Summary of the doctoral thesis

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Summary

This thesis is concerned with studying the Quantitative Fatou Property (QFP) and the ε-approximability, as well as notions such as the nontangential maximal function, the area function and Carleson measures. Let us briefly describe QFP and ε-approximability. Suppose that Ω is a domain in a space that we are interested in and suppose there is a function $u : \Omega \to \mathbb{R}$. The ε-approximability states that there is a function which is sufficiently regular such that it is ε close to u in L^∞ norm and such that the norm of its gradient gives rise to a Carleson measure. QFP usually follows from ε-approximability. QFP states that a function counting oscillations of u is in the space $L^1_{loc}(\partial\Omega)$. It is a property stronger than Fatou theorem which reads that for a harmonic function there exists a nontangential limit at almost every point of the boundary of Ω. The thesis focuses on extending the results known for harmonic functions in the Euclidean setting. The conducted research contains results pertaining to not necessarily harmonic function in the Euclidean setting and to harmonic functions in settings that are not Euclidean. To be precise, these non Euclidean settings are Riemanninan manifolds and Heisenberg groups. The thesis is based on three papers, [AGG], [Gr], [AdGr].

Firstly, in Chapter 3 based on [AGG], the case of not necessarily harmonic functions in the Euclidean setting is dealt with. We show that for Lipschitz-graph domains, i.e. superlevel sets of Lipschitz functions, a certain class of functions satisfies QFP. This class contains harmonic functions, but it is broader as nonnegative subharmonic functions are also elements of this class. We first show that for such functions ε-approximability holds and then how QFP follows from it.

Next chapter, that is Chapter 4 based on [Gr], handles the case of harmonic functions in Riemannian manifolds. We deal with Lipschitz domains. We prove ε-approximability of harmonic functions, and more generally A-harmonic functions. Then we proceed with the proof of QFP.

Finally, in the last Chapter 5 based on [AdGr], we work in the setting of Heisenberg groups with nontangentially accessible domains (NTA) and domains admissible for Dirichlet problem (ADP). We prove several theorems concerning the Carleson measures, the nontangential maximal functions and the area functions. We say that a measure μ defined on Ω is a Carleson measure if a measure

of a ball intersected with Ω, i.e. $\mu(\Omega \cap B(x,r))$ for $x \in \partial\Omega$, is comparable with the measure of a boundary ball, i.e. $\sigma(\partial\Omega \cap B(x,r))$, where σ denotes a surface measure. The nontangential maximal function of function u is the supremum over a cone with vertex at the boundary of Ω of the absolute value of u. The area function of u at point $x \in \partial\Omega$ is the integral over a cone with vertex at x of the square of the norm of the gradient of u multiplied by the distance to $\partial\Omega$ raised to the appropriate power. First, we prove the characterization of Carleson measures in the first Heisenberg group $\mathbb{H}^1$ using the nontangential maximal function for regular enough domains. Then, we prove characterization of Carleson measures on balls. We then prove that for a harmonic function u on NTA domain Ω with boundary data f the L^2 norm of the area function is bounded by the L^2 norm of f. We also prove the Carleson type estimate, saying that the squared norm of the gradient of a harmonic function multiplied with the Green function defines a Carleson measure. Lastly, we prove a refined version of the Fatou theorem. The refinement lies in the fact that we prove that the set where the nontangential limit does not exist is of capacity zero, rather than of measure zero.
Keywords: Quantitative Fatou Property, ε-approximability, nontangential maximal function, area function, Carleson measure.
AMS Subject Classification 2024: 58J05, 35J05, 35R01, 35R03, 31B05, 31C05.

References

[AGG] T. Adamowicz, M. J. González and M. Gryszówka, *ε-Approximability and Quantitative Fatou Property on Lipschitz-graph domains for a class of non-harmonic functions*, arXiv:2412.13072.

[AdGr] T. Adamowicz and M. Gryszówka, *Carleson measures on domains in Heisenberg groups*, to appear in Mathematische Nachrichten.

[Gr] M. Gryszówka, *ε-Approximability and Quantitative Fatou Theorem on Riemannian manifolds*, J. Geom Anal 35, 138 (2025).