(\phi)(\omega) > \frac{\lambda}{C}\right\}\right) \\
&\leq C\sigma\left(\left\{\omega \in \partial\Omega \,:\, M_\sigma|\phi|^\beta(\omega) > \left(\frac{\lambda}{C}\right)^\beta\right\}\right) \\
&\leq \frac{C^\beta}{\lambda^\beta}\|\phi\|^\beta_{L^\beta(\partial\Omega, \mathrm{d}\sigma)} \lesssim C\sigma(\partial(B(x_0, 3r) \cap \Omega)) \lesssim r^3.
\end{aligned} \tag{5.12}$$

Since $y \in \Omega$ is arbitrary and any two harmonic measures ω^y and $\omega^{y'}$ are comparable for any $y, y' \in \Omega$ with the constant depending on the diameter $\operatorname{diam}\Omega < \infty$, it follows that μ is Carleson in Ω, as the constants in (5.12) do not depend on the choice of r.

□

5.3 Carleson measures and Möbius-type transformations on the unit gauge ball

The purpose of this chapter is to show Theorem 1.5.2, a counterpart of Lemma 3.3 in Section 3, Chapter VI in [G] characterizing the Carleson measures on the unit disk $\mathbb{D}$ in terms of the canonical Möbius transformations on $\mathbb{D}$. Namely, the lemma stays that *a positive measure μ on $\mathbb{D}$ is a Carleson measure if and only if the following holds:*

$$\sup_{z_0 \in \mathbb{D}} \int_{\mathbb{D}} \frac{1 - |z_0|^2}{|1 - \bar{z}_0 z|^2} \mathrm{d}\mu(z) = M < \infty. \tag{5.13}$$

Moreover, the constant M is comparable to the Carleson constant, i.e. $M \approx \gamma_\mu$ with absolute constants. Notice that, for a given $z_0 \in \mathbb{D}$, the integrand in (5.13) satisfies the following

$$\frac{1 - |z_0|^2}{|1 - \bar{z}_0 z|^2} = \frac{1 - \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|^2}{1 - |z|^2} = \frac{1 - |T_{z_0}(z)|^2}{1 - |z|^2}, \tag{5.14}$$

where $T_{z_0}(z) = e^{-i\theta_0}\frac{z - z_0}{1 - \bar{z}_0 z}$, for $z_0 = r_0 e^{i\theta_0}$ is the Möbius self-mapping of $\mathbb{D}$ with the property $T_{z_0}(z_0) = 0$. Such family of conformal mappings, and its n-dimensional counterpart, play an important role in the studies of quasiconformal and quasiregular mappings and related Hardy spaces, see e.g. [AK, AG1, AG2] and [Ah] for basic properties of such mappings. The relation between expressions in (5.14) and the Carleson condition becomes more apparent once we observe that for small enough radii $r > 0$, any $\omega \in \partial\mathbb{D}$ and $z \in \mathbb{D} \cap B(\omega, r)$ it holds that

$$\frac{1 - |T_{z_0}(z)|^2}{1 - |z|^2} \approx \frac{1 - |T_{z_0}(z)|}{1 - |z|} \approx \frac{1}{r}, \tag{5.15}$$

see Lemma 2.2 in [AG2] and Formula (33) in Ch. II in [Ah]. Hence, (5.15) together with (5.13) imply the Carleson condition for μ:

$$\mu(\mathbb{D} \cap B(\omega, r)) = \int_{\mathbb{D}\cap B(\omega,r)} \frac{r}{r} \mathrm{d}\mu \lesssim Mr.$$

The class of the Möbius transformations T_{z_0} has no direct counterpart in the Heisenberg setting as a class of the conformal maps from a unit Korányi–Reimann ball into itself due to lack of rotational symmetry of Korányi-Reimann ball. Nevertheless, recently in [AF, Section 4.1] the notion of class T_{z_0} has been extended to the subriemannian setting in the following way.

Recall that the Korányi-Reimann unit ball in $\mathbb{H}^1$ is defined as follows:

$$B(0, 1) := \{x \in \mathbb{H}^1 \,:\, d_{\mathbb{H}^n}(0, x) < 1\}.$$

Recall the *Korányi inversion* in the Korányi unit sphere centered at the origin defined as follows: $I(y) = -\frac{1}{\|y\|^4}\left(y_z(|y_z|^2 + iy_t), y_t\right)$, where $y = (y_z, y_t) \in \mathbb{H}^1 \setminus \{0\}$. It is the restriction of a conformal self-map of the compactification $\hat{\mathbb{H}}^1$, with $I(0) = \infty$ and $I(\infty) = 0$. Moreover, if y lies in the complex plane (i.e. $y_t = 0$), the inversion I agrees with the well-known inversion in the unit circle.

Let us fix $x \in \mathbb{H}^1$, $a \in \mathbb{H}^1 \setminus \{x\}$, and $\rho > 0$. Define the map $T := T_{x,a,\rho} : \hat{\mathbb{H}}^1 \to \hat{\mathbb{H}}^1$ as follows

$$T(y) := \delta_\rho\left(\left[I(a^{-1}\cdot x)\right]^{-1} \cdot \left[I(a^{-1}\cdot y)\right]\right), \tag{5.16}$$

where δ_ρ denotes the Heisenberg dilation by ρ, i.e. $\delta_\rho(x) = \delta_\rho(x_1, x_2, x_3) := (\rho x_1, \rho x_2, \rho^2 x_3)$.

Below we collect some properties of maps $T_{x,a,\rho}$ proven in Proposition 4.2, Corollaries 4.9 and 4.11 and Proposition 4.13 in [AF]:

(1) The mapping
$$T|_{\mathbb{H}^1\setminus\{a\}} : \mathbb{H}^1 \setminus \{a\} \to \mathbb{H}^1 \setminus \{\delta_\rho\left([I(a^{-1}\cdot x)]^{-1}\right)\}$$
is 1-quasiconformal. Moreover, $T(x) = 0$, $T(a) = \infty$, $T(\infty) = \delta_\rho\left([I(a^{-1}\cdot x)]^{-1}\right)$. Recall that 1-quasiconformal maps are conformal both in $\mathbb{R}^n$, see [V], [Geh], [IM], and in $\mathbb{H}^1$, see [CC], [CCLDO].

(2) For all $y, y' \in \mathbb{H}^1 \setminus \{a\}$, it holds that
$$d(T(y), T(y')) = \rho\frac{d(y, y')}{d(a, y)d(a, y')}, \quad \|T(y)\| = \rho\frac{d(x, y)}{d(a, y)d(a, x)}, \quad J_T(y) = \frac{\rho^4}{d(a, y)^8},$$
where J_T denotes the Jacobian of map T.

(3) Let any $x \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$ and $\rho > 0$, be such that

$$\rho \lesssim \min\{d(x, \partial B), d(a, \partial B)\} \quad \text{and} \quad \rho \approx d(a, x). \tag{5.17}$$

Then the map $T = T_{x,a,\rho}$ satisfies

$$B(0, m) \subset T(B) \subset B(0, M). \tag{5.18}$$

for radii m and M depending on x, a, and ρ only through the implicit multiplicative constants in the inequalities in (5.17). This property is a reflection of the similar one for the Möbius self-maps of a unit ball in $\mathbb{R}^n$, see e.g. Lemma 2.2 in [AG2]

(4) Let $\omega \in \partial B$, $x \in B$, and $\rho > 0$. Assume that $a \in \mathbb{H}^1 \setminus \overline{B}$ and $r > 0$ are such that $d(a, \omega) \lesssim r$ and $d(a, B) \geq Cr$, for a constant $C > 1$. Then, the map $T = T_{x,a,\rho}$ satisfies

$$\frac{d(T(y), \partial T(B))}{d(y, \partial B)} \approx_C \frac{\rho}{d(y, a)^2}, \quad \text{for all } y \in B(\omega, r) \cap B. \tag{5.19}$$

The similar property holds in $\mathbb{R}^n$, see Lemma 2.2 in [AG2].

After the above preparatory observations we are in a position to formulate and prove the main result of this chapter. For the reader's convenience we recall the statement of Theorem 1.5.2.

Theorem. 1.5.2 [cf. Lemma 3.3 in [G]] *A measure μ on the Korányi-Reimann unit ball $B := B(0, 1) \subset \mathbb{H}^1$ is a Carleson measure if and only if*

$$\int_B \left(\frac{d(T_{x,a,\rho}(y), \partial T_{x,a,\rho}(B))}{d(y, \partial B)} \right)^3 \mathrm{d}\mu(y) = M < \infty, \tag{5.20}$$

for all $x \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$, and $\rho > 0$ such that $\rho \lesssim \min\{d(x, \partial B), d(a, \partial B)\}$ and $\rho \approx d(a, x)$.

Basing on (5.15), one could expect that the corresponding hypotheses (5.20) of the above theorem in $\mathbb{H}^1$ should involve the Kóranyi norms of points in $y \in B$ and their images $T(y)$. Indeed, by property (5.18) we have that

$$\left(\frac{m+1}{2}\right) \frac{1 - \|T_{x,a,\rho}(y)\|}{1 - \|y\|} \leq \frac{1 - \|T_{x,a,\rho}(y)\|^2}{1 - \|y\|^2} \leq (M + 1) \frac{1 - \|T_{x,a,\rho}(y)\|}{1 - \|y\|}.$$

However, due to the geometry of balls in $\mathbb{H}^1$ it is more convenient to work with the distances to the corresponding boundaries of y and $T(y)$, see the proof below.

Proof of Theorem 1.5.2. Set $T := T_{x,a,\rho}$ and observe that by (5.19) and by assumptions the following holds for any $y \in B(\omega, r) \cap B$:

$$\frac{d(T(y), \partial T(B))}{d(y, \partial B)} \approx_C \frac{\rho}{d(y, a)^2} \approx \frac{d(a, x)}{d(y, a)^2}. \tag{5.21}$$

Choose x such that $x \in B(\omega, r) \cap B$ and $d(\omega, x) = \frac{r}{2}$. Then, for a choosen as in (5.19), it holds that

$$r \leq d(a, y) \leq d(a, x) + d(x, y) \approx d(x, \partial B) + \frac{3}{2}r \approx \frac{5}{2}r,$$
$$d(a, x) \approx d(x, \partial B) \approx d(x, \omega) \approx r.$$

Hence, by applying these estimates in (5.21) we obtain that

$$\frac{d(T(y), \partial T(B))}{d(y, \partial B)} \approx \frac{1}{r} \approx \frac{1}{d(x, \partial B)}. \tag{5.22}$$

In order to show the necessity part of the assertion, let us assume that (5.20) holds. Then, for any $\omega \in \partial B$ and $r > 0$ we have

$$\begin{aligned}\mu(B(\omega, r) \cap B) &= \int_{B(\omega,r)\cap B} \frac{d(x, \partial B)^3}{d(x, \partial B)^3} \mathrm{d}\mu(y) \\ &\approx d(x, \partial B)^3 \int_{B(\omega,r)\cap B} \left(\frac{d(T(y), \partial T(B))}{d(y, \partial B)} \right)^3 \mathrm{d}\mu(y) \\ &\lesssim M d(x, \partial B)^3 \approx r^3.\end{aligned}$$

In order to show the opposite implication in the assertion of the theorem let us consider two cases for points $x \in B$ in the definition of maps $T = T_{x,a,\rho}$: (1) $d(x, \partial B) > \frac{1}{4}$, and (2) $d(x, \partial B) \leq \frac{1}{4}$. In the first case by (5.22), we trivially have that

$$\int_{B(\omega,r)\cap B} \left(\frac{d(T(y), \partial T(B))}{d(y, \partial B)} \right)^3 \mathrm{d}\mu(y) \lesssim_C \mu(B) \leq C\gamma_\mu < \infty.$$

Therefore, we may assume that points $x \in B$ satisfy $d(x, \partial B) \leq \frac{1}{4}$ and mimic the approach in the proof of Lemma 3.3 in [G, Section 3, Chapter VI]. However, we need to take into account the differences between the Euclidean and the Heisenberg settings.

Recall that the Euclidean radial curves need not be horizontal in $\mathbb{H}^1$ and hence may have an infinite subriemannian length. However, by works [KR1] and [BT], see also the discussion in Section 2.1.2 in [AF], we have that the following formula describes the radial curves given by the horizontal curves joining the origin with the point $\omega = (z, t)$ belonging to the boundary $\partial B \setminus \{z = 0\}$:

$$\gamma(s, (z, t)) = \left(sze^{-\mathrm{i}\frac{t}{|z|^2}\log s}, s^2 t \right), \quad (z, t) \in \partial B \setminus \{z = 0\}. \tag{5.23}$$

It is easy to compute that $\|\gamma(s)\| = s$. Moreover, given $x \in B$ we may find a point $\omega \in \partial B$ corresponding to $x = (x_z, x_t)$, in a sense that $x = \gamma(s, \omega)$ for some $0 < s < 1$, by solving (5.23) for z and t. Namely, we have that

$$t = \frac{x_t}{\|x\|^2}, \quad z = \frac{x_z}{\|x\|} e^{\mathrm{i}\frac{x_t}{|x_z|^2}\log \|x\|}.$$

We denote such point by ω_x and define the following family of subsets in B:

$$E_n := \{y \in B : d(y, \omega_x) < 2^n d(x, \partial B)\} \quad n = 1, 2, \dots.$$

Therefore, since μ is assumed to be a Carleson measure in B, we find that

$$\mu(E_n) = \mu\big(B(\omega_x, 2^n d(x, \partial B)) \cap B\big) \leq \gamma_\mu 2^{3n} d(x, \partial B)^3, \quad n = 1, 2, \dots .$$

Hence, by appealing to (5.21), we obtain the following estimate

$$\begin{aligned}
&\int_{B(\omega,r)\cap B} \left(\frac{d(T(y), \partial T(B))}{d(y, \partial B)}\right)^3 \mathrm{d}\mu(y) \\
&\qquad \leq \int_{E_1} \left(\frac{d(T(y), \partial T(B))}{d(y, \partial B)}\right)^3 \mathrm{d}\mu(y) + \sum_{n=2}^{\infty} \int_{E_n \setminus E_{n-1}} \left(\frac{d(T(y), \partial T(B))}{d(y, \partial B)}\right)^3 \mathrm{d}\mu(y). \qquad (5.24)
\end{aligned}$$

Since, by assumption $d(a, x) \approx d(x, \partial B)$ and for $y \in E_n \setminus E_{n-1}$ it holds that

$$2^{n-1} d(x, \partial B) < d(y, \omega_x) < 2^n d(x, \partial B),$$

we have that

$$\frac{d(a, x)}{d(y, a)^2} \lesssim \frac{d(x, \partial B)}{d(y, \omega_x)^2} \lesssim \frac{1}{2^{2n} d(x, \partial B)},$$

as $d(y, a) > d(y, \omega_x)$ due to the assumption that $a \in \mathbb{H}^1 \setminus \overline{B}$. We are in a position to complete the above estimate (5.24) as follows (cf. (5.21)):

$$\begin{aligned}
&\int_{E_1} \left(\frac{d(T(y), \partial T(B))}{d(y, \partial B)}\right)^3 \mathrm{d}\mu(y) + \sum_{n=2}^{\infty} \int_{E_n \setminus E_{n-1}} \left(\frac{d(T(y), \partial T(B))}{d(y, \partial B)}\right)^3 \mathrm{d}\mu(y) \\
&\leq \frac{\mu(E_1)}{2^6 d(x, \partial B)^3} + \sum_{n=2}^{\infty} \int_{E_n \setminus E_{n-1}} \frac{1}{2^{6n} d(x, \partial B)^3} \mathrm{d}\mu(y) \\
&\lesssim \sum_{n=1}^{\infty} \frac{1}{2^{6n} d(x, \partial B)^3} \mu(E_n) \lesssim_{\gamma_\mu} \sum_{n=1}^{\infty} \frac{1}{2^{3n}} < \infty.
\end{aligned}$$

This completes the sufficiency part of the proof and thus, the whole proof is completed as well. □

Remark 5.3.1. We observe that since the Korányi inversion can be defined in groups $\mathbb{H}^n$, see e.g. [KR2], so is the class of maps $T_{x,a,\rho}$. Moreover, the horizontal curves (5.23) exist not only in $\mathbb{H}^1$, but also in $\mathbb{H}^n$ (in fact, in polarizable groups, see Section 3 in [BT]). Therefore, Theorem 1.5.2 has a counterpart in $\mathbb{H}^n$. However, for the sake of simplicity of the presentation and in order to avoid repeating similar construction of maps $T_{x,a,\rho}$ presented in [AF, Section 4.1], we decided to state the theorem only in the $\mathbb{H}^1$ setting.

5.4 Square function and Carleson measures for L^2 and BMO boundary data

The purpose of this chapter is to prove main results of this chapter, namely Theorems 1.5.3 and 1.5.4, which generalize, respectively, Theorem 9.1 and Theorem 9.6 in [JK]. Theorem 9.1 provides the bound for the L^2-norm of the square function in terms of the L^2-norm of the boundary data on NTA domains.

Theorem 9.6 gives a Carleson-measure estimate for a subelliptic harmonic function defined by the integral of a BMO function with respect to harmonic measure on the NTA domain. Such estimates in $\mathbb{R}^n$ are essential, for example, when proving the ε-approximability for harmonic functions. We refer to Chapters 3.2 and 4.3 of this thesis for proofs of ε-approximability. Recall that in Chapter 3.2 we proved ε-approximability for a class of nonharmonic functions on Lipschitz-graph domains and in Chapter 4.3 we proved ε-approximability on Riemannian manifolds.

Recall Definition 5.1.10 of the square function (also known in the literature as the area function, depending on the authors and the context)

$$S_\alpha u(x)^2 := \int_{\Gamma_\alpha(x)} |\nabla_H u(y)|^2 d(y, \partial\Omega)^{2-Q} \mathrm{d}y.$$

For the reader's convenience we recall our main results.

Theorem. 1.5.3(L^2-boundedness of the square function) *Let $\Omega \subset \mathbb{H}^n$ be a bounded NTA domain. Let further $f \in L^2(\mathrm{d}\omega)$ and $u(x) := \int_{\partial\Omega} f(y)\mathrm{d}\omega^x(y)$. Then, the following estimate holds for the square function S_α of a subelliptic harmonic function u in Ω*

$$\|S_\alpha u\|_{L^2(\mathrm{d}\omega)} \leq C\|f\|_{L^2(\mathrm{d}\omega)},$$

where the constant C depends on n, M, constant from the Harnack inequality, α and Ω.

The corresponding Euclidean result in [JK], cf. Theorem 9.1, is proven for any $1 < p < \infty$. However, for us the case $p = 2$ is the most interesting, as it is the one that we would like to use to prove ε-approximability for harmonic functions. According to our best knowledge, the result is new in the subriemannian setting.

Theorem. 1.5.4 (Carleson measure estimate) *Let $\Omega \subset \mathbb{H}^n$ be a bounded NTA domain and u be subelliptic harmonic in Ω such that $u(x) = \int_{\partial\Omega} f(y)\mathrm{d}\omega^x(y)$ for some $f \in BMO(\partial\Omega)$. Then, for any $D > 1$ there exists a constant $C > 0$ such that for any ball $B(x_0, r)$ centered at $x_0 \in \partial\Omega \setminus \Sigma_\Omega$ with any $0 < r < \frac{r_0}{DM} \leq \frac{1}{DM}\min\{1, \frac{d(x_0,\Sigma_\Omega)}{M}\}$ it holds that*

$$\int_{B(x_0,r)\cap\Omega} |\nabla_H u|^2 G(x, A_{DMr}(x_0))\mathrm{d}x \leq C\omega(B(x_0, r) \cap \partial\Omega),$$

where constant C depends on D, n, M, r_0 and $\|f\|_{BMO(\partial\Omega)}$ and G denotes the Green function of Ω.

Let us remark that the assertion of the theorem can be formulated equivalently in a way similar to the bottom of page 3 in [HT], i.e. by using the supremum over radii and the averaged integral.

In the theorem, ω stands for a harmonic measure with respect to any but fixed point $y \in \Omega \setminus B(x_0, ar)$ and so $\omega := \omega^y$. However, for the sake of convenience of the presentation in what follows we omit the reference points. The constant a can be taken equal to M, see [CGN, Section 5].

Upon strengthening the regularity assumptions of the boundary, the following consequence of Theorem 1.5.4 holds, as the ADP condition allows us to compare the harmonic measure with the surface measure, see (A) in the discussion following Definition 5.2.2.

Corollary 5.4.1. *Under the assumptions of Theorem 1.5.4 if we additionally assume that Ω is a smooth ADP domain, then it holds that for any ball $B(x_0, r)$ centered at $x_0 \in \partial\Omega \setminus \Sigma_\Omega$ with radius $0 < r < \frac{r_0}{DM} \leq \frac{1}{DM}\min\{1, \frac{d(x_0,\Sigma_\Omega)}{M}\}$ that*

$$\int_{B(x_0,r)\cap\Omega} |\nabla_H u|^2 G(x, A_{DMr}(x_0))\mathrm{d}x \leq Cr^{Q-1},$$

where C depends on D, n, M, r_0 and $\|f\|_{BMO(\partial\Omega)}$.

An important class of examples of NTA domains in $\mathbb{H}^n$ is the one of (Euclidean) $C^{1,1}$ domains, see [CG, GP]. Since the Green function $G(\cdot, A_{DMr}(x_0))$ is nonnegative subelliptic harmonic in $\Omega \setminus \{A_{DMr}(x_0)\}$ we may use Theorem 1.2 in [GP] to get the following lower boundary Harnack-type estimate for a (Euclidean) $C^{1,1}$ domain $\Omega \subset \mathbb{H}^n$, any $x_0 \in \partial\Omega \setminus \Sigma_\Omega$ and

$$0 < r' < \frac{1}{DM}\min\{1, d(x_0, \Sigma_\Omega)/C, d(A_r(x_0), \partial\Omega)\},$$

where C is a constant from Theorem 1.2 in [GP], and for all $x \in B(x_0, r') \cap \Omega$:

$$G(x, A_r(x_0)) \geq C(n, \Omega)G(A_{r'}(x_0), A_r(x_0))\frac{d(x, \partial\Omega)}{r'} \gtrsim_{C(n,\Omega)} G(A_{r'}(x_0), A_r(x_0))d(x, \partial\Omega).$$

Moreover, notice that by building a chain of balls joining $A_{r'}(x_0)$ with fixed, but any $y \in \Omega$ such that $\mathrm{dist}(y, \partial\Omega) > r_0$, for r_0 as in the definition of the NTA domains, and by the standard iteration of the Harnack inequality on metric balls (see e.g. [BLU, Corollary 5.7.3]), we obtain that

$$G(A_{r'}(x_0), A_r(x_0)) \geq C^N G(y, A_r(x_0)),$$

where the length of the Harnack chain N depends on $\operatorname{diam}\Omega$ and r and the constant C depends on the distance d and the geometric parameters of $\mathbb{H}^n$ and Δ_H, cf. [BLU].

The above discussion and the Proposition 5.6 in [GP] implies the following Carleson-type estimate.

Corollary 5.4.2. *Let $\Omega \subset \mathbb{H}^n$ be a (Euclidean) $C^{1,1}$ domain which also satisfies the uniform outer ball condition. Then, under the assumptions of Theorem 1.5.4, it holds for any ball $B(x_0, r)$ centered at $x_0 \in \partial\Omega \setminus \Sigma_\Omega$ with radius $0 < r < \frac{r_0}{DM} \leq \frac{1}{DM}\min\{1, \frac{d(x_0,\Sigma_{Om})}{M}\}$ that*

$$\int_{B(x_0,r)\cap\Omega} |\nabla_H u|^2 G(A_{r'}(x_0), A_{DMr}(x_0))d(x, \partial\Omega)\mathrm{d}x \leq Cr^{Q-1}, \tag{5.25}$$

where C depends on $D, n, M, r_0, \operatorname{diam}\Omega$ and $\|f\|_{BMO(\partial\Omega)}$ and the radius r' satisfies

$$0 < r' < \frac{1}{DM}\min\{1, d(x_0, \Sigma_\Omega)/C, d(A_r(x_0), \partial\Omega)\}.$$

Let us observe some further consequences of Theorem 1.5.4. First, we explain how it corresponds to Garnett's result, see [G, Theorem 3.4]. Then, in Example 5.4.4 we show how for the gauge unit ball, a special but important case of the NTA domain in $\mathbb{H}^n$, the estimate in the theorem takes simpler and convenient form.

Remark 5.4.3. Theorem 1.5.4 generalizes part of the following characterization of the Carleson measures on the unit disc in the plane to the setting of NTA domains in $\mathbb{H}^n$, see Theorem 3.4 in [G]:

Let $\phi \in L^1(\mathbb{S}^1)$ and u be a Poisson extension of ϕ to the unit disc in $\mathbb{R}^2$. Let

$$\mathrm{d}\lambda_\phi := |\nabla\phi(z)|^2 \ln\frac{1}{|z|}\,\mathrm{d}x\mathrm{d}y.$$

Then $\phi \in BMO(\mathbb{S}^1)$ if and only if λ_ϕ is a Carleson measure. Moreover, the Carleson constant of λ_ϕ is comparable to $\|\phi\|^2_{BMO}$.

Recall that, up to the constant $\frac{1}{2\pi}$, the function $\ln\frac{1}{|z|}$ is the Green function of the planar unit disc with a pole at 0; also that Euclidean balls are NTA domains. Moreover, the harmonic measure $\omega \approx \sigma$, where σ stands for the surface measure on $\mathbb{S}^1$, see Exc. 3, Ch. I in [G]. Therefore, the sufficiency part of Remark 5.4.3 corresponds in $\mathbb{R}^2$ to the assertion of Theorem 1.5.4.

The next consequence of Theorem 1.5.4 addresses the fact that for some NTA domains in $\mathbb{H}^n$ the Green functions can be found explicitly and so the Carleson condition in Theorem 1.5.3 can be refined.

Example 5.4.4. Let $\Omega = B(0,1)$ be a unit gauge ball in $\mathbb{H}^n$. By Corollary 4 in [CG] such balls are NTA domains. Below, we show that on Ω it is possible to refine the estimate for $G(A_{r'}(x_0), A_r(x_0))$ and obtain the following more natural Carleson estimate.

Recall that for the unit gauge ball in $\mathbb{H}^n$ the set of characteristic points Σ_Ω consists of the north and south poles only.

Fix $\delta \in (0,1)$ and let u be as in Theorem 1.5.4, i.e. a subelliptic harmonic function on Ω with the boundary data in BMO. Then, for all points $x_0 = (z,t) \in \partial B \setminus \{z : |z| \leq \delta\}$ and all radii $0 < r < r_0 \leq \min\{1, \frac{d(x_0,\Sigma_\Omega)}{M}\}$ it holds

$$\int_{B(x_0,r)\cap\Omega} |\nabla_H u|^2 d(x,\partial\Omega)\mathrm{d}x \leq Cr^{Q-1}. \tag{5.26}$$

Here, the constant C is as in Corollary 5.4.2 and, additionally, depends on δ.

In order to show this estimate, we appeal to the horizontal curves joining the origin with the point $x_0 = (z,t)$ in the boundary $\partial B \setminus \{z = 0\}$, see (5.23). The proofs of Lemmas A.2 and A.4 in [AF] show the following properties of curves γ_{x_0}:

$$d(\gamma_{x_0}(s), \gamma_{x_0}(s')) \leq \mathrm{length}(\gamma_{x_0}(\cdot)|_{[s,s']}) = \frac{s'-s}{|z|}, \quad 0 < s < s' \leq 1,$$
$$d(\gamma_{x_0}(s), \partial B) \gtrsim \frac{1-s}{|z|}, \quad \text{if} \quad 1-s \leq |z|.$$

These properties allow us to choose s such that $1-s = r|z|$ and obtain a point on γ_{x_0} which satisfies the definition of a corkscrew point $A_r(x_0)$ in the interior corkscrew condition in Definition 5.1.1. Choose r' close enough to r (i.e. $|r-r'| \ll 1$) and the corresponding s' with $1-s' = r'|z|$. Therefore, we get that

$$d(A_r(x_0), A_{r'}(x_0)) = d(\gamma_{x_0}(s), \gamma_{x_0}(s')) \leq \frac{|s'-s|}{|z|} \leq \frac{1}{2}r' \lesssim_M \frac{1}{2}d(A_{r'}(x_0), \partial B),$$

where M stands for the NTA constant of a gauge ball in $\mathbb{H}^n$. In a consequence, we may estimate function G from below as follows:

$$G(A_{r'}(x_0), A_r(x_0)) \gtrsim \frac{1}{d(A_r(x_0), A_{r'}(x_0))^{Q-2}} \gtrsim \left(\frac{|z|}{|s-s'|}\right)^{Q-2} \gtrsim |z|^{Q-2},$$

and the proof of the first inequality follows by repeating the steps of the corresponding proof of Property (1.9) in Theorem 1.1 in [GW], see Proposition A.1 in Appendix. From this, the estimate (5.26) follows immediately, by applying Corollary 5.4.2 upon noticing that under our assumption on x_0, it holds that $|z|^{2-Q}$ remains bounded from above by δ^{2-Q}.

5.4.1 Proof of Theorem 1.5.3

Before proving Theorem 1.5.3 we need to show a counterpart of the Euclidean result in Theorem 5.14 in [JK], but in $\mathbb{H}^n$. Here we present it in a weaker form, i.e. only one implication, cf. [JK].

Proposition 5.4.5. *Let $\Omega \subset \mathbb{H}^n$ be an NTA domain. Let $f \in L^2(\partial\Omega, \mathrm{d}\omega^z)$ for some $z \in \Omega$ and such that $\int_{\partial\Omega} f \mathrm{d}\omega^z = 0$. Then a function $u(x) := \int_{\partial\Omega} f(y)\mathrm{d}\omega^x(y)$ satisfies the following identity*

$$\int_\Omega |\nabla_H u|^2 G(x,z)\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega} f(y)^2 \mathrm{d}\omega^z(y) < \infty. \tag{5.27}$$

Proof. The proof closely follows the corresponding one in [JK] and, therefore, we discuss only the main steps. The key tool used in [JK] is the Riesz representation theorem for subharmonic functions in $\mathbb{R}^n$, whose subriemannian counterpart is given by Theorem 9.4.7 in [BLU], applied to $-u$ in the notation of [BLU]; see also Definitions 9.4.1 and 9.3.1 in [BLU]. Namely, the following holds.

Let v be a subharmonic function in a domain $\Omega \subset \mathbb{H}^n$. Then $\int_\Omega G(x,z)\Delta_H u(x)\mathrm{d}x < \infty$ for some $z \in \Omega$ if and only if v has a harmonic majorant. Moreover, if h denotes the least harmonic majorant of v, then it holds

$$v(x) = h(x) - \int_\Omega G(x,y)\Delta_H v(y)\mathrm{d}y. \tag{5.28}$$

For the proof of Proposition 5.4.5 one defines a subharmonic function $v = u^2$, as $\Delta_H v = 2|\nabla_H u|^2 \geq 0$ and applies the above representation theorem. Moreover, $v(z) = u^2(z) = 0$ by assumptions of the proposition. Since Green's function is zero at the boundary of Ω we have, by (5.28), that $h \equiv f^2$ on $\partial\Omega$, and so $h(z) = \int_{\partial\Omega} f(y)^2\mathrm{d}\omega^z(y)$. Upon collecting these observations we obtain (5.27). Then, as in [JK], we assume that $f \in L^2(\partial\Omega, \mathrm{d}\omega^z)$ with $\int_{\partial\Omega} f\mathrm{d}\omega^z = 0$ and approximate f in the $L^2(\partial\Omega, \mathrm{d}\omega^z)$-norm by the sequence of continuous functions. We can approximate L^2-functions with continuous functions e.g. by Proposition 3.3.49 in [HKST]. Let $f_j \in C(\partial\Omega)$ be such that f_j converges to f in L^2 norm. Denote by u and u_j harmonic extensions of f and f_j, respectively. Then $\nabla_H u_j$ approaches $\nabla_H u$ uniformly on compact subsets of Ω. Hence,

$$\int_\Omega |\nabla_H u|^2 |G(x,z)|\mathrm{d}x \leq \sup_K \lim_{j\to\infty} \int_K |\nabla_H u_j|^2 |G(x,z)|\mathrm{d}x \leq \sup_j \frac{1}{2}\int_{\partial\Omega} f_j(y)^2\mathrm{d}\omega^z(y) < \infty.$$

A limit procedure implies assertion (5.27). $\square$

Proof of Theorem 1.5.3. Since an $L^2(\partial\Omega, \mathrm{d}\omega)$ function can be approximated by $C(\partial\Omega)$ functions we may assume that $f \in C(\partial\Omega)$. Moreover, without the loss of generality, we may also assume that $f > 0$, as otherwise we split f into a positive and negative parts and consider two cases separately.

In what follows we will employ the Green function G of domain Ω and so, in order to avoid problems with the pole of G, we need to bring on stage the truncated square function, i.e. the operator $S_\alpha u(x)$ considered with respect to truncated cones $\Gamma_\alpha^h(x) := \Gamma_\alpha(x) \cap B(x,h)$, for $x \in \partial\Omega$ for h small enough, so that the pole of G at $z \in \Omega$ does not belong to any of such truncated cones. Thus, we split $S_\alpha u(x)$ as follows

$$S_\alpha u(x)^2 = \int_{\Gamma_\alpha^h(x)} + \int_{\Gamma_\alpha(x)\setminus\Gamma_\alpha^h(x)}. \tag{5.29}$$

The second integral can be handled by the gradient estimates for subelliptic harmonic function u (see [LU, Proposition 2.1]) and by the Harnack inequality as follows:

$$\begin{aligned}\int_{\Gamma_\alpha(x)\setminus\Gamma_\alpha^h(x)} |\nabla_H u(y)|^2 d(y,\partial\Omega)^{2-Q}\mathrm{d}y &\leq c(n)\int_{\Gamma_\alpha(x)\setminus\Gamma_\alpha^h(x)} \frac{1}{d(y,\partial\Omega)^2}\Big(\sup_{B(y,\frac{1}{4}d(y,\partial\Omega))}|u|\Big)^2 d(y,\partial\Omega)^{2-Q}\,\mathrm{d}y\\ &\lesssim_{n,M,C} u^2(z)\int_{\Gamma_\alpha(x)\setminus\Gamma_\alpha^h(x)} \frac{1}{d(y,\partial\Omega)^Q}\,\mathrm{d}y \\ &\lesssim_{n,M,C} u^2(z)\left(\frac{h}{1+\alpha}\right)^{-Q}|\Omega|.\end{aligned} \tag{5.30}$$

The Harnack inequality (see e.g. [BLU, Corollary 5.7.3]) is used in the second estimate: we choose big enough compact subset of Ω containing points z and y. Since they both are enough far away from the boundary, there exists a Harnack chain of finite length, depending on M, joining z and y. We iterate the Harnack estimate along that chain and obtain (5.30) with constant C coming from the constants in the Harnack inequality.

Therefore, since $u(z) = \int_{\partial\Omega} f(y)\mathrm{d}\omega^z(y)$, we get the estimate

$$\int_{\Gamma_\alpha(x)\setminus\Gamma_\alpha^h(x)} |\nabla_H u(y)|^2 d(y,\partial\Omega)^{2-Q}\mathrm{d}y \lesssim_{n,M,C} \left(\frac{h}{1+\alpha}\right)^{-Q}|\Omega|\|f\|^2_{L^2(\omega^z)}.$$

In order to get the $L^2(\partial\Omega)$-norm estimate we integrate both sides of the above inequality and use the fact that harmonic measure is a probability measure to obtain

$$\begin{aligned}\int_{\partial\Omega}\int_{\Gamma_\alpha(x)\setminus\Gamma_\alpha^h(x)} |\nabla_H u(y)|^2 d(y,\partial\Omega)^{2-Q}\mathrm{d}y\mathrm{d}\omega^z &\lesssim_{n,M,C} \left(\frac{h}{1+\alpha}\right)^{-Q}|\Omega|\|f\|^2_{L^2(\omega^z)}\\ &\lesssim_{n,M,C,\alpha,\operatorname{diam}\Omega,h} \|f\|_{L^2(\omega^z)}.\end{aligned} \tag{5.31}$$

We now proceed to estimate the first integral in (5.29). For any point $y \in \Omega$ let us denote by q_y a point at which the distance $d(y,\partial\Omega)$ is attained. Next, observe that a point $y \in \Gamma_\alpha(x)$ if and only if $x \in S(y)$ the shadow of point y, defined as $S(y) := \partial\Omega \cap B(y,(1+\alpha)d(y,\partial\Omega))$. For any $z \in S(y)$ it holds that

$$d(x,z) \leq d(x,y) + d(y,z) \leq 2(1+\alpha)d(y,\partial\Omega).$$

Therefore, $\left\{x \in \partial\Omega \,:\, y \in \Gamma_\alpha^h(x)\right\} \subset \Delta(q_y, 2(1+\alpha)d(y,\partial\Omega))$, see (5.1) for a definition of $\Delta(q_y, 2(1+\alpha)d(y,\partial\Omega))$.

We set $\Omega_h := \{y \in \Omega : d(y, \partial\Omega) < h\}$. An application of the Fubini theorem together with the Dahlberg-type estimate, see Theorem 5.1.4, give us that

$$\begin{aligned}
\int_{\partial\Omega} \int_{\Gamma_\alpha^h(x)} &|\nabla_H u(y)|^2 d(y, \partial\Omega)^{2-Q} \mathrm{d}y \mathrm{d}\omega^z(x) \\
&= \int_{\Omega_h} |\nabla_H u(y)|^2 d(y, \partial\Omega)^{2-Q} \omega^z \left(\left\{ x \in \partial\Omega : y \in \Gamma_\alpha^h(x) \right\} \right) \mathrm{d}y \\
&\leq \int_{\Omega_h} |\nabla_H u(y)|^2 d(y, \partial\Omega)^{2-Q} \omega^z \left(\Delta(q_y, 2(1+\alpha) d(y, \partial\Omega)) \right) \mathrm{d}y \qquad (5.32) \\
&\leq \int_{\Omega_h} |\nabla_H u(y)|^2 G(y, z) \, \mathrm{d}y. \qquad (5.33)
\end{aligned}$$

The last inequality follows by standard reasoning which, however, deserves some details.

Let $z \in \Omega \setminus B(q_y, 2M(1+\alpha)d(y, \partial\Omega))$. Then by Theorem 5.1.4 we have (see also Section 5 in [CGN] to see why constant a in Theorem 5.1.4 can be taken equal to M):

$$\begin{aligned}
\omega^z \left(\Delta(q_y, 2(1+\alpha) d(y, \partial\Omega)) \right) &\approx \frac{|B(q_y, 2(1+\alpha)d(y, \partial\Omega))|}{(2(1+\alpha)d(y, \partial\Omega))^2} G(z, A_{2(1+\alpha)d(y,\partial\Omega)}(q_y)) \\
&\approx_\alpha d(y, \partial\Omega)^{Q-2} G(z, A_{2(1+\alpha)d(y,\partial\Omega)}(q_y)). \qquad (5.34)
\end{aligned}$$

By taking h small enough we ensure that $2(1+\alpha)d(y, \partial\Omega) < r_0$ so that we can use Theorem 5.1.4. Notice that since $A_{2(1+\alpha)d(y,\partial\Omega)}(q_y)$ is a corkscrew point we know by ithe interior corkscrew condition, see Definition 5.1.1 (1), that

$$d \left(A_{2(1+\alpha)d(y,\partial\Omega)}(q_y), \partial\Omega \right) \geq \frac{2(1+\alpha)d(y, \partial\Omega)}{M}.$$

Moreover, $d(A_{2(1+\alpha)d(y,\partial\Omega)}(q_y), y) \leq 4(1+\alpha)d(y, \partial\Omega)$, as both y and $A_{2(1+\alpha)d(y,\partial\Omega)}(q_y)$ lie in a ball $B(q_y, 2(1+\alpha)d(y, \partial\Omega))$.

Set $\varepsilon := \min\{d(y, \partial\Omega), \frac{2(1+\alpha)d(y,\partial\Omega)}{M}\} \approx d(y, \partial\Omega)$. Then $d(A_{2(1+\alpha)d(y,\partial\Omega)}(q_y), y) \leq C\varepsilon$ with constant C depending only on α and M and independent of y. Therefore, there is a Harnack chain joining y and $A_{2(1+\alpha)d(y,\partial\Omega)}(q_y)$ with length independent of y and hence by the Harnack inequality

$$G(z, A_{2(1+\alpha)d(y,\partial\Omega)}(q_y)) \approx_{\alpha,M,C} G(z, y). \qquad (5.35)$$

Thus, by combining (5.34) and (5.35) and applying them at (5.32), we obtain (5.33), as desired.

Finally, we apply (5.27) to arrive at

$$\int_{\partial\Omega} \int_{\Gamma_\alpha^h(x)} |\nabla_H u(y)|^2 d(y, \partial\Omega)^{2-Q} \mathrm{d}y \mathrm{d}\omega^z(x) \leq \frac{1}{2} \int_{\partial\Omega} |f(y) - u(z)|^2 \mathrm{d}\omega^z(y) \leq 2 \int_{\partial\Omega} |f(y)|^2 \mathrm{d}\omega^z(y).$$

Adding up together this estimate and (5.31) we obtain the assertion of the theorem. □

5.4.2 Proof of Theorem 1.5.4

The structure of the proof follows the corresponding one for the proof of Theorem 9.6 in [JK]. However, the subriemannian setting of $\mathbb{H}^n$ requires applying different tools.

CLAIM 1. *Let $x_0 \in \partial\Omega$, $r < r_0$ and $A_r(x_0) \in \Omega$ be an internal corkscrew point as in Definition 5.1.1 such that it satisfies $\frac{r}{M} < d(A_r(x_0), x_0) \leq r$. Then it holds that*

$$v(A_r(x_0)) := \int_{\partial\Omega\setminus\Delta_1} |f(x) - f_{\Delta_1}| \mathrm{d}\omega^{A_r(x_0)}(x) \leq C\|f\|_{BMO(\partial\Omega)}, \tag{5.36}$$

where C is independent of r and f and $\Delta_1 := B(x_0, 2r) \cap \partial\Omega$ the surface ball.

In order to the prove the claim, we repeat the reasoning from the proof of Lemma 9.7 in [JK], see also the proof of Lemma on pg. 35 in [FB].

Fix $y \in \Omega$ and consider the harmonic measure ω^y. Define $\Delta_j := B(x_0, 2^j r) \cap \partial\Omega$ and the related ring domains $R_j := \Delta_j \setminus \Delta_{j-1}$ for $j = 1, 2, \ldots$. Recall the notation $f_{\Delta_j} := ⨍_{\Delta_j} f \mathrm{d}\omega^y$.

Recall that similarly to the Euclidean case, also in the setting of Heisenberg groups one can define a kernel function associated with the boundary point P_0, $K : \Omega \times \partial\Omega \to \mathbb{R}_+ \cup \{\infty\}$. Function K is normalized at $y_0 \in \Omega$, i.e. $K(y_0, P_0) = 1$, and moreover, $K(\cdot, P_0)$ is a solution to $\Delta_H u = 0$ in Ω and $K(\cdot, P_0)$ vanishes continuously on $\partial\Omega \setminus \{P_0\}$. We refer to Definition 2 in [CG] for details.

One of the key properties of such defined kernel function is that given a point $y \in \Omega$, there is always a unique kernel function at $P \in \partial\Omega$, normalized at y, see Theorem 11 in [CG].

Therefore, we have that for a kernel $K(A_r(x_0), x)$ it holds

$$\begin{aligned}
v(A_r(x_0)) &\leq \sum_{j\geq 2} \int_{R_j} |f(x) - f_{\Delta_j}| \mathrm{d}\omega^{A_r(x_0)}(x) + \sum_{j\geq 2} \int_{R_j} |f_{\Delta_1} - f_{\Delta_j}| \mathrm{d}\omega^{A_r(x_0)}(x) \\
&\leq \sum_{j\geq 2} \int_{R_j} |f(x) - f_{\Delta_j}| K(A_r(x_0), x) \mathrm{d}\omega^y(x) + \sum_{j\geq 2} |f_{\Delta_1} - f_{\Delta_j}| \int_{R_j} K(A_r(x_0), x) \mathrm{d}\omega^y(x) \\
&\leq \sum_{j\geq 2} \frac{C2^{-\kappa j}}{\omega^y(\Delta_j)} \int_{\Delta_j} |f(x) - f_{\Delta_j}| \mathrm{d}\omega^y(x) + \left(\sup_{k\geq 1} |f_{\Delta_k} - f_{\Delta_{k-1}}| \right) \sum_{j\geq 2} jC2^{-\kappa j} \frac{\omega^y(R_j)}{\omega^y(\Delta_j)},
\end{aligned}$$

where in the last step we appeal also to the growth estimate for kernel functions, provided in Proposition 6 in [CG]. Namely, for $x \in \partial\Omega$, $0 < r < r_0$, there exist constants $C, C_j, \kappa > 0$, with $C_j \leq C2^{\kappa j}$, such that

$$\sup\{K(A_r(x_0), x) : x \in R_j\} \leq \frac{C_j}{\omega^y(\Delta_j)}.$$

Finally, the standard argument involving mean value integrals gives us that

$$\begin{aligned}
\left| f_{\Delta_k} - f_{\Delta_{k-1}} \right| &= \frac{\omega^y(\Delta_k)}{\omega^y(\Delta_{k-1})} \left(\frac{1}{\omega^y(\Delta_k)} \int_{\Delta_k} |f(x) - f_{\Delta_k}| \mathrm{d}\omega^y(x) \right) \\
&\leq C \left(\frac{1}{\omega^y(\Delta_k)} \int_{\Delta_k} |f(x) - f_{\Delta_k}| \mathrm{d}\omega^y(x) \right).
\end{aligned}$$

Here we also appeal to the doubling property of ω^y, see Theorem 2 in [CG]. We remark that Theorem 2 in [CG] is proven for k and r small enough so that radii $2^k r < r_0$. In order to obtain the doubling property for large k we use the Harnack inequality and Corollary 3 in [CG], cf. the discussion following Lemma 4.9 in [JK] and the proof of Lemma 4.2 therein.

Namely, there exists $j \in \mathbb{N}$ such that $2^{-j-1} r \leq r_0$. The Harnack inequality allows us to change the reference point of the harmonic measure. Therefore, using it we may change the corkscrew point

$A_r(\Delta(x_0, s))$ for $s < r_0$ to any point $y \in \Omega$. We change it because in Corollary 3 in [CG] a reference point is a corkscrew point. Hence, the Harnack inequality and the mentioned Corollary 3 give:

$$\frac{\omega^y\big(\Delta(x_0, 2r)\big)}{\omega^y\big(\Delta(x_0, r)\big)} \leq \frac{\omega^y\big(\Delta(x_0, 2r)\big)}{\omega^y\big(\Delta(x_0, 2^{-j}r)\big)} \leq \frac{1}{C}.$$

Hence, also for big r, the harmonic measure is doubling.

By applying the definition of the seminorm in $BMO(\partial\Omega)$, cf. Definition 5.1.12, we obtain

$$v(A_r(x_0)) \leq \|f\|_{BMO(\partial\Omega)}\Big(C + C\sum_{j\geq 2} j2^{-\kappa j}\Big) \leq C\|f\|_{BMO(\partial\Omega)}$$

and Claim 1 is proven.

CLAIM 2. *Let $x_0 \in \partial\Omega$ and $A_{DMr}(x_0) \in \Omega$ be an internal corkscrew point as in Definition 5.1.1 (1), with constant $D > 1$. Denote by $\Delta(x_0, r) := B(x_0, r) \cap \partial\Omega$ the surface ball. Then it holds that*

$$\frac{r^{2(Q-2)}}{(\omega^{A_{DMr}(x_0)}(\Delta(x_0, r)))^3} \int_{B(x_0,r)\cap\Omega} \frac{G^3(x, A_{DMr}(x_0))}{d(x, \partial\Omega)^2} \mathrm{d}x \leq C, \tag{5.37}$$

where C depends only on n, the geometry of $\mathbb{H}^n$ and r_0 and M (the NTA parameters of Ω).

We again follow the corresponding proof of Lemma 9.8 in [JK], although observe that instead of the dyadic Whitney cubes we need a different family of sets covering the set $B(x_0, r) \cap \Omega$. Such a family can be constructed by the direct modification of the proof of Proposition 4.1.15 in [HKST] as follows:

There exists a countable family of balls in $B(x_0, r) \cap \Omega$ denoted by

$$\mathcal{F} := \left\{ B(x_i, \frac{1}{8}d(x_i, \partial\Omega)) \right\}, \quad x_i \in \Omega,$$

such that each ball $B(x_i)$ has a non-empty intersection with set $B(x_0, r) \cap \Omega$ and, moreover, $\sum_i \chi_{2B(x_i)} \leq 2N^5$, where N stands for the doubling constant in $\mathbb{H}^n$.

We can find such a family because Heisenberg group $\mathbb{H}^n$ with a standard measure is a doubling space. Then the existence of family $\mathcal{F}$ follows from the 5-covering lemma.

Let us define the following subfamily of $\mathcal{F}$:

$$\mathcal{F}_k := \left\{ B(x_i) \in \mathcal{F} \ : \ 2^{-k} \leq \frac{1}{8}d(x_i, \partial\Omega) \leq 2^{-k+1} \right\}, \quad k = -\lceil \log_2 r_0 \rceil - 1, \dots, 0, 1, 2, \dots .$$

Notice that we do not need to take exponents bigger than $\lceil \log_2 r_0 \rceil + 1$ because any x_i has to satisfy

$$d(x_i, \partial\Omega) \leq d(x_i, x_0) \leq \frac{8}{7}r \leq \frac{8}{7}r_0.$$

In order to prove the second inequality above let us set $d(x_i, x_0) = (1+t)r$, for some $t \in \mathbb{R}$ and notice that $d(x_i, x_0) \leq r + \frac{1}{8}d(x_i, \partial\Omega)$.Thus, we have

$$(1+t)r = d(x_i, x_0) \leq r + \frac{1}{8}d(x_i, \partial\Omega) \leq r + \frac{1}{8}d(x_i, x_0) = r + \frac{1}{8}(1+t)r,$$

and so $t \leq \frac{1}{7}$.

With the above introduced notation we may reduce the estimate in (5.37) to the estimate over the balls in $\mathcal{F}$:

$$\int_{B(x_0,r)\cap\Omega} \frac{G^3(x, A_{DMr}(x_0))}{d(x,\partial\Omega)^2} \mathrm{d}x \leq 2N^5 \sum_k \sum_{B(x_i)\in\mathcal{F}_k} \int_{B(x_i)} \frac{G^3(x, A_{DMr}(x_0))}{d(x,\partial\Omega)^2} \mathrm{d}x. \tag{5.38}$$

For a fixed k and given ball $B(x_i) \in \mathcal{F}_k$ let $x_i^* \in \partial\Omega$ denote a point such that $d(x_i, x_i^*) = d(x_i, \partial\Omega)$. That such a point exists is a consequence of compactness of $\overline{\Omega}$, as Ω is a bounded domain.

Set

$$\Delta_i := \Delta(x_i^*, 2^{-k+1}) = B(x_i^*, 2^{-k+1}) \cap \partial\Omega \quad \text{a surface ball.}$$

Notice that any point $A_{DMr}(x_0)$ by the Definition 5.1.1 lies outside the set $B(x_0, r) \cap \Omega$ and hence the function $G(\cdot, A_{DMr}(x_0))$ is harmonic in $B(x_0, r) \cap \Omega$. By the Harnack inequality for a harmonic function $G(\cdot, A_{DMr}(x_0))$ applied on a ball $B(x_i)$ we have that $G(x, A_{DMr}(x_0)) \approx_C G(x_i, A_{DMr}(x_0))$ and, thus,

$$\int_{B(x_i)\in\mathcal{F}_k} \frac{G^3(x, A_{DMr}(x_0))}{d(x,\partial\Omega)^2} \mathrm{d}x \approx G^3(x_i, A_{DMr}(x_0)) 2^{2k} \left(\frac{1}{8} d(x_i, \partial\Omega)\right)^Q \approx G^3(x_i, A_{DMr}(x_0)) 2^{k(2-Q)}. \tag{5.39}$$

Indeed, if $x \in B(x_i) \in \mathcal{F}_k$, then $d(x_i, \partial\Omega) \lesssim d(x, \partial\Omega) \leq d(x_i, \partial\Omega) + d(x, x_i) \leq 16 \cdot 2^{-k+1}$ and so $d(x, \partial\Omega) \approx 2^{-k}$.

Next we show that we may consider points x_i as the corkscrew points in Definition 5.1.1, so that $x_i := A_{2^{-k+4}}(x_i^*)$. Since $B(x_i) \in \mathcal{F}_k$ we have

$$d(x_i, x_i^*) = d(x_i, \partial\Omega) \leq 2^{-k+4}.$$

On the other hand

$$d(x_i, x_i^*) = d(x_i, \partial\Omega) \geq \frac{2^{-k+4}}{2} \geq \frac{2^{-k+4}}{M}$$

for any $M \geq 2$. However, in the definition of NTA domains we only have existence of some constant M. If it happens that $M < 2$, we can always make it larger without losing anything in the said definition. Hence, without loss of generality, we can assume $M \geq 2$. This shows that indeed $x_i = A_{2^{-k+4}}(x_i^*)$. Now let us choose a point $y_0 \in \Omega \setminus B(x_i^*, a2^{-k+4})$. In fact, one can take $a = M$, see Section 5 Theorem 5.4 [CGN] and, moreover, assume that $y_0 := A_{\widetilde{C}r}(x_i^*)$ with $\widetilde{C} = \widetilde{C}(M)$.

We apply Theorem 5.1.4 to obtain the following estimate

$$\begin{aligned} G(x_i, A_{\widetilde{C}r}(x_i^*)) &= G(A_{\widetilde{C}r}(x_i^*), A_{2^{-k+4}}(x_i^*)) \\ &\approx \frac{2^{2(-k+4)}}{|B(A_{\widetilde{C}r}(x_i^*), 2^{-k+4})|} \omega^{A_{\widetilde{C}r}(x_i^*)}(\Delta(x_i^*, 2^{-k+4})) \\ &\lesssim 2^{(-k+4)(2-Q)} \omega^{A_{\widetilde{C}r}(x_i^*)}(\Delta_i), \end{aligned} \tag{5.40}$$

where in the last inequality we use the doubling property of the harmonic measure. Furthermore, observe that

$$G(x_i, A_{\widetilde{C}r}(x_i^*)) = G(A_{\widetilde{C}r}(x_i^*), x_i) \approx_{C,M,D} G(A_{DMr}(x_0), x_i) = G(x_i, A_{DMr}(x_0)), \tag{5.41}$$

and the change of point $A_{\widetilde{C}r}(x_i^*)$ to $A_{DMr}(x_0)$ in the middle estimate requires explanation: Indeed, choose $A_{\widetilde{C}r}(x_i^*) \in \Omega \setminus B(x_i^*, a2^{-k+4})$ and observe that, by the above discussion, we can consider the same $y_0 = A_{\widetilde{C}r}(x_i^*)$ for any point x_i^*. We also recall that $d(y_0, \partial\Omega) \geq \frac{r}{M}$. Moreover, we can assume that $d(A_{DMr}(x_0), A_{\widetilde{C}r}(x_i^*)) \leq Cr$, where $C = C(M, D)$. Therefore, points $A_{DMr}(x_0)$ and $A_{\widetilde{C}r}(x_i^*)$ can be joined by a Harnack chain of length depending only on $C(M, D)$. This observation together with the Harnack inequality allow us to replace in (5.41) point $A_{\widetilde{C}r}(x_i^*)$ with $A_{DMr}(x_0)$ by the price of possibly increasing constants.

Upon applying estimates (5.40) and (5.41) in (5.39) we obtain the following:

$$\int_{B(x_i)\in\mathcal{F}_k} \frac{G^3(x, A_{DMr}(x_0))}{d(x, \partial\Omega)^2} dx \approx G^2(x_i, A_{DMr}(x_0))\omega^{A_{\widetilde{C}r}(x_i^*)}(\Delta_i). \tag{5.42}$$

In order to estimate the expression on the right-hand side, we appeal to the Carleson-type estimate Theorem 5.1.5. Recall that $G \geq 0$ in Ω and $G(\cdot, A_{DMr}(x_0)) \equiv 0$ on $\partial\Omega$, also that $G(\cdot, A_{DMr}(x_0))$ is subelliptic harmonic in $\Omega \setminus \{A_{DMr}(x_0)\}$. Moreover, notice that $\Delta(x_0, 2r) \subset \Delta(x_i^*, Cr)$. Indeed, let $y \in \Delta(x_0, 2r)$. Then,

$$d(y, x_i^*) \leq d(y, x_0) + d(x_0, x_i) + d(x_i, x_i^*) \leq r + (r + r2^{-k+1}) + 2^{-k+1} \lesssim Cr.$$

The last step requires that $2^{-k} \leq Cr$ and under our assumptions this restriction is sufficient. Since, otherwise suppose that $2^{-k} \geq r$. Then for any x_i it holds that $d(x_i, \partial\Omega) \geq 8 \cdot 2^{-k} \geq 8r$. On the other hand,

$$d(x_i, \partial\Omega) \leq d(x_i, x_0) \leq r + 2r_i \leq r + \frac{1}{4} d(x_i, \partial\Omega).$$

Hence $d(x_i, \partial\Omega) \leq \frac{4}{3}r$, giving the contradiction.

We apply Theorem 5.1.5 on $B(x_i^*, Cr)$ to get that, for an exponent $\beta > 0$ as in Theorem 5.1.5, the following inequality holds

$$G^2(x_i, A_{DMr}(x_0)) \leq C(M, r_0) \left(\frac{d(x_i, x_i^*)}{Cr} \right)^{2\beta} \left(\sup_{x\in\partial B(x_i^*, Cr)\cap\Omega} G(x, A_{DMr}(x_0)) \right)^2. \tag{5.43}$$

Denote by $z \in \partial B(x_i^*, Cr) \cap \Omega$ a point, where the function $G(\cdot, A_{DMr}(x_0))$ attains its maximum. (Notice that this maximum cannot be obtained at a point in $B(x_i^*, Cr) \cap \partial\Omega$, as then it would be zero, as $G(\cdot, A_{DMr}(x_0)) \equiv 0$ on $\partial\Omega$ and by the maximum principle G would be zero on whole $B(x_i^*, Cr) \cap \Omega$. Therefore, the Carleson estimate in Theorem 5.1.7 gives us that

$$G(z, A_{DMr}(x_0)) \lesssim_{M, r_0} CG(A_{Cr}(x_i^*), A_{DMr}(x_0))$$

for all $z \in \Omega \cap B(x_i^*, Cr)$.

Here, in order to apply Theorem 5.1.7, we need to slightly increase constant C on the right-hand side of the estimate, so that point z belongs to $B(x_i^*, Cr) \cap \Omega$. Moreover, in the case $A_{DMr}(x_0) \in B(x_i^*, 2Cr) \cap \Omega$ one needs additional chaining argument, by the definition of NTA domains, to join $A_{DMr}(x_0)$ with the point $A_{Cr}(x_0) \notin B(x_i^*, 2Cr) \cap \Omega$. This however, can be done by the price of increasing again constant C. From this discussion and (5.43) we infer, by the Dahlberg-type estimate in Theorem 5.1.4, that

$$G^2(x_i, A_{DMr}(x_0)) \lesssim_C \left(\frac{8 \cdot 2^{-k+1}}{Cr} \right)^{2\beta} r^{2(2-Q)} \left(\omega^{A_{Cr}(x_i^*)}(\Delta(x_0, r)) \right)^2.$$

Then we apply the last estimate in (5.42) and in (5.38) to arrive at the following inequality

$$\int_{B(x_0,r)\cap\Omega} \frac{G^3(x, A_{DMr}(x_0))}{d(x,\partial\Omega)^2} \mathrm{d}x \lesssim_C 2N^5 \sum_k \left(\frac{2^{-k}}{r}\right)^{2\beta} r^{2(2-Q)} \left(\omega^{A_{Cr}(x_i^*)}(\Delta(x_0,r))\right)^2 \omega^{A_{\widetilde{C}r}(x_i^*)}(\Delta_i).$$

Finally, recall that by the discussion following (5.41) we may join points $A_{\widetilde{C}r}(x_i^*)$ and $A_{DMr}(x_0)$ by the Harnack chain whose length depends only on M and D. By applying this observation, we conclude that

$$\omega^{A_{DMr}(x_0)}(\Delta(x_0,r)) \approx_{C,M,D} \omega^{A_{Cr}(x_i^*)}(\Delta(x_0,r)).$$

Hence, (5.38) becomes

$$\int_{B(x_0,r)\cap\Omega} \frac{G^3(x, A_{DMr}(x_0))}{d(x,\partial\Omega)^2} \mathrm{d}x \leq 2N^5 \sum_k \left(\frac{2^{-k}}{r}\right)^{2\beta} r^{2(2-Q)} (\omega^{A_{DMr}(x_0)}(\Delta(x_0,r)))^3$$

and thus the proof of Claim 2 is completed.

Continuation of the proof of Theorem 1.5.4.

We are now in a position to complete the proof of Theorem 1.5.4. Suppose that $f \in BMO(\partial\Omega, \mathrm{d}\omega)$ and u is the harmonic function in Ω such that $u(x) = \int_{\partial\Omega} f(y) \mathrm{d}\omega^x(y)$.

Let $B(x_0, r)$ be a ball centered at $x_0 \in \partial\Omega$ and a radius $r < \min\{1, r_0\}$ and denote by $\Delta_1 = 2\Delta := B(x_0, 2r) \cap \partial\Omega$. We modify the boundary data as follows:

$$f_1 := (f - f_{\Delta_1})\chi_{c\Delta_1}, \qquad f_2 := (f - f_{\Delta_1})\chi_{\partial\Omega \setminus c\Delta_1}$$

and, as in [JK] we let u_1 and u_2 be their harmonic extensions, respectively, i.e.

$$u_i(x) = \int_{\partial\Omega} f_i(y) \mathrm{d}\omega^x(y), \quad i = 1, 2.$$

By direct application of Proposition 5.4.5 to f_1 and u_1 we obtain that

$$\frac{1}{\omega^{A_{DMr}(x_0)}(\Delta)} \int_{B(x_0,r)\cap\Omega} |\nabla_H u_1|^2 G(x, A_{DMr}(x_0)) \mathrm{d}x$$

$$\leq \frac{1}{\omega^{A_{DMr}(x_0)}(\Delta)} \int_{\Omega} |\nabla_H u_1|^2 G(x, A_{DMr}(x_0)) \mathrm{d}x \tag{5.44}$$

$$= \frac{1}{2} \frac{1}{\omega^{A_{DMr}(x_0)}(\Delta)} \int_{\partial\Omega} |(f(y) - f_{\Delta_1})\chi_{c\Delta_1}|^2 \mathrm{d}\omega^z(y) < \|f\|^2_{BMO(\partial\Omega, \mathrm{d}\omega)}, \tag{5.45}$$

where in the last inequality we use the John–Nirenberg theorem to get the equivalent definition of the BMO spaces in terms of the $L^2(\partial\Omega, \mathrm{d}\omega)$ functions with L^2-integrable means, well-known in the Euclidean setting. Indeed, such an equivalent definition holds, as by Theorem 2 in [CG], the harmonic measure ω^z is doubling in Ω for z enough away from the boundary of Ω and we may repeat the appropriate part of the standard reasoning of Proposition 3.19 [BB], as long as the John-Nirenberg lemma holds for the surface balls Δ. However, this follows by direct application of Theorem 5.2 in [AKBY]:

Let $\Delta = B(x_0, r) \cap \partial\Omega$ be a surface ball and $f \in BMO(\partial\Omega, \mathrm{d}\omega^z)$. Then for all $\lambda > 0$

$$\omega^z(\{z \in \Delta : |f(x) - f_\Delta| > \lambda\}) \leq 2\omega^z(\Delta) e^{-\frac{c\lambda}{\|f\|_{BMO(\partial\Omega, \mathrm{d}\omega^z)}}},$$

where c depends only on the doubling constant of ω^z.

The proof of John-Nirenberg lemma in [AKBY] requires only that the measure is doubling. By applying this result to the metric space $(\partial\Omega, d|_{\partial\Omega}, \mathrm{d}\omega^z)$, we conclude that indeed (5.45) holds.

By the gradient estimate for harmonic functions (see Proposition 2.2 in [LU]) and by the Harnack inequality on B-balls, we have that for any $x \in \Omega$

$$|\nabla_H u_2(x)| \leq \frac{4c(n)}{d(x, \partial\Omega)} \sup_{B(x, \frac{1}{4}d(x, \partial\Omega))} |u_2| \leq \frac{C}{d(x, \partial\Omega)} \int_{\partial\Omega \setminus c\Delta_1} |f - f_{\Delta_1}| \mathrm{d}\omega^x.$$

We denote by $v(x)$ the last integral on the right-hand side, i.e. $v(x) = \int_{\partial\Omega \setminus c\Delta_1} |f - f_{\Delta_1}| \mathrm{d}\omega^x$, and note that v is a positive part of u_2. Next, we apply the (local) boundary Harnack inequality to u_2 and $G(\cdot, A_{DMr}(x_0))$, see Theorem 5.1.6, followed by the use of (5.36) in Claim 1 and the Dahlberg-type estimate in Theorem 5.1.4, to arrive at the following estimate holding for $x \in B(x_0, r) \cap \Omega$

$$\begin{aligned}
v(x) &\lesssim_{n,M,r_0} \frac{v(A_{2Mr}(x_0))}{G(A_{2Mr}(x_0), A_{DMr}(x_0))} G(x, A_{DMr}(x_0)) \\
&\lesssim_{n,M,r_0} \|f\|_{BMO(\partial\Omega, \mathrm{d}\omega)} \frac{r^{Q-2}}{\omega^{A_{2Mr}(x_0)}(\Delta(x_0, DMr))} G(x, A_{DMr}(x_0)) \\
&\lesssim_{n,M,r_0} \|f\|_{BMO(\partial\Omega, \mathrm{d}\omega)} \frac{r^{Q-2}}{\omega^{A_{2Mr}(x_0)}(\Delta(x_0, r))} G(x, A_{DMr}(x_0)).
\end{aligned}$$

Since Ω in an NTA domain, we may apply the Harnack chain condition to join points $A_{DMr}(x_0)$ and $A_{2Mr}(x_0)$ with the chain of at most C balls and invoke the Harnack inequality to conclude that

$$\omega^{A_{2Mr}(x_0)}(\Delta(x_0, r)) \approx_{C,M,D} \omega^{A_{DMr}(x_0)}(\Delta(x_0, r)).$$

We apply this observation together with estimates for v and $|\nabla_H u_2|$ and apply (5.37) in Claim 2 to obtain that

$$\begin{aligned}
&\frac{1}{\omega^{A_{DMr}(x_0)}(\Delta)} \int_{B(x_0,r)\cap\Omega} |\nabla_H u_2|^2 G(x, A_{DMr}(x_0)) \mathrm{d}x \\
&\qquad \leq C\|f\|^2_{BMO(\partial\Omega, \mathrm{d}\omega)} \frac{r^{2(Q-2)}}{(\omega^{A_{DMr}(x_0)}(\Delta))^3} \int_{B(x_0,r)\cap\Omega} \frac{G(x, A_{DMr}(x_0))^3}{d(x, \partial\Omega)^2} \mathrm{d}x \\
&\qquad \leq C\|f\|^2_{BMO(\partial\Omega, \mathrm{d}\omega)}.
\end{aligned}$$

Finally, we combine this estimate with the previous one for $|\nabla_H u_1|$, see (5.44), and note that $\nabla_H u = \nabla_H u_1 + \nabla_H u_2$. From this the assertion of Theorem 1.5.4 follows. □

5.5 The Fatou theorem

The goal of this chapter is to prove a version of the harmonic Fatou theorem in the Heisenberg setting. As mentioned in the Preliminaries, the studies of such theorems have led to several important notions and results to which our manuscript appeals to, for instance, the NTA domains, the area function and the nontangential maximal function, see e.g. [JK, Section 1]. Recall that the classical Fatou theorem asserts that a bounded harmonic function defined on the half-space in $\mathbb{R}^n$ has nontangential limits at almost every point of the boundary, see e.g. [S, Theorem 2, Ch. VII], see also [Car2] for the local version. For the NTA domains in the Euclidean spaces, the Fatou theorem with respect to the harmonic measure is due to [JK, theorem 6.4]. In the subriemannian setting the analogous results are proven in [CG, Theorem 4] for bounded NTA domains.

We show a counterpart of the Fatou theorem for (ε, δ)-domains and, thus, for more general domains than the NTA ones, see the discussion below. Moreover, we are able to show the refinement of classical results, namely that nontangential limits of a harmonic function u exist outside a set of p-capacity zero, not only zero measure. This, however, is obtained under stronger assumption on the global L^p-integrability of the gradient of harmonic function.

We will now recall necessary definitions.

Definition 5.5.1 (cf. Definition 2.7 [Nh1])**.** We say that a bounded domain $\Omega \subset \mathbb{H}^n$ is an (ε, δ)-domain if for all $x, y \in \Omega$ such that $d(x, y) < \delta$ there exists a rectifiable curve $\gamma \subset \Omega$ joining x and y satisfying

$$l(\gamma) \leq \frac{1}{\varepsilon} d(x, y),$$

and

$$d(z, \partial\Omega) \geq \varepsilon \min\{d(x, z), d(y, z)\} \quad \text{for all } z \text{ on } \gamma.$$

The definition of the (ε, δ)-domains in the Euclidean setting was first given in [Jon] and a fact that such domains are uniform and hence, John domains, is observed in Remark 4.2 in [V]. The above definition has also a counterpart in more general Carnot groups, see Definition 4.1 in [Nh2], and leads to an extension theorem applied in the proof of Theorem 6.1 below, see Theorem 1.1 in [Nh2]. It is also known that a large class of NTA domains in $\mathbb{H}^n$ satisfies the definition of (ε, δ)-domains, see Theorem 1.2 in [Nh2] and the discussion following it. Moreover, bounded (ε, δ)-domains are uniform, see also [CT].

The results below employ the following notion of the Sobolev p-capacity.

Definition 5.5.2. Let $1 \leq p < \infty$. The Sobolev p-capacity of set $E \subset \mathbb{H}^n$ is:

$$\mathrm{C}_p(E) = \inf \int_{\mathbb{H}^n} \left(|u|^p + |\nabla_H u|^p\right) \mathrm{d}X,$$

where the infimum is taken over all functions $u \in HW^{1,p}(\mathbb{H}^n)$ such that $u \geq 1$ on E outside a p-exceptional set of measure zero.

The importance of this notion comes from the fact that p-capacity is more refined than a measure. There exist sets of measure zero such that their capacity is not zero. However, every set that has capacity equal to zero, has also measure equal to zero.

There is a vast literature on the topic of p-capacities in the Euclidean and metric measure spaces settings, see e.g. [EG], [HKST], [BB].

For the definition and basic properties of p-Sobolev capacities we refer to [HKST, Chapter 7.2].

Theorem. 1.5.5 *Let $\Omega \subset \mathbb{H}^n$ be a bounded (ε, δ)-domain and let further u be subelliptic harmonic in Ω. If $\int_\Omega |\nabla_H u|^p < \infty$ for some $1 < p \leq 2n+2$, then u has nontangential limits on $\partial\Omega$ along horizontal curves in Ω outside the set of p-Sobolev capacity zero.*

The proof of the theorem employs among other results the following auxiliary observations. The proof of the first one is new in the literature, due to applying results of [AW].

Lemma 5.5.3. *Let u be subelliptic harmonic function in $\Omega \subset \mathbb{H}^n$. Then, for any Korányi–Reimann ball $B(x, r) \subset B(x, 2r) \subset \Omega$ and all $c \in \mathbb{R}$ we have that for any $p > 1$*

$$\sup_{B(x,r)} |u(y) - c| \leq C(p, n) \left(\fint_{B(x,2r)} |u(y) - c|^p \mathrm{d}y \right)^{\frac{1}{p}}.$$

The result is well-known in the Euclidean setting and for $\mathcal{A}$-harmonic functions, see Lemma 3.4 in [HKM].

Proof. We apply the mean-value theorem in $\mathbb{H}^n$, see Theorem 4.4 in [AW] (cf. Theorem 5.5.4 in [BLU]). By Definition 5.5.1 in [BLU] and pg. 253 we know that $|\nabla_H d|^2(x) \leq 1$ for any $x \neq 0$ and so, we have that for any point $y \in B(x, r)$

$$|u(y)| \leq \fint_{B(y,r)} |u(z)| \mathrm{d}z.$$

By the Hölder inequality and the fact that if u is harmonic then so is $u - c$, for any constant $c \in \mathbb{R}$, we obtain that

$$|u(y) - c| \leq \left(\fint_{B(y,r)} |u(z) - c|^p \mathrm{d}z \right)^{\frac{1}{p}}.$$

Since for any $y \in B(x, r)$ it holds that $B(y, r) \subset B(x, 2r)$, the claim follows by the doubling property of the Lebesgue measure. □

Lemma 5.5.4. *Let $u \in HW^{1,p}(\mathbb{H}^n, \mathbb{R})$ for some $1 < p < 2n + 2$. Then*

$$\lim_{r \to 0} \fint_{B_{CC}(x,r)} |u(x) - u(y)|^p \mathrm{d}x = 0,$$

for all points $x \in \mathbb{H}^n$ except for a set $E \subset \mathbb{H}^n$ of p-Sobolev capacity zero, where B_{CC} denotes a ball in Carnot-Carathéodory distance.

The result is a counterpart of Lemma 3.2 in [KMV] and Theorem 3.10.2 in [Zr] proven for $\mathbb{R}^n$ and the Bessel capacity. The proof follows from more general results for complete metric measure spaces supporting the p-Poincaré inequality, see Theorem 4.5 in [KL] and also Theorem 9.2.8 in [HKST]. The metric space $(\mathbb{H}^n, d_{CC}, \mathrm{d}x)$ satisfies the assumption of these theorems, see e.g the discussion on pg. 400-403 in [HKST].

In the proof below we also need the following non-local version of the p-Poincaré inequality for a John domain $\Omega \subset \mathbb{H}^n$, see Theorem 2.31 in [Fr].

$$\int_\Omega |f - f_\Omega|^q \mathrm{d}x \leq C_\Omega \int_\Omega |\nabla_H f|^p \mathrm{d}x, \tag{5.46}$$

where $1 \leq p < Q$ for $Q = 2n+2$, $q = \frac{pQ}{Q-p} = 2 + \frac{2}{n}$ and f is a Lipschitz function. Moreover, the constant C_Ω is independent of f. Here, we specialize the statement in [Fr] to our setting. In particular, observe that the balance condition in [Fr, Theorem 2.14] is with our p, q and Q equivalent to the so-called relative lower volume decay, cf. (9.1.14) in [HKST]. This in turn holds if an underlying measure is doubling, which is the case for the Lebesgue measure in $\mathbb{H}^n$.

Proof of Theorem 1.5.5. In the proof we follow the steps of the corresponding Euclidean result, cf. [KMV, Theorem 3.1]. Since $\int_\Omega |\nabla_H u|^p < \infty$, it holds by the Poincaré inequality (5.46) that $u \in HW^{1,p}(\Omega)$, as u is subelliptic harmonic in Ω and so analytic, in particular Lipschitz in a bounded domain Ω.

We apply an extension result, see Theorem 1.1 in [Nh2] with $G = \mathbb{H}^n$ and $\mathcal{L}^{1,p} = HW^{1,p}$ allowing us to conclude that $u \in HW^{1,p}(\mathbb{H}^n)$ provided that Ω is an (ε, δ)-domain. Notice that in the notation of [Nh1], it holds that $0 < \operatorname{rad}(\Omega) < \operatorname{diam}\Omega$, as Ω is connected and bounded, cf. Definition 4.2 in [Nh2] and also [Nh1].

Let us consider a cone $\Gamma_\alpha(x_0)$ at any $x_0 \in \partial\Omega \setminus E$, where E is the set in Lemma 5.5.4. Hence, for any $x \in \Gamma_\alpha(x_0)$ we have that

$$d(x, x_0) \leq (1+\alpha) d(x, \partial\Omega).$$

Therefore, it holds that

$$B_{CC}\left(x, \frac{1}{2} d(x, \partial\Omega)\right) \subset B_{CC}\left(x_0, \left(1 + \alpha + \frac{1}{2}\right) d(x, \partial\Omega)\right).$$

Recall that the Koranyi–Reimann distance and the subriemannian distance are equivalent in $\mathbb{H}^n$ with a constant depending on n, see Chapter 5.1.1, and thus we have that

$$B\left(x, c\frac{1}{2} d(x, \partial\Omega)\right) \subset B_{CC}\left(x, \frac{1}{2} d(x, \partial\Omega)\right) \subset B_{CC}\left(x_0, \left(1+\alpha+\frac{1}{2}\right) d(x, \partial\Omega)\right) \subset B\left(x_0, \frac{1}{c}\left(1+\alpha+\frac{1}{2}\right) d(x, \partial\Omega)\right).$$

We apply Lemma 5.5.3 with $a = u(x_0)$ to get that

$$\begin{aligned} |u(x) - u(x_0)| &\leq C(p,n) \left(\fint_{B\left(x, \frac{1}{2} d(x,\partial\Omega)\right)} |u(y) - u(x_0)|^p \mathrm{d}y \right)^{\frac{1}{p}} \\ &\leq C(p,n,c) \left(\fint_{B\left(x_0, \frac{1}{c}(1+\alpha+\frac{1}{2}) d(x,\partial\Omega)\right)} |u(y) - u(x_0)|^p \mathrm{d}y \right)^{\frac{1}{p}}, \end{aligned}$$

where in the last step we also use a consequence of the doubling property (the relative lower volume decay (9.1.14) in [HKST]):

$$\frac{\left| B\left(x_0, \frac{1}{c}(1+\alpha+\frac{1}{2}) d(x, \partial\Omega)\right)\right|}{\left| B\left(x, \frac{1}{2} d(x, \partial\Omega)\right)\right|} \lesssim_n \left(\frac{2d(x, \partial\Omega) + 1}{c} \right)^{2n+2}.$$

The assertion of the theorem now follows from Lemma 5.5.4 by letting $d(x, \partial\Omega) \to 0$. $\square$

Appendix. The lower bound for a Green function

The following result, applied in Example 5.1, is of independent interest and to best of our knowledge did not yet appear in the literature on Green functions in the subriemannian setting.

Proposition 5.5.5 (cf. (1.9) in Theorem (1.1), [GW]). *Let $\Omega \subset \mathbb{H}^n$ be a domain and $G : \Omega \times \Omega \to \mathbb{R}$ be a Green function of Ω associated with the Laplacian Δ_H. Then, it holds*

$$G(z, y) \geq c(n, \Delta_H) d(z, y)^{2-Q},$$

for all $z, y \in \Omega$ satisfying $d(z, y) \leq \frac{1}{2} d(y, \partial\Omega)$.

Proof. We follow the steps of the corresponding proof in [GW]. Recall that $G \geq 0$, $G(x, \cdot) = 0$ for $x \in \partial\Omega$, $G(x, y) = G(y, x)$ and, moreover, for any fixed $y \in \Omega$, the following representation formula holds: $G(\cdot, y) = \Gamma(\cdot, y) - h_y(\cdot)$, where Γ is the fundamental solution Γ with the pole at $y \in \Omega$ (in the Perron-Brelot-Wiener sense, PWB for short). A function is a solution in PWB sense if it is the largest subharmonic function with boundary values below the desired values. Thus, $\Delta_H^x G(x, y) = -\delta_y(x)$ which in the weak sense reads:

$$\int_\Omega \langle \nabla_H G(x, y), \nabla_H \phi(x) \rangle \mathrm{d}x = \phi(y), \quad \text{for any } \phi \in C_0^\infty(\Omega). \tag{5.47}$$

Let $z, y \in \Omega$ satisfy the assumption $d(z, y) \leq \frac{1}{2} d(y, \partial\Omega)$ and set $r := d(z, y)$. Define the test function $\phi \in C_0^\infty(\Omega)$ such that:

$$0 \leq \phi \leq 1 \text{ in } \Omega, \quad \phi \equiv 1|_{B(y, \frac{r}{2})}, \quad \phi \equiv 0|_{\Omega \setminus B(y,r)} \text{ and also } |\nabla_H \phi| \leq \frac{C}{r}.$$

Then by applying (5.47) with the above ϕ, we obtain that

$$1 \leq \frac{C}{r} \int_{B(y,r) \setminus B(y, \frac{r}{2})} |\nabla_H G(x, y)| \mathrm{d}x, \tag{5.48}$$

for all $x \in B(y, \frac{r}{2})$. Similarly, we consider another test function $\eta(x) := G(x, y)\psi^2(x)$, where $\psi \in C_0^\infty(\Omega)$ is such that

$$\begin{aligned} &0 \leq \psi \leq 1 \text{ in } \Omega, \quad \psi \equiv 1|_{B(y,r) \setminus B(y, \frac{r}{2})}, \\ &\psi \equiv 0 \text{ on } B(y, \frac{1}{4} r) \text{ and outside the ball } B(y, \frac{3}{2} r), \\ &\text{also } |\nabla_H \psi| \leq \frac{C}{r}. \end{aligned}$$

Since $\nabla_H \eta(x) = \nabla_H G(x, y)\psi^2(x) + 2G(x, y)\psi(x)\nabla_H \psi(x)$, upon substituting this expression into (5.47), we obtain the following equation:

$$\int_\Omega |\nabla_H G(x, y)|^2 \psi^2(x) \mathrm{d}x + 2 \int_\Omega \langle \nabla_H G(x, y), \nabla_H \psi(x) \rangle G(x, y)\psi(x) \mathrm{d}x = 0,$$

as the support of ψ does not contain y. Now, the standard inequality $2ab \leq \frac{1}{4}a^2 + 4b^2$ for $a, b \in \mathbb{R}$ together with the Hölder inequality imply that

$$(1 - \frac{1}{4}) \int_{B(y,r)\setminus B(y,\frac{r}{2})} |\nabla_H G(x, y)|^2 \mathrm{d}x \lesssim \frac{C^2}{r^2} \sup_{B(y,\frac{3r}{2})\setminus B(y,\frac{r}{4})} G^2(x, y) r^Q.$$

We combine this estimate with (5.48) to arrive at the following inequality

$$\begin{aligned}
1 &\leq \frac{C}{r} \left(\int_{B(y,r)\setminus B(y,\frac{r}{2})} |\nabla_H G(x, y)|^2 \mathrm{d}x \right)^{\frac{1}{2}} r^{\frac{Q}{2}} \\
&\leq C r^{\frac{Q}{2}-1} \left(C^2 r^{Q-2} \sup_{B(y,\frac{3r}{2})\setminus B(y,\frac{r}{4})} G^2(x, y) \right)^{\frac{1}{2}} \\
&\lesssim_{n,\Delta_H} C^2 r^{Q-2} G(z, y),
\end{aligned}$$

where in the last step we also appeal to the Harnack inequality for harmonic function $G(\cdot, y)$. Thus, the proof is completed upon recalling that $r = d(z, y)$. $\square$

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