A report on the PhD thesis

Mean value property approach to various notions of harmonicity on Euclidean spaces, Carnot groups and metric measure spaces

by Antoni Kijowski

The PhD thesis of Antoni Kijowski consists of four different chapters including the introduction. The main results of the thesis are mainly based on the two joint papers. As the title indicates, Kijowski studies mean value properties as a way of defining harmonic functions in different contexts.

The topics of this thesis has its roots in the classical studies dating back all the way to works of Gauss in 1840. It is nowadays well known that harmonic functions can be characterized in terms of mean value properties. Nonetheless, there are many active research directions stemming from these classical results.

The starting point of the thesis is that the mean value property makes immediately formally sense in a metric measure space. Moreover, it has recently been observed in connection to stochastic games that in a suitable asymptotic sense generalizations of mean value theorems can be used to characterize solutions to the nonlinear equations, for example the p-harmonic functions. These generalizations also formally make sense in the metric measure spaces and can be taken as a starting point of a theory.

Chapter 1, the introduction, gives a nice overall review on the topic, the state of art, and the main results. In Chapter 2, author studies so called strongly harmonic functions in Euclidean domains with general distance functions induced by a norm, not necessarily the Euclidean one, and weighted Lebesgue measures. The strongly harmonic function is a function satisfying the mean value property

$$u(x) = \fint_{B(x,r)} u(y)d\mu(y)$$

for all balls contained in a domain. Over the past few years strongly harmonic functions have been studied in the context of metric spaces for example by Gaczkowski and Górka, 2009, as well as Adamowicz, Gaczkowski, and Górka, 2019. Kijowski establishes a necessary condition and a sufficient condition for a function being strongly harmonic in the spirit of Bose, Flatto, Friedman, Littman, Zalcman and others. Necessary condition is described by Theorem 2.2, which states that a strongly harmonic functions in this context is also a weak solution to a system of partial differential equations, where the number of equations in the system depends on the regularity of the weight. The proof is based on using the Fourier transform.

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Theorem 2.3 states the sufficient condition under the assumption that the weight is analytic. Then the infinite amount of equations in the system implies that a function is strongly harmonic. Kijowski also proves several useful regularity results (Proposition 2.18, 2.20 and Lemma 2.25) under different regularity assumptions on the weight. In Section 2.6, the author gives several nice applications or examples of Theorems 2.2 and 2.3. For example, the results allow him to give very explicit example that the space of strongly harmonic functions in a special case is spanned by 8 linearly independent functions.

Chapter 3 deals with asymptotically p-harmonic functions on Carnot groups. In 2010 it was observed by Manfredi, Parviainen and Rossi that a generalization of the mean value property characterizes p-harmonic functions when interpreted correctly. This gives a generalization of the classical result of Privaloff. The idea for such a generalization stems from stochastic game theory: the generalization of the mean value theorem can be understood as the dynamic programming principle for the corresponding stochastic game. In addition to the above authors, the generalized mean value theorems and/or the related stochastic games have been studied by Arroyo, Lewicka, Llorente, Lindqvist, Peres, Schramm, Sheffield, Wilson and many others. Ferrari, Liu and Manfredi characterized p-harmonic functions in terms of generalized mean value properties in the context of Heisenberg groups, and in the context of Carnot groups this was established by Ferrari and Pinamonti.

A different generalization of the mean value property was established by Ishiwata, Magnanini, and Wadade, 2017, in terms of so called p-mean. To be more precise, they characterize viscosity solutions to the normalized or game theoretic p-Laplace equation, $p \in [1, \infty]$, in terms of the asymptotic generalized mean value property. Kijowski extends the result of Ishiwata et. al. in Theorem 3.1 to the setting of Carnot groups. The proof of Theorem 3.1 relies on Lemma 3.15 which describes a suitable asymptotic expansion of a quadratic function on a Carnot group. Once this is obtained, the proof follows in a straightforward manner by applying the expansion to suitable test functions. In the thesis, the proof of the key lemma is presented in two special cases (Heisenberg group and Carnot group of step 2). Compared to the original proof of Ishiwata et. al., the author needs to take care of several subtle points due to the geometry of the situation.

The last chapter of the thesis discusses, among other things, the asymptotically mean value (AMV) harmonic functions in doubling metric measure spaces. To be more precise, the author proves that functions of locally finite amv-norm are locally α-Hölder, where α depends on p and the doubling exponent, and that a strongly amv-harmonic function is locally α-Hölder continuous for any $\alpha \in (0, 1)$. As for the other results in the last chapter, Theorem 4.8 improves the known regularity of strongly harmonic functions and establishes that strongly harmonic functions are locally Lipschitz if the underlying space is complete and locally doubling. The idea in the proof is consider a refined averaging process, i.e. to integrate the usual average with respect to the radius. As a corollary, this also gives a result related to the Phragmen-Lindelöf theorem. The author also studies implications of growth bounds, blow-ups and obtains the analyticity of strongly amv-harmonic functions on a Heisenberg group. Finally, in Section 4.5, the author returns to the framework of Euclidean spaces with norm induced metric and considers both the weighted and unweighted case. In this context, he studies a connection of functions of finite amv-norm with appropriate class of Sobolev functions.

The thesis of Kijowski is a timely, and well written, contribution to this rapidly expanding research area. The thesis clearly shows that Antoni Kijowski has a good knowledge of mathematical analysis on metric measure spaces, and that he masters the technically challenging modern methods. He has obtained new, interesting and nontrivial results in an area of actual interest in contemporary mathematics. The first article on which the thesis is based on has already appeared. In addition, the candidate demonstrates a good knowledge of the literature in the field.

To summarize, this is a very good thesis and clearly fulfills the requirements. Taking all this into account, I am pleased to recommend to the PhD committee to support the motion to confer the degree of doctor of mathematical sciences upon MSc Antoni Kijowski.

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