# Mean value property approach to various notions of harmonicity on Euclidean spaces, Carnot groups and metric measure spaces

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Oświadczam, że niniejsza rozprawa została napisana przeze mnie samodzielnie.

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Niniejsza rozprawa jest gotowa do oceny przez recenzentów.

dr hab. Tomasz Adamowicz, prof. IMPAN

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# Contents

# Streszczenie

W rozprawie zajmujemy się nowym podejściem do funkcji harmonicznych na przestrzeniach metrycznych z miarą używając przy ich definiowaniu własności wartości średniej. Badamy trzy typy funkcji i związanych z nimi zagadnień: funkcje silnie harmoniczne, funkcje $p$-harmoniczne i ich nieliniową asymptotyczną własność wartości średniej oraz funkcje asymptotycznie średnio harmoniczne. Badania powyższych pojęć prezentujemy odpowiednio w trzech przypadkach: ważonych przestrzeni Euklidesowych, grup Carnot–Carathéodory'ego i przestrzeni metrycznych z miarą podwajającą.

Na początku charakteryzujemy funkcje silnie harmoniczne określone na otwartych podzbiorach przestrzeni Euklidesowej z ważoną miarą Lebesgue'a oraz z metryką indukowaną przez normę. Warunkiem koniecznym na silną harmoniczność funkcji jest jej bycie słabym rozwiązaniem układu eliptycznych równań różniczkowych cząstkowych, którego liczba równań zależy od regularności wagi. Warunek dostateczny jest udowodniony przy użyciu wzoru Pizzettiego i stanowi, że każde rozwiązanie wyżej wymienionego układu równań jest silnie harmoniczne. Wzór Pizzettiego jest prawdziwy tylko dla funkcji analitycznych, dlatego zakładamy analityczną regularność wagi. Jedną z konsekwencji przeprowadzonej analizy są wyniki o regularności funkcji silnie harmonicznych. Dowodzimy, że dla wagi z przestrzeni Sobolewa funkcje silnie harmoniczne należą do przestrzeni Sobolewa oraz, że dla analitycznej wagi funkcje silnie harmoniczne również są analityczne. Przeprowadzona analiza została zilustrowana w przypadku planarnym z metryką indukowaną przez normę $l^p$. Dla $p = 2$ oraz gładkiej wagi przedstawiamy w możliwie najprostszy sposób wyżej wymieniony układ równań różniczkowych cząstkowych charakteryzujący harmoniczność. Ponadto, dla stałej wagi oraz pozostałych wykładników $p \in [1, \infty] \setminus \{2\}$ wykazujemy, że wymiar przestrzeni funkcji silnie harmonicznych wynosi 8.

W rozdziale trzecim charakteryzujemy ciągłe rozwiązania lepkościowe równania znormalizowanego subeliptycznego $p$-Laplasjanu na grupach Carnot jako funkcje o asymptotyczej $p$-własności wartości średniej w sensie lepkościowym.

W ostatniej części pracy badamy funkcje asymptotycznie średnio harmoniczne na przestrzeniach metrycznych z lokalnie podwajającą miarą. Używając metody uśredniania dowodzimy, że funkcje ze skończoną amv-normą należą do ułamkowych przestrzeni Hajłasza–Sobolewa oraz, że funkcje asymptotycznie średnio harmoniczne są $\alpha$-Hölderowsko ciągłe z dowolnym wykładnikiem $0 < \alpha < 1$. Konsekwencją zastosowania metody uśredniania jest udowodnienie lokalnej ciągłości Lipschitzowskiej dla funkcji silnie harmonicznych przy założeniach słabszych niż znane w literaturze. Ponadto, dowodzimy skończoności wymiaru przestrzeni funkcji silnie harmonicznych o wzroście wielomianowym o ile miara ma własność zanikania na pierścieniach. Twierdzenie Blaschke–Privaloffa–Zaremby zostało uogólnione na grupę Heisenberga $\mathbb{H}_1$. Używając metody blow-up'ów na przestrzeni metrycznej wykazujemy, że funkcje styczne do tych ze skończoną amv-normą są silnie harmoniczne na przestrzeni stycznej. W ważonych przestrzeniach Euklidesowych, gdy waga jest lokalnie ciągła w sensie Lipschitza, dowodzimy, że funkcje asymptotycznie średnio harmoniczne są rozwiązaniami eliptycznego równania różniczkowego cząstkowego.

**Słowa kluczowe:** analiza na przestrzeniach metrycznych, własność wartości średniej, funkcja harmoniczna, funkcja silnie harmoniczna, funkcja asymptotycznie średnio harmoniczna, funkcja $p$-harmoniczna, grupa Carnot, ważona miara Lebesgue'a, $p$-średnia.

# Abstract

In the thesis we study a recent approach to harmonic functions on metric measure spaces defined via the mean value property. Namely, we investigate three types of functions and related problems: strongly harmonic functions, $p$-harmonic functions in connections to nonlinear asymptotic mean value property and asymptotically mean value harmonic functions. Our analysis is divided into three settings: weighted Euclidean domains with a norm induced metric, Carnot–Carathéodory groups and doubling metric measure spaces, respectively.

First, we present a characterization of strongly harmonic functions on Euclidean spaces equipped with a weighted Lebesgue measure and a norm induced metric. The necessary condition says, that any strongly harmonic function is a solution to a system of elliptic partial differential equations, where the number of equations in a system depends on the regularity of the weight. The sufficient condition is proved using the Pizzetti formula and shows that every solution to previously described system of equations is strongly harmonic. The result holds for analytic weights. As an outcome of the discussion we obtain the Sobolev/analytic regularity of strongly harmonic functions assuming Sobolev/analytic regularity of the weight, respectively. The discussion is illustrated by distance functions induced by $l^p$ norm for planar domains. We demonstrate the aforementioned system for a smooth weight and $p = 2$ and show, that for a constant weight and $p \in [1, \infty] \setminus \{2\}$ the space of strongly harmonic functions has dimension 8.

In the second part of the dissertation we work with normalized subelliptic $p$-Laplace equation in Carnot groups. We show a characterization of continuous viscosity solutions via an asymptotic $p$-mean value property understood in the viscosity sense.

Finally, we investigate asymptotically mean value harmonic functions in locally doubling metric measure spaces. We employ a refined averaging to prove fractional Hajłasz–Sobolev regularity of functions with finite amv-norm and $\alpha$-Hölder regularity of strongly amv-harmonic functions for all $0 < \alpha < 1$. An outcome of the discussion is local Lipschitz regularity for strongly harmonic functions obtained under weaker set of assumptions than those known in the literature. Moreover, we show that the space of strongly harmonic functions with polynomial growth has finite dimension whenever the measure has $\delta$-annular decay property. Moreover, we prove Blaschke–Privaloff–Zaremba theorem in the Heisenberg group $\mathbb{H}_1$. We also study blow-ups of functions with finite amv-norm proving, that a tangent function at almost every point is strongly harmonic on the tangent space at that point. In the end, we show that amv-harmonic functions on weighted Euclidean domains with locally Lipschitz weights are solutions to an elliptic partial differential equation.

**Keywords:** analysis on metric spaces, mean value property, harmonic function, strongly harmonic function, asymptotically mean value harmonic function, $p$-harmonic function, Carnot group, weighted Lebesgue measure, $p$-mean.

**AMS Subject Classification:** 31C05, 30L99, 31E05; Secondary: 35R03, 53C23, 35J99, 35H20.
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# Chapter 1

# Introduction

This thesis is devoted to investigation of a recent concept of harmonic functions on metric measure spaces defined via various kinds of the mean value property. In what follows, we describe the subject and draw a road map of the thesis. We begin with presenting historical background of relations between harmonic functions and the mean value property focusing on the classical results and placing its place in the development of analysis of Partial Differential Equations. Then, we divide the discussion with respect to three viewpoints:

1. strongly harmonic functions,
2. nonlinear averages of $p$-harmonic functions,
3. asymptotic mean value property.

In those parts we address recent results, which lead to our research. At the end of this chapter we present most important results obtained throughout this thesis.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say, that a function $u \in C^2(\Omega)$ is harmonic on $\Omega$ if the Laplace operator $\Delta u(x) := \frac{\partial^2 u}{\partial x_1^2}(x) + \ldots + \frac{\partial^2 u}{\partial x_n^2}(x) = 0$ for all $x \in \Omega$.

The standard questions concerning harmonic functions are the existence and the uniqueness of solutions to the Dirichlet problem: given an open bounded smooth $\Omega \subset \mathbb{R}^n$ and a boundary data $\varphi \in C(\partial\Omega)$ find $u : \Omega \to \mathbb{R}$ which satisfies the following conditions

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

One of the most important features of harmonic functions is the Dirichlet principle saying, that $u$ is a solution to the Dirichlet problem if and only if it is a minimizer of the Dirichlet energy

$$E(u) := \int_\Omega |\nabla u(y)|^2 \, dy$$

in the class of all $W^{1,2}(\Omega)$ functions with fixed trace $\varphi$ on the boundary $\partial\Omega$.

In the setting of a metric measure space $(X, d, \mu)$, where $d$ is a distance function and $\mu$ is a Borel measure, the lack of the linear structure of $X$ makes the notion of a partial derivative of $u$ not well defined and a pointwise definition of metric Laplace operator is not accessible in general. Observe, that in its very matter, the Dirichlet energy does not use the full information about behaviour of a gradient in different directions, but only its length. Suitable counterpart to length of the gradient of a function have been studied from many perspectives. Below, we present two significant ideas which allowed analysis on metric spaces to flourish and enabled the development of the metric differentiation and, hence, the theory of Sobolev spaces:

1. A weak upper gradient $g : X \to [0, \infty]$ of a function $u$ is a function, which controls the growth of $u$ over almost every curve (in the sense of the modulus of a curve family) subsequent to the Newton–Leibniz theorem, i.e.

$$|u(x) - u(y)| \leq \int_\gamma g,$$

where $\gamma$ is a curve joining $x$ and $y$. This approach led to the construction of Newtonian spaces as those consisting of $L^p(X)$ functions, for which there exists a $p$-integrabe weak upper gradient. For more information see [Sha00; Hei+15; BB11].

2. A Hajłasz gradient $g$ of function $u$ is a counterpart of the Hardy–Littlewood maximal function of the length of a gradient in the sense of satisfying the following estimate

$$|u(x) - u(y)| \leq d(x, y)[g(x) + g(y)]$$

for $\mu$-a.e. $x, y \in X$. Analogously to the Newtonian spaces, the Hajłasz–Sobolev space consists of all $L^p(X)$ functions, for which there exists a $p$-integrable Hajłasz gradient, see [HK00; Haj96].

Using the first concept of metric gradients one may define harmonic functions as minimizers of a metric counterpart of the Dirichlet energy

$$E(u) = \inf_g \int_\Omega g^2 d\mu, \qquad \Omega \subset X,$$

where the above infimum is considered over all $g \in L^2(\Omega)$ which are weak upper gradients of $u$. Then, we say that $u$ is harmonic in $\Omega$ whenever it minimizes energy $E$ over all functions from the Newtonian space on $\Omega$ having the same trace at $\partial\Omega$ as $u$.

To our best knowledge, the mean value property of a function was firstly associated with the Laplace operator by Gauss [Gau40], who proved that for a harmonic function $u : \Omega \to \mathbb{R}$ and every ball $B(x, r) \Subset \Omega$ there holds

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)dy =: \fint_{B(x,r)} u(y)dy. \tag{1.1}$$

The converse result, and hence a characterization of harmonic functions by the mean value property, was observed by Koebe [Koe06], who proved that if $u : \Omega \to \mathbb{R}$ is continuous and for every ball $B(x, r) \Subset \Omega$ there holds (1.1), then $u$ is harmonic in $\Omega$. The mean value property is a tool used in proving such properties of harmonic functions as the maximum principle, Harnack inequality, analytic regularity and other potential analytic properties. For an interesting survey on the mean value property and harmonic functions see [NV94].

The aforementioned Gauss–Koebe characterization of harmonic functions has also been investigated on Riemannian manifolds and led to establishing the notion of harmonic manifolds. Recall, that harmonic functions on Riemannian manifold $(M, g)$ are solutions of the Beltrami–Laplace equation, which is defined as follows $\Delta_{BL} u := (\det g)^{-1/2} \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x_j} u) = 0$. Let us restrict our discussion only to manifolds which are complete. The theory of harmonic manifolds appeared for the first time in the dissertation of Ruse in 1930. He developed harmonic analysis on general Riemannian manifolds by using a solution to the Beltrami–Laplace equation which only depended on the geodesic distance from some fixed point (a counterpart of the fundamental solution). It was not until 1939 when he realised that such solutions do not necessarily exist on general manifolds. Ruse defined harmonic manifolds as the class of manifolds which locally support such radial fundamental solutions. This theory was later developed by Lichnerowicz, see [Lic44]. He observed that one can equivalently define harmonic manifolds as those whose density function $\omega$ expressed in normal coordinates at any point $p$, i.e. $\omega_p(q) = \sqrt{\det g_q}$, depends only on the geodesic distance between $p$ and $q$. Lichnerowicz proved in [Lic44] that all harmonic manifolds of dimension non-greater than 3 are either flat or rank one symmetric and conjectured that the same holds in higher dimensions. The converse implication is always true. The Lichnerowicz conjecture was confirmed in the case of dimension 4 by Walker [Wal49] and dimension 5 by Nikolayevsky [Nik05]. Later on, Szabó [Sza90] proved the conjecture for compact simply connected manifolds of all dimensions. On the other hand Damek–Ricci constructed a class of simply connected noncompact harmonic manifolds with negative curvature being nonsymmetric, see [DR92a; DR92b], where the smallest dimension of this type of counterexample to Lichnerowicz conjecture is 7. As it turned out due to Heber [Heb06] the class of harmonic manifolds which are additionally homogeneous consists only of flat, rank one symmetric and Damek–Ricci counterexamples. The Lichnerowicz conjecture remains unsettled for dimension 6 and in the class of nonhomogeneous harmonic manifolds. From our point of view harmonic manifolds possess a very refined depiction due to Willmore [Wil50]: A Riemannian manifold $M$ is harmonic if and only if for any function $u$ on $M$ being harmonic is equivalent to having the spherical mean value property over any geodesic sphere.

A different approach to the mean value property is due to Blaschke–Privaloff–Zaremba, who proved independently the following result: Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C(\Omega)$. Suppose that for every $x \in \Omega$ there holds

$$\lim_{r\to 0} \frac{⨍_{B(x,r)} u(y)dy - u(x)}{r^2} = 0. \tag{1.2}$$

Then $u$ is harmonic in $\Omega$. Observe, that the converse is always true, since if $u$ is harmonic, then the numerator in (1.2) is constantly 0. The main reason, why the Blaschke–Privaloff–Zaremba theorem holds true is the following observation, which is well explained in the introduction to [Llo15]. Fix a function $u \in C^2(\Omega)$ and a point $x \in \Omega$. Then by the Taylor expansion there holds for all $y \in B(x,r)$ that

$$u(y) = u(x) + \langle \nabla u(x), y - x\rangle + \frac{1}{2}\langle \nabla^2 u(x)(y-x), y-x\rangle + o(r^2).$$

This, upon taking the mean integral over $B(x,r)$ on both sides and observing that the linear term vanishes, reads

$$⨍_{B(x,r)} u(y)dy = u(x) + \frac{1}{2(n+2)} r^2 \Delta u(x) + o(r^2).$$

Therefore, for a function $u \in C^2(\Omega)$ the expression

$$\Delta_r u(x) := \frac{⨍_{B(x,r)} u(y)dy - u(x)}{r^2} \tag{1.3}$$

converges to $\Delta u(x)$ up to a constant. The above equality or (1.2) is often called the asymptotic mean value property. In order to discuss the difference between conditions (1.1) and (1.2) notice that property (1.1) can be expressed as a condition on the function $M_x(r) = ⨍_{B(x,r)} u(y)dy$. By the Lebesgue differentiation theorem we extend the domain of $M_x$ to an interval containing 0 and set $M_x(r) = u(x)$. Then, (1.1) can be equivalently stated as follows: for each $x \in \Omega$ the function $M_x(r)$ is constant for all $r \in [0, \operatorname{dist}(x, \partial\Omega))$. On the other hand, condition (1.2) can be expressed equivalently that $M_x$ around $r = 0$ has at most quadratic rate of change, which is more extensive than (1.1). One of the most important features of (1.2) is that it holds also for harmonic functions on general Riemannian manifolds, in contrast to (1.1) which holds only on the class of harmonic manifolds being a subclass of all Riemannian manifolds. Moreover, even beyond the setting of Riemannian manifolds the space of functions with property (1.2) has richer structure than the space of functions with property (1.1). As we will see in further parts of the dissertation in weighted Euclidean setting (1.2) is equivalent to a partial differential equation, while (1.1) is characterized with a system of PDEs. Moreover, in the Heisenberg group (1.2) is equivalent to being harmonic and in general Carnot group a weak version of property (1.2) is equivalent to harmonicity, while functions with (1.1) are a proper subclass of those. Finally, in non-collapsed RCD spaces with vanishing metric measure boundary every harmonic function has (1.2) in a weak sense. For more information we refer to Chapter 4 and [AKS20].

Let us define one of the main objects studied in Chapter 2 of this thesis.

## 1.1 Strongly harmonic functions

Let us observe, that the mean value property (1.1) makes sense formally on a metric measure space $(X, d, \mu)$, where the notion of a ball and an integral is available. In order to write down the mean value we only need to assume, that the measure of every ball is positive and finite and that a function $u$ is locally integrable on $X$. Therefore, from now on we denote by a *metric measure space* a metric space $(X, d)$ equipped with a Borel regular measure $\mu$, which assigns to every ball a positive and finite value. This observation and the historical discussion concerning Gauss–Koebe theorem and related results motivated Gaczkowski–Górka [GG09] and Adamowicz–Gaczkowski–Górka [AGG19] to formulate a novel approach in metric theory of harmonic functions: Let $\Omega \subset X$ be open and function $u \in L^1_{loc}(\Omega)$. We say, that $u$ is *strongly harmonic* on $\Omega$ if for every point $x \in \Omega$ and every radius $r > 0$ for which $B(x, r) \Subset \Omega$ there holds

$$u(x) = \fint_{B(x,r)} u(y) d\mu(y). \tag{1.4}$$

The class of all strongly harmonic functions on $\Omega$ is denoted by $\mathcal{H}(\Omega, d, \mu)$ and often abbreviated to $\mathcal{H}(\Omega)$, when the metric and the measure are clear from the context. Notice, that this approach to harmonic functions on metric measure spaces is more straightforward than by minimization of Dirichlet energy, since it does not require the use of metric gradients and Sobolev spaces. Moreover, by our previous discussion, strongly harmonic functions agree with harmonic functions on Euclidean domains and on harmonic manifolds.

The class of strongly harmonic functions has been widely examined: in general metric measure spaces [AGG19; GG09], in case of metric space being homogeneous graph in [Zuc02; PW89] and in case of metric space being Carnot group in [AW20; BLU07]. Let us shortly discuss those results.

Gaczkowski–Górka [GG09] showed that in metric measure spaces, where the measure is continuous with respect to metric strongly harmonic functions are continuous. Moreover, they proved the strong maximum and minimum principle and in spaces with precompact balls showed the Harnack inequality and compactness of locally bounded subfamilies of $\mathcal{H}(X)$.

To our best knowledge [GG09] is the first paper which deals with harmonicity defined via the mean value property (1.4) in such a generality. In their further paper Adamowicz–Gaczkowski–Górka [AGG19] studied the class $\mathcal{H}(X)$ more deeply and also from different perspectives. Among results let us mention further Harnack estimates, weak and strong maximum principles, local Hölder regularity on metric spaces with measures satisfying the $\delta$-annular decay property for some $\delta \in (0, 1]$ (the latter meaning that there exists constant $C > 0$ such that for all $x \in X$, $r > 0$ and $\varepsilon > 0$ there holds $\mu(B(x, r) \setminus B(x, (1 - \varepsilon)r)) \leq C\varepsilon^\delta \mu(B(x, r))$). The Hölder exponent equals $\delta$ and if $\delta = 1$ the regularity raises to locally Lipschitz. The authors employed Cheeger's result [Che99] to show that if the space supports a Poincaré inequality and measure $\mu$ is either $Q$-regular or posseses the 1-annular decay property, then every strongly harmonic function has the minimal weak upper gradient. Additionally, the Liouville theorems for entire harmonic functions on metric measure spaces were obtained. Finally, the authors employed the Perron method to study existence of solutions to Dirichlet problem, see [AGG19][Section 6].

Apart from strongly harmonic functions, the authors studied in [AGG19] a class of the so-called weakly harmonic functions. Since it is largely connected to strongly harmonic function, we are going to briefly sketch this notion. Its origin goes back to studies by Kellogg, Koebe, Littlewood and Volterra and grows from the attempt of weakening the mean value property so that it still implies the harmonicity in the sense of the Laplace equation. There are two key ways to approach the task:

1. by reducing the assumption that (1.1) holds on every ball $B(x, r) \Subset \Omega$ and assume instead that for every point $x \in \Omega$ there exists a nonempty collection of radii $\{r^x_\alpha\}_{\alpha \in A^x}$ with $B(x, r^x_\alpha) \Subset \Omega$ for every $\alpha \in A^x$,

2. by reducing the set of points $x \in \Omega$ for which the mean value property holds.

Let us illustrate the above discussion with presenting one of results by Hansen–Nadirashvili [HN93] which is often called the 1-radius theorem: *Let* $\Omega$ *be an open bounded subset of* $\mathbb{R}^n$, $u \in C(\Omega) \cap L^\infty(\Omega)$ *be such that for every* $x \in \Omega$ *there exists* $0 < r^x \leq \operatorname{dist}(x, \partial\Omega)$ *with the property* $u(x) = \fint_{B(x,r^x)} u(y)dy$. *Then* $u$ *is Laplace harmonic in* $\Omega$. On the other hand, the sufficient number of radii, for which the mean value property must hold to imply the harmonicity of the function whose domain is the whole $\mathbb{R}^n$ is 2. This type of result is often called the 2-radius theorem and was first observed by Delsarte [Del58] in $\mathbb{R}^n$ and later on generalized by Berenstein–Zalcman [BZ80] to rank one symmetric spaces and by Peyerimhoff–Samiou [PS15] to noncompact harmonic manifolds.

The aforementioned relation between harmonicity and the weaker variant of the mean value property leads to formulating a relaxed version of the strong harmonicity: Let $\Omega \subset X$ be an open set in a metric measure space $(X, d, \mu)$. We call a locally integrable function $u : \Omega \to \mathbb{R}$ *weakly harmonic* in $\Omega$ if for all points $x \in \Omega$ there exists at least one radius $0 < r^x < \operatorname{dist}(x, \partial\Omega)$ with the following property $u(x) = \fint_{B(x,r^x)} u(y)d\mu(y)$. For further information about properties of weakly harmonic functions we refer the reader to [AGG19].

Picardello–Woess in [PW89] studied relations between the discrete Laplacian on graphs and the mean value property. Given a graph $G = (V, E)$, where $V$ is the set of vertices and $E$ the set of unoriented edges between vertices in $V$ we define the *graph Laplacian* of a function $u : V \to \mathbb{R}$ at vertex $x \in V$ as follows $\Delta_G u(x) := \frac{1}{\deg x} \sum_{y \sim x} (u(y) - u(x))$, where $\deg x$ is the number of neighbours of $x$ and we write $y \sim x$ whenever $y$ is adjacent to $x$. A graph $G$ can be viewed as a metric measure space $(V, d, \#)$, where $d(x, y)$ is the infimum of number of edges joining $x$ to $y$ and $\#$ is the counting measure. Picardello–Woess proved that in any homogeneous tree $T_k$ for $k \geq 3$ harmonic functions in the sense of the graph Laplacian possess the mean value property (1.4). Conversely, any function on a homogeneous tree $T_k$, $k \geq 3$ having (1.4) at every vertex $x \in V$ with one radius $r = r(x) \in \mathbb{N}$, $r(x) \geq 1$, is graph harmonic assuming a Lipschitz-type growth on $x \mapsto r(x)$. For a connection to Markov processes we refer to [Zuc02].

Let us complete this part with a short discussion on the results obtained by Adamowicz–Warhurst in [AW20]. The authors studied strongly harmonic functions on Carnot groups. Results of [AW20] encompass the smoothness of strongly harmonic functions and the fact that they solve the sub-Laplace equation. The converse need not be true, but the authors found in the Heisenberg group that a class of spherical harmonic polynomials is both strongly harmonic and solves the sub-Laplace equation.

Next, we describe relations between $p$-harmonic functions and a nonlinear mean value property, which is a subject of Chapter 3.

## 1.2 Asymptotic mean value characterization for $p$-harmonic functions

Let us fix an open set $\Omega \subset \mathbb{R}^n$. One of the classical generalizations of harmonic functions originates from allowing in the Dirichlet energy exponents different that 2: $E_p(u) := \int_\Omega |\nabla u|^p$ for any $p \in [1, \infty)$. The Euler–Lagrange equation arising from the minimization problem of energy $E_p$ is the $p$-Laplace equation. The $p$-Laplace operator is defined as follows

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u),$$

which for $p = 2$ coincides with the Laplace operator, and solutions to the equation $\Delta_p u = 0$ are called $p$-harmonic functions. The infinity Laplacian is defined as follows

$$\Delta_\infty u(x) = \sum_{i,j=1}^{n} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j} = \langle \nabla^2 u(x) \nabla u(x), \nabla u(x) \rangle.$$

Among many properties of the $p$-Laplace operator is the equivalence between continuous weak solutions and viscosity solutions whenever $p \in (1, \infty)$, see [JLM01]. First, let us describe shortly the concept of viscosity solutions, for more rigorous discussion see [CIL92; Koi04]. The idea originates from a geometric approach to monotone operators. By saying that $\Delta_p$ is monotone we mean here, that for a function $u \in C^2(\Omega)$, a point $x \in \Omega$ and any twice differentiable test function $\varphi$ touching $u$ at $x$ from below (the latter meaning that $u - \varphi$ has a local minimum at $x$ equal to 0 and $\nabla\varphi(x) \neq 0$) it holds that $\Delta_p \varphi(x) \leq \Delta_p u(x)$. Observe, that if $\varphi$ touches $u$ at $x$ from above, then $u$ touches $\varphi$ at $x$ from below and by the monotonicity of $\Delta_p$ the inequality between operators is the following $\Delta_p u(x) \leq \Delta_p \varphi(x)$. Notice, that formally in order to say that a test function touches $u$ (from below or above) it is enough to assume that $u$ is continuous. This idea is employed in order to shift calculating the value of $\Delta_p u$ (about which we want to assume the least possible a priori regularity) to $C^2$ test functions touching $u$ from below and from above. Therefore, we say, that a continuous function $u$ is a viscosity solution to the $p$-Laplace equation if:

1. for every point $x$ and any twice differentiable $\varphi$ touching $u$ from below at $x$ there holds $\Delta_p \varphi(x) \geq 0$ and
2. for every point $x$ and any twice differentiable $\varphi$ touching $u$ from above at $x$ there holds $\Delta_p \varphi(x) \leq 0$.

One of the most important features of viscosity solutions to the $p$-Laplace equation is that they allow suitable generalization of the Blaschke–Privaloff–Zaremba theorem, which was observed by Manfredi–Parviainen–Rossi, see [MPR10]. The authors proved that given $p \in (1, \infty]$ a function $u$ is a viscosity solution to the $p$-Laplace equation in $\Omega \subset \mathbb{R}^n$ if and only if $u$ has the following nonlinear asymptotic mean value property

$$u(x) = \frac{\alpha}{2}\left(\min_{B(x,r)} u + \max_{B(x,r)} u\right) + \beta \fint_{B(x,r)} u(y)dy + o(r^2) \tag{1.5}$$

holding in the viscosity sense as $r \to 0$, for all $x \in \Omega$ and constants $\alpha = \frac{p-2}{n+p}$, $\beta = \frac{n+2}{n+p}$. Notice, that $\alpha + \beta = 1$, hence two first terms on the right-hand side of (1.5) converge to $u(x)$ as $r \to 0$ and (1.5) in fact implies that the rate of this convergence is at least quadratic. For $p = 2$ we obtain the Blaschke–Privaloff–Zaremba theorem, since for $p \in (1, \infty)$ viscosity solutions coincide with weak solutions. The nonlinear asymptotic mean value property (1.5) is strongly connected to the so-called tug-of-war games, which is the two players game in an open set $\Omega \subset \mathbb{R}^n$, with a step $\varepsilon > 0$ and a continuous payoff function $F : \partial\Omega \to \mathbb{R}$. The starting point is a fixed $x_0 \in \Omega$. Then, at $k$-th turn a fair coin is tossed and one of the players wins a chance of moving from a point $x_{k-1}$ to $x_k$. If $\operatorname{dist}(x_{k-1}, \partial\Omega) > \varepsilon$, then the player chooses a direction $v_k \in \mathbb{R}^n$ with $|v_k| \leq \varepsilon$ and sets $x_k = x_{k-1} + v_k + y_k$, where $y_k$ is a random noise vector. The game stops when $\operatorname{dist}(x_{k-1}, \partial\Omega) \leq \varepsilon$ for the first time. The active player can choose the final point $x_k \in \partial\Omega$ at the distance $|x_k - x_{k-1}| \leq \varepsilon$ and receive $F(x_k)$ payoff from the other player. Both players get zero payoff if $\operatorname{dist}(x_k, \partial\Omega) > \varepsilon$ for all $k \in \mathbb{N}$. In this way we obtain a function $x_0 \mapsto u_\varepsilon(x_0)$ describing the payoff for a game starting at $x_0$. In [PS08] for $1 < p < \infty$ and in [Per+09] for $p = \infty$ the authors proved, that for a regular domain $\Omega$ the sequence $(u_\varepsilon)$ converges to a $p$-harmonic function with boundary values $u|_{\partial\Omega} = F$ as $\varepsilon \to 0$. The min-max term in (1.5) corresponds to the choice of the direction by an active player during the tug-of-war game and the mean value term corresponds to the random noise vector.

The study of results in the spirit of those in [MPR10] has developed in the following way. In [KMP12] the authors studied the case of $p = 1$. The planar case is investigated in [LM16], where it is proved that (1.5) in the pointwise sense is equivalent to $p$-harmonicity of $u$ whenever $1 < p < p_0$. In [AL16a] this result was extended to the full range of $p \in (1, \infty)$. On the other hand, in [MPR13] the authors defined a class of $p$-harmonious functions, i.e. those functions $u_r$ for which

$$u_r(x) = \frac{\alpha}{2}\left(\min_{B(x,r)} u_r + \max_{B(x,r)} u_r\right) + \beta \fint_{B(x,r)} u_r$$

holds for a fixed $r > 0$, and showed that $p$-harmonious functions approximate $p$-harmonic functions as $r \to 0$ and $p \in [2, \infty)$. The nonlinear asymptotic mean value property of viscosity $p$-harmonic functions has been studied also outside the Euclidean setting, for instance in the Heisenberg group $\mathbb{H}_1$, see [FLM14] and in Carnot groups, see [FP15].

A variant of the $p$-Laplace operator, better cut our for studying the viscosity solutions is the so-called *normalized $p$-Laplacian* defined as follows

$$\Delta_p^N u = \frac{\Delta_p u}{|\nabla u|^{p-2}} \text{ for } 1 \le p < \infty \quad \text{and} \quad \Delta_\infty^N u = \left\langle \nabla^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right\rangle \text{ for } p = \infty. \tag{1.6}$$

Notice, that for a function $u$ with $\nabla u \neq 0$ being a solution of the $p$-Laplace equation is equivalent to being a solution of the normalized $p$-Laplace equation. Therefore, being a viscosity solution of the $p$-Laplace equation is equivalent to being a viscosity solution to normalized $p$-Laplace equation, since test functions in the definition of viscosity solution omit critical points.

A recent result by Ishiwata–Magnanini–Wadade [IMW17] deals with the whole range of $p \in [1, \infty]$ and a different version of (1.5), which seems to suit better the problem of asymptotic mean value property for $p$-harmonic functions. Namely, for $\Omega \subset \mathbb{R}^n$ the authors work with the so-called *$p$-mean* of a continuous function $u : \Omega \to \mathbb{R}$, denoted by $\mu_p(r, u)(x)$ and defined as a unique number $\lambda$ which minimizes $\|u - \lambda\|_{L^p(B(x,r))}$, where $B(x, r) \subset \Omega$. The $p$-means form a large class of averages, as for $p = 1$ we retrieve a median of $u$ on $B(x, r)$, the 2-mean coincides with $⨍_{B(x,r)} u$ and for $p = \infty$ is equal to $\frac{1}{2}(\min_{B(x,r)} u + \max_{B(x,r)})$. The main result of [IMW17] is the following: Let $\Omega \subset \mathbb{R}^n$ be open, $p \in [1, \infty]$ and $u \in C(\Omega)$. Then the following conditions are equivalent:

1. $u$ is a viscosity solution to $\Delta_p^N u = 0$,
2. $u(x) = \mu_p(r, u)(x) + o(r^2)$ as $r \to 0$ in the viscosity sense for every $x \in \Omega$.

Now, let us define the last of the main objects of the thesis, which is investigated in Chapter 4.

## 1.3 Strongly amv-harmonic functions

As mentioned in the beginning of this introduction, see (1.2) and (1.3), the Blaschke–Privaloff–Zaremba theorem suggests yet one more approach to the notion of harmonicity. Suppose that $(X, d, \mu)$ is a metric measure space and a function $u \in L^1_{loc}(X)$. We define the *$r$-laplacian* of $u$ as follows

$$\Delta_r u(x) := \frac{⨍_{B(x,r)} u(y) d\mu(y) - u(x)}{r^2} \tag{1.7}$$

for $x \in X$. In order to generalize harmonic functions into metric measure spaces using the operator (1.7) one should assert that it converges to 0 in some sense. It turns out that, for example, for a jump function $f$ on the real line $\Delta_r f \to 0$ as $r \to 0$ pointwise, but neither in any $L^p$ norm nor almost uniformly. In the sense of measures $\Delta_r f$ converges to a $\delta$ distribution at a jump point.

The properties of the operator $\Delta_r$ has been first studied, to our best knowledge, by Burago–Ivanov–Kurylev in [BIK19] in the context of spectral stability and by Córdoba–Ocáriz in [CO20] from the perspective of minimal surfaces. Recently, Minne–Tewodrose studied in [MT19] pointwise limits of $\Delta_r u$ for $u$ being twice differentiable in weighted Euclidean spaces and on Riemannian manifolds. Moreover, they proved maximum principle and a Green-type identity in general metric measure spaces.

On the other hand, Adamowicz–K–Soultanis in [AKS20] defined the class of strongly and weakly amv-harmonic functions, but also a class of functions with finite amv-norm in the following way. We say, that a function $u \in L^1_{loc}(X)$ is *strongly amv-harmonic* if $\Delta_r u$ converges to 0 almost uniformly in $X$. Moreover, we say that $u$ is *weakly amv-harmonic* whenever $\Delta_r u$ converges to 0 as a measure on $X$. Since it is hard to determine what class of functions one should consider as the domain of the operator $\Delta_r$, we consider a class of functions with *finite amv-norm*:

$$\mathrm{AMV}^p(X) = \{u \in L^p(X) : \limsup_{r \to 0} \|\Delta_r u\|_{L^p(X)} < \infty\}.$$

Additionally, we consider functions with *locally finite amv-norm*, denoted $\mathrm{AMV}^p_{loc}(X)$ defined by changing $L^p(X)$ to $L^p_{loc}(X)$ in the above formulation.

In the last chapter of the introduction we gather results presented in this dissertation.

## 1.4 Main results

Let us briefly describe the results of this thesis. The discussion is divided into three chapter with respect to their settings: Euclidean spaces equipped with a weighted Lebesgue measure and a norm induced metric in Chapter 2, Carnot groups in Chapter 3 and general metric measure spaces in Chapter 4.

In Chapter 2 we prove a characterization of strongly harmonic functions on a metric measure space $(X, d, \mu)$, where $X = \Omega \subset \mathbb{R}^n$ is an open set, $d$ is induced by a norm on $\mathbb{R}^n$ and the measure $d\mu = wdx$ for a positive almost everywhere weight $w \in L^1_{loc}(\Omega)$. The characterization of $\mathcal{H}(\Omega, d, wdx)$ is divided into two theorems: a necessary condition and a sufficient condition, the second one obtained under the analyticity assumption on $w$.

The necessary condition is described by Theorem 2.2 and states that every function $u \in \mathcal{H}(\Omega, d, wdx)$ is a weak solution of the following system of partial differential equations

$$\sum_{|\alpha|=j} A_\alpha \left(D^\alpha(uw) - uD^\alpha w\right) = 0, \qquad \text{for } j = 2, 4, \ldots, 2m. \tag{1.8}$$

Here, the number of equations $m$ depends on the regularity of the weight $w$ in the following way: if $w \in W^{2,2}_{loc}(\Omega)$, then $m = 1$ and if $w \in C^{2k-1,1}_{loc}(\Omega)$ for some natural number $k \in \mathbb{N}$, $k > 1$, then $m = k$. The coefficients $A_\alpha$ in the system are defined via the $\alpha$-moments of the Lebesgue measure on a unit ball with respect to metric $d$, i.e. $A_\alpha := \binom{|\alpha|}{\alpha} \int_{B^d(0,1)} x^\alpha dx$.

The sufficient condition is presented in Theorem 2.3. It asserts that, assuming the analyticity of $w$, every solution $u$ to system (1.8) for $m = \infty$, in the sense that there are infinitely many equations solved by $u$, is strongly harmonic in $(\Omega, d, wdx)$. The assumption of analyticity of the weight function appears here due to the Pizzetti formula, which is used in the proof of Theorem 2.3.

Moreover, as an outcome of the discussion in Chapter 2 we examine regularity of strongly harmonic functions obtaining the following three results: In Proposition 2.18 we show that if $w \in W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ for some $1 < p < \infty$, then $\mathcal{H}(\Omega, d, wdx) \subset W^{1,p}_{loc}(\Omega)$. Then, in the lines of proof of Theorem 2.2, we prove Proposition 2.20 raising regularity of strongly harmonic functions to $W^{2m,2}_{loc}(\Omega)$, whenever $w \in C^{2m-1,1}_{loc}(\Omega)$ for some $m > 1$, $m \in \mathbb{N}$. Finally, in Lemma 2.25 we show that if $w$ is analytic, then every strongly harmonic function is analytic as well.

We apply the aforementioned characterizations in the case of metric $d$ induced by $l^p$-norm for $p \in [1, \infty]$. We observe, that system (1.8) for $p = 2$ and a smooth weight $w$ reads

$$\Delta u \Delta^j w + 2\langle \nabla u, \nabla(\Delta^j w)\rangle = 0 \quad \text{for } j = 0, 1, \ldots$$

Furthermore, in the case of $p \in [1, \infty] \setminus \{2\}$, a constant weight and a planar domain $\Omega \subset \mathbb{R}^2$ we show that $\mathcal{H}(\Omega, l^p, dx)$ consists of 8 linearly independent functions $1, x, y, xy, x^2 - y^2, xy^2 - \frac{x^3}{3}, xy^3 - x^3y, x^2y - \frac{y^3}{3}$.

In Chapter 3 we prove a generalization of Ishiwata–Magnanini–Wadade [IMW17] result to the setting of Carnot groups. More precisely, in Theorem 3.1 we prove that for an open subset $\Omega$ of a Carnot group $\mathbb{G}$, $p \in [1, \infty]$ and a continuous function $u \in C(\Omega)$ the following two conditions are equivalent:

1. $u$ is a viscosity solution to the normalized $p$-Laplace equation $\Delta^N_{p,\mathbb{G}} u = 0$,
2. $u(x) = \mu_p(r, u)(x) + o(r^2)$ as $r \to 0$ in the viscosity sense for every $x \in \Omega$, and $\mu_p$ denotes the $p$-mean.

The proof of Theorem 3.1 relies on an auxiliary result (Lemma 3.15) describing the asymptotic behaviour of a quadratic function on a Carnot group $\mathbb{G}$. We present the proof of Lemma 3.15 in two cases: the Heisenberg group $\mathbb{H}_1$ in Lemma 3.21 and a two-step Carnot group in Lemma 3.22.

In the last chapter we discuss strongly amv-harmonic functions on general metric measure spaces $(X, d, \mu)$. We begin with employing a refined average of a function

$$A^r u(x) := \frac{2}{r} \int_{r/2}^{r} \fint_{B(x,t)} u(y) d\mu(y) dt$$

to prove the regularity of functions with locally finite amv-norm, see Theorem 4.15. The claim is the following: Suppose that $X$ is a complete locally doubling metric measure space, $\Omega \subset X$ is open, $p > 1$ and a function $u \in \mathrm{AMV}^p_{loc}(\Omega)$. Then, $u$ belongs to the fractional Hajłasz–Sobolev space $M^{\alpha,p}_{loc}(\Omega)$ for every $\alpha \in (0,1)$. Upon applying the fractional Morrey embedding we show in Theorem 4.18 that functions with locally finite amv-norm are locally $\alpha$-Hölder for every exponent $\alpha < 1 - Q/p$, where $Q$ is the doubling exponent. Moreover, for strongly amv-harmonic functions we show the local $\alpha$-Hölder regularity for every exponent $\alpha \in (0,1)$. An outcome of the discussion is an improvement in regularity of strongly harmonic functions obtained in [AGG19] formulated in Theorem 4.8, which says that strongly harmonic functions are locally Lipschitz if the underlying space $X$ is complete and locally doubling. Additionally, the space of strongly harmonic functions of polynomial growth is examined following the Yau's finite dimension conjecture and Colding–Minicozzi results. The result is presented in Proposition 4.22 and states that for a fixed $m > 0$ and a complete doubling metric measure space $X$ with the $\alpha$-annular decay property, the space of all strongly harmonic functions for which there exists $C > 0$ and $p \in X$ such that $|u(x)| \leq C(2 + d(p,x))^m$ is of finite dimension. Moreover, in the Heisenberg group $\mathbb{H}_1$ we prove an analogue of the Blaschke–Privaloff–Zaremba theorem, which, in particular, implies the analyticity of strongly amv-harmonic functions on $\mathbb{H}_1$.

In Chapter 4.4 we study blow-ups of functions with finite amv-norm in the sense of Gromov–Hausdorff limits of a rescaling around a fixed point. We develop auxiliary results used in the proof of Theorem 4.41: Let $\Omega \subset X$ be an open subset of a proper locally doubling space $X$, which is additionally a length space. Suppose, that $p \in (1,\infty)$ and $u \in M^{1,p}_{loc}(\Omega) \cap \mathrm{AMV}^p_{loc}(\Omega)$. Then, the tangent function at $\mu$-almost every point $x$ is strongly harmonic on the tangent space at $x$.

Finally, in Chapter 4.5 we focus on the setting of the weighted Euclidean spaces, which is the framework in which we characterize strongly harmonic functions in Chapter 2. Recall, that we consider an open set $\Omega \subset \mathbb{R}^n$ equipped with a norm induced metric and a weighted Lebesgue measure. We divide the discussion into unweighted (i.e. $w \equiv 1$) and weighted case, where we consider weights which are locally Lipschitz and positive in $\Omega$.

In the unweighted case we prove in Proposition 4.47 that for $p \in (1,\infty)$ the space of functions with locally finite amv-norm $\mathrm{AMV}^p_{loc}(\Omega)$ coincides with the Sobolev space $W^{2,p}_{loc}(\Omega)$. Moreover, the $r$-laplacian operator converges to an elliptic operator $\frac{1}{2}\mathrm{div}(M\nabla u)$ in $L^p_{loc}(\Omega)$ as $r \to 0$. The matrix $M = (m_{ij})$ is defined as the matrix of second moments of the Lebesgue measure on the unit ball, i.e. $m_{ij} := \fint_{B^d(0,1)} y_i y_j dy$ for $1 \leq i,j \leq n$.

For the weighted case we prove in Theorem 4.45 that $\mathrm{AMV}^p_{loc}(\Omega_w)$, where $\Omega_w := (\Omega, d, wdx)$, coincides with the Sobolev space $W^{2,p}_{loc}(\Omega)$. Moreover, the $r$-laplacian operator converges to an operator $\frac{1}{2}\mathrm{div}(M\nabla u) + \langle \nabla \ln w, M\nabla u\rangle$ in $L^p_{loc}(\Omega)$ as $r \to 0$.

# Chapter 2

# Strongly harmonic functions in Euclidean domains

## 2.1 Introduction

Let $(X, d, \mu)$ be a metric measure space equipped with a metric $d$ and a measure $\mu$. Fix $x \in X$ and $r > 0$ and denote the open ball by $B(x, r) := \{y \in X : d(x, y) < r\}$. In what follows we will assume that that $\mu$ is a Borel regular measure with $0 < \mu(B) < \infty$ for each ball $B \subset X$. We recall the following class of functions.

**Definition 2.1** (Definition 3.1 in [AGG19])**.** Suppose, that $(X, d, \mu)$ is a metric measure space and $\Omega \subset X$ be an open set. We say that a locally integrable function $u : \Omega \to \mathbb{R}$ is *strongly harmonic* in $\Omega$ if for all balls $B(x, r) \Subset \Omega$ there holds

$$u(x) = \fint_{B(x,r)} u(y)d\mu(y) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(y)d\mu(y).$$

We call a radius $r > 0$ *admissible* at some $x \in \Omega$ whenever $B(x, r) \Subset \Omega$. The space of all strongly harmonic functions in $\Omega$ is denoted by $\mathcal{H}(\Omega, d, \mu)$. In what follows we will omit writing the set, metric or measure whenever they are clear from the context.

The main subject of this chapter is a characterization of strongly harmonic functions on a certain class of metric measure spaces. Namely, we consider an open subset $\Omega \subset \mathbb{R}^n$ equipped with a norm induced metric $d$ and a weighted Lebesgue measure

$$d\mu = wdx, w \in L^1_{loc}(\Omega), w > 0 \text{ a.e.}$$

Bose, Flatto, Friedman, Littman, Zalcman studied the mean value property in the Euclidean setting, see [Bos65; Bos66; Bos68; Fla61; Fla63; Fla65; FL62; Zal73]. We extended their appropriate results with our main result, see Theorem 2.2 below. It generalizes results in [FL62] (see Theorem 2.10 below) and in [Bos68] (see Theorem 2.14 below) in the following ways:

(1) we consider general distance functions induced by a norm, not necessarily the Euclidean one,

(2) we allow more general measures, i.e. the weighted Lebesgue measures $d\mu = wdx$, under the appropriate assumptions on $w$ (see the discussion in Chapter 2.2).

**Theorem 2.2.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let further $(\Omega, d, \mu)$ be a metric measure space equipped with a norm induced metric $d$ and a weighted Lebesgue measure $d\mu = wdx$, $w \in L^1_{loc}(\Omega)$, $w > 0$ a.e. Suppose that there exists $m \in \mathbb{N}$ such that if $m = 1$ then $w \in W^{2,2}_{loc}(\Omega)$, and if $m > 1$ then the* *weight* $w \in C^{2m-1,1}_{loc}(\Omega)$. *Then it holds that every function* $u \in \mathcal{H}(\Omega, d, wdx)$ *is a weak solution to the following system of partial differential equations*

$$\sum_{|\alpha|=j} A_\alpha \left(D^\alpha(uw) - uD^\alpha w\right) = 0, \qquad \text{for } j = 2, 4, \dots, 2m. \tag{2.1}$$

*Coefficients* $A_\alpha$ *are defined as follows:*

$$A_\alpha := \binom{|\alpha|}{\alpha} \int_{B^d(0,1)} x^\alpha dx = \frac{|\alpha|\,!}{\alpha_1! \cdot \ldots \cdot \alpha_n!} \int_{B^d(0,1)} x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} dx,$$

*where* $B^d(0,1)$ *is a unit ball in metric* $d$.

For the sake of simplicity of notation in what follows we will denote $B^d(0,1) = B(0,1)$.

Let us briefly compare Theorem 2.2 for the Euclidean distance $d = l^2$ to Bose's results [Bos65; Bos66; Bos68]. In order to prove the necessary condition (Theorem 2.14 below) for being strongly harmonic, Bose assumes the regularity of weight $w \in C^{m-1}(\Omega)$, whereas our methods for showing Theorem 2.2 require that $w \in C^{m-1,1}(\Omega)$. Nevertheless, if $d = l^2$ we retrieve the same system of PDEs as Bose, however this observation needs additional calculations presented in Chapter 2.6.1. On the other hand, in order to prove the sufficient condition for being strongly harmonic Bose assumes that the weight $w$ is an generalized eigenfunction of the laplacian, see Proposition 2.15. In Theorem 2.3 we assume analyticity of weight $w$ in order to prove the sufficient condition. Our assumption is more general than Bose's, which is illustrated by Lemma 2.24.

In order to prove Theorem 2.2 we need to establish regularity result which is stated as Proposition 2.18. Roughly speaking, Proposition 2.18 shows that if weight $w$ is locally bounded and belongs to the space $W^{1,p}_{loc}$, then all strongly harmonic functions are in $W^{1,p}_{loc}$. The discussion demonstrating the way how Theorem 2.2 generalizes Theorem 2.14 requires computations. We present them after the proof of Theorem 2.2, in Chapter 2.6.1.

Our second main result is the following converse to Theorem 2.2.

**Theorem 2.3.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $(\Omega, d, \mu)$ *be a metric measure space equipped with a norm induced metric* $d$ *and a weighted Lebesgue measure* $d\mu = wdx$. *Suppose that weight* $w$ *is analytic and positive in* $\Omega$. *Then, any solution* $u$ *to system of equations* (2.1) *is strongly harmonic in* $\Omega$.

Another, perhaps most surprising results are presented in Chapter 2.6 where we illustrate Theorem 2.2 with the following observations:

*If* $p \neq 2$ *and* $n = 2$, *then the space* $\mathcal{H}(\Omega, l^p, dx)$ *is spanned by 8 linearly independent harmonic polynomials.*

We already know that for any $n \geq 1$ the space $\mathcal{H}(\Omega, l^2, dx)$ consists of all harmonic functions in $\Omega$, and is infinitely dimensional. The result describing $\dim \mathcal{H}(\Omega, l^p, dx)$ for $p \neq 2$ in dimension $n = 3$ is due to Łysik [Łys18a], who computed it to be equal to 48. The problem for $n > 3$ is open. It is also worthy mentioning here, that the dimensions 8 for $n = 2$ and 48 for $n = 3$ coincide with the number of linear isometries of the normed space $(\mathbb{R}^n, l^p)$, which is $2^n n!$ and is computed in [AB12]. For more information see Chapter 2.6.

In Chapter 2.2 we present a historical background of the topic. We focus on the results by Friedman–Littman [FL62] and Bose [Bos65; Bos66; Bos68]. The fact that Theorem 2.2 and Theorem 2.3 generalize those by Friedman–Littman and Bose is presented in Remark 2.23. In Chapter 2.3 we study regularity of strongly harmonic functions. We prove continuity of strongly harmonic functions in Proposition 2.17 for general weights $w \in L^1_{loc}$ and Sobolev $W^{1,p}$ regularity in Proposition 2.18 for weights $w \in W^{1,p}_{loc} \cap L^\infty_{loc}$. Chapter 2.4 is devoted to proving Theorem 2.2. An additional outcome of the proof is Proposition 2.20, which says that strongly harmonic functions are Sobolev $W^{2m,2}_{loc}$ regular whenever the weight $w \in C^{2m-1,1}_{loc}$ for some natural number $m > 1$. In Chapter 2.5 we discuss the proof of Theorem 2.3 and recall the Pizzetti formula. We show, that if the weight $w$ is analytic, then strongly harmonic functions are analytic as well, see Lemma 2.25.

Finally, in Chapter 2.6 we demonstrate applications of our main results Theorems 2.2 and 2.3 in the case of $\Omega \subset \mathbb{R}^2$, a metric induced by $l^p$-norm for $p \in [1, \infty]$ and weight $w \equiv 1$. We manifest the main difference between cases $p = 2$ and $p \in [1, \infty] \setminus \{2\}$ by calculating the coefficients $A_\alpha$ defined in Theorem 2.2 and explaining how it affects system (2.1). We also write down the system (2.1) for $p = 2$ and an analytic weight $w$ and show that our results are more general than those of Bose.

At the end of introduction to this chapter let us define basic notions and definitions used throughout the thesis.

### 2.1.1 Basic notions and definitions

In this chapter we outline basic notions and definitions used below.

Let $V$ be a linear space over the real numbers. We say, that a function $n : V \to \mathbb{R}$ is a norm on $V$, if it satisfies the following conditions:

1. for every $x \in X$ there holds $n(x) \geq 0$ and $n(x) = 0$ if and only if $x = 0$,
2. for every $x \in X$, $a > 0$ there holds $n(ax) = |a|\, n(x)$,
3. for every $x, y \in X$ there holds $n(x + y) \leq n(x) + n(y)$.

The following notion of a distance function generalizes the notion of a norm. We call a pair $(X, d)$ *metric space*, if the *distance function* $d : X \times X \to \mathbb{R}$ satisfies the following conditions:

1. for every $x, y \in X$ there holds $d(x, y) \geq 0$ and the equality $d(x, y) = 0$ holds if and only if $x = y$.
2. for every $x, y \in X$ there holds $d(x, y) = d(y, x)$,
3. for every $x, y, z \in X$ there holds $d(x, z) \leq d(x, y) + d(y, z)$.

Recall, that every norm $n$ on $V$ induces a metric $d$ on $V$ via the relation $d(x, y) := n(x - y)$. If $X \subset V$, then we say that $d$ is a *norm induced metric*, if there exists a norm $n$ on $V$ such that for every $x, y \in X$ there holds $n(x - y) = d(x, y)$.

Throughout this work we use the multi-index notation: $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Moreover, for two multi-indices $\alpha, \beta \in \mathbb{N}^n$ we say that $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, \ldots, n$ and $\beta < \alpha$ if $\beta \leq \alpha$ and there exists $i = 1, \ldots, n$ such that $\beta_i < \alpha_i$. For $\beta \leq \alpha$ we will write that $\binom{\alpha}{\beta} := \frac{\alpha_1! \cdot \ldots \cdot \alpha_n!}{\beta_1! \cdot \ldots \cdot \beta_n!} = \frac{\alpha!}{\beta!}$ and for $k \in \mathbb{N}$ we denote $\binom{k}{\beta} = \frac{k!}{\beta!}$. For more information see Appendix A in the Evans' book [Eva98].

Next, let us consider a function $f : \mathbb{R}^n \to \mathbb{R}$. For $x, h \in \mathbb{R}^n$ we define the *difference quotient* of $f$ at $x$ as follows

$$\Delta_h f(x) := \frac{f(x + h) - f(x)}{|h|}.$$

We use difference quotients to prove regularity of strongly harmonic functions in Proposition 2.18. Therefore, we present below a characterization of Sobolev functions via difference quotients.

**Theorem 2.4** (Theorem 3, p. 277 in [Eva98])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set.*

1. *Suppose that $1 \leq p < \infty$, $f \in W^{1,p}(\Omega)$. Then for each $K \Subset \Omega$*

$$\|\Delta_h f\|_{L^p(K)} \leq C \,\|\nabla f\|_{L^p(\Omega)},$$

*for some constant $C > 0$ and all $h \in \mathbb{R}^n$, $0 < 2\,|h| < \operatorname{dist}(K, \partial\Omega)$.*

2. *Suppose that $1 < p < \infty$, $K \Subset \Omega$, function $f \in L^p(K)$ and there exists constant $C > 0$ such that*

$$\|\Delta_h f\|_{L^p(K)} \leq C$$

*for all $h \in \mathbb{R}^n$, $0 < 2\,|h| < \operatorname{dist}(K, \partial\Omega)$. Then $f \in W^{1,p}(K)$.*

Let us recall, that the Fourier transform of a function $u \in L^1(\mathbb{R}^n)$ is defined in the following way

$$\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^n} u(y) e^{-i\langle \xi, y\rangle} dy.$$

For every function $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ its Fourier transform $\mathcal{F}(f) \in L^2(\mathbb{R}^n)$ and for every pair of functions $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ the Parseval identity holds true

$$\int_{\mathbb{R}^n} f(x)g(x)dx = \int_{\mathbb{R}^n} \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)}d\xi.$$

Moreover, if $f \in W^{k,2}(\mathbb{R}^n)$, then for every multi-index $\alpha$ with $|\alpha| \leq k$ there holds

$$(i\xi)^\alpha \mathcal{F}(f)(\xi) = \mathcal{F}(D^\alpha f)(\xi).$$

## 2.2 Historical background

Properties of strongly and weakly harmonic functions were broadly studied in [GG09; AGG19] and in [AW20] in the setting of Carnot groups. Below, we list out some of those properties especially important for further considerations.

**Proposition 2.5** (Proposition 4.1 in [AGG19])**.** *Suppose that measure $\mu$ is* continuous with respect to metric *$d$, i.e. for all $r > 0$ and $x \in X$ there holds $\lim_{d(x,y)\to 0} \mu(B(x,r) \triangle B(y,r)) = 0$, where we denote by $E \triangle F := (E \setminus F) \cup (E \setminus F)$ the symmetric difference of $E$ and $F$. Then $\mathcal{H}(\Omega, d, \mu) \subset C(\Omega)$.*

Moreover, the Harnack inequality and the strong maximum principle hold for strongly harmonic functions as well as the local Hölder continuity and even local Lipschitz continuity under more involved assumptions, see [AGG19] and Theorem 4.8. It is important to mention here that similar type of problems were studied for a more general, nonlinear mean value property by Manfredi–Parvainen–Rossi, Arroyo–Llorente and Ferrari–Pinamonti, see [MPR13; Llo15; AL18; AL16b; FP15].

We know that $\mathcal{H}$ is a linear space, but verifying by using the definition whether some function satisfies the mean value property might be a complicated computational challenge. From that comes the need for finding a handy characterization of class $\mathcal{H}$, or some necessary and sufficient conditions for being strongly harmonic.

In what follows we are interested in extending results by Flatto [Fla61; Fla63], Friedman–Litmann [FL62], Bose [Bos65; Bos66; Bos68] and Zalcman [Zal73]. Below, we briefly discuss these results. According to our best knowledge, the investigation in this area originate from a work by Flatto [Fla61]. He considered functions with the following property:

Let us fix an open set $\Omega \subset \mathbb{R}^n$ and a bounded set $K \subset \mathbb{R}^n$. Moreover, let $\mu$ be a probabilistic measure on $K$ such that all continuous functions on $K$ are $\mu$-measurable and for all hyperplanes $V \subset \mathbb{R}^n$ it holds that $\mu(K \cap V) < 1$, i.e. $\mu$ is not concentrated on a hyperplane. We will say that a continuous function $u \in C(\Omega)$ has the mean value property in the sense of Flatto, if

$$u(x) = \int_K u(x + ry)d\mu(y) \tag{2.2}$$

for all $x \in \Omega$ and radii $r > 0$ such that $x + r \cdot K := \{x + ry : y \in K\} \subset \Omega$. Let us observe that for $K = B(0,1)$ a unit ball in a given norm induced metric $d$ and $\mu$ being the normalized Lebesgue measure on $K$ (the latter meaning that $d\mu = \frac{1}{|K|}dx$), property (2.2) is equivalent to the strong harmonicity of $u$ in $\Omega$ by the following formula

$$u(x) = \fint_{B(x,r)} u(z)dz = \fint_{B(0,1)} u(x + ry)dy = \int_K u(x + ry)d\mu(y). \tag{2.3}$$

This holds exactly for homogeneous and translation invariant metrics, because only then

$$B(x,r) = x + r \cdot B(0,1) = \{x + ry : y \in B(0,1)\}.$$

For such distance functions one can obtain any ball $B(x,r)$ from $B(0,1)$ by using the change of variables $y = \frac{z-x}{r}$. In relation to homogeneous and translation invariant distance let us recall the following lemma, which is likely a part of the mathematical folklore. However, in what follows we will not appeal to this observation.

**Lemma 2.6.** *If $d$ is a translation invariant and homogeneous metric on $\mathbb{R}^n$, then there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that for all $x, y \in \Omega$ there holds that $d(x,y) = \|x - y\|$.*

We recall also a characterization of all such metrics on $\mathbb{R}^n$ by using the Minkowski functional, see [Sch14a]. Recall, that a set $K \subset \mathbb{R}^n$ is *symmetric* if $-y \in K$ for every $y \in K$. For any nonempty convex set $K$ we consider the Minkowski functional.

**Lemma 2.7** (p.54 in [Sch14a])**.** *Suppose that $K$ is a symmetric convex bounded subset of $\mathbb{R}^n$, containing the origin as an interior point. Then, its Minkowski functional $n_K$ defines a norm on $\mathbb{R}^n$. Moreover, if $\|\cdot\|$ is a norm on $\mathbb{R}^n$, then the Minkowski functional $n_K$, where $K$ is a unit ball with respect to $\|\cdot\|$, is equal to that norm.*

**Example 2.8.** Among many examples of norm induced metrics on $\mathbb{R}^n$ are $l^p$ distances for $1 \leq p \leq \infty$. Moreover, let us fix numbers $a_i > 0$ for $i = 1, \ldots, n$, set $a := (a_1, \ldots, a_n)$ and $1 \leq p < \infty$ and define

$$\|x\|_p^a := \left( \sum_{i=1}^{n} \left( \frac{|x_i|}{a_i} \right)^p \right)^{\frac{1}{p}}.$$

In case $p = 2$ all balls with respect to $\|\cdot\|_p^a$ are ellipsoids with the length of semi-axes equal to $a_i$ in $x_i$'s axes direction respectively.

**Remark 2.9.** Let us observe that by Lemma 2.7 there is the injective correspondence between norms on $\mathbb{R}^n$ and a class of all symmetric convex open bounded subsets $K$ of $\mathbb{R}^n$. More specifically, every $K$ defines a norm on $\mathbb{R}^n$ through the Minkowski functional and vice versa, given a norm on $\mathbb{R}^n$ the unit ball $B(0,1)$ is a symmetric convex open bounded set, hence provides an example of $K$. This can be expressed in one more way, namely that all norms can be distinguished by their unit balls, so to construct a norm we only need to say what is its unit ball. Therefore, further examples of norms can be constructed for any $n$-dimensional symmetric convex polyhedron $K$. All balls with respect to $n_K$ will be translated and dilated copies of $K$.

The formula (2.3) is true only if the measure of a ball scales with the $n$-th power of its radius, the same which appears in the Jacobian from the change of variables formula $z = x + ry$. This is true only for measures which are constant multiples of the Lebesgue measure. Note that (2.2) does not coincide, in general, with the mean value property presented in our work, since the Flatto's mean value is calculated always with respect to the same fixed reference set $K$ and measure $\mu$, whose support is being shifted and scaled over $\Omega$. Whereas, in Definition 2.1 the measure is defined on the whole space, and as $x$ and $r$ vary, the mean value is calculated with respect to different weighted measures. Indeed, in order to see that this case is not covered by the Flatto's (2.2), let us rewrite the condition from Definition 2.1 in the following way

$$u(x) = \int_X u(y) \frac{d\mu|_{B(x,r)}}{\mu(B(x,r))}.$$

This mean value property cannot be written as an integral with respect to one fixed measure for different pairs of $x$ and $r$, even when (2.3) holds.

Flatto discovered that functions satisfying (2.2) are solutions to a second order elliptic equation, see [Fla61]. However, from the point of view of our discussion, more relevant is the following later result.

**Theorem 2.10** (Friedman–Littman, Theorem 1 in [FL62])**.** *Suppose that u has property* (2.2) *in* $\Omega \subset \mathbb{R}^n$*. Then u is analytic in* $\Omega$ *and satisfies the following system of partial differential equations*

$$\sum_{|\alpha|=j} A_\alpha D^\alpha u = 0 \qquad \text{for } j = 1, 2, \ldots \tag{2.4}$$

*The coefficients* $A_\alpha$ *are moments of measure* $\mu$ *and are defined by* $A_\alpha := \binom{|\alpha|}{\alpha} \int_K x^\alpha d\mu(x)$*. Moreover, any function* $u \in C^\infty(\Omega)$ *solving system* (2.4) *is analytic and has property* (2.2)*.*

**Remark 2.11.** Theorem gives full characterization of $\mathcal{H}(\Omega, d)$ for $d$ being induced by a norm. Theorem 3.1 in [Fla61] states that all functions having property (2.2) are harmonic with respect to variables obtained from $x$ by using an orthogonal transformation and dilations along the axes of the coordinate system. On the other hand the proof of Theorem 2.10 shows that the equation in system (2.4) corresponding to $j = 2$ is always elliptic with constant coefficients from which the analyticity follows.

Flatto as well as Friedman and Littman described in their works the space of functions possessing property (2.2). We present appropriate results below.

**Proposition 2.12** (Friedman–Littman, Theorem 2 in [FL62])**.** *The space of solutions to system* (2.4) *is finitely dimensional if and only if the system of algebraic equations* $\sum_{|\alpha|=j} A_\alpha z^\alpha = 0$ *for* $j = 1, 2, \ldots$ *has the unique solution* $z = (z_1, \ldots, z_n) = 0$*, where* $z_i \in \mathbb{C}$*.*

**Remark 2.13.** From the proof of Proposition 2.12 it follows that if there exists a nonpolynomial solution to (2.4), then the solution space is infinitely dimensional. If the dimension is finite, then all strongly harmonic functions are polynomials.

A rather different approach to the mean value property and its consequences was studied by Bose, see [Bos65; Bos66; Bos68]. He considered strongly harmonic functions on $\Omega \subset \mathbb{R}^n$ equipped with non-negatively weighted measure $\mu = w dx$, for a weight $w \in L^1_{loc}(\Omega)$ being a.e. positive in $\Omega$ and only a metric $d$ induced by the $l_2$-norm. Under the higher regularity assumption of weight $w$, Bose proved the following result.

**Theorem 2.14** (Bose, Theorem 1 in [Bos68])**.** *If there exists* $m \in \mathbb{N}$ *such that* $w \in C^{2m+1}(\Omega)$*, then every* $u \in \mathcal{H}(\Omega, w)$ *solves the following system of partial differential equations*

$$\Delta u \Delta^j w + 2\langle \nabla u, \nabla \left(\Delta^j w\right)\rangle = 0, \qquad \text{for } j = 0, 1, \ldots, m, \tag{2.5}$$

*where* $\Delta^j$ *stands for the* $j$*th composition of the Laplace operator* $\Delta$ *with* $\Delta^0 w \equiv w$*. If* $w$ *is smooth, then equations* (2.5) *hold true for all* $j \in \mathbb{N}$*.*

The converse is not true for smooth weights in general, see counterexamples on p. 479 in [Bos65]. Furthermore, Bose proved in [Bos68] the following result, by imposing further assumptions on $w$.

**Proposition 2.15** (Bose, Theorem 2 in [Bos68])**.** *Let* $w \in C^{2m}(\Omega)$ *for some* $m \in \mathbb{N}$*,* $m \geq 1$*. Suppose that there exist* $a_0, \ldots, a_{m-1} \in \mathbb{R}$ *such that*

$$\Delta^m w = a_0 w + a_1 \Delta w + \ldots + a_{m-1} \Delta^{m-1} w.$$

*Then any* $C^2$ *solution* $u$ *to* (2.5) *for all* $j = 0, 1, \ldots, m-1$ *is strongly harmonic, i.e.* $u \in \mathcal{H}(\Omega, w)$*.*

The following result by Bose contributes to the studies of the dimension of the space $\mathcal{H}(\Omega, l^2, w)$ under certain additional assumption on the weight (in particular, assuming that $w$ is an eigenfunction for the laplacian).

**Proposition 2.16** (Bose, Corollary 2 in [Bos65])**.** *Suppose that* $\Omega \subset \mathbb{R}^n$ *for* $n > 1$*,* $w \in C^2(\Omega)$ *and there exists* $\lambda \in \mathbb{R}$ *such that* $\Delta w = \lambda w$*. Then* $\dim \mathcal{H}(\Omega, w) = \infty$*.*

## 2.3 Regularity of strongly harmonic functions in the weighted case

In order to prove Theorem 2.2 we need to establish regularity of strongly harmonic functions, see Proposition 2.18. There, we show the Sobolev regularity for functions in $\mathcal{H}(\Omega, d, wdx)$ depending on the Sobolev regularity of weight $w$.

*From now on we a priori assume that a function $w \in L^1_{loc}(\Omega)$ and $w > 0$ almost everywhere in $\Omega$.*

Let us begin with noting that strongly harmonic functions in such setting are continuous.

**Proposition 2.17.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then $\mathcal{H}(\Omega, d, wdx) \subset C(\Omega)$.*

*Proof.* Observe that $\mu(\partial B(x,r)) = \int_{\partial B(x,r)} w(y)dy = 0$. Therefore, by Lemma 2.1 from [GG09] measure $\mu$ is continuous with respect to metric. This completes the proof by Proposition 2.5. □

Let us observe that the proof of continuity of strongly harmonic functions works for all weights $w \in L^1_{loc}(\Omega)$. However, in order to show existence and integrability of weak derivatives we need to assume Sobolev regularity of $w$.

**Proposition 2.18.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $d$ be a norm induced metric and a weight $w \in W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ for some $1 < p < \infty$. Then $\mathcal{H}(\Omega, d, wdx) \subset W^{1,p}_{loc}(\Omega)$.*

Before we present the proof of Proposition 2.18 let us comment on the

**Remark 2.19.** The necessity of the assumption on regularity of the weight $w$ in Proposition 2.18 is not settled. Notice, that the space $\mathcal{H}(\Omega, d, wdx)$ always contains constant functions. When considering examples of weights $w$ which are neither weakly differentiable nor bounded, the space $\mathcal{H}(\Omega, d, wdx)$ turns out to consist of only constant functions. Nevertheless, we did not find any example of a weight, for which there would exist some strongly harmonic non-differentiable functions.

*Proof.* Fix a compact set $K \Subset \Omega$. Moreover, let $r = \frac{1}{4}\mathrm{dist}(K, \partial\Omega)$. Fix $h \in \mathbb{R}^n$ with $|h| < r$. Denote by $K' := \{z \in \Omega : \mathrm{dist}(z, K) \leq 2r\}$. Let us observe that due to the first assertion of Lemma 2.1 in [AGG19], i.e. that continuity of $\mu$ with respect to $d$ implies that the map $x \mapsto \mu(B(x,r))$ is continuous in $d$, there exists $0 < M := \inf_{x \in K'} \mu(B(x,r))$. The difference quotient of $u$ at $x \in K$ reads

$$|\Delta_h u(x)| = \frac{|u(x+h) - u(x)|}{|h|} = \frac{1}{|h|}\left| \frac{\int_{B(x+h,r)} uw}{\int_{B(x+h,r)} w} - \frac{\int_{B(x,r)} uw}{\int_{B(x,r)} w} \right|,$$

where we used the mean value property of $u \in \mathcal{H}(\Omega, d, wdx)$. Now we add and subtract a term $\frac{\int_{B(x,r)} uw}{\int_{B(x+h,r)} w}$ and use the triangle inequality to get

$$|h|\, |\Delta_h u(x)| \leq \left| \frac{\int_{B(x+h,r)} uw}{\int_{B(x+h,r)} w} - \frac{\int_{B(x,r)} uw}{\int_{B(x+h,r)} w} \right| + \left| \frac{\int_{B(x,r)} uw}{\int_{B(x+h,r)} w} - \frac{\int_{B(x,r)} uw}{\int_{B(x,r)} w} \right|. \tag{2.6}$$

The first term can be estimated as follows

$$\left| \frac{\int_{B(x+h,r)} uw - \int_{B(x,r)} uw}{\int_{B(x+h,r)} w} \right| = \frac{1}{\int_{B(x+h,r)} w} \left| \int_{B(x+h,r)} uw - \int_{B(x,r)} uw \right| \leq \frac{1}{M} \int_{B(x+h,r) \triangle B(x,r)} |uw|$$
$$\leq \frac{\|uw\|_{L^\infty(K')}}{M} |B(x+h,r) \triangle B(x,r)|. \tag{2.7}$$

In order to manage this term we refer to Theorem 3 in [Sch14b] to get that

$$|B(x+h,r) \triangle B(x,r)| \leq |h|\, |\partial B(x,r)| = |h| c_{n,d} r^{n-1}, \tag{2.8}$$

where in the last term the constant $c_{n,d}$ stands for the $(n-1)$-dimensional Lebesgue measure of the unit sphere with respect to the metric $d$. Computation of $c_{n,d}$ is highly nontrivial and for a general distance $d$ we only know that $0 < c_{n,d} < \infty$.

The second term of (2.6) reads

$$\begin{aligned}
&\left| \frac{\int_{B(x,r)} u(y)w(y)}{\int_{B(x+h,r)} w(y)} - \frac{\int_{B(x,r)} u(y)w(y)}{\int_{B(x,r)} w(y)} \right| \\
&\leq \frac{\int_{B(x,r)} |uw|(y)}{\int_{B(x+h,r)} w(y) \int_{B(x,r)} w(y)} \left| \int_{B(x+h,r)} w(y)dy - \int_{B(x,r)} w(y)dy \right| \\
&\leq \frac{\|uw\|_{L^\infty(K')} |B(x,r)|}{M^2} \left| \int_{B(x,r)} (w(y+h) - w(y))dy \right| \\
&\leq \frac{\|uw\|_{L^\infty(K')} C_{n,d} r^n}{M^2} \int_{B(x,r)} |\Delta_h w(y)|\, dy, \qquad (2.9)
\end{aligned}$$

where in the second inequality we used the translation invariance of the metric $d$ and by $C_{n,d} := |B(0,1)|$ we denote the $n$-dimensional Lebesgue measure of the unit ball with respect to the metric $d$. By gathering together estimates of both terms (2.6), (2.7), (2.8), (2.9) and applying the standard inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$ we obtain the following

$$\int_K |\Delta_h u(x)|^p\, dx \leq 2^{p-1} \|uw\|^p_{L^\infty(K')} \int_K \left[ \frac{c_n^p r^{p(n-1)}}{M^p} + \frac{C^p_{n,d} r^{pn}}{M^{2p}} \left( \int_{B(x,r)} |\Delta_h w(y)|\, dy \right)^p \right] dx.$$

The first term above is bounded, therefore we only need to take care of the second one. For the sake of simplicity we omit writing the constant $2^{p-1} M^{-2p} \|uw\|^p_{L^\infty(K')} C^p_{n,d} r^{pn}$. Upon applying the Jensen inequality and Theorem 2.4 (for $\Omega = K'$) the following estimate holds true

$$\begin{aligned}
\left( \int_{B(x,r)} |\Delta_h w(y)|\, dy \right)^p dydx &\leq (C_{n,d} r^n)^{p-1} \int_K \int_{B(x,r)} |\Delta_h w(y)|^p\, dydx \\
&\leq C C^{p-1}_{n,d} r^{n(p-1)} |K| \, \|\nabla w\|^p_{L^p(K')}.
\end{aligned}$$

This integral is finite by the assumptions on regularity of $w$ and Theorem 2.4 applied to weight $w$ with an observation that $w \in W^{1,p}(K')$. Hence, the following estimate holds true

$$\int_K |\Delta_h u(x)|^p\, dx \leq 2^{p-1} \|uw\|^p_{L^\infty(K')} |K| \left( \frac{c_n^p r^{p(n-1)}}{M^p} + \frac{C C^{2p-1}_{n,d} r^{n(2p-1)} \|\nabla w\|^p_{L^p(K')}}{M^{2p}} \right) < \infty.$$

We apply Theorem 2.4 to $u$ and obtain that $u \in W^{1,p}(K)$, which completes the proof. □

We are now in a position to present the proof of Theorem 2.2.

## 2.4 Proof of Theorem 2.2

Before we present the proof of Theorem 2.2 let us discuss the equations of system (2.1). First of all, by Remark 2.9 we know that $B(0,1)$ is symmetric with respect to the origin. If $|\alpha|$ is an odd number, then $x^\alpha$ is an odd function, hence $A_\alpha = 0$. Therefore only evenly indexed equations of (2.1) are nontrivial, although we will prove them for all $j \leq 2l$. In fact, the proof of Theorem 2.2 can be applied to functions with the mean value property over any compact set $K \subset \mathbb{R}^n$, which does not necessarily need to be a unit ball with respect to a norm on $\mathbb{R}^n$, i.e. to functions with the following property

$$u(x) = \frac{1}{\int_K w(x+ry)dy} \int_K u(x+ry)w(x+ry)dy,$$

which holds for all $x \in \Omega$ and radii $0 < r$ such that $x + rK \subset \Omega$. In that case in the analogue of system (2.1) appear also equations with odd indices.

If the unit ball is symmetric with respect to all coordinate axes, the coefficient $A_\alpha$ is zero whenever some $\alpha_i$ is odd. Therefore, in the $j$-th equation of (2.1) occur only differential operators acting evenly on each of variables. Examples of norms for which $B(0,1)$ is symmetric with respect to all coordinate axes include the $l^p$ norms for $p \in [1, \infty]$, but also by Lemma 2.7 one can produce more examples.

*Proof of Theorem 2.2.* Let $\Omega \subset \mathbb{R}^n$ be an open set, metric space $(\Omega, d, wdx)$ be as in assumptions of Theorem 2.2, $m \in \mathbb{N}$. Then, if $m = 1$ then $w \in W^{2,2}_{loc}(\Omega)$ and if $m > 1$ then $w \in C^{2m-1,1}_{loc}(\Omega)$. Then, following the reasoning of (2.3), for $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$ there holds

$$u(x) \int\limits_{B(x,r)} w(y)dy = u(x) \int\limits_{B(0,1)} w(x+ry)r^n dy = \int\limits_{B(0,1)} u(x+ry)w(x+ry)r^n dy = \int\limits_{B(x,r)} u(y)w(y)dy,$$

where the middle equality holds true by the mean value property of $u$. Without the loss of generality we may assume that $B(0,1) := \{x : d(x,0) < 1\}$ satisfies $B(0,1) \subset \{x : \|x\|_2 \leq 1\}$, since we will consider only small enough admissible radii in the mean value property. The assertion is a local property, therefore we may restrict our considerations to the analysis of the behaviour of $u$ on a ball $B' \subset \Omega$ with $\operatorname{dist}(B', \partial\Omega) = \varepsilon > 0$ for some fixed $\varepsilon > 0$. Furthermore, let $B$ be a ball concentric with $B'$ with $2\varepsilon$ distance from $\partial\Omega$. We redefine $u$ and $w$ in the following way

$$\bar{u}(x) = u(x)\chi_{B'}(x) \qquad \bar{w}(x) = w(x)\chi_{B'}(x).$$

The function $\bar{u}$ is continuous in $B$ and if $m = 1$ then the weight $\bar{w}$ is in the space $W^{2,2}(B)$ since $B \Subset \Omega$. Analogously if $m > 1$ then $w \in C^{2m-1,1}(B)$. Let $\varphi \in C_0^\infty(B)$. Then for all $x \in B$, $y \in B(0,1)$ and $0 < r < \varepsilon$ it holds $u(x + ry) = \bar{u}(x + ry)$. Since $\varphi(x) = 0$ outside of $B$ we have that for all $x \in \mathbb{R}^n$ there holds

$$\bar{u}(x)\varphi(x) \int\limits_{B(0,1)} \bar{w}(x + ry)dy = \varphi(x) \int\limits_{B(0,1)} \bar{u}(x + ry)\bar{w}(x + ry)dy. \tag{2.10}$$

For the sake of simplicity below we still use symbols $u$ and $w$ to denote $\bar{u}$ and $\bar{w}$, respectively. We integrate both sides of (2.10) with respect to $x \in \mathbb{R}^n$ to obtain

$$\int\limits_{\mathbb{R}^n} u(x)\varphi(x)\Bigg( \int\limits_{B(0,1)} w(x + ry)dy\Bigg)dx = \int\limits_{\mathbb{R}^n} \varphi(x)\Bigg( \int\limits_{B(0,1)} u(x + ry)w(x + ry)dy\Bigg)dx. \tag{2.11}$$

Observe, that the Fourier transform of functions

$$\varphi(x)u(x), \quad \int_{B(0,1)} w(x + ry)dy, \quad \int_{B(0,1)} u(x + ry)w(x + ry)dy$$

exist and the latter two are $L^2(\mathbb{R}^n)$ integrable in variable $x$. Therefore, we apply the Parseval identity in (2.11) and obtain

$$\int\limits_{\mathbb{R}^n} \overline{\mathcal{F}(\varphi u)(\xi)}\mathcal{F}\Bigg( \int\limits_{B(0,1)} w(\cdot + ry)dy\Bigg)(\xi)d\xi = \int\limits_{\mathbb{R}^n} \overline{\mathcal{F}(\varphi)(\xi)}\mathcal{F}\Bigg( \int\limits_{B(0,1)} u(\cdot + ry)w(\cdot + ry)dy\Bigg)(\xi)d\xi. \tag{2.12}$$

Here $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} e^{-i\langle \xi, y\rangle} f(y)dy$ stands for the Fourier transform of $f$ at $\xi \in \mathbb{R}^n$. The following formula holds for any $f \in L^1_{loc}(\Omega)$:

$$\mathcal{F}\left( \int_{B(0,1)} f(\cdot + ry)dy \right)(\xi) = \mathcal{F}(f)(\xi) \int_{B(0,1)} e^{ir\langle y, \xi\rangle} dy. \tag{2.13}$$

Indeed, upon applying the Fubini Theorem and the change of variables $z := x + ry$ we obtain

$$\begin{aligned}\mathcal{F}\left(\int_{B(0,1)} f(\cdot + ry)dy\right)(\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\left(\int_{B(0,1)} f(x+ry)dy\right)dx \\ &= \int_{B(0,1)}\left(\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x+ry)dx\right)dy \\ &= \int_{B(0,1)} e^{i\langle ry,\xi\rangle}\left(\int_{\mathbb{R}^n} e^{-i\langle x+ry,\xi\rangle} f(x+ry)dx\right)dy \\ &= \int_{B(0,1)} e^{ir\langle y,\xi\rangle}\left(\int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle} f(z)dz\right)dy \\ &= \int_{B(0,1)} e^{ir\langle y,\xi\rangle}\mathcal{F}(f)(\xi)dy \\ &= \mathcal{F}(f)(\xi)\int_{B(0,1)} e^{ir\langle y,\xi\rangle}dy.\end{aligned}$$

We apply formula (2.13) twice: for $f = w$ and $f = uw$ and employ respectively to the left- and the right-hand side in (2.12) to arrive at the following identity:

$$\int\limits_{\mathbb{R}^n}\overline{\mathcal{F}(\varphi u)(\xi)}\mathcal{F}(w)(\xi)\Bigg(\int\limits_{B(0,1)} e^{ir\langle y,\xi\rangle}dy\Bigg)d\xi = \int\limits_{\mathbb{R}^n}\overline{\mathcal{F}(\varphi)(\xi)}\mathcal{F}(uw)(\xi)\Bigg(\int\limits_{B(0,1)} e^{ir\langle y,\xi\rangle}dy\Bigg)d\xi. \quad (2.14)$$

Let us observe that both sides of (2.14) are smooth functions when considered with respect to $r$ and this allows us to calculate the appropriate derivatives by differentiating under the integral sign. Namely, we differentiate (2.14) with respect to $r$ by $j$ times ($j \le 2l$):

$$\int\limits_{\mathbb{R}^n}\overline{\mathcal{F}(\varphi u)(\xi)}\mathcal{F}(w)(\xi)\Bigg(\int\limits_{B(0,1)}(i\langle\xi,y\rangle)^j e^{ir\langle y,\xi\rangle}dy\Bigg)d\xi = \int\limits_{\mathbb{R}^n}\overline{\mathcal{F}(\varphi)(\xi)}\mathcal{F}(uw)(\xi)\Bigg(\int\limits_{B(0,1)}(i\langle\xi,y\rangle)^j e^{ir\langle y,\xi\rangle}dy\Bigg)d\xi.$$

For $r = 0$ this identity reads

$$\int\limits_{\mathbb{R}^n} i^j\overline{\mathcal{F}(\varphi u)(\xi)}\mathcal{F}(w)(\xi)\Bigg(\int\limits_{B(0,1)}\langle\xi,y\rangle^j dy\Bigg)d\xi = \int\limits_{\mathbb{R}^n} i^j\overline{\mathcal{F}(\varphi)(\xi)}\mathcal{F}(uw)(\xi)\Bigg(\int\limits_{B(0,1)}\langle\xi,y\rangle^j dy\Bigg)d\xi. \quad (2.15)$$

Note that

$$\int\limits_{B(0,1)}\langle\xi,y\rangle^j dy = \int\limits_{B(0,1)}(\xi_1 y_1 + \ldots + \xi_n y_n)^j dy = \int\limits_{B(0,1)}\sum_{|\alpha|=j}\binom{|\alpha|}{\alpha}\xi^\alpha y^\alpha dy = \sum_{|\alpha|=j} A_\alpha \xi^\alpha. \quad (2.16)$$

Using the above observations, equation (2.15) transforms to

$$\int\limits_{\mathbb{R}^n}\sum_{|\alpha|=j} A_\alpha (i\xi)^\alpha\overline{\mathcal{F}(\varphi u)(\xi)}\mathcal{F}(w)(\xi)d\xi = \int\limits_{\mathbb{R}^n}\sum_{|\alpha|=j} A_\alpha (i\xi)^\alpha\overline{\mathcal{F}(\varphi)(\xi)}\mathcal{F}(uw)(\xi)d\xi. \quad (2.17)$$

Let us focus on equation (2.17) for $j = 2$:

$$\int\limits_{\mathbb{R}^n}\sum_{|\alpha|=2} A_\alpha (i\xi)^\alpha\overline{\mathcal{F}(\varphi u)(\xi)}\mathcal{F}(w)(\xi)d\xi = \int\limits_{\mathbb{R}^n}\sum_{|\alpha|=2} A_\alpha (i\xi)^\alpha\overline{\mathcal{F}(\varphi)(\xi)}\mathcal{F}(uw)(\xi)d\xi,$$

which can be rewritten in the following way

$$\int\limits_{\mathbb{R}^n}\sum_{|\alpha|=2} A_\alpha\overline{\mathcal{F}(\varphi u)(\xi)}\cdot(i\xi)^\alpha\mathcal{F}(w)(\xi)d\xi = \int\limits_{\mathbb{R}^n}\sum_{|\alpha|=2} A_\alpha\overline{(i\xi)^\alpha\mathcal{F}(\varphi)(\xi)}\cdot\mathcal{F}(uw)(\xi)d\xi.$$

Hence

$$\int\limits_{\mathbb{R}^n} \sum_{|\alpha|=2} A_\alpha \overline{\mathcal{F}(\varphi u)(\xi)} \cdot \mathcal{F}(D^\alpha w)(\xi) d\xi = \int\limits_{\mathbb{R}^n} \sum_{|\alpha|=2} A_\alpha \overline{\mathcal{F}(D^\alpha \varphi)(\xi)} \cdot \mathcal{F}(uw)(\xi) d\xi. \tag{2.18}$$

We apply the Parseval identity in (2.18) and move the expression on the left-hand side to the right-hand side to arrive at

$$\int\limits_{\mathbb{R}^n} \sum_{|\alpha|=2} A_\alpha \left( D^\alpha \varphi(x) \cdot u(x) w(x) - \varphi(x) u(x) D^\alpha w(x) \right) dx = 0. \tag{2.19}$$

Notice, that (2.19) is a weak formulation in $B$ of the equation

$$Lu(x) := \sum_{|\alpha|=2} A_\alpha \Big( D^\alpha (uw)(x) - u(x) D^\alpha w(x) \Big) = 0, \tag{2.20}$$

where the operator $L$ is defined by (2.20) and in the case $m = 1$ this observation ends the proof.

If $m > 1$, then in order to complete the proof we need to etablish higher regularity for function $u$. We intend to employ Theorem 8.10 in [GT01]. Let us observe, that by Proposition 2.18 function $u \in W^{1,2}_{loc}(\Omega)$. The following holds true

$$\begin{aligned} Lu(x) &= \sum_{|\alpha|=2} A_\alpha \Big( D^\alpha (uw)(x) - u(x) D^\alpha w(x) \Big) \\ &= \sum_{|\alpha|=2} A_\alpha w(x) D^\alpha u(x) + \sum_{|\alpha|=2} \sum_{\substack{\beta < \alpha \\ |\beta|=1}} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} w(x). \end{aligned}$$

Observe, that by the hypothesis on the regularity of $w$, coefficients appearing in the operator $L$ are in $C^{2m-2,1}(\Omega)$. Moreover, $L$ is strongly elliptic: Indeed, take $\xi \in \mathbb{R}^n$ and consider the second order terms of $L$. By (2.16) we obtain for all $y \in \Omega$ that

$$\sum_{|\alpha|=2} A_\alpha w(y) \xi^\alpha = w(y) \int_{B(0,1)} \langle x, \xi \rangle^2 dx \geq w(y) \int_{\|x\|_2 \leq \varepsilon} \langle x, \xi \rangle^2 dx, \tag{2.21}$$

where the last estimate holds with some $\varepsilon > 0$ since $d$ is equivalent to the Euclidean distance, as every norm on $\mathbb{R}^n$. Next, observe that

$$\int_{\|x\|_2 \leq \varepsilon} \langle x, \xi \rangle^2 dx = \|\xi\|_2^2 \int_{\|x\|_2 \leq \varepsilon} \left\langle x, \frac{\xi}{\|\xi\|_2} \right\rangle^2 dx = \theta \|\xi\|_2^2,$$

where $\theta > 0$ is defined by the above equality and does not depend on $\xi$ due to the symmetry of the Euclidean ball. Indeed, let us apply the change of variables $z = Rx$, where $R$ is a rotation matrix such that $R^T \frac{\xi}{\|\xi\|_2} = e_1$. Then,

$$\theta := \int_{\|x\|_2 \leq \varepsilon} \left\langle x, \frac{\xi}{\|\xi\|_2} \right\rangle^2 dx = \int_{\|z\|_2 \leq \varepsilon} z_1^2 dx.$$

Therefore (2.21) takes form:

$$\sum_{|\alpha|=2} A_\alpha w(y) \xi^\alpha \geq \theta w(y) \|\xi\|_2^2, \quad y \in \Omega. \tag{2.22}$$

Therefore the operator $L$ is strongly elliptic and we are allowed to apply Theorem 8.10 in[GT01] to obtain, that $u \in W^{2m,2}_{loc}(\Omega)$.

Now we are in a position to complete the proof. Observe, that in (2.20) we showed that $u$ solves the equation of system (2.1) for $j = 2$ and we need to show that $u$ is as well a solution to remaining equations of (2.1), i.e. for $j = 4, 6, \ldots, 2m$. Let us analyze (2.17) similarly to (2.18) by applying the Parseval identity and move the expression on the left-hand side to the right-hand side in order to recover the following equation

$$\int_{\mathbb{R}^n} \sum_{|\alpha|=j} A_\alpha \varphi(x) \Big( D^\alpha (uw)(x) - u(x) D^\alpha w(x) \Big) dx = 0 \quad \text{for } \varphi \in C_0^\infty(B),$$

which is a weak formulation of the following equation

$$\sum_{|\alpha|=j} A_\alpha \Big( D^\alpha (uw) - u D^\alpha w \Big) = 0.$$

The proof of Theorem 2.2 is, therefore, completed. □

One of the immediate consequences of the proof of Theorem 2.2 is the following regularity result.

**Proposition 2.20.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $d$ be a norm induced metric and $w$ be a weight such that $w \in C^{2m-1,1}_{loc}(\Omega)$ for some $m \in \mathbb{N}$, $m > 1$. Then $\mathcal{H}(\Omega, d, w dx) \subset W^{2m,2}_{loc}(\Omega)$.*

## 2.5 Theorem 2.3: The converse of Theorem 2.2

Since both Theorems 2.10, 2.14 and Proposition 2.15 give not only the necessary, but also the sufficient condition for the mean value properties in the sense of Flatto and Bose, respectively, our next goal is to find an appropriate counterpart of these results. In case of nonconstant weights Proposition 2.15 imposes an additional PDE condition on $w$, hence we expect an analogous condition. From the point of view of our further considerations, the following *generalized Pizzetti formula* introduced by Zalcman in [Zal73], will be vital.

**Theorem 2.21** (Theorem 1, [Zal73])**.** *Let $\mu$ be a finite Borel measure on $\mathbb{R}^n$ with compact support and $\mathcal{F}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, y\rangle} d\mu(y)$ be the Fourier transform of the measure $\mu$. Suppose that $f$ is an analytic function on a domain $\Omega \subset \mathbb{R}^n$. Then the following equality holds*

$$\int_{\mathbb{R}^n} f(x + ry) d\mu(y) = [\mathcal{F}(-rD) f](x), \tag{2.23}$$

*for all $x \in \Omega$ and $r > 0$ such that the left-hand side exists and the right-hand side converges. The symbol $D$ is given by $D := -i \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$.*

**Remark 2.22.** Formula (2.23) is the main tool used in the proof of Theorem 2.3, hence we need to assume analyticity of weight $w$. Due to a result by Łysik [Łys18b] the Pizzetti formula on Euclidean balls is valid exactly for analytic functions. Therefore, dropping the analyticity assumption of $w$ would require finding a different proof of Theorem 2.3.

**Remark 2.23.** Theorem 2.10 by Friedman–Littman is a special case of our Theorems 2.2 and 2.3 for $w \equiv 1$. Proposition 2.15 by Bose is generalized by Theorem 2.3 due to the following lemma.

**Lemma 2.24.** *Suppose that $w \in C^{2l}(\Omega)$ solves the following equation*

$$\Delta^l w + a_{l-1} \Delta^{l-1} w + \ldots + a_1 \Delta w + a_0 w = 0, \tag{2.24}$$

*where all $a_i \in \mathbb{C}$ for $i = 0, 1, \ldots, l-1$ and $\Delta^i$ is the $i$-th composition of the laplacian $\Delta$ and $i = 1, 2, \ldots$. Then $w$ is analytic in $\Omega$.*

*Proof.* We prove the lemma by the mathematical induction with respect to $l$. Recall the following fact (see p. 57 in [Joh55]): Suppose that $w \in C^2(\Omega)$ solves the following equation

$$Lw + \lambda w = \varphi, \tag{2.25}$$

where $L$ is elliptic with analytic coefficients and $\varphi$ is analytic in $\Omega$, $\lambda \in \mathbb{C}$. Then $w$ is analytic in $\Omega$.

If $l = 1$, then we use the above regularity fact with $L = \Delta$, $a_0 = \lambda$ and $\varphi \equiv 0$. Now let us assume that the assertion holds for $l-1$ and consider $w$ as in (2.24). By adding and subtracting the appropriate terms we may rewrite this equation as follows with any $\lambda \in \mathbb{C}$ and given $a_0, a_1, \ldots, a_{l-1} \in \mathbb{C}$:

$$0 = \Delta^{l-1}(\Delta w + \lambda w) + (a_{l-1} - \lambda)\Delta^{l-2}(\Delta w + \lambda w) + (a_{l-2} - \lambda(a_{l-1} - \lambda))\Delta^{l-3}(\Delta w + \lambda w) + \ldots$$
$$+ (a_1 - \lambda(a_2 - \lambda(\ldots)))(\Delta w + \lambda w) + \Big(a_0 - \lambda(a_1 - \lambda(a_2 - \lambda(\ldots)))\Big)w.$$

Since the factor in the last $w$-term is a complex polynomial in $\lambda$, one can choose such $\lambda$, so that this last factor standing by $w$ in the equation vanishes (e.g. take $\lambda$ to be one of the roots of $w$). We use the assumption for $l-1$ to obtain that $\Delta w + \lambda w$ is an analytic function, denoted by $\varphi$, i.e. $\Delta w + \lambda w = \varphi$. This observation together with the regularity observation allow us to conclude the proof. □

**Lemma 2.25.** *Suppose that $\Omega \subset \mathbb{R}^n$ is open, $w$ is a positive analytic function and $d$ is induced by norm. Then any $u \in \mathcal{H}(\Omega, d, wdx)$ is analytic as well.*

*Proof.* By Theorem 2.2 function $u$ is a weak solution to the equation for $j = 2$ of system (2.1). In (2.22) we show, that this equation is strongly elliptic. We apply the regularity result (2.25) to obtain that $u$ is analytic. □

Now we are in a position to prove Theorem 2.3.

*Proof of Theorem 2.3.* We need to show the following equality

$$u(x)\int_{B(0,1)} w(x+ry)dy = \int_{B(0,1)} u(x+ry)w(x+ry)dy, \tag{2.26}$$

where $B(0,1)$ is a unit ball in metric $d$. In order to prove (2.26) we use the generalized Pizzetti formula for a measure $\mu$ being the normalized Lebesgue measure on the unit ball. Then

$$\mathcal{F}(\xi) = \int_{B(0,1)} e^{-i\langle \xi, y\rangle}dy = \sum_{j=0}^{\infty}\frac{(-i)^j}{j!}\int_{B(0,1)}\langle \xi, y\rangle^j dy \overset{(2.16)}{=} \sum_{j=0}^{\infty}\frac{(-i)^j}{j!}\sum_{|\alpha|=j} A_\alpha \xi^\alpha = \sum_{\alpha\in\mathbb{N}^n}\frac{(-i)^{|\alpha|}}{|\alpha|!}A_\alpha\xi^\alpha,$$

where $A_\alpha = \binom{|\alpha|}{\alpha}\int_{B(0,1)} y^\alpha dy$. We apply Theorem 2.21 twice: to $w$ and $uw$ to obtain

$$\int_{B(0,1)} w(x+ry)dy = \sum_{\alpha\in\mathbb{N}^n}\frac{r^{|\alpha|}}{|\alpha|!}A_\alpha D^\alpha w(x), \tag{2.27}$$

$$\int_{B(0,1)} u(x+ry)w(x+ry)dy = \sum_{\alpha\in\mathbb{N}^n}\frac{r^{|\alpha|}}{|\alpha|!}A_\alpha D^\alpha (u(x)w(x)), \tag{2.28}$$

Multiply (2.27) by $u(x)$ and subtract from it (2.28) to obtain the following:

$$u(x)\int_{B(0,1)} w(x+ry)dy - \int_{B(0,1)} u(x+ry)w(x+ry)dy$$
$$= \sum_{\alpha\in\mathbb{N}^n}\frac{r^{|\alpha|}}{|\alpha|!}A_\alpha\left(u(x)D^\alpha(w(x)) - D^\alpha(u(x)w(x))\right)$$
$$= \sum_{j=0}^{\infty}\frac{r^j}{j!}\sum_{|\alpha|=j}A_\alpha\left(u(x)D^\alpha(w(x)) - D^\alpha(u(x)w(x))\right) = 0,$$

where in the last step we appeal to (2.1). Thus $u$ satisfies the weighted mean value property and the proof is completed. $\square$

## 2.6 Applications of Theorem 2.2 and Theorem 2.3

In this chapter we illustrate Theorem 2.2 and Theorem 2.3 by determining the space $\mathcal{H}(\Omega, d, dx)$ in case of the distance function $d$ being induced by the $l^p$ norm and a constant weight $w = 1$. Our goal is to show that whenever $p \neq 2$ and $n = 2$, the space $\mathcal{H}(\Omega, l^p, dx)$ consists of 8 linearly independent harmonic polynomials. We already know that $\mathcal{H}(\Omega, l^2, dx)$ consists of all harmonic functions in $\Omega$, which differs significantly from the previous case. Moreover, in this chapter we describe system (2.1) for $p = 2$ and smooth $w$ and compare with the equations from Theorem 2.14. Our computations are new both for $\mathcal{H}(\Omega, l^p, dx)$ with $p \neq 2$ and for $p = 2$ and a smooth weight.

Let us consider the space $\mathbb{R}^n$ with the distance $l^p$ for $1 \leq p < \infty$ and a smooth weight $w$. First, we calculate coefficients $A_\alpha$. By the first paragraph of Chapter 2.4 we only need to consider multi-indices $\alpha$ with even components. The integral formula (called the Dirichlet Theorem), see p. 157 in [Edw22] and also Lemma 3.16, allows us to infer that

$$A_\alpha = 2^n \binom{|\alpha|}{\alpha} \int\limits_{\{\sum x_i^p < 1, x_i \geq 0\}} x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \, dx = \left(\frac{2}{p}\right)^n \binom{|\alpha|}{\alpha} \frac{\prod\limits_{i=1}^{n} \Gamma\left(\frac{\alpha_i+1}{p}\right)}{\Gamma\left(\frac{|\alpha|+n+p}{p}\right)}, \tag{2.29}$$

where $\Gamma$ stands for the gamma function. Notice, that coefficients $A_\alpha$ for $j = 2$ are constant by symmetry of balls in the $l^p$ norm. Therefore, the equation of system (2.1) for $j = 2$ translates to

$$\sum_{i=1}^{n} \left( \frac{\partial^2}{\partial x_i^2}(uw) - u \frac{\partial^2}{\partial x_i^2}(w) \right) = 0,$$

or equivalently to

$$w\Delta u + 2\nabla u \nabla w = 0. \tag{2.30}$$

Let us recall that since (2.30) is an elliptic equation with smooth coefficients, then every weak solution is smooth and solves (2.30) in a classical way. Therefore, $\mathcal{H}(\Omega, l^p, w) \subset C^\infty(\Omega)$ and the system (2.1) can be understood in the classical sense. In order to describe further equations we need to divide our calculations into more specific instances: $p = 2$, $p = \infty$ and remaining values of $1 \leq p < \infty$.

### 2.6.1 The case of weighted $l^2$ distance

In this chapter we intend to show, that Theorem 2.2 is a generalization of Theorem 2.14. In order to demonstrate this we show that for $p = 2$ system (2.1) is equivalent to (2.5), see Theorem 2.14. Recall that the coefficients $A_\alpha$ in (2.29) take the following form (including the case $j = 2$ discussed in the beginning of Chapter 2.6)

$$A_\alpha = \binom{|\alpha|}{\alpha} \frac{\prod_{i=1}^{n} \Gamma\left(\frac{\alpha_i+1}{2}\right)}{\Gamma\left(\frac{|\alpha|+n+2}{2}\right)}. \tag{2.31}$$

Furthermore, recall the following two formulas concerning the gamma function: For any $k \in \mathbb{N}$ there holds

$$\Gamma\left(\frac{k}{2}\right) = \sqrt{\pi} \frac{(k-2)!!}{2^{\frac{k-1}{2}}} \quad \text{and} \quad \Gamma\left(k + \frac{1}{2}\right) = \sqrt{\pi} \frac{(2k)!}{4^k k!}.$$

We use the first formula in the denominator of (2.31) and second in the numerator of (2.31) to obtain that

$$A_\alpha = \binom{|\alpha|}{\alpha} \frac{2^{\frac{|\alpha|+n+1}{2}}}{\sqrt{\pi}(|\alpha|+n)!!} \prod_{i=1}^{n} \left( \sqrt{\pi} \frac{\alpha_i!}{2^{\alpha_i}(\frac{\alpha_i}{2})!} \right) = \binom{|\alpha|}{\alpha} \frac{\pi^{\frac{n-1}{2}} 2^{\frac{n+1}{2}}}{(|\alpha|+n)!!} \prod_{i=1}^{n} \frac{\alpha_i!}{(\frac{\alpha_i}{2})!}.$$

Therefore, the $j$-th equation of system (2.1) can be written in the following form

$$\begin{aligned} 0 &= \sum_{|\alpha|=j, \alpha_i \in 2\mathbb{N}} A_\alpha \left( D^\alpha(uw) - uD^\alpha w \right) \\ &= \sum_{|\alpha|=j, \alpha_i \in 2\mathbb{N}} \frac{j!}{\alpha_1! \ldots \alpha_n!} \frac{\pi^{\frac{n-1}{2}} 2^{\frac{n+1}{2}}}{(j+n)!!} \prod_{i=1}^{n} \frac{\alpha_i!}{(\frac{\alpha_i}{2})!} (D^\alpha(uw) - uD^\alpha w) \\ &= \frac{j!\, \pi^{\frac{n-1}{2}} 2^{\frac{n+1}{2}}}{(\frac{j}{2})!\, (j+n)!!} \sum_{|\beta|=j/2} \binom{\frac{j}{2}}{\beta} (D^{2\beta}(uw) - uD^{2\beta} w). \end{aligned} \tag{2.32}$$

Next, observe that for any $f \in C^{2l}(\Omega)$ its $l$-th Laplace operator can be written in the following form

$$\Delta^l f = \left( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right)^l f = \sum_{|\beta|=l} \frac{l!}{\beta_1! \ldots \beta_n!} D^{2\beta} f, \tag{2.33}$$

where the multinomial formula has been applied. Finally by (2.32) and (2.33) we conclude that in the $l^2$-case system (2.1) is equivalent to system

$$\Delta^l(uw) = u\Delta^l w, \qquad \text{for } l = 1, 2, \ldots \tag{2.34}$$

In fact (2.34) is equivalent to (2.5). To that end observe that $\Delta(uw) = w\Delta u + 2\nabla u \nabla w + u\Delta w$. Upon joining this with the equation of (2.34) for $l = 1$ we obtain the first equation of (2.5). Further equations of (2.5) follow from (2.34) and the following computation:

$$u\Delta^{l+1} w = \Delta(\Delta^l(uw)) = \Delta(u\Delta^l w) = \Delta u \Delta^l w + 2\langle \nabla u, \nabla(\Delta^l w) \rangle + u\Delta^{l+1} w.$$

Therefore,

$$\Delta u \Delta^l w + 2\langle \nabla u, \nabla(\Delta^l w) \rangle = 0 \qquad \text{for } l = 0, 1, 2, \ldots$$

and we end this part of discussion by concluding, that by above considerations our Theorem 2.2 is a generalization of Bose's result, see Theorem 2.14.

Moreover, by Theorems 2.2 and 2.3 we know that $\mathcal{H}(\Omega, l^2, wdx)$ consists exactly of solutions to the following system of equations

$$\Delta u \Delta^j w + 2\langle \nabla u, \nabla(\Delta^j w) \rangle = 0, \qquad \text{for } j = 0, 1, \ldots. \tag{2.35}$$

Let us observe, that $u$ solves also infinitely many other systems of equations, obtained from (2.35) by excluding $l \in \mathbb{N}$ initial equations

$$\Delta u \Delta^{j+l} w + 2\langle \nabla u, \nabla(\Delta^{j+l} w) \rangle = \Delta u \Delta^j(\Delta^l w) + 2\langle \nabla u, \nabla(\Delta^j(\Delta^l w)) \rangle = 0, \qquad \text{for } j = 0, 1, \ldots.$$

Therefore, $u$ is strongly harmonic in countably many metric measure spaces $(\Omega, l^2, \Delta^l w dx)$ for all $l \in \mathbb{N}$. In other words, function $u$ has infinitely many mean value properties, with respect to different weighted Lebesgue measures $d\mu = \Delta^l w dx$ for all $l \in \mathbb{N}$, whenever $\Delta^l w$ are positive.

### 2.6.2 The case of $l^p$ distance for $p \notin \{2, \infty\}$

Strongly harmonic functions on $\Omega \subset \mathbb{R}^n$ equipped with the $l^p$-distance and the Lebesgue measure behave quite differently for $p \notin \{2, \infty\}$ than for $p = 2$. In what follows we demonstrate that only finitely many equations of system (2.1) are nontrivial, and that in fact all of functions in

$\mathcal{H}(\Omega, l^p, dx)$ are harmonic polynomials. For the sake of simplicity we consider case $n = 2$, and $u$ depending on two variables $x := x_1$ and $y := x_2$.

We now focus our attention on equations of system (2.1) for $j > 2$ since the equation for $j = 2$ is described in (2.30). We examine the differential operator $R_j := \sum_{|\alpha|=j} A_\alpha D^\alpha$. We already showed that for $p = 2$ operator $R_2$ is equal to $\Delta$ up to a multiplicative constant. Recall formula (2.29) for $n = 2$:

$$A_\alpha = \binom{|\alpha|}{\alpha} \left(\frac{2}{p}\right)^2 \frac{\prod_{i=1}^{2} \Gamma\left(\frac{\alpha_i+1}{p}\right)}{\Gamma\left(\frac{|\alpha|+2+p}{p}\right)}.$$

Let us observe, that for $|\alpha|= 4$ those coefficients attain only two different values:

(1) $A_\alpha = \left(\frac{2}{p}\right)^2 \frac{\Gamma\left(\frac{5}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{p+6}{p}\right)}$, whenever $\alpha = (4, 0)$ or $\alpha = (0, 4)$. This coefficient stands by $\frac{\partial^4}{\partial x^4}$ and $\frac{\partial^4}{\partial y^4}$ in $R_4$,

(2) $A_\alpha = 6\left(\frac{2}{p}\right)^2 \frac{\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{p+6}{p}\right)}$, if $\alpha = (2, 2)$. This coefficient appears by $\frac{\partial^4}{\partial x^2 \partial y^2}$ in operator $R_4$.

Therefore, $R_4$ takes a form

$$R_4 = \left(\frac{2}{p}\right)^2 \Gamma\left(\frac{p+6}{p}\right)^{-1} \left[\Gamma\left(\frac{5}{p}\right)\Gamma\left(\frac{1}{p}\right)\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right) + 6\Gamma\left(\frac{3}{p}\right)^2 \left(\frac{\partial^4}{\partial x^2 \partial y^2}\right)\right],$$

which, up to a multiplicative constant, reduces to operator $\Delta^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}$ if and only if the following function $f$ (and so, a parameter $p$) satisfies condition

$$f(p) := \frac{\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{5}{p}\right)\Gamma\left(\frac{1}{p}\right)} = \frac{1}{3}.$$

By the previous considerations this holds true for $p = 2$. We will show that $f(p) \neq \frac{1}{3}$ for other values of $p \in [1, \infty)$. Let us differentiate $f$ with respect to $p$. Recall, that the formula for derivative of the gamma function stays:

$$\Gamma'(z) = \Gamma(z)\left(-\frac{1}{z} - \gamma - \sum_{k=1}^{\infty}\left(\frac{1}{k+z} - \frac{1}{k}\right)\right) = \Gamma(z)\Psi(z),$$

where $\gamma$ is the Euler constant and $\Psi$ is the digamma function defined by the above equality, for more details see [AS64]. We use this identity to compute the following

$$\begin{aligned} f'(p) &= \frac{2\Gamma\left(\frac{3}{p}\right)^2 \Psi\left(\frac{3}{p}\right)\left(-\frac{3}{p^2}\right)\Gamma\left(\frac{5}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{5}{p}\right)^2 \Gamma\left(\frac{1}{p}\right)^2} \\ &- \frac{\Gamma\left(\frac{3}{p}\right)^2 \left[\Gamma\left(\frac{5}{p}\right)\Psi\left(\frac{5}{p}\right)\left(-\frac{5}{p^2}\right)\Gamma\left(\frac{1}{p}\right) + \Gamma\left(\frac{5}{p}\right)\Gamma\left(\frac{1}{p}\right)\Psi\left(\frac{1}{p}\right)\left(-\frac{1}{p^2}\right)\right]}{\Gamma\left(\frac{5}{p}\right)^2 \Gamma\left(\frac{1}{p}\right)^2} \\ &= f(p)\left(-\frac{6}{p^2}\Psi\left(\frac{3}{p}\right) + \frac{5}{p^2}\Psi\left(\frac{5}{p}\right) + \frac{1}{p^2}\Psi\left(\frac{1}{p}\right)\right). \end{aligned}$$

Since $f$ is positive for $p \in [1, \infty)$, we only need to investigate the sign of the second factor in the above formula:

$$\begin{aligned}
&-\frac{6}{p^2}\Psi\left(\frac{3}{p}\right)+\frac{5}{p^2}\Psi\left(\frac{5}{p}\right)+\frac{1}{p^2}\Psi\left(\frac{1}{p}\right)\\
&=-\frac{6}{p^2}\left(-\frac{p}{3}-\gamma-\sum_{k=1}^{\infty}\left(\frac{1}{k+\frac{3}{p}}-\frac{1}{k}\right)\right)+\frac{5}{p^2}\left(-\frac{p}{5}-\gamma-\sum_{k=1}^{\infty}\left(\frac{1}{k+\frac{5}{p}}-\frac{1}{k}\right)\right)\\
&+\frac{1}{p^2}\left(-p-\gamma-\sum_{k=1}^{\infty}\left(\frac{1}{k+\frac{1}{p}}-\frac{1}{k}\right)\right)=-\frac{1}{p^2}\sum_{k=1}^{\infty}\left(\frac{-6}{k+\frac{3}{p}}+\frac{5}{k+\frac{5}{p}}+\frac{1}{k+\frac{1}{p}}\right)\\
&=\frac{1}{p^2}\sum_{k=1}^{\infty}\frac{8k}{p(k+\frac{3}{p})(k+\frac{5}{p})(k+\frac{1}{p})}>0.
\end{aligned}$$

Therefore, $f$ is monotonically increasing on $[1, \infty)$ and attains value $1/3$ exactly at $p = 2$. We conclude our computations with the following:

$$R_4 = \begin{cases} \left(\frac{2}{p}\right)^2 \Gamma\left(\frac{p+6}{p}\right)^{-1} \Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)\Delta^2 & \text{for } p = 2,\\ \left(\frac{2}{p}\right)^2 \Gamma\left(\frac{p+6}{p}\right)^{-1} \Gamma\left(\frac{5}{p}\right)\Gamma\left(\frac{1}{p}\right)\left(\Delta^2 + \left(\frac{6\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{5}{p}\right)\Gamma\left(\frac{1}{p}\right)} - 2\right)\frac{\partial^4}{\partial x^2 \partial y^2}\right) & \text{for } p \neq 2. \end{cases}$$

We are now in a position to apply Theorem 2.2 and Theorem 2.3. Function $u \in \mathcal{H}(\Omega, l^p, dx)$ if and only if it satisfies the system of equations (2.1) with $w = 1$. Therefore, (2.30) reads

$$\Delta u = 0, \tag{2.36}$$

hence $u$ is harmonic, and its bilaplacian vanishes. Moreover, $u$ has to satisfy equation of system (2.1) for $j = 4$, i.e. $R_4 u = 0$. Since bilaplacian of $u$ vanishes, therefore, $u$ is in fact solution to $u_{xxyy} = 0$. Let us observe, that differentiating twice $\Delta u$ with respect to $x$ and $y$ respectively we obtain

$$u_{xxxx} + u_{xxyy} = 0 \quad \text{and} \quad u_{xxyy} + u_{yyyy} = 0.$$

Therefore, both $u_{xxxx} = 0$ and $u_{yyyy} = 0$, which means that for each fixed value of $y$ function $u(x, y)$ is a polynomial in $x$ of degree at most 3 and analogously for a fixed $x$ function $u(x, y)$ is a polynomial in $y$ of degree at most 3. Then there exist $a_i(y)$ and $b_i(x)$ for $i = 0, 1, 2, 3$ such that

$$u(x,y) = a_0(y) + a_1(y)x + a_2(y)x^2 + a_3(y)x^3 = b_0(x) + b_1(x)y + b_2(x)y^2 + b_3(x)y^3. \tag{2.37}$$

In what follows we omit writing the arguments of $a_i$ and $b_i$. Simple calculations give us that

$$u_{xxxx} = b_0^{(4)} + b_1^{(4)}y + b_2^{(4)}y^2 + b_3^{(4)}y^3 = 0, \tag{2.38}$$

and

$$u_{yyyy} = a_0^{(4)} + a_1^{(4)}x + a_2^{(4)}x^2 + a_3^{(4)}x^3 = 0. \tag{2.39}$$

Now at each fixed $x$ in (2.38) the polynomial in $y$ has to have all coefficients equal to 0 due to the Equality of Polynomials Theorem, hence $b_i^{(4)} = 0$ for $i = 1, 2, 3, 4$. Similarly, at (2.39) we set that $a_i^{(4)} = 0$ for all $i = 1, 2, 3, 4$. Therefore, all of $a_i$ and $b_i$ are polynomials of degree at most 3. Moreover, we know that $u_{xxyy} = 0$. We calculate this derivative in (2.37) to get

$$0 = u_{xxyy} = 2a_2'' + 6xa_3'' = 2b_2'' + 6yb_3''.$$

Thus, once again we obtain that $a_i'' = 0$ and $b_i'' = 0$ for $i = 2, 3$, so $a_2$, $a_3$, $b_2$ and $b_3$ are in fact of degree at most 1. By the above considerations we conclude that $u$ is a linear combination of the monomials

$$1, x, y, xy, x^2, x^3, xy^2, xy^3, x^2y, x^3y, y^2, y^3, \tag{2.40}$$

which solves equation (2.36). Therefore, $u$ has to be a harmonic polynomial of the form described by (2.40). The part of $u$ generated by $\{1, x, y, xy\}$ is already harmonic and for that reason we only need to consider $u$ being a combination of the remaining monomials in (2.40), i.e.

$$u = c_1x^2 + c_2x^3 + c_3xy^2 + c_4xy^3 + c_5x^2y + c_6x^3y + c_7y^2 + c_8y^3.$$

Inserting $u$ to (2.36) we get the following

$$0 = 2(c_1 + c_7) + 2x(3c_2 + c_3) + 6xy(c_4 + c_6) + 2y(c_5 + 3c_8),$$

and once again by appealing to the Equality of Polynomials Theorem we obtain that $u \in \mathcal{H}(\Omega, l^p, dx)$ if and only if

$$u \in \operatorname{span}\left\{1, x, y, xy, x^2 - y^2, xy^2 - \frac{x^3}{3}, xy^3 - x^3y, x^2y - \frac{y^3}{3}\right\}. \tag{2.41}$$

Finally, let us observe that in equations of system (2.1) for $j = 6$ there appear only the following operators

$$\frac{\partial^6}{\partial x^6}, \frac{\partial^6}{\partial x^4 \partial y^2}, \frac{\partial^6}{\partial x^2 \partial y^4}, \frac{\partial^6}{\partial y^6},$$

which all vanish on $u$ in the form as in (2.41). The triviality of equations for $j > 6$ follows immediately. Therefore, we summarize our discussion with the following inclusion:

$$\mathcal{H}(\Omega, l^p, dx) = \operatorname{span}\left\{1, x, y, xy, x^2 - y^2, xy^2 - \frac{x^3}{3}, xy^3 - x^3y, x^2y - \frac{y^3}{3}\right\}. \tag{2.42}$$

Now let us discuss the case $p = \infty$.

### 2.6.3 The case of $l^\infty$ distance

In order to complete our illustration of Theorem 2.2 and Theorem 2.3 we need to consider the remaining case, i.e. characterize functions $u$ in $\mathcal{H}(\Omega, l^p, dx)$ for $p = \infty$. In this case $B(0, 1) = [-1, 1]^n$ in $l^\infty$ norm. Therefore, we obtain the following formula for the coefficients $A_\alpha$ in (2.1):

$$A_\alpha = \binom{|\alpha|}{\alpha} \int_{-1}^{1} x_1^{\alpha_1} \cdot \ldots \int_{-1}^{1} x_n^{\alpha_n} = \binom{|\alpha|}{\alpha} \frac{2^n}{\prod_{i=1}^{n}(\alpha_i + 1)}.$$

Then, after inserting $A_\alpha$ and dividing by the $2^n$ factor, system (2.1) converts to the following

$$\sum_{\substack{|\alpha|=j \\ \alpha_i \text{ even}}} \binom{|\alpha|}{\alpha} \frac{1}{(\alpha_1 + 1)! \cdot \ldots \cdot (\alpha_n + 1)!} D^\alpha u = 0.$$

As in the previous chapter we restrict our attention to case $n = 2$ and write out the equation for $j = 2$: $\frac{1}{6}(u_{xx} + u_{yy}) = 0$. Hence $u$ is a harmonic function. Equation for $j = 4$ is the following

$$\frac{1}{120}(u_{xxxx} + u_{yyyy}) + \frac{1}{6}u_{xxyy} = 0,$$

and can be reduced to $\Delta^2 u + 20u_{xxyy} = 0$. This, combined with an analogous discussion to the one ending the previous chapter leads us to the conclusion that (2.42) holds true also for $p = \infty$.

Finally, in the remark below we discuss the case of dimensions $n > 2$ and a general question of $\dim \mathcal{H}(\Omega, l^p, dx)$.

**Remark 2.26.** Let us consider an open connected set $\Omega \subset \mathbb{R}^2$, metric $d$ induced by the $l^p$ norm for $1 \leq p \leq \infty$, $p \neq 2$ and $\mu$ being the Lebesgue measure. Due to computations summarized in (2.42) we know that:

$$\mathcal{H}(\Omega, l^p, dx) = \operatorname{span}\left\{1, x, y, xy, x^2 - y^2, xy^2 - \frac{x^3}{3}, xy^3 - x^3y, x^2y - \frac{y^3}{3}\right\}.$$

Notice, that the dimension of $\mathcal{H}(\Omega, l^p, dx)$ is equal to 8. As mentioned in the introduction, in $\mathbb{R}^3$ Łysik [Łys18a] computed $\dim \mathcal{H}(\Omega, l^p, dx) = 48$. Moreover, in case of $p = 1$ and $p = \infty$, when the unit ball is cube-shaped, Iwasaki [Iwa12] proved that $\dim \mathcal{H}(\Omega, l^p, dx) = 2^n n!$. Those numbers coincide with $2^n n!$ - the number of linear isometries of $(\mathbb{R}^n, l^p)$, which is discovered in [AB12]. We believe that there is a link between the dimension of the space $\mathcal{H}(\Omega, d, \mu)$ and the number of linear isometries, still to be examined.

# Chapter 3

# Asymptotically $p$-harmonic functions on Carnot groups of step 2

## 3.1 Introduction

In Chapter 2 we studied functions with the mean value property and its consequences, whereas in this Chapter we will focus on an so-called asymptotic mean value property. In the last decade there has been a growing interest in studying a generalized mean-value property originating in [MPR10] and [MPR12], called the asymptotic mean-value property or amv-property for short. It allows to characterize solutions to harmonic, $p$-harmonic and more general equations of elliptic and parabolic types. Related are applications of $p$-harmonic functions in statistical Tug-of-War games, see for instance [MPR10] and [PS08]. The studies in [MPR10] allow, in the simplest case, to weaken the classical characterizations of a harmonic function $u$ in $\mathbb{R}^n$ as follows:

$$u(x) = \fint_{B(x,\varepsilon)} u \, + \, o(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

It is important from the point of view of our studies below, that the amv-property can be shown to hold for the viscosity solutions to the normalized $p$-harmonic equation $\Delta_p^N u = 0$ in $\mathbb{R}^n$ for all $1 \leq p \leq \infty$. Namely, in [MPR10] it is proven that $u(x) = \mu_p^*(\varepsilon, u) + o(\varepsilon^2)$, as $\varepsilon \to 0$, where $\mu_p^*(\varepsilon, u)$ is the linear combination of the mean value and the *min-max mean*:

$$\mu_p^*(\varepsilon, u) = \frac{n+2}{n+p} \fint_{B(x,\varepsilon)} u + \frac{1}{2}\frac{p-2}{n+p}\left(\max_{B(x,\varepsilon)} u + \min_{B(x,\varepsilon)} u\right).$$

Similar means characterizing $p$-harmonic functions have been found in [HR11; HR13], by using the *median* of a function, see also [KMP12]. The results in [HR11] yield the amv-property for all $p$ but for $n = 2$ only, while results of [KMP12] provide the amv-property for $n \geq 2$. Moreover, the mean-value property for solutions to general elliptic equations with nonsmooth coefficients is studied in [CT76].

The amv-property has also been investigated beyond the Euclidean setting, see [FLM14] for results in the first Heisenberg group $\mathbb{H}_1$, [LY13] for the higher order Heisenberg groups $\mathbb{H}_n$ and [FP15] for the setting of general Carnot groups.

A new approach to the asymptotic mean-value property has been recently proposed in [IMW17] (see also [BM19] for relations with statistical games). More precisely, in [IMW17], the authors proved that every viscosity solution $u$ to the normalized $p$-laplacian in an open set $\Omega \subset \mathbb{R}^n$ for a given $1 \leq p \leq \infty$ (Definition 3.10), can be characterized using an asymptotic mean-value property in terms of the function $\mu_p(\varepsilon, u)(x)$, defined as the unique minimizer of the following variational problem

$$\|u - \mu_p(\varepsilon, u)\|_{L^p(\overline{B(x,\varepsilon)})} = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(\overline{B(x,\varepsilon)})},$$

where $B(x, \varepsilon) \subset \Omega$ denotes the ball centered at $x$ with radius $\varepsilon$. This notion encompasses the median, the mean-value and the min-max mean of a continuous function, see [IMW17] for details.

In this chapter we present generalization of the results of [IMW17] to the setting of an arbitrary Carnot group of step 2, including the first Heisenberg group $\mathbb{H}_1$.

Let $\mathbb{G}$ be a Carnot group of step $k$ (Definition 3.2). Denote by $\Delta^N_{p,\mathbb{G}}$ the subelliptic normalized $p$-Laplacian (see (3.4) and (3.5)) and by $\mu_p(\varepsilon, u)$ the generalized median of a function $u$ defined uniquely as in (3.7). The theorem below stays that a viscosity solution of $\Delta^N_{p,\mathbb{G}} u = 0$ can be characterized asymptotically by the minimum $\mu_p(\varepsilon, u)$. This provides one more, intrinsic, way to characterize $p$-harmonic functions via a variant of the asymptotic mean-value property.

**Theorem 3.1.** *Let $1 \leq p \leq \infty$ and let $\Omega \subset \mathbb{G}$ be open. For a function $u \in C^0(\Omega)$ the following are equivalent:*

*(i) $u$ is a viscosity solution of $\Delta^N_{p,\mathbb{G}} u = 0$ in $\Omega$;*

*(ii) $u(x) = \mu_p(\varepsilon, u)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$, in the viscosity sense for every $x \in \Omega$.*

We present the proof of this theorem in two special cases: (1) for $\mathbb{G}$ being the Heisenberg group and (2) for any two-step group $\mathbb{G}$, see Chapter 3.3. The proof in a general case is presented in [Ada+20]. The main tool used in the proof is Lemma 3.15, where the asymptotic behavior of minimizers $\mu_p$ is described for quadratic polynomials on balls, see Chapters 3.4 and 3.5 for the proofs of Lemma 3.15 in the setting of the Heisenberg group and Carnot group of step 2, respectively. As presented in Remark 3.18, our results generalize those obtained in the Euclidean setting in [IMW17].

## 3.2 Carnot groups

In what follows we briefly recall the definition and some standard facts on Carnot groups, see [BLU07; Cap+07; Gro96; Mon02] for a more detailed treatment.

**Definition 3.2.** A finite dimensional Lie algebra $\mathfrak{g}$, is said to be stratified of step $k \in \mathbb{N}$, if there exist linear subspaces $V_1, \ldots, V_k$ of $\mathfrak{g}$ such that:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k \text{ and } [V_1, V_i] = V_{i+1} \quad i = 1, \ldots, k-1; \quad [V_1, V_k] = \{0\}.$$

The symbol $[\cdot, \cdot]$ stands for the Lie bracket in $\mathfrak{g}$. We denote by $v_i$ the dimension of $V_i$ for $i = 1, 2, \ldots, k$.

A connected and simply connected Lie group $(\mathbb{G}, *)$ is a Carnot group if its Lie algebra $\mathfrak{g}$ is finite dimensional and stratified. We also set

$$h_0 := 0, \quad h_i := \sum_{j=1}^{i} v_j \quad \text{and} \quad m := h_k. \tag{3.1}$$

Observe, that any stratified Lie algebra is nilpotent. Every Carnot group $\mathbb{G}$ of step $k$ is isomorphic via the exponential map $\operatorname{Exp} : \mathfrak{g} \to \mathfrak{g}$ as a Lie group to $(\mathbb{R}^m, \circ)$ where $\circ$ is the group operation given by the Baker-Campbell-Hausdorff formula, see Definition 2.2.11 and Theorem 2.2.13 in [BLU07]. More precisely, the Baker-Campbell-Hausdorff formula in Carnot groups solves equation $\operatorname{Exp}(X) * \operatorname{Exp}(Y) = \operatorname{Exp}(Z)$ in $Z$ for any given $X, Y \in \mathfrak{g} \cong \mathbb{R}^m$. The solution $Z$ defines a group operation $\circ$ on $\mathbb{R}^m$ which depends only on finite number of compositions of Lie brackets of $X$ and $Y$ and can be expressed as follows

$$X \circ Y := \sum_{n=1}^{k} \frac{(-1)^n}{n} \sum_{\substack{p_i + q_i \geq 1 \\ 1 \leq i \leq n}} \frac{(\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \ldots (\operatorname{ad} X)^{p_n} (\operatorname{ad} Y)^{q_n - 1} Y}{p_1!\, q_1! \cdot \ldots \cdot p_n!\, q_n! \sum_{j=1}^{n} (p_j + q_j)}, \tag{3.2}$$

where $(\operatorname{ad} X)Y := [X,Y]$. Terms up to 4-th order on the right-hand side of (3.2) are the following

$$X \circ Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) - \frac{1}{24}[Y,[X,[X,Y]]] + \dots$$

The inverse element to $X$ with respect to $\circ$ is $-X$ and the neutral element of $\circ$ is 0.

From now on we use the above construction of isomorphisms without mentioning it explicitly. Every element $x$ of group $\mathbb{G}$ is identified with appropriate element $X$ of the Lie algebra $\mathfrak{g}$ and also with an appropriate $x \in \mathbb{R}^m$, since the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R}^m$. For each $x \in \mathbb{G}$ we define left the translation $\tau_x : \mathbb{G} \longrightarrow \mathbb{G}$ by the formula

$$\tau_x(y) := x \circ y.$$

For each $\lambda > 0$ we define a dilation $\delta_\lambda : \mathbb{G} \longrightarrow \mathbb{G}$ by the formula

$$\delta_\lambda(x) = \delta_\lambda(x_1, \dots, x_m) := (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_k} x_m),$$

where each $\sigma_i \in \mathbb{N}$ is called the homogeneity of the variable $x_i$ in $\mathbb{G}$ and it is defined by $\sigma_j := i$, whenever $h_{i-1} < j \leq h_i$, cf. (3.1). Observe, that vector basis of $\mathbb{R}^m$ can be constructed from the vector bases of subspaces $V_i$ for $i = 1, \dots, k$ so that the homogeneity of variable $x_j$, which is a coefficient of the element of the basis of $V_i$, is equal to the index $i$.

We have discussed the group structure of Carnot groups, now let us define the metric structure in Carnot groups. For this purpose, let us recall the following notion of pseudonorm, see Definition 5.1.1 in [BLU07].

**Definition 3.3.** We call a function $\mathcal{N} : \mathbb{G} \to [0,\infty)$ a *pseudonorm* on Carnot group $\mathbb{G}$, if the following conditions hold true

1. $\mathcal{N}$ is continuous with respect to the Euclidean topology,
2. $\mathcal{N}(\delta_\lambda(x)) = \lambda \mathcal{N}(x)$ for every $\lambda > 0$ and $x \in \mathbb{G}$,
3. $\mathcal{N}(x) > 0$ if and only if $x \neq 0$,
4. $\mathcal{N}(x) = \mathcal{N}(x^{-1})$ for every $x \in \mathbb{G}$.

We say, that a function $d : \mathbb{G} \times \mathbb{G} \to [0,\infty)$ is a *pseudodistance* on Carnot group $\mathbb{G}$, if the following conditions hold true

1. $d(x,y) = 0$ if and only if $x = y$,
2. $d(x,y) = d(y,x)$ for all $x, y \in \mathbb{G}$,
3. there exists a constant $C > 0$ such that for all $x, y, z \in \mathbb{G}$ there holds *pseudo-triangle inequality*
$$d(x,y) \leq C(d(x,z) + d(z,y)).$$

We endow Carnot group $\mathbb{G}$ with a pseudonorm inducing pseudodistance by defining

$$|x|_{\mathbb{G}} := |(x^{(1)}, \dots, x^{(k)})|_{\mathbb{G}} := \Big( \sum_{j=1}^{k} \left\| x^{(j)} \right\|^{\frac{2k!}{j}} \Big)^{\frac{1}{2k!}} \tag{3.3}$$

$$d(x,y) := |y^{-1} \circ x|_{\mathbb{G}},$$

where $x^{(j)} := (x_{h_{j-1}+1}, \dots, x_{h_j})$ and $\left\| x^{(j)} \right\|$ denotes the standard Euclidean norm in $\mathbb{R}^{h_j - h_{j-1}}$. For more information on the pseudo-triangle inequality we refer to Proposition 5.1.8 in [BLU07]. We define the pseudoball centered at $x \in \mathbb{G}$ of radius $R > 0$ by

$$B(x,R) := \{ y \in \mathbb{G} : \ |y^{-1} \circ x|_{\mathbb{G}} < R \}.$$

We illustrate the concept of Carnot groups with the following important examples.

**Example 3.4** (The Euclidean space $\mathbb{R}^n$)**.** The Euclidean space is an abelian group, hence all Lie brackets are trivial and $\mathbb{R}^n$ is a 1-step Carnot group. Therefore, the Lie group multiplication $*$ and the operation $\circ$ described above coincide and are the same as the standard addition $+$ of vectors in $\mathbb{R}^n$. Analogous observation applies to any group which is a linear space endowed with a the Lie bracket equal zero for all pairs of vectors.

**Example 3.5** (The Heisenberg groups $\mathbb{H}_n$)**.** The $n$-dimensional Heisenberg group $\mathbb{G} = \mathbb{H}_n$, is the Carnot group with a 2-step Lie algebra and the orthonormal basis $\{X_1, \ldots, X_{2n}, X_{2n+1}\}$ such that

$$\mathfrak{g}_1 = \mathrm{Span}\,\{X_1, \ldots, X_{2n}\}, \quad \mathfrak{g}_2 = \mathrm{Span}\,\{X_{2n+1}\},$$

and the only nontrivial brackets are $[X_i, X_{n+i}] = X_{2n+1}$ for $i = 1, \ldots, n$.

In particular, if $n = 1$, then the Heisenberg group $\mathbb{H}_1$ is often presented by using coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$ and multiplication $\circ$ is defined on $\mathbb{R}^3$ by

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1)).$$

The pseudonorm given by

$$\|(x_1, x_2, x_3)\| = ((x_1^2 + x_2^2)^2 + x_3^2)^{1/4}$$

gives rise to a left invariant distance defined by $d_{\mathbb{H}_1}(p, q) = \|p^{-1}q\|$ which is called the Heisenberg distance. A dilation by $r > 0$ is defined by $\delta_r(x_1, x_2, x_3) = (rx_1, rx_2, r^2x_3)$ and the left invariant Haar measure $\lambda$ is simply the 3-dimensional Lebesgue measure, moreover $\delta_r^* d\lambda = r^4 d\lambda$. It follows that the Hausdorff dimension of the metric measure space $(\mathbb{H}_1, d_{\mathbb{H}_1}, \lambda)$ is 4, and the space is 4-Ahlfors regular, i.e., there exists a positive constant $c$ such that for all balls $B$ with radius $r$, we have $\frac{1}{c}r^4 \leq \mathcal{H}^4(B) \leq cr^4$, where $\mathcal{H}^4$ denotes the 4-dimensional Hausdorff measure induced by $d_{\mathbb{H}_1}$. For further discussion on the Heisenberg group see Chapter 4.3.3.

Proposition 1.3.21 proved in [BLU07], shows that the Lebesgue measure is the Haar measure on Carnot groups.

**Proposition 3.6.** *Let $\mathbb{G} = (\mathbb{R}^m, \circ)$ be a Carnot group. Then the Lebesgue measure on $\mathbb{R}^m$ is invariant with respect to the left and the right translations on $\mathbb{G}$. Precisely, if we denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}^m$, then for all $x \in \mathbb{G}$ we have that $|x \circ E| = |E| = |E \circ x|$. Moreover, for all $\lambda > 0$ it holds $\delta_\lambda(E)| = \lambda^Q |E|$, where $Q := \sum_{j=1}^m v_j \sigma_j$.*

A basis $X = \{X_1, \ldots, X_m\}$ of $\mathfrak{g}$, is called *the Jacobian basis* if $X_j = J(e_j)$ where $(e_1, \ldots, e_m)$ is the canonical basis of $\mathbb{R}^m$ and $J : \mathbb{R}^m \longrightarrow \mathfrak{g}$ is defined by $J(\eta)(x) := \mathcal{J}_{\tau_x}(0) \cdot \eta$, where $\mathcal{J}_{\tau_x}$ denotes the Jacobian matrix of the left-translation $\tau_x$.

Let us recall the following classical proposition describing the Jacobian basis on Carnot groups, see [BLU07, Corollary 1.3.19] for a proof.

**Proposition 3.7.** *Let $\mathbb{G} = (\mathbb{R}^m, \circ)$ be a Carnot group of step $k \in \mathbb{N}$. Then the elements of the Jacobian basis $\{X_1, \ldots, X_m\}$ have polynomial coefficients and if $h_{l-1} < j \leq h_l$, $1 \leq l \leq k$, then*

$$X_j(x) = \partial_j + \sum_{i > h_l}^{m} a_i^{(j)}(x)\partial_i,$$

*where $a_i^{(j)}(x) = a_i^{(j)}(x_1, \ldots, x_{h_{l-1}})$ when $h_{l-1} < i \leq h_l$, and $a_i^{(j)}(\delta_\lambda(x)) = \lambda^{\sigma_i - \sigma_j} a_i^{(j)}(x)$.*

The following definition is one of the key concepts of the analysis on Carnot groups. Let $X = \{X_1, \ldots, X_m\}$ be a Jacobian basis of $\mathbb{G} = (\mathbb{R}^m, \circ)$. For any function $u \in C^1(\mathbb{R}^m)$, we define its *horizontal gradient* by the formula

$$\nabla_{V_1} u := \sum_{i=1}^{h_1} (X_i u) X_i$$

and the *intrinsic divergence* of $u$ as

$$\mathrm{div}_{V_1} u := \sum_{i=1}^{h_1} X_i u.$$

**Remark 3.8.** In the setting of the Heisenberg group we follow the notation convention and denote

$$\nabla_H u := \nabla_{V_1} u.$$

Moreover, for $2 \leq j \leq k$, we set $\nabla_{V_j} u := \sum_{h_{j-1} < i \leq h_j} (X_i u) X_i$. The horizontal Laplacian $\Delta_{\mathbb{G}} u$ of a function $u : \mathbb{G} \longrightarrow \mathbb{R}$ is defined by the following

$$\Delta_{\mathbb{G}} u := \sum_{i=1}^{h_1} X_i^2 u.$$

A priori, one studies solutions to the Laplace equation under the $C^2$-regularity assumption. However, as in the Euclidean setting, it is natural to weaken the required degree of regularity and consider weak solutions belonging to the so-called horizontal Sobolev space. For further details we refer to e.g. [CDG96; MM07].

The following result describes the Taylor expansion formula in the Carnot groups, see [BLU07, Proposition 20.3.11] .

**Proposition 3.9.** *Let $\Omega \subset \mathbb{G}$ be an open neighbourhood of* $0$ *and let $u \in C^\infty(\Omega)$. Then, the following Taylor formula holds for any point $P = (x^{(1)}, x^{(2)}, \ldots, x^{(k)}) \in \Omega$:*

$$u(P) = u(0) + \langle \nabla_{V_1} u(0), x^{(1)} \rangle_{\mathbb{R}^{h_1}} + \langle \nabla_{V_2} u(0), x^{(2)} \rangle_{\mathbb{R}^{h_2}} + \frac{1}{2} \langle D^{2,*}_{V_1} u(0) x^{(1)}, x^{(1)} \rangle_{\mathbb{R}^{h_1}} + o(\|P\|^2)$$

*where*

$$D^{2,*}_{V_1} u := \left( \frac{(X_i X_j + X_j X_i) u}{2} \right)_{1 \leq i,j \leq h_1}$$

*is the so-called* symmetrized horizontal Hessian *of $u$.*

Next, we recall the definition of the main differential operator studied in this chapter. For $p \in [1, +\infty]$ and a function $u \in C^2$ the *subelliptic normalized $p$-Laplace operator* is defined at points where $\nabla_{V_1} u \neq 0$ in the following way

$$\Delta^N_{p,\mathbb{G}} u := \frac{\mathrm{div}_{V_1}(|\nabla_{V_1} u|^{p-2} \nabla_{V_1} u)}{|\nabla_{V_1} u|^{p-2}} \quad \text{if} \quad 1 \leq p < \infty \tag{3.4}$$

and

$$\Delta^N_{\infty,\mathbb{G}} u := \left\langle D^{2,*}_{V_1} u \, \frac{\nabla_{V_1} u}{|\nabla_{V_1} u|}, \frac{\nabla_{V_1} u}{|\nabla_{V_1} u|} \right\rangle. \tag{3.5}$$

In this chapter we work with viscosity solutions to the subelliptic normalized $p$-Laplace equation discussed in Definition 3.10. Notice, that we use there the definition of the subelliptic normalized $p$-Laplace operator at those points, where the horizontal gradient of the function to which $\Delta^N_{p,\mathbb{G}}$ is applied is nonzero.

Note that for $p = 2$, $\Delta^N_{2,\mathbb{G}} u = \Delta_{\mathbb{G}} u$ is the so called Kohn-Laplace operator in $\mathbb{G}$. Thus, the $p$-Laplace operator is the natural generalization of the Laplacian. Furthermore, the $\infty$-Laplacian can be viewed as a limit of $p$-Laplacians in the appropriate sense for $p \to \infty$. Among its applications, let us mention best Lipschitz extensions, image processing and mass transport problems, see e.g. the presentation in [MPR10] and references therein.

In the case of the non-renormalized $p$-Laplacian, notions of a viscosity solution and a weak solution agree for $1 < p < \infty$, see [JLM01] for the result in the Euclidean setting and [Bie06] for the Heisenberg group. Since the normalized $p$-Laplacian is in the non-divergence form, the concept of viscosity solutions is more handy than weak solutions. Let us now introduce this notion.

**Definition 3.10.** Fix a value of $p \in [1, \infty]$ and consider the subelliptic normalized $p$-Laplace equation

$$\Delta^N_{p,\mathbb{G}} u = 0 \qquad \text{in} \qquad \Omega \subset \mathbb{G}. \tag{3.6}$$

(i) A lower semi-continuous function $u$, is a viscosity supersolution of (3.6), if for every $x_0 \in \Omega$, and every $\phi \in C^2(\Omega)$ such that $\nabla_{V_1}\phi(x_0) \neq 0$ and $u - \phi$ has a strict minimum at $x_0 \in \Omega$, we have $\Delta^N_{p,\mathbb{G}}\phi \leq 0$ in $\Omega$.

(ii) An upper semi-continuous function $u$, is a viscosity subsolution of (3.6), if for every $x_0 \in \Omega$, and every $\phi \in C^2(\Omega)$ such that $\nabla_{V_1}\phi(x_0) \neq 0$ and $u - \phi$ has a strict maximum at $x_0 \in \Omega$, we have $\Delta^N_{p,\mathbb{G}}\phi \geq 0$ in $\Omega$.

(iii) A continuous function $u$ is a viscosity solution of of (3.6), if it is both a viscosity supersolution and a viscosity subsolution in $\Omega$.

To our best knowlege the concept of viscosity solutions was first introduced by Crandall and Lions. The main idea comes from regularizing PDE by adding a viscosity term $\varepsilon\Delta u$ in order to regularize the equation and then letting $\varepsilon \to 0$. For an comprehensive survey on the topic see [CIL92] and for a more recent begginer's guide see [Koi04].

Next, we define one of the central objects of this chapter. Fix an open set $\Omega \subset \mathbb{G}$, let $1 \leq p \leq \infty$ and let $u$ be a real-valued continuous function in $\Omega$. For a given $x \in \Omega$, choose $\varepsilon > 0$ so that $\overline{B(x,\varepsilon)} \subset \Omega$, we define the number $\mu_p(\varepsilon, u)(x)$ (or simply $\mu_p(\varepsilon, u)$ if the point $x$ is clear from the context) as the unique real number satisfying

$$\|u - \mu_p(\varepsilon, u)\|_{L^p(\overline{B(x,\varepsilon)})} = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(\overline{B(x,\varepsilon)})}. \tag{3.7}$$

The following properties of $\mu_p(\varepsilon, u)(x)$ have been proved in [IMW17] for the setting of compact topological spaces $X$, equipped with a positive Radon measure $\nu$ such that $\nu(X) < \infty$. Here we apply results from [IMW17] to $X = \overline{B(x,\varepsilon)} \subset \mathbb{G}$ and $\nu$ the Lebesgue measure, cf. Proposition 3.6.

In Theorem 3.11 below, we summarize results proven in Theorems 2.1, 2.4 and 2.5 in [IMW17].

**Theorem 3.11.** *Let $1 \leq p \leq \infty$ and $u \in C(\overline{B(x,\varepsilon)})$.*

*(1) There exists a unique real valued $\mu_p(\varepsilon, u)$ such that*

$$\|u - \mu_p(\varepsilon, u)\|_{L^p(\overline{B(x,\varepsilon)})} = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(\overline{B(x,\varepsilon)})}.$$

*Furthermore, for $1 \leq p < \infty$, $\mu_p(\varepsilon, u)$ is characterized by the equation*

$$\int_{B(x,\varepsilon)} |u(y) - \mu_p(\varepsilon, u)|^{p-2}\,(u(y) - \mu_p(\varepsilon, u))\, dy = 0, \tag{3.8}$$

*where for $1 \leq p < 2$ we assume that the integrand is zero if $u(y) - \mu_p(\varepsilon, u) = 0$. For $p = \infty$ we have the following equality:*

$$\mu_\infty(\varepsilon, u) = \frac{1}{2}\left(\min_{\overline{B(x,\varepsilon)}} u + \max_{\overline{B(x,\varepsilon)}} u\right). \tag{3.9}$$

*(2) If $1 \leq p \leq \infty$ then it follows that*

$$\left| \|u - \mu_p(\varepsilon, u)\|_{L^p(\overline{B(x,\varepsilon)})} - \|v - \mu_p(\varepsilon, v)\|_{L^p(\overline{B(x,\varepsilon)})} \right| \leq \|u - v\|_{L^p(\overline{B(x,\varepsilon)})}$$

*for any $u, v \in L^p(\overline{B(x,\varepsilon)})$. Moreover, if $u_n \to u$ in $L^p(\overline{B(x,\varepsilon)})$ for $1 \leq p \leq \infty$ and $u_n, u \in C^0(\overline{B(x,\varepsilon)})$ for $p = 1$, then $\mu_p(\varepsilon, u_n) \to \mu_p(\varepsilon, u)$ as $n \to \infty$, the same is true for any $p \in [1, \infty]$ if $\{u_n\} \subset C^0(\overline{B(x,\varepsilon)})$ converges uniformly on $\overline{B(x,\varepsilon)}$ as $n \to \infty$.*

(3) *Let $u$ and $v$ be two functions which, in the case $1 < p \leq \infty$, belong to $L^p(B(x,\varepsilon))$, and in the case $p = 1$, belong to $C^0(\overline{B(x,\varepsilon)})$. If $u \leq v$ a.e. in $\overline{B(x,\varepsilon)}$, then $\mu_p(\varepsilon, u) \leq \mu_p(\varepsilon, v)$.*

(4) *$\mu_p(\varepsilon, u + c) = \mu_p(\varepsilon, u) + c$ for every $c \in \mathbb{R}$.*

(5) *$\mu_p(\varepsilon, cu) = c\mu_p(\varepsilon, u)$ for every $c \in \mathbb{R}$.*

Observe, that by (3.8) there holds $\mu_2(\varepsilon, u) = \fint_{B(x,\varepsilon)} u$ and $\mu_1(\varepsilon, u)$ is a median of $u$ over a ball $B(x, \varepsilon)$. Recall, that $\lambda \in \mathbb{R}$ is a median of function $u$ over a set $A$ if the measures of sub- and super-level sets at level $\lambda$ are equal.

The following is a generalization of [IMW17, Corollary 2.3] in Carnot groups of step $k$:

**Corollary 3.12.** *Let $u \in L^p(B(x,\varepsilon))$, for $1 < p \leq \infty$, or in $C^0(\overline{B}(x,\varepsilon))$ for $p = 1$. Let $u_\varepsilon(z) = u(x\delta_\varepsilon(z))$ for $z \in \overline{B}(0,1)$, then*

$$\mu_p(\varepsilon, u)(x) = \mu_p(1, u_\varepsilon)(0).$$

*Proof.* For every $\lambda \in \mathbb{R}$ and $1 \leq p < \infty$ it holds:

$$\begin{aligned} \|u - \lambda\|^p_{L^p(B(x,\varepsilon))} &= \int_{B(x,\varepsilon)} |u(\xi) - \lambda|^p \, d\xi \\ &= \varepsilon^{\sigma_1 + \cdots + \sigma_k} \int_{B(0,1)} |u_\varepsilon(\xi) - \lambda|^p \, d\xi \\ &= \varepsilon^{v_1 + 2v_2 + \cdots + kv_k} \|u_\varepsilon - \lambda\|^p_{L^p(B(0,1))} \end{aligned}$$

and

$$\|u - \lambda\|_{L^\infty(B(x,\varepsilon))} = \|u_\varepsilon - \lambda\|_{L^\infty(B(0,1))}$$

and the conclusion follows by (1) in Theorem 3.11. □

Next we state carefully what is meant by the statement that the asymptotic expansion of the function $u$ in terms of $\mu_p$ holds in the viscosity sense, see (3.7) and Definition 3.14. First, we need the following auxiliary definition.

**Definition 3.13.** Let $h$ be a real valued function defined in a neighbourhood of zero. We say that

$$h(x) \leq o(x^2) \text{ as } x \to 0^+$$

if any of the three equivalent conditions is satisfied:

a) $\displaystyle\limsup_{x \to 0^+} \frac{h(x)}{x^2} \leq 0,$

b) there exists a nonnegative function $g(x) \geq 0$ such that $h(x) + g(x) = o(x^2)$ as $x \to 0^+$,

c) $\displaystyle\lim_{x \to 0^+} \frac{h^+(x)}{x^2} \leq 0.$

A similar definition is given for $h(x) \geq o(x^2)$ as $x \to 0^+$ by reversing the inequalities in a) and c), requiring that $g(x) \leq 0$ in b) and replacing $h^+$ by $h^-$ in c)[1].

Let $f$ and $g$ be two real valued functions defined in a neighbourhood of $x_0 \in \mathbb{R}$. We say that *$f$ and $g$ are asymptotic functions for $x \to x_0$*, if there exists a function $h$ defined in a neighbourhood $V_{x_0}$of $x_0$ such that:

(i) $f(x) = g(x)h(x)$ for all $x \in V_{x_0} \setminus \{x_0\}$.

(ii) $\lim_{x \to x_0} h(x) = 1$.

If $f$ and $g$ are asymptotic for $x \to x_0$, then we simply write $f \sim g$ as $x \to x_0$.

[1] As usual, we denote by $h^+(x) := \max\{h(x), 0\}$ and $h^-(x) := -\min\{h(x), 0\}$.

**Definition 3.14.** A continuous function defined in a neighbourhood of a point $x \in \mathbb{G}$, satisfies

$$u(x) = \mu_p(\varepsilon, u)(x) + o(\epsilon^2),$$

as $\epsilon \to 0^+$ in the viscosity sense, if the following conditions hold:

(i) for every continuous function $\phi$ defined in a neighbourhood of a point $x$ such that $u - \phi$ has a strict minimum at $x$ with $u(x) = \phi(x)$ and $\nabla_{V_1}\phi(x) \neq 0$ , we have

$$\phi(x) \geq \mu_p(\varepsilon, \phi)(x) + o(\epsilon^2), \quad \text{as } \epsilon \to 0^+.$$

(ii) for every continuous function $\phi$ defined in a neighbourhood of a point $x$ such that $u - \phi$ has a strict maximum at $x$ with $u(x) = \phi(x)$ and $\nabla_{V_1}\phi(x) \neq 0$, then

$$\phi(x) \leq \mu_p(\varepsilon, \phi)(x) + o(\epsilon^2), \quad \text{as } \epsilon \to 0^+.$$

## 3.3 The proof of Theorem 3.1

In order to prove Theorem 3.1, we need the following key lemma. The proof of the lemma in full generality is presented in [Ada+20]. In the next part of this thesis we present statements and proofs of special cases of this lemma in two cases: $\mathbb{G}$ being the Heisenberg group and a general Carnot group $\mathbb{G}$ of step 2. The Heisenberg group is a model example of a Carnot group of step 2, hence understanding the proof in this case is a first step towards the more general case.

**Lemma 3.15** (cf. Lemma 3.1 in [Ada+20])**.** *Let $\mathbb{G}$ be a Carnot group of step $k$. Moreover, let $\Omega \subset \mathbb{G}$ be an open set and $x \in \Omega$ be a point such that $B(x, \varepsilon) \subset \Omega$ for all small enough $\varepsilon \leq \varepsilon_0(x)$. Let $1 \leq p \leq \infty$ and $\xi \in \mathbb{R}^{v_1} \setminus \{0\}$, $\eta \in \mathbb{R}^{v_2}$. Let further $A$ be a symmetric $v_1 \times v_1$ matrix with trace $\operatorname{tr}(A)$. Moreover, consider the quadratic function $q : B(x, \varepsilon) \to \mathbb{R}$ given by*

$$q(y) = q(x) + \langle \xi, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^{v_1}} + \langle \eta, (x^{-1}y)^{(2)} \rangle_{\mathbb{R}^{v_2}} + \frac{1}{2}\langle A(x^{-1}y)^{(1)}, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^{v_1}}, \quad y \in B(x, \varepsilon), \tag{3.10}$$

*where $(x^{-1}y)^{(1)}$ and $(x^{-1}y)^{(2)}$ are the horizontal and the vertical components of $x^{-1}y$, respectively and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{v_1}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{v_2}}$ denote the Euclidean scalar products on $\mathbb{R}^{v_1}$ and $\mathbb{R}^{v_2}$, respectively. Then it follows that*

$$\mu_p(\varepsilon, q)(x) = q(x) + \varepsilon^2 c \left( \operatorname{tr}(A) + (p-2)\frac{\langle A\xi, \xi \rangle_{\mathbb{R}^{v_1}}}{|\xi|^2} \right) + o(\varepsilon^2), \tag{3.11}$$

*where*

$$c := c(p, v_1, \ldots, v_k) = \frac{1}{2(p+v_1)} \frac{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{p+\sum_{j=1}^{k-1} jv_j}{2(k-1)!} + 1\right)}{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{p-2+\sum_{j=1}^{k-1} jv_j}{2(k-1)!} + 1\right)} \prod_{j=2}^{k-1} \frac{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{p+\sum_{i=1}^{j-1} iv_i}{2k!} + 1\right)}{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{p-2+\sum_{i=1}^{j-1} iv_i}{2k!} + 1\right)}$$

*and $\mathcal{B}(x, y)$ denotes the Beta function $\mathcal{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$ for all $x, y > 0$. Furthermore, if $u \in C^2(\Omega)$ with $\nabla_{V_1}u(x) \neq 0$, then*

$$\mu_p(\varepsilon, u)(x) = u(x) + c\Delta^N_{p,\mathbb{G}}u(x)\varepsilon^2 + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0^+. \tag{3.12}$$

In the proof of Lemma 3.15 we employ the following integral formula.

**Lemma 3.16.** *Let $\alpha_1, \ldots, \alpha_n$ be real numbers such that $\alpha_i > -1$ for $i = 1, \ldots, n$. It then follows that*

$$\int_{T_n} x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} dx = \frac{1}{2^n} \frac{\prod_{i=1}^n \Gamma(\frac{\alpha_i+1}{2})}{\Gamma(\frac{n+2+\sum \alpha_i}{2})} \tag{3.13}$$

*where $T_n := \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 < 1,\ x_i \geq 0 \text{ for } i = 1, \ldots, n\right\}$ and $\Gamma$ denotes the gamma function.*

*Proof of Lemma 3.16.* Let $a,b>-1$. Upon applying the change of variables $t=\sin^2 x$, we obtain the following equation:

$$\int_0^{\frac{\pi}{2}} \sin^a x \cos^b x dx = \int_0^1 t^{\frac{a}{2}}(1-t)^{\frac{b}{2}} \frac{1}{2\sqrt{t}\sqrt{1-t}} dt = \frac{1}{2}\int_0^1 t^{\frac{a-1}{2}}(1-t)^{\frac{b-1}{2}} dt = \frac{1}{2}\mathcal{B}\left(\frac{a+1}{2}, \frac{b+1}{2}\right),$$

where $\mathcal{B}$ stands for the beta function.

Now we are in a position to calculate the left-hand side of (3.13). We apply the spherical coordinates

$$\begin{cases} x_1 = r\cos\varphi_1 \\ x_2 = r\sin\varphi_1\cos\varphi_2 \\ x_3 = r\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ \vdots \\ x_{n-1} = r\sin\varphi_1\sin\varphi_2\cdot\ldots\cdot\cos\varphi_{n-1} \\ x_n = r\sin\varphi_1\sin\varphi_2\cdot\ldots\cdot\sin\varphi_{n-1} \end{cases}$$

with the Jacobian determinant $|J|= r^{n-1}\sin^{n-2}\varphi_1\sin^{n-3}\varphi_2\cdot\ldots\cdot\sin\varphi_{n-2}$ and the spherical coordinates varying as follows: $r\in(0,1)$, $\varphi_i\in(0,\pi/2)$ for $i=1,\ldots,n-2$. The result is

$$\begin{aligned}\int_{T_n} x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n} dx &= \int_0^1\int_0^{\frac{\pi}{2}}\ldots\int_0^{\frac{\pi}{2}}\Big[r^{\sum_{i=1}^n\alpha_i+n-1}\cdot\cos^{\alpha_1}\varphi_1(\sin\varphi_1)^{\sum_{i=2}^n\alpha_i+n-2} \\ &\quad\cdot\cos^{\alpha_2}\varphi_2(\sin\varphi_2)^{\sum_{i=3}^n\alpha_i+n-3}\cdot\ldots\cdot\cos^{\alpha_{n-1}}\varphi_{n-1}\sin^{\alpha_n}\varphi_{n-1}\Big]d\varphi_1\ldots d\varphi_{n-1}dr \\ &= \frac{1}{n+\sum_{i=1}^n\alpha_i}\frac{1}{2}\mathcal{B}\left(\frac{\sum_{i=2}^n\alpha_i+n-1}{2}, \frac{\alpha_1+1}{2}\right)\frac{1}{2}\mathcal{B}\left(\frac{\sum_{i=3}^n\alpha_i+n-2}{2}, \frac{\alpha_2+1}{2}\right) \\ &\quad\cdot\ldots\cdot\frac{1}{2}\mathcal{B}\left(\frac{\alpha_n+1}{2}, \frac{\alpha_{n-1}+1}{2}\right),\end{aligned}$$

which is equal to the right-hand side of (3.13) upon using the well-known formula $\mathcal{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. □

Let us comment about the differences between the above Lemma 3.15 and [IMW17, Lemma 3.1].

**Remark 3.17.** (1) The quadratic polynomial $q$ in formula (3.10) is defined for any Carnot group of step $k$ and differs from the original one studied in $\mathbb{R}^n$, cf. [IMW17, Lemma 3.1]. The formula for $q$ reflects the dependence of $q$ on the first two layers of $\mathbb{G}$.

(2) The geometry of gauge balls in Carnot groups is far from Euclidean and nontrivial in comparision: balls are flattened at the characteristic points (at poles) and possess less symmetry than balls in $\mathbb{R}^n$. A noticable difference in comparison with [IMW17] is the appearance of the Beta function which is not present in the Euclidean case and can be viewed as consequence of the stratification in the geometry.

(3) Our proof for the case $p=\infty$ differs from the corresponding one in [IMW17], as it requires appealing to results in [FP15]. Indeed, the geometry of gauge balls in general Carnot groups makes obtaining limits in (3.27) and (3.50) a subtle and highly nontrivial task, see the proof of Lemma 1.6 in [FP15] and the discussion following its formulation in [FP15] on pg. 207.

**Remark 3.18.** The formula describing the constant $c(p,v_1,\ldots,v_k)$ is complicated and not easily simplified using the properties of the Beta function.

**Example 3.19** (The Euclidean space $\mathbb{R}^N$)**.** If $\mathbb{G}$ is the Euclidean space $\mathbb{R}^N$ then $c(p,v_1,\ldots,v_k)$ agrees with the constant computed in [IMW17], namely

$$c(p,N)=\frac{1}{2(p+N)}.$$

When the Lemma 3.15 is proven, the proof of Theorem 3.1 relies on careful use of Definition 3.10 and Definition 3.14.

*The proof of Theorem 3.1.* Let $B(x,r) \subset \Omega$ be ball and let us fix $u \in C^0(\Omega)$ and $\phi \in C^2(B(x,r))$ with $\nabla_{V_1}\phi(x) \neq 0$. The asymptotic formula (3.12) implies that

$$\phi(x) = \mu_p(\varepsilon,\phi)(x) - c(p,v_1,\dots,v_k)\Delta^N_{p,\mathbb{G}}\phi(x)\varepsilon^2 + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0. \tag{3.14}$$

Suppose that $u$ is a viscosity solution, in the sense of Definition 3.10, to the equation $\Delta^N_{p,\mathbb{G}}u = 0$ in $\Omega$. Thus, in particular, $u$ satisfies parts (i) and (ii) of Definition 3.10. Since $u$ is a viscosity supersolution of $\Delta^N_{p,\mathbb{G}}u = 0$ in $\Omega$, then at point $x$, for $\phi$ as above such that $u - \phi$ has a strict minimum at $x$ and $u(x) = \phi(x)$, it holds that $\Delta^N_{p,\mathbb{G}}\phi(x) \leq 0$. Therefore, from (3.14) we obtain

$$\phi(x) \geq \mu_p(\varepsilon,u)(x) + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0,$$

which proves that $\phi$ at $x$ satisfies part (i) of Definition 3.14. By using the fact that $u$ is also a viscosity subsolution (and so $u$ satisfies part (ii) of Definition 3.10) we show that inequality in part (ii) of Definition 3.14 holds as well. This proves that $u(x) = \mu_p(\varepsilon,u)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$ in the viscosity sense.

Now we will prove the converse. Suppose, that $u(x) = \mu_p(\varepsilon,u)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$ in the viscosity sense. If $u - \phi$ attains a strict minimum at $x$, then by Definition 3.14, it follows that $\phi(x) \geq \mu_p(\varepsilon,\phi)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$. Using this result in (3.14), we get

$$\Delta^N_{p,\mathbb{G}}\phi(x) = \frac{\mu_p(\varepsilon,\phi)(x) - \phi(x)}{c(p,v_1,\dots,v_k)\varepsilon^2} + o(1) \leq o(1),$$

as $\varepsilon \to 0$, and hence $\Delta^N_{p,\mathbb{G}}\phi(x) \leq 0$. We apply a similar reasoning in the case $u - \phi$ has a strict maximum at $x$. This proves, that $u$ is a viscosity solution of $\Delta^N_{p,\mathbb{G}}u = 0$ in $\Omega$. □

**Remark 3.20.** Mean value formulas similar to the ones proved in Theorem 3.1 have been used in [LMR20] to study random walks and random tug of war in the Heisenberg group. In [LMR20], the authors implemented the approach of Peres-Sheffield [PS08] to provide a game-theoretical interpretation of the $p$-Laplacian in the Heisenberg group, they also characterized its viscosity solutions via an asymptotic mean value expansion similar to the one proved in [MPR10]. We expect that our result could be used to generalize [LMR20] to general Carnot groups.

## 3.4 Lemma 3.15 in the Heisenberg group $\mathbb{H}_1$

In this chapter we state and prove the special case of Lemma 3.15 when the Carnot group $\mathbb{G}$ is assumed to be the Heisenberg group.

**Lemma 3.21.** *Let $\Omega \subset \mathbb{H}_1$ be an open set and $x \in \Omega$ be a point such that a ball $B(x,\varepsilon) \subset \Omega$ for all small enough radii $\varepsilon \leq \varepsilon_0(x)$. Let $1 \leq p \leq \infty$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$. Let further $A$ be a symmetric $2 \times 2$ matrix with real coefficients. Moreover, consider the quadratic function $q : B(x,\varepsilon) \to \mathbb{R}$ given by*

$$q(y) = q(x) + \langle \xi, (x^{-1}y)_h \rangle + w(x^{-1}y)_v + \frac{1}{2}\langle A(x^{-1}y)_h, (x^{-1}y)_h \rangle, \quad y \in B(x,\varepsilon), \tag{3.15}$$

*where $(x^{-1}y)_h$ and $(x^{-1}y)_v$ are the horizontal and the vertical components of $x^{-1}y$, respectively and $w \in \mathbb{R}$ is fixed. Then it holds that*

$$\mu_p(\varepsilon,q) = q(x) + \varepsilon^2 C(p)\left(\operatorname{tr}(A) + (p-2)\frac{\langle A\xi,\xi\rangle}{|\xi|^2}\right) + o(\varepsilon^2), \tag{3.16}$$

*for $C(p) := \frac{2}{(p+2)(p+4)}\left(\frac{\Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+4}{4}\right)}\right)^2$. Furthermore, if $u \in C^2(\Omega)$ with the horizontal gradient $\nabla_{V_1}u(x) = \nabla_H u(x) \neq 0$, then it holds*

$$\mu_p(\varepsilon,u)(x) = u(x) + C(p)\Delta^N_{p,\mathbb{H}_1}u(x)\varepsilon^2 + o(\varepsilon^2), \quad \textit{as} \quad \varepsilon \to 0. \tag{3.17}$$

*Proof.* In the proof we follow the steps of the proof of Lemma 3.1 in [IMW17]. However, since the setting of Carnot groups differs from the Euclidean one, the computations are to some extent, more demanding and nontrivial.

We begin with computing $\mu_p(\varepsilon, q)$. For $z = (z_1, z_2, z_3) \in B(0,1) =: B$ we introduce the following

$$q_\varepsilon(z) = q(x\delta_\varepsilon(z)), \quad v_\varepsilon(z) = \frac{q_\varepsilon(z) - q(x)}{\varepsilon} \quad \text{and} \quad v(z) = \langle \xi, (z_1, z_2) \rangle := \langle \xi, z_h \rangle.$$

We know that $\mu_p(\varepsilon, q)(x) = \mu_p(1, q_\varepsilon)(0)$ by Corollary 3.12. Then, by parts (4) and (5) of Theorem 3.11, we see that

$$\frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon} = \mu_p(1, v_\varepsilon)(0).$$

Let us observe, that

$$v_\varepsilon(z) = \frac{1}{\varepsilon}\left( \langle \xi, \delta_\varepsilon(z)_h \rangle + \frac{1}{2}\langle A\delta_\varepsilon(z)_h, \delta_\varepsilon(z)_h \rangle + a\delta_\varepsilon(z)_3 \right) = \langle \xi, z_h \rangle + \frac{\varepsilon}{2}\langle Az_h, z_h \rangle + w\varepsilon z_3, \tag{3.18}$$

which shows that $v_\varepsilon$ converges uniformly to $v$ as $\varepsilon \to 0$ on $\overline{B}$. We appeal to the second part of claim (2) in Theorem 3.11 to obtain that $\mu_p(1, v_\varepsilon)(0) \to \mu_p(1, v)(0)$ as $\varepsilon \to 0$. Recall that the characterization of $\lambda = \mu_p(1, v)(0)$ given by (3.8) in Theorem 3.11 states that if $p \in [1, \infty)$, then $\lambda$ is the unique number such that

$$\int_B |\langle \xi, y_h \rangle - \lambda|^{p-2}(\langle \xi, y_h \rangle - \lambda) dy = 0.$$

On the other hand

$$\int_B |\langle \xi, y_h \rangle|^{p-2}(\langle \xi, y_h \rangle) dy = 0,$$

which follows from the symmetry of the unit ball and the following natural change of variables

$$\Phi(y_1, y_2, y_3) = (-y_1, y_2, y_3), \quad |J_\Phi| = 1, \quad \Phi(B) = B.$$

It now follows that $\mu_p(1, v)(0) = \lambda = 0$.

If $p = \infty$, then by (3.9):

$$\mu_\infty(1, v)(0) = \frac{1}{2}\left( \min_{\overline{B}} \langle \xi, y_h \rangle + \max_{\overline{B}} \langle \xi, y_h \rangle \right) = \frac{1}{2}\left( -|\xi| + |\xi| \right) = 0.$$

Next, we split the discussion into the cases depending on the value of $p$. Let us define

$$\gamma_\varepsilon = \frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon^2}.$$

### 3.4.1 Case 1: $1 < p < \infty$

For the sake of brevity we introduce a function $f(s) = |s|^{p-2}s$. Then, upon applying (3.8) to $\mu_p(1, v_\varepsilon)(0) = \varepsilon\gamma_\varepsilon$ we obtain

$$\int_B f(v_\varepsilon(z) - \varepsilon\gamma_\varepsilon) dz = 0.$$

By using (3.18) this can be transformed to the following expression:

$$\int_B f\left( \langle \xi, z_h \rangle + \varepsilon\left( \frac{1}{2}\langle Az_h, z_h \rangle - \gamma_\varepsilon + wz_3 \right) \right) dz = 0. \tag{3.19}$$

Without loss of generality we may assume that $|\xi| = 1$, since otherwise we can consider the quadratic function $\tilde{q} = q/|\xi|$. Let us apply the change of variables $z = (z_1, z_2, z_3) = (R(y_1, y_2), y_3)$ in (3.19), where $R$ is a $2 \times 2$ rotation matrix

$$R = \begin{bmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{bmatrix}.$$

Notice that $R^T\xi = e_1 = (1,0,0)$. Set $C = R^T AR$, then (3.19) reads

$$\int_B f\left(y_1 + \varepsilon\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right) dy = 0.$$

Since $\int_B f(y_1)dy = 0$, it follows that for all $\varepsilon > 0$, we have:

$$\frac{1}{\varepsilon}\int_B f\left(y_1 + \varepsilon\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right) - f(y_1)dy = 0.$$

Therefore, by the Fundamental Theorem of Calculus, we have:

$$\int_B \left[\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right) dt\right]\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right) dy = 0. \quad (3.20)$$

Equality (3.20) implies that $\gamma_\varepsilon$ is a weighted mean value of the function $\frac{1}{2}\langle Cy_h, y_h\rangle + wy_3$ over $B$ with respect to a weighted Lebesgue measure $\omega(y)dy$ for

$$\omega(y) := \int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right) dt, \quad y \in B.$$

The weight function $w$ is nonnegative since $f'(s) = (p-1)|s|^{p-2} \geq 0$. Therefore, $\gamma_\varepsilon$ is bounded by $c := \left\|\frac{1}{2}\langle Cy_h, y_h\rangle + wy_3\right\|_{L^\infty(B)}$.

Let us consider any subsequence of $(\gamma_\varepsilon)$ converging to $\gamma_0$ as $\varepsilon \to 0^+$, which for the sake of brevity, we also denote by $(\gamma_\varepsilon)$. Let us consider two cases. If $2 \leq p < \infty$, then for all $y \in B$ we obtain

$$\begin{aligned}
&\left|\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right) dt\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right| \\
&\leq 2c(p-1)\int_0^1 \left|y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right|^{p-2} dt \leq 2c(p-1)(1+2c\varepsilon).
\end{aligned}$$

Therefore, by the dominated convergence theorem the sequence $(\gamma_\varepsilon)$ converges to

$$\gamma_0 := \lim_{\varepsilon\to 0}\gamma_\varepsilon = \frac{\int_B |y_1|^{p-2}\left(\frac{1}{2}\langle Cy_h, y_h\rangle + wy_3\right) dy}{\int_B |y_1|^{p-2}dy}. \quad (3.21)$$

Let now $1 < p < 2$. Fix $0 < \theta < 1$ and split the integral (3.20) into two parts: over the set $G_\theta := B \cap \{|y_1| > \theta\}$ and $F_\theta := B \cap \{|y_1| \leq \theta\}$. Observe that for all $y \in G_\theta$ and for all $\varepsilon > 0$ satisfying $2c\varepsilon < \theta$, we have the following:

$$\begin{aligned}
&\left|\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right) dt\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right| \\
&\leq 2c\,||y_1| - 2c\varepsilon|^{p-2}.
\end{aligned}$$

Moreover,

$$\lim_{\varepsilon\to 0}\int_{G_\theta} ||y_1| - 2c\varepsilon|^{p-2}\, dy = \int_{G_\theta} |y_1|^{p-2}dy < \int_B |y_1|^{p-2}dy, \quad (3.22)$$

where the inequality holds uniformly for all $\theta \in (0,1)$. Furthermore, the last integral turns out to be finite which can be seen from the explicit calculation below in (3.23). Hence, by applying Theorem 5.4 in [IMW17] to $X = G_\theta$ with $\nu$ being the Lebesgue measure, we obtain the following:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{G_\theta}\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right)\right) dt\left(\frac{1}{2}\langle Cy_h, y_h\rangle - \gamma_\varepsilon + wy_3\right) dy \\
&= \int_{G_\theta} (p-1)|y_1|^{p-2}\left(\frac{1}{2}\langle Cy_h, y_h\rangle + wy_3 - \gamma_0\right).
\end{aligned}$$

Observe that here the upper bound in (3.22) allows us to conclude that the limit as $\theta \to 0^+$ is finite. We now focus on the part of the integral in (3.20) involving the set $F_\theta$. Since $|F_\theta|= \int_{F_\theta} 1dy$, then upon writing this integral as in (3.23), one sees that $|F_\theta|= c(k)\theta$, and so $|F_\theta|\to 0$, as $\theta \to 0^+$. Moreover, it suffices to consider $\theta = 2c\varepsilon$ and the related $\int_{F_{2c\varepsilon}} ||y_1|-2c\varepsilon|^{p-2}\, dy$. We again appeal to integral (3.23) and reduce our computations to finding

$$\int_{D(0,r)\cap\{|y_1|\leq 2c\varepsilon\}} (2c\varepsilon - |y_1|)^{p-2}\, dy_h,$$

$D(0,r)$ denotes the disc centered at 0 with radius $r$. However, direct computation shows that this integral is of order $\varepsilon^{p-1}$, which then allows us to let $\varepsilon \to 0^+$, and in turn conclude (3.21).

In order to approach the proof of (3.11), we first need to compute integrals in (3.21). We begin with computing the denominator of (3.21). Once this is completed, the computation of the numerator will be more straightforward.

$$I = \int_B |y_1|^{p-2} dy = \int_{-1}^{1} \left( \int_{D(0,\sqrt[4]{1-y_3^2})} |y_1|^{p-2} dy_1 dy_2 \right) dy_3, \tag{3.23}$$

where $B = \{(y_1, y_2, y_3) : \sqrt{y_1^2 + y_2^2} \leq \sqrt[4]{1 - y_3^2}\}$. In general we have

$$\begin{aligned}
\int_{D(0,r)} |y_1|^{p-2} dy_1 dy_2 &= \int_{-r}^{r} \int_{-\sqrt{r^2-y_2^2}}^{\sqrt{r^2-y_2^2}} |y_1|^{p-2} dy_1 dy_2 = 4 \int_0^r \int_0^{\sqrt{r^2-y_2^2}} y_1^{p-2} dy_1 dy_2 \\
&= \frac{4}{p-1} \int_0^r (r^2 - y_2^2)^{\frac{p-1}{2}} dy_2 = \frac{4r^p}{p-1} \int_0^1 (1 - z^2)^{\frac{p-1}{2}} dz \qquad (3.24)\\
&= \frac{2r^p}{p-1} \int_0^1 t^{-\frac{1}{2}} (1-t)^{\frac{p-1}{2}} dt = \frac{2r^p}{p-1} \mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right),
\end{aligned}$$

where $\mathcal{B}$ stands for the beta-function. Here we also use the change of variables: $y_2 = rz$ in the second line and $z^2 = t$ in the last line. Inserting this into $I$ we obtain

$$\begin{aligned}
I &= \frac{2}{p-1} \mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right) \int_{-1}^{1} (1 - y_3^2)^{\frac{p}{4}} dy_3 = \frac{4}{p-1} \mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right) \int_0^1 (1 - y_3^2)^{\frac{p}{4}} dy_3 \\
&= \frac{2}{p-1} \mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right) \int_0^1 t^{-\frac{1}{2}} (1-t)^{\frac{p}{4}} dt = \frac{2}{p-1} \mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right) \mathcal{B}\left(\frac{1}{2}, \frac{p+4}{4}\right).
\end{aligned}$$

Next we consider the integral in the numerator of (3.21), namely

$$J := \int_B |y_1|^{p-2} \left( \frac{1}{2} \langle Cy_h, y_h \rangle + wy_3 \right) dy.$$

Notice, that $\int_B y_3 |y_1|^{p-2} = 0$. Let $C = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, then $\langle Cy_h, y_h \rangle = ay_1^2 + 2by_1y_2 + cy_2^2$. Therefore,

$$2J = a \underbrace{\int_B |y_1|^p dy}_{J_1} + 2b \underbrace{\int_B |y_1|^{p-2} y_1 y_2 dy}_{J_2} + c \underbrace{\int_B |y_1|^{p-2} y_2^2 dy}_{J_3}.$$

Observe, that by the symmetry of $B$ the middle integral $J_2 = 0$. We deal with $J_1$ and $J_3$ analogously to $I$ computing the following integral

$$\begin{aligned}
\int_{D(0,r)} |y_1|^{p-2} y_2^2 dy_1 dy_2 &= 4 \int_0^r y_2^2 \int_0^{\sqrt{r^2-y_2^2}} y_1^{p-2} dy_1 dy_2 = \frac{4}{p-1} \int_0^r y_2^2 (r^2 - y_2^2)^{\frac{p-1}{2}} dy_2 \\
&= \frac{4r^{p+2}}{p-1} \int_0^1 z^2 (1 - z^2)^{\frac{p-1}{2}} dz = \frac{2r^{p+2}}{p-1} \int_0^1 t^{\frac{1}{2}} (1-t)^{\frac{p-1}{2}} dt \\
&= \frac{2r^{p+2}}{p-1} \mathcal{B}\left(\frac{3}{2}, \frac{p+1}{2}\right),
\end{aligned}$$

where again we used the change of variables $y_2 = rz$ and $z^2 = t$. Notice, that (3.24) works for an arbitrary $p > 1$. We use this observation to obtain that $\int_{D(0,r)} |y_1|^p dy_1 dy_2 = \frac{2r^{p+2}}{p+1} \mathcal{B}\left(\frac{1}{2}, \frac{p+3}{2}\right)$. Next, we focus our attention on $J_1$ and $J_3$:

$$
\begin{aligned}
J_1 &= \frac{2}{p+1} \mathcal{B}\left(\frac{1}{2}, \frac{p+3}{2}\right) \int_{-1}^{1} (1-y_3^2)^{\frac{p+2}{4}} dy_3 = \frac{4}{p+1} \mathcal{B}\left(\frac{1}{2}, \frac{p+3}{2}\right) \int_{0}^{1} (1-y_3^2)^{\frac{p+2}{4}} dy_3 \\
&= \frac{2}{p+1} \mathcal{B}\left(\frac{1}{2}, \frac{p+3}{2}\right) \int_{0}^{1} z^{-\frac{1}{2}} (1-z)^{\frac{p+2}{4}} dz = \frac{2}{p+1} \mathcal{B}\left(\frac{1}{2}, \frac{p+3}{2}\right) \mathcal{B}\left(\frac{1}{2}, \frac{p+6}{4}\right).
\end{aligned}
$$

Similarly,

$$
J_3 = \frac{2}{p-1} \mathcal{B}\left(\frac{3}{2}, \frac{p+1}{2}\right) \int_{-1}^{1} (1-y_3^2)^{\frac{p+2}{4}} dy_3 = \frac{2}{p-1} \mathcal{B}\left(\frac{3}{2}, \frac{p+1}{2}\right) \mathcal{B}\left(\frac{1}{2}, \frac{p+6}{4}\right).
$$

We sum up our calculations and upon dividing $J$ by $I$ we arrive at the following:

$$
\gamma_0 = \frac{J}{I} = \mathcal{B}\left(\frac{1}{2}, \frac{p+6}{4}\right) \frac{a\frac{2}{p+1}\mathcal{B}\left(\frac{1}{2}, \frac{p+3}{2}\right) + c\frac{2}{p-1}\mathcal{B}\left(\frac{3}{2}, \frac{p+1}{2}\right)}{2\frac{2}{p-1}\mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right)\mathcal{B}\left(\frac{1}{2}, \frac{p+4}{4}\right)}. \tag{3.25}
$$

In order to simplify the fraction in (3.25) we need to recall the following property of the beta function: $\mathcal{B}(x, y+1) = \frac{y}{x+y}\mathcal{B}(x,y)$ and $\mathcal{B}(x+1, y) = \frac{x}{x+y}\mathcal{B}(x,y)$ which follows from the relation between the Beta and Gamma function and the identity $\Gamma(x+1) = x\Gamma(x)$. We apply these identities to get

$$
\mathcal{B}\left(\frac{1}{2}, \frac{p+3}{2}\right) = \frac{p+1}{p+2}\mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right), \quad \mathcal{B}\left(\frac{3}{2}, \frac{p+1}{2}\right) = \frac{1}{p+2}\mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right).
$$

We apply these formulas in the numerator of (3.25) to obtain the following

$$
\begin{aligned}
\gamma_0 &= \mathcal{B}\left(\frac{1}{2}, \frac{p+6}{4}\right) \mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right) \frac{\frac{a}{p+2} + \frac{c}{(p-1)(p+2)}}{\frac{2}{p-1}\mathcal{B}\left(\frac{1}{2}, \frac{p+1}{2}\right)\mathcal{B}\left(\frac{1}{2}, \frac{p+4}{4}\right)} \\
&= \frac{a(p-1)+c}{2(p+2)} \frac{\mathcal{B}\left(\frac{1}{2}, \frac{p+6}{4}\right)}{\mathcal{B}\left(\frac{1}{2}, \frac{p+4}{4}\right)} = \frac{a(p-1)+c}{2(p+2)} \frac{\Gamma\left(\frac{p+6}{4}\right)^2}{\Gamma\left(\frac{p+4}{4}\right)\Gamma\left(\frac{p+8}{4}\right)} \\
&= 2\frac{a(p-1)+c}{(p+2)(p+4)} \left(\frac{\Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+4}{4}\right)}\right)^2.
\end{aligned}
$$

In order to finish this part of the proof, we express coefficients $a$ and $c$ of matrix $C$ in terms of matrix $A$ and the horizontal vector $\xi$. Recall that $C = R^T A R$, which implies that

$$
a = \xi_1^2 a_{11} + 2\xi_1\xi_2 a_{12} + \xi_2^2 a_{22} \quad \text{and} \quad c = \xi_2^2 a_{11} - 2\xi_1\xi_2 a_{12} + \xi_1^2 a_{22}.
$$

Therefore, $a = \langle A\xi, \xi\rangle$ and $c = \operatorname{tr}(C) - a$. Noting that $\operatorname{tr}(C) = \operatorname{tr}(A)$, we conclude that

$$
\gamma_0 = 2\frac{(p-2)\langle A\xi, \xi\rangle + \operatorname{tr}(A)}{(p+2)(p+4)} \left(\frac{\Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+4}{4}\right)}\right)^2.
$$

Then, upon substituting $\xi$ with $\xi/|\xi|$ we arrive at the assertion (3.16).

We now consider the second assertion of the lemma, namely the asymptotic formula (3.17) for $\mu_p(\varepsilon, u)$ and $u \in C^2(\Omega)$. Suppose $\varepsilon > 0$ is chosen so that $\overline{B(x,\varepsilon)} \subset \Omega$. Consider the function $q(y)$ as in (3.15), with

$$
q(x) = u(x), \quad \xi = \nabla_H u(x), \quad A = \nabla_H^2 u(x), \quad \text{and} \quad \eta = 2\frac{\partial u}{\partial x_3}(x).
$$

Notice that with this notation (and by the assumption $\xi \neq 0$), it holds that

$$\Delta^N_{p,\mathbb{H}_1} u(x) = \mathrm{tr}(A) + (p-2)\frac{\langle A\xi, \xi\rangle}{|\xi|^2}.$$

Set $u_\varepsilon(z) = u(x\delta_\varepsilon(z))$ and $q_\varepsilon(z) = q(x\delta_\varepsilon(z))$. Since $u \in C^2(\Omega)$, it follows that for all $t > 0$, there exists $\varepsilon(t) > 0$ such that for every $z \in \overline{B}$ and all $\varepsilon \in (0, \varepsilon(t))$ it holds $|u_\varepsilon(z) - q_\varepsilon(z)| < t\varepsilon^2$. Furthermore, by claims (4) and (5) of Theorem 3.11 we have $\mu_p(\varepsilon, q \pm t\varepsilon^2)(x) = \mu_p(\varepsilon, q)(x) \pm t\varepsilon^2$. These observations together with Corollary 3.12 and Part (3) of Theorem 3.11 allow us to obtain the following estimates:

$$\frac{\mu_p(\varepsilon, q) - u(x)}{\varepsilon^2} - t \leq \frac{\mu_p(\varepsilon, u) - u(x)}{\varepsilon^2} \leq \frac{\mu_p(\varepsilon, q) - u(x)}{\varepsilon^2} + t.$$

Applying (3.16) we obtain

$$\begin{aligned} C(p)\Delta^N_{p,\mathbb{H}_1} u(x) - t &\leq \liminf_{\varepsilon\to 0} \frac{\mu_p(\varepsilon, q) - u(x)}{\varepsilon^2} \\ &\leq \limsup_{\varepsilon\to 0} \frac{\mu_p(\varepsilon, q) - u(x)}{\varepsilon^2} \leq C(p)\Delta^N_{p,\mathbb{H}_1} u(x) + t, \end{aligned}$$

which implies the assertion (3.17) for $1 < p < \infty$.

### 3.4.2 Case 2: $p = \infty$

We need to show that there exists the limit of the following expression

$$\begin{aligned} \gamma_\varepsilon &= \frac{\mu_\infty(\varepsilon, q) - q(x)}{\varepsilon^2} \qquad (3.26) \\ &= \frac{1}{2\varepsilon}\left( \min_{y\in\overline{B}} \left[ \langle \xi, y_h\rangle + \varepsilon\left( wy_3 + \frac{1}{2}\langle Ay_h, y_h\rangle \right) \right] + \max_{y\in\overline{B}} \left[ \langle \xi, y_h\rangle + \varepsilon\left( wy_3 + \frac{1}{2}\langle Ay_h, y_h\rangle \right) \right] \right). \end{aligned}$$

Let us define a function $g : \mathbb{H}_1 \to \mathbb{R}$ with $g(y) = \langle \xi, y_h\rangle + wy_3 + \frac{1}{2}\langle Ay_h, y_h\rangle$. Observe further, that by $z := \delta_\varepsilon(y)$ there holds

$$\min_{y\in\overline{B}} \left[ \langle \xi, y_h\rangle + \varepsilon\left( wy_3 + \frac{1}{2}\langle Ay_h, y_h\rangle \right) \right] = \frac{1}{\varepsilon} \min_{z\in\overline{B(0,\varepsilon)}} g(z),$$

and

$$\max_{y\in\overline{B}} \left[ \langle \xi, y_h\rangle + \varepsilon\left( wy_3 + \frac{1}{2}\langle Ay_h, y_h\rangle \right) \right] = \frac{1}{\varepsilon} \max_{z\in\overline{B(0,\varepsilon)}} g(z),$$

and it follows that

$$\gamma_\varepsilon = \frac{1}{2\varepsilon^2}\left( \min_{z\in\overline{B(0,\varepsilon)}} g(z) + \max_{z\in\overline{B(0,\varepsilon)}} g(z) \right).$$

Furthermore, notice that $\nabla_H g(0) = \xi \neq 0$. Therefore, we apply Lemma 3.1 and 3.2 in [FLM14] to obtain, that for all small enough $\varepsilon$, there exist points $P^M_\varepsilon = (x^M_\varepsilon, y^M_\varepsilon, t^M_\varepsilon)$ and $P^m_\varepsilon = (x^m_\varepsilon, y^m_\varepsilon, t^m_\varepsilon)$ in $\partial B(0, \varepsilon)$ with the following properties:

$$\max_{\overline{B(0,\varepsilon)}} g = g(x^M_\varepsilon, y^M_\varepsilon, t^M_\varepsilon), \quad \min_{\overline{B(0,\varepsilon)}} g = g(x^m_\varepsilon, y^m_\varepsilon, t^m_\varepsilon).$$

Moreover,

$$\lim_{\varepsilon\to 0} \frac{(x^M_\varepsilon, y^M_\varepsilon)}{\varepsilon} = \frac{\xi}{|\xi|}, \quad \lim_{\varepsilon\to 0} \frac{(x^m_\varepsilon, y^m_\varepsilon)}{\varepsilon} = -\frac{\xi}{|\xi|}. \qquad (3.27)$$

We use these to estimate (3.26) in the following way

$$\frac{1}{2\varepsilon^2}\left(g(P_\varepsilon^m)+g(-P_\varepsilon^m)\right)\leq\frac{1}{2\varepsilon^2}\left(\min_{z\in\overline{B(0,\varepsilon)}}g(z)+\max_{z\in\overline{B(0,\varepsilon)}}g(z)\right)\leq\frac{1}{2\varepsilon^2}\left(g(P_\varepsilon^M)+g(-P_\varepsilon^M)\right). \tag{3.28}$$

Compute

$$\begin{aligned}\frac{1}{2\varepsilon^2}(g(P_\varepsilon^M)+g(-P_\varepsilon^M))&=\frac{1}{4\varepsilon^2}\left(\langle A(x_\varepsilon^M,y_\varepsilon^M),(x_\varepsilon^M,y_\varepsilon^M)\rangle+\langle A(-x_\varepsilon^M,-y_\varepsilon^M),(-x_\varepsilon^M,-y_\varepsilon^M)\rangle\right)\\&=\frac{1}{2}\langle A\left(\frac{x_\varepsilon^M}{\varepsilon},\frac{y_\varepsilon^M}{\varepsilon}\right),\left(\frac{x_\varepsilon^M}{\varepsilon},\frac{y_\varepsilon^M}{\varepsilon}\right)\rangle\xrightarrow{\varepsilon\to0}\frac{1}{2}\frac{\langle A\xi,\xi\rangle}{|\xi|^2}.\end{aligned}$$

We treat the left-hand side of (3.28) similarly to conclude that

$$\mu_\infty(\varepsilon,q)=q(x)+\frac{\varepsilon^2}{2}\frac{\langle A\xi,\xi\rangle}{|\xi|^2}+o(\varepsilon^2).$$

We are now in a position to show the second assertion of the lemma, namely the asymptotic formula (3.17) for $\mu_p(\varepsilon,u)$.

Let $\varepsilon>0$ be such that $\overline{B(x,\varepsilon)}\subset\Omega$. Consider function $q(y)$ as in (3.15) with

$$q(x)=u(x),\quad \xi=\nabla_H u(x),\quad A=\nabla_H^2u(x),\quad w=2\frac{\partial u}{\partial x_3}(x).$$

Notice that with this notation

$$\Delta^N_{p,\mathbb{H}_1}u(x)=\operatorname{tr}(A)+(p-2)\frac{\langle A\xi,\xi\rangle}{|\xi|^2}.$$

Set $u_\varepsilon(z)=u(x\delta_\varepsilon(z))$ and $q_\varepsilon(z)=q(x\delta_\varepsilon(z))$. Since $u\in C^2(\Omega)$ it holds that for all $\eta>0$ there is $\varepsilon=\varepsilon(\eta)>0$ such that for every $z\in\overline{B}$ and all $\varepsilon\in(0,\varepsilon(\eta))$ it holds

$$|u_\varepsilon(z)-q_\varepsilon(z)|<\eta\varepsilon^2.$$

Furthermore, by parts (4) and (5) of Theorem 3.11 we have $\mu_p(\varepsilon,q\pm\eta\varepsilon^2)(x)=\mu_p(\varepsilon,q)(x)\pm\eta\varepsilon^2$. These observations together with Corollary 3.12 and Part (3) of Theorem 3.11 allow us to obtain the following estimates:

$$\frac{\mu_p(\varepsilon,q)-u(x)}{\varepsilon^2}-\eta\leq\frac{\mu_p(\varepsilon,u)-u(x)}{\varepsilon^2}\leq\frac{\mu_p(\varepsilon,q)-u(x)}{\varepsilon^2}+\eta.$$

By applying (3.16) we obtain

$$C(p)\Delta^N_{p,\mathbb{H}_1}u(x)-\eta\leq\liminf_{\varepsilon\to0}\frac{\mu_p(\varepsilon,q)-u(x)}{\varepsilon^2}\leq\limsup_{\varepsilon\to0}\frac{\mu_p(\varepsilon,q)-u(x)}{\varepsilon^2}\leq C(p)\Delta^N_{p,\mathbb{H}_1}u(x)+\eta,$$

where $C(p):=\frac{2}{(p+2)(p+4)}\left(\frac{\Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+4}{4}\right)}\right)^2$.

### 3.4.3 Case 3: $p=1$

Recall, that by the discussion at the beginning of the proof of Lemma 3.21 (cf. (3.18)), the unique number $\gamma_\varepsilon$ for $p=1$ is defined with the following equation

$$|\{z\in B:\langle\xi,z_h\rangle+\frac{\varepsilon}{2}\langle Az_h,z_h\rangle+w\varepsilon z_3<\varepsilon\gamma_\varepsilon\}|=|\{z\in B:\langle\xi,z_h\rangle+\frac{\varepsilon}{2}\langle Az_h,z_h\rangle+w\varepsilon z_3>\varepsilon\gamma_\varepsilon\}|.$$

We apply the same change of variables via the matrix $R$, as described in the paragraph following formula (3.19) (for the sake of simplicity we still use the variable $z$) and divide both inequalities by $\varepsilon$ to arrive at

$$|\{z \in B : \frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz_h, z_h\rangle + wz_3 < \gamma_\varepsilon\}| = |\{z \in B : \frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz_h, z_h\rangle + wz_3 > \gamma_\varepsilon\}|. \tag{3.29}$$

We again assume that $|\xi| = 1$ and let $C = R^T AR$, where $R$ denotes the rotation matrix as defined in the discussion following (3.19). Equation (3.29) means that for each fixed $\varepsilon > 0$, $\gamma_\varepsilon$ is the median $\mu_1(1, h) =: \mu_1(h)$ of the function $h : \overline{B} \to \mathbb{R}$ defined with the following formula

$$h(z) := \frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz_h, z_h\rangle + wz_3.$$

Denote by $c' := \left\|\frac{1}{2}\langle Cz_h, z_h\right\|_{L^\infty(B)} < \infty$. Let us observe, that by monotonicity of $\mu_1$ and property (4) in Theorem 3.11, we obtain the following estimates

$$\begin{aligned} \gamma_\varepsilon &= \mu_1\left(\frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz_h, z_h\rangle + wz_3\right) \\ &\leq \mu_1\left(\frac{z_1}{\varepsilon} + wz_3 + c'\right) \\ &= \mu_1\left(\frac{z_1}{\varepsilon} + wz_3\right) + c', \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} \gamma_\varepsilon &= \mu_1\left(\frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz_h, z_h\rangle + wz_3\right) \\ &\geq \mu_1\left(\frac{z_1}{\varepsilon} + wz_3 - c'\right) \\ &= \mu_1\left(\frac{z_1}{\varepsilon} + wz_3\right) - c'. \end{aligned} \tag{3.31}$$

Let us observe, that for all $\varepsilon > 0$ we have

$$|\{z \in B : \frac{z_1}{\varepsilon} + wz_3 < 0\}| = |\{z \in B : \frac{z_1}{\varepsilon} + wz_3 > 0\}|.$$

since the two quantities are equivalent under the change of variables $z \mapsto -z$. It then follows that

$$\mu_1\left(\frac{z_1}{\varepsilon} + wz_3\right) = 0,$$

and estimates (3.30) and (3.31) reads $-c' \leq \gamma_\varepsilon \leq c'$. Hence $\gamma_\varepsilon$ is bounded, and after passing to a subsequence, there exists $\gamma_0 := \lim_{\varepsilon\to 0} \gamma_\varepsilon$.

Now let us apply the following change of variables to both sides of (3.29)

$$(z_1, z_2, z_3) \mapsto (\varepsilon z_1, z_2, z_3) =: \varepsilon z_1 e_1 + \tilde{z},$$

where $\tilde{z} := (0, z_2, z_3)$. The Jacobian of this transformation is constant, hence it cancels out on both sides and (3.29) becomes

$$\begin{aligned} &|\{z \in \mathbb{R}^3 : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{H}_1} < 1, \quad z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1, z_2), (\varepsilon z_1, z_2)\rangle + wz_3\right) < \gamma_\varepsilon\}| \\ &= |\{z \in \mathbb{R}^3 : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{H}_1} < 1, \quad z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1, z_2), (\varepsilon z_1, z_2)\rangle + wz_3\right) > \gamma_\varepsilon\}|. \end{aligned} \tag{3.32}$$

Let us denote by $\tilde{B} := \{(z_2, z_3) \in \mathbb{R}^2 : |(0, z_2, z_3)|_{\mathbb{H}_1} < 1\}$ and consider a function $F : \{z \in \mathbb{R}^3 : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{H}_1} < 1\} \to \mathbb{R}$ defined by

$$F(z) := z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1, z_2), (\varepsilon z_1, z_2)\rangle + wz_3\right).$$

For small $\varepsilon$, we are going to represent the intersection of the boundaries of sets in (3.32), i.e., the surface $\{F(z) = \gamma_\varepsilon \,:\, |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{H}_1} < 1\}$, as the graph of a function of the form $\tilde{z} \to g_\varepsilon(\tilde{z})e_1 + \tilde{z}$ where $g_\varepsilon : \tilde{B} \to \mathbb{R}$.

Let us observe, that the derivative $F_{z_1}$ can be estimated from below:

$$F_{z_1}(z) = 1 + \varepsilon^2 c_{11} z_1 + \varepsilon c_{12} z_2 > \frac{1}{2}$$

for $\varepsilon$ sufficiently small. This follows from $|\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{H}_1} < 1$ and the fact that

$$-\varepsilon(|c_{11}| + |c_{12}|) \leq \varepsilon^2 c_{11} z_1 + \varepsilon c_{12} z_2 \leq \varepsilon(|c_{11}| + |c_{12}|).$$

Hence, for a fixed $\tilde{z} \in \tilde{B}$ the function $z_1 \to F(z_1 e_1 + \tilde{z})$ is monotone increasing and, therefore, has an inverse denoted $h_{\varepsilon,\tilde{z}}(t)$. It follows that $F(h_{\varepsilon,\tilde{z}}(t)e_1 + \tilde{z}) = t$ and $g_\varepsilon(\tilde{z}) = h_{\varepsilon,\tilde{z}}(\gamma_\varepsilon)$ is a point in the intersection of the boundaries of sets in (3.32). Furthermore, let us observe that, possibly after passing to a subsequence, the following limit exists for all $\tilde{z} \in \tilde{B}$

$$g_\varepsilon(\tilde{z}) \to \gamma_0 - \frac{1}{2} c_{22} z_2^2 - w z_3 \quad \text{as } \varepsilon \to 0^+. \tag{3.33}$$

Indeed, for all $\tilde{z} \in \tilde{B}$ the equation $F(g_\varepsilon(\tilde{z})e_1 + \tilde{z}) = \gamma_\varepsilon$ equivalently reads:

$$g_\varepsilon(\tilde{z}) + \frac{1}{2}\langle C(\varepsilon g_\varepsilon(\tilde{z}), z_2), (\varepsilon g_\varepsilon(\tilde{z}), z_2)\rangle + w z_3 = \gamma_\varepsilon.$$

From this we get that

$$g_\varepsilon(\tilde{z}) + \frac{1}{2}\left(\varepsilon^2 c_{11} g_\varepsilon^2(\tilde{z}) + 2\varepsilon c_{12} g_\varepsilon(\tilde{z}) z_2 + c_{22} z_2^2\right) + w z_3 = \gamma_\varepsilon,$$

which for fixed $\tilde{z}$ and $c_{11} \neq 0$ is the following quadratic equation in $g_\varepsilon(\tilde{z})$:

$$g_\varepsilon^2(\tilde{z})\frac{\varepsilon^2 c_{11}}{2} + g_\varepsilon(\tilde{z})\left(1 + 2\varepsilon c_{12} z_2\right) + \frac{1}{2} c_{22} z_2^2 + w z_3 - \gamma_\varepsilon = 0.$$

Therefore, $g_\varepsilon(\tilde{z})$ has to be either equal to

$$g_\varepsilon(\tilde{z}) = \frac{-1 - 2\varepsilon c_{12} z_2 + \sqrt{(1 + 2\varepsilon c_{12} z_2)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2} c_{22} z_2^2 + w z_3 - \gamma_\varepsilon\right)}}{\varepsilon^2 c_{11}},$$

or equal to

$$g_\varepsilon(\tilde{z}) = \frac{-1 - 2\varepsilon c_{12} z_2 - \sqrt{(1 + 2\varepsilon c_{12} z_2)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2} c_{22} z_2^2 + w z_3 - \gamma_\varepsilon\right)}}{\varepsilon^2 c_{11}}.$$

Observe, that the second solution is of order $\varepsilon^{-2}$ and hence does not lie in the set

$$\{z : |\varepsilon g_\varepsilon(\tilde{z})e_1 + \tilde{z}|_{\mathbb{H}_1} < 1\}$$

for $\varepsilon \to 0^+$. We consider the first solution, which after simplification reads

$$g_\varepsilon(\tilde{z}) = \frac{2\left(\gamma_\varepsilon - \frac{1}{2} c_{22} z_2^2 - w z_3\right)}{\sqrt{(1 + 2\varepsilon c_{12} z_2)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2} c_{22} z_2^2 + w z_3 - \gamma_\varepsilon\right)} + 1 + 2\varepsilon c_{12} z_2}.$$

If $c_{11} = 0$, then

$$g_\varepsilon(\tilde{z})\left(1 + 2\varepsilon c_{12} z_2\right) = \gamma_\varepsilon - \frac{1}{2} c_{22} z_2^2 - w z_3.$$

Therefore, we conclude (3.33). Thus, we can represent the measures of the sets appearing in (3.32) as integrals, and obtain the following

$$\int_{\tilde{B}} \left[\min\left\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\right\} + G_\varepsilon(\tilde{z})\right] \, d\tilde{z} = \int_{\tilde{B}} \left[G_\varepsilon(\tilde{z}) - \max\left\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\right\}\right] \, d\tilde{z}, \tag{3.34}$$

where

$$G_\varepsilon(\tilde{z}) := \frac{1}{\varepsilon}\sqrt{\sqrt{1-z_3^2} - z_2^2}.$$

The function $G_\varepsilon$ is the non-negative solution $z_1$ to the equation $|\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{H}_1} = 1$ describing the boundary of $B$. Observe, that (3.34) is equivalent to

$$\int_{\tilde{B}} \left[\min\left\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\right\} + \max\left\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\right\}\right] \, d\tilde{z} = 0. \tag{3.35}$$

Applying the dominated convergence theorem to the case $\varepsilon \to 0^+$ in (3.35) we obtain that

$$\int_{\tilde{B}} \left(\gamma_0 - \frac{1}{2}c_{22}z_2^2 - wz_3\right) d\tilde{z} = 0. \tag{3.36}$$

The symmetry of $\tilde{B}$ shows that $\int_{\tilde{B}} wz_3 = 0$ and so (3.36) becomes

$$\gamma_0 = \frac{c_{22}}{2} ⨍_{\tilde{B}} z_2^2 d\tilde{z}. \tag{3.37}$$

Let us calculate the above integral

$$\begin{aligned}\int_{\tilde{B}} z_2^2 d\tilde{z} &= \int_{\{z_2^4 + z_3^2 < 1\}} z_2^2 dz_2 dz_3 = \int_{-1}^{1}\int_{-(1-z_3^2)^{\frac{1}{4}}}^{(1-z_3^2)^{\frac{1}{4}}} z_2^2 dz_2 dz_3 = \int_{-1}^{1} \frac{2}{3}(1-z_3^2)^{\frac{3}{4}} dz_3 \\ &= \frac{4}{3}\int_0^1 (1-z_3^2)^{\frac{3}{4}} dz_3 \overset{t:=z_3^2}{=} \frac{2}{3}\int_0^1 (1-t)^{\frac{3}{4}} t^{-\frac{1}{2}} dt = \frac{2}{3}\mathcal{B}\left(\frac{7}{4}, \frac{1}{2}\right).\end{aligned}$$

We follow the above reasoning to compute the measure of $\tilde{B}$ in the following way

$$|\tilde{B}| = \int_{-1}^{1}\int_{-(1-z_3^2)^{\frac{1}{4}}}^{(1-z_3^2)^{\frac{1}{4}}} 1 dz_2 dz_3 = \int_{-1}^{1} 2(1-z_3^2)^{\frac{1}{4}} dz_3 \overset{t:=z_3^2}{=} 2\int_0^1 (1-t)^{\frac{1}{4}} t^{-\frac{1}{2}} dt = 2\mathcal{B}\left(\frac{5}{4}, \frac{1}{2}\right).$$

We sum up the above calculations to rewrite (3.37) in the following way

$$\gamma_0 = \frac{c_{22}}{6}\frac{\mathcal{B}\left(\frac{7}{4}, \frac{1}{2}\right)}{\mathcal{B}\left(\frac{5}{4}, \frac{1}{2}\right)}.$$

Therefore,

$$\gamma_0 = \frac{c_{22}}{6}\frac{\mathcal{B}\left(\frac{7}{4}, \frac{1}{2}\right)}{\mathcal{B}\left(\frac{5}{4}, \frac{1}{2}\right)}\left(\operatorname{tr}(A) - \frac{\langle A\xi, \xi\rangle}{|\xi|^2}\right),$$

which follows from the same argument used in the case $1 < p < \infty$, and the same reasoning allows us to conclude (3.11) and (3.12) for $p = 1$ as well. Thus, the proof of Lemma 3.15 is completed for all $1 \le p \le \infty$. □

## 3.5 Lemma 3.15 in the Carnot group of step 2

In what follows we are going to prove Lemma 3.15 in the setting of Carnot groups of step 2. In order to obtain this result, we need to find a generalization of Lemma 3.21 for $\mathbb{H}_1$. Observe, that a reasonable counterpart of the formula for the quadratic function $q$ would arise from the Taylor expansion in step 2 Carnot groups. In the next lemma we deal with a step 2 Carnot group $\mathbb{G}$ (recall Definition 3.2 for $k = 2$). For the sake of brevity let us denote by $n := v_1$ and $k := v_2$ the dimensions of subspaces $V_1$ and $V_2$, respectively. Recall, that we compute the distance to 0 of an element $x = (x^{(1)}, x^{(2)}) = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) \in \mathbb{G}$ by using formula (3.3):

$$d(0,x) = \left( \left( x_1^2 + \ldots + x_n^2 \right)^2 + x_{n+1}^2 + \ldots + x_{n+k}^2 \right)^{\frac{1}{4}} = \left( \|x^{(1)}\|_{\mathbb{R}^n}^4 + \|x^{(2)}\|_{\mathbb{R}^k}^2 \right)^{\frac{1}{4}}.$$

Then, the following result is a generalization of Lemma 3.21 to the Carnot groups of step 2.

**Lemma 3.22.** *Let $\mathbb{G}$ be a Carnot group of step 2. Moreover, let $\Omega \subset \mathbb{G}$ be an open set and $x \in \Omega$ be a point such that ball $B(x, \varepsilon) \subset \Omega$ for all small enough radii $\varepsilon \leq \varepsilon_0(x)$. Let $1 \leq p \leq \infty$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{R}^k$. Let further $A$ be a symmetric $n \times n$ matrix with real coefficients. Moreover, consider the quadratic function $q : B(x, \varepsilon) \to \mathbb{R}$ given by*

$$q(y) = q(x) + \langle \xi, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^n} + \langle \eta, (x^{-1}y)^{(2)} \rangle_{\mathbb{R}^k} + \frac{1}{2} \langle A(x^{-1}y)^{(1)}, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^n}, \quad y \in B(x, \varepsilon), \tag{3.38}$$

*where $(x^{-1}y)^{(1)}$ and $(x^{-1}y)^{(2)}$ are the horizontal and the vertical components of $x^{-1}y$, respectively. Then it holds that*

$$\mu_p(\varepsilon, q) = q(x) + \varepsilon^2 C(p, n, k) \left( \operatorname{tr}(A) + (p-2) \frac{\langle A\xi, \xi \rangle}{|\xi|^2} \right) + o(\varepsilon^2), \tag{3.39}$$

*for $C(p,n,k) := \frac{1}{2(n+p)} \frac{\mathcal{B}\left(\frac{k}{2}, \frac{n+p+4}{4}\right)}{\mathcal{B}\left(\frac{k}{2}, \frac{n+p+2}{4}\right)}$. Furthermore, if $u \in C^2(\Omega)$ with $\nabla_{V_1} u(x) \neq 0$, then it holds*

$$\mu_p(\varepsilon, u)(x) = u(x) + C(p) \Delta_{p,\mathbb{G}}^N u(x) \varepsilon^2 + o(\varepsilon^2), \quad as \quad \varepsilon \to 0. \tag{3.40}$$

*Proof.* The proof goes verbatim to the proof of Lemma 3.21. We begin with computing $\mu_p(\varepsilon, q)$. For $z = (z^{(1)}, z^{(2)}) \in B := B(0,1)$, where $B$ denotes the unit open ball in $\mathbb{G}$:

$$B = \left\{ z \in \mathbb{R}^{n+k} : (z_1^2 + \ldots + z_n^2)^2 + z_{n+1}^2 + \ldots + z_{n+k}^2 < 1 \right\}$$

we introduce the following

$$q_\varepsilon(z) = q(x\delta_\varepsilon(z)), \quad v_\varepsilon(z) = \frac{q_\varepsilon(z) - q(x)}{\varepsilon} \quad \text{and} \quad v(z) = \langle \xi, (z_1, \dots, z_n) \rangle_{\mathbb{R}^n} := \langle \xi, z^{(1)} \rangle_{\mathbb{R}^n}.$$

We know that $\mu_p(\varepsilon, q)(x) = \mu_p(1, q_\varepsilon)(0)$ by Corollary 3.12. Then, by points (4) and (5) of Theorem 3.11, we see that

$$\frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon} = \mu_p(1, v_\varepsilon)(0).$$

Let us observe, that

$$\begin{aligned} v_\varepsilon(z) &= \frac{1}{\varepsilon} \left( \langle \xi, \delta_\varepsilon(z)^{(1)} \rangle + \frac{1}{2} \langle A\delta_\varepsilon(z)^{(1)}, \delta_\varepsilon(z)^{(1)} \rangle + \langle \eta, \delta_\varepsilon(z)^{(2)} \rangle \right) \\ &= \langle \xi, z^{(1)} \rangle + \frac{\varepsilon}{2} \langle Az^{(1)}, z^{(1)} \rangle + \varepsilon \langle \eta, z^{(2)} \rangle. \end{aligned} \tag{3.41}$$

Therefore, $v_\varepsilon$ converges uniformly to $v$ as $\varepsilon \to 0$ on $\overline{B}$. We appeal to the second part of claim (2) in Theorem 3.11 to obtain, that $\mu_p(1, v_\varepsilon)(0) \to \mu_p(1, v)(0)$ as $\varepsilon \to 0$. Recall that the characterization of $\lambda = \mu_p(1, v)(0)$ given by (3.8) in Theorem 3.11 states that if $p \in [1, \infty)$, then $\lambda$ is the unique number such that

$$\int_B |\langle \xi, y^{(1)} \rangle - \lambda|^{p-2} (\langle \xi, y^{(1)} \rangle - \lambda) dy = 0.$$

On the other hand

$$\int_B |\langle \xi, y^{(1)}\rangle|^{p-2}(\langle \xi, y^{(1)}\rangle)dy = 0,$$

by symmetry of the unit ball and the change of variables

$$\Phi(y^{(1)}, y^{(2)}) = (-y_1, -y_2, \ldots, -y_n, y_{n+1}, \ldots, y_{n+k}), \quad |J_\Phi| = 1, \quad \Phi(B) = B.$$

Therefore, $\mu_p(1, v)(0) = \lambda = 0$.

If $p = \infty$, then by (3.9):

$$\mu_\infty(1, v)(0) = \frac{1}{2}\left(\min_{\overline{B}}\langle \xi, y^{(1)}\rangle + \max_{\overline{B}}\langle \xi, y^{(1)}\rangle\right) = \frac{1}{2}(-|\xi| + |\xi|) = 0.$$

Subsequently, we define

$$\gamma_\varepsilon = \frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon^2}.$$

### 3.5.1 Case 1: $1 < p < \infty$

For the sake of brevity let us introduce a function $f(s) = |s|^{p-2}s$. Then, upon applying (3.8) to $\mu_p(1, v_\varepsilon)(0) = \varepsilon\gamma_\varepsilon$ we obtain

$$\int_B f(v_\varepsilon(z) - \varepsilon\gamma_\varepsilon)dz = 0.$$

By using (3.41) this can be transformed to the following expression:

$$\int_B f\left(\langle \xi, z^{(1)}\rangle + \varepsilon\left(\frac{1}{2}\langle Az^{(1)}, z^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, z^{(2)}\rangle\right)\right)dz = 0. \tag{3.42}$$

Without loss of generality we may assume that $|\xi| = 1$, since otherwise we can consider the quadratic function $\tilde{q} = q/|\xi|$. Let us apply the change of variables $z = (z^{(1)}, z^{(2)}) = (Ry^{(1)}, y^{(2)})$ in (3.42), where $R$ is a $n \times n$ rotation matrix with $R^T\xi = e_1$. Set $C = R^TAR$, then (3.42) reads

$$\int_B f\left(y_1 + \varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right)dy = 0.$$

Therefore, by the Fundamental Theorem of Calculus, we have:

$$\int_B \left[\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right)dt\right]\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)dy = 0. \tag{3.43}$$

Equality (3.43) implies that $\gamma_\varepsilon$ is a weighted mean value of the function $\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle$ over $B$ with respect to a weighted Lebesgue measure $\omega(y)dy$ for

$$\omega(y) := \int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right)dt, \quad y \in B.$$

The weight function $w$ is nonnegative since $f'(s) = (p-1)|s|^{p-2} \geq 0$. Therefore, $\gamma_\varepsilon$ is bounded by $c := \left\|\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle\right\|_{L^\infty(B)}$.

Let us consider any subsequence of $(\gamma_\varepsilon)$ converging to $\gamma_0$ as $\varepsilon \to 0^+$, which for the sake of brevity, we also denote by $(\gamma_\varepsilon)$. Let us consider two cases. If $2 \leq p < \infty$, then for all $y \in B$ we obtain

$$\begin{aligned}&\left|\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right)dt\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right| \\ &\leq 2c(p-1)\int_0^1 \left|y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right|^{p-2}dt \leq 2c(p-1)(1 + 2c\varepsilon).\end{aligned}$$

Therefore, by the dominated convergence theorem the sequence $(\gamma_\varepsilon)$ converges to

$$\gamma_0 := \lim_{\varepsilon\to 0}\gamma_\varepsilon = \frac{\int_B |y_1|^{p-2}\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle\right)dy}{\int_B |y_1|^{p-2}dy}. \tag{3.44}$$

Let now $1 < p < 2$. Fix $0 < \theta < 1$ and split the integral (3.43) into two parts: over the set $G_\theta := B\cap\{|y_1|> \theta\}$ and $F_\theta := B\cap\{|y_1|\leq \theta\}$. Observe that for all $y\in G_\theta$ and for all $\varepsilon > 0$ satisfying $2c\varepsilon < \theta$, we have the following:

$$\left|\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right)dt\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right|$$
$$\leq 2c\,||y_1|-2c\varepsilon|^{p-2}.$$

Moreover,

$$\lim_{\varepsilon\to 0}\int_{G_\theta} ||y_1|-2c\varepsilon|^{p-2}\,dy = \int_{G_\theta}|y_1|^{p-2}dy < \int_B |y_1|^{p-2}dy, \tag{3.45}$$

where the inequality holds uniformly for all $\theta\in(0,1)$. Furthermore, the last integral turns out to be finite which can be seen from the explicit calculation below in (3.46). Hence, by applying Theorem 5.4 in [IMW17] to $X = G_\theta$ with $\nu$ being the Lebesgue measure, we obtain the following:

$$\lim_{\varepsilon\to 0}\int_{G_\theta}\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right)dt\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)dy$$
$$= \int_{G_\theta}(p-1)|y_1|^{p-2}\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle - \gamma_0\right).$$

Observe that here the upper bound in (3.45) allows us to conclude that the limit as $\theta\to 0^+$ is finite. We now focus on the part of the integral in (3.43) involving the set $F_\theta$. Since $|F_\theta|= \int_{F_\theta} 1dy$, then upon writing this integral as in (3.46), one sees that $|F_\theta|= c(k,n,k)\theta$, and so $|F_\theta|\to 0$, as $\theta\to 0^+$. Moreover, it suffices to consider $\theta = 2c\varepsilon$ and the related $\int_{F_{2c\varepsilon}} ||y_1|-2c\varepsilon|^{p-2}\,dy$. We again appeal to integral (3.46) and reduce our computations to finding

$$\int_{B_n(0,R_1)\cap\{|y_1|\leq 2c\varepsilon\}}(2c\varepsilon - |y_1|)^{p-2}\,dy^{(1)}.$$

However, direct computation shows that this integral is of order $\varepsilon^{p-1}$, which then allows us to let $\varepsilon\to 0^+$, and in turn conclude (3.44).

In order to complete the proof, we only need to compute the above two integrals. We begin with the denominator of (3.44), cf. (3.23):

$$I = \int_B |y_1|^{p-2}dy = \int_{B_k(0,1)}\left(\int_{B_n(0,\sqrt[4]{1-\|y^{(2)}\|^2})}|y_1|^{p-2}dy^{(1)}\right)dy^{(2)}, \tag{3.46}$$

where $B_l(0,r)$ stands for a ball in $\mathbb{R}^l$ for $l\in\{k,n\}$ centered at 0 with radius $r>0$. Upon applying the change of variables and Lemma 3.16 with $\alpha_1 = p-2$ and $\alpha_i = 0$ for $i = 2,\dots,n$ we have

$$\int_{B_n(0,r)}|y_1|^{p-2}dy^{(1)} = r^{n+p-2}\int_{B_n(0,1)}|y_1|^{p-2}dy^{(1)} = r^{n+p-2}2^n\int_{T_n} y_1^{p-2}dy^{(1)}$$
$$= r^{n+p-2}\frac{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{1}{2}\right)^{n-1}}{\Gamma\left(\frac{n+p}{2}\right)}. \tag{3.47}$$

We apply (3.47) in $I$ with $r = \sqrt[4]{1-\|y^{(2)}\|^2}$ to obtain

$$I = \frac{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{1}{2}\right)^{n-1}}{\Gamma\left(\frac{n+p}{2}\right)}\int_{B_k(0,1)}\left(1-\left\|y^{(2)}\right\|^2\right)^{\frac{n+p-2}{4}}dy^{(2)}.$$

Since the integrand is a radial function, we apply the spherical coordinates and obtain that

$$
\begin{aligned}
I &= \frac{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{1}{2}\right)^{n-1}}{\Gamma\left(\frac{n+p}{2}\right)}\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\int_0^1 (1-r^2)^{\frac{n+p-2}{4}} r^{k-1}dr \\
&= \frac{2\sqrt{\pi}^{k+n-1}\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{n+p}{2}\right)\Gamma\left(\frac{k}{2}\right)}\frac{1}{2}\int_0^1 (1-t)^{\frac{n+p-2}{4}} t^{\frac{k-2}{2}}dt \qquad\qquad (t := r^2) \\
&= \frac{\sqrt{\pi}^{k+n-1}\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{n+p}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathcal{B}\left(\frac{k}{2}, \frac{n+p+2}{4}\right).
\end{aligned}
$$

Next we consider the integral in the numerator of (3.44), namely

$$
J := \int_B |y_1|^{p-2}\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle\right) dy.
$$

Notice, that $\int_B \langle \eta, y^{(2)}\rangle |y_1|^{p-2} = 0$. Let us denote the coefficients of matrix $C$ as follows $C = [c_{ij}]_{i,j=1,\dots,n}$, then

$$
2J = \underbrace{c_{11}\int_B |y_1|^p dy}_{J_1} + \underbrace{\sum_{i\neq j} c_{ij}\int_B |y_1|^{p-2} y_i y_j dy}_{J_2} + \underbrace{\sum_{i=2}^{n} c_{ii}\int_B |y_1|^{p-2} y_i^2 dy}_{J_3}.
$$

Observe, that by the symmetry of $B$ every integral term of the sum $J_2$ vanishes. We will handle $J_1$ and $J_3$ analogously to $I$. First, we compute the following integrals

$$
\int_{B_n(0,r)} |y_1|^{p-2} y_i^2 dy^{(1)} = r^{n+p}\frac{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)^{n-2}}{\Gamma\left(\frac{p+n+2}{2}\right)} = r^{n+p}\frac{\sqrt{\pi}^{n-1}\Gamma\left(\frac{p-1}{2}\right)}{2\Gamma\left(\frac{p+n+2}{2}\right)}, \quad \text{for } i = 2, \dots, n \tag{3.48}
$$

where we again use Lemma 3.16 for $\alpha_1 = p-2, \alpha_i = 2$ and $\alpha_j = 0$ for the remaining $j \neq i$; we also apply familiar property of $\Gamma$ functions: $\Gamma(1+s) = s\Gamma(s)$ for $s = \frac{1}{2}$. Moreover, notice that (3.47) works for an arbitrary $p > 1$. We use this observation to obtain that

$$
\int_{B_n(0,r)} |y_1|^p dy^{(1)} = \frac{r^{n+p}\sqrt{\pi}^{n-1}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p+2}{2}\right)}.
$$

We are in a position to complete the computations for $J_1$ and $J_3$:

$$
\begin{aligned}
J_1 &= c_{11}\frac{\sqrt{\pi}^{n-1}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p+2}{2}\right)}\int_{B_k(0,1)} \left(1 - \left\|y^{(2)}\right\|^2\right)^{\frac{n+p}{4}} dy^{(2)} \\
&= c_{11}\frac{\sqrt{\pi}^{n-1}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p+2}{2}\right)}\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\int_0^1 (1-r^2)^{\frac{n+p}{4}} r^{k-1}dr \\
&= c_{11}\frac{\sqrt{\pi}^{n+k-1}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p+2}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathcal{B}\left(\frac{k}{2}, \frac{n+p+4}{4}\right),
\end{aligned}
$$

where in the second line we used the fact that the integrand is radial and the spherical coordinates can be applied simplifying the integrand. Similarly, by (3.48) we get

$$
\begin{aligned}
J_3 &= \sum_{i=2}^{n} c_{ii}\frac{\sqrt{\pi}^{n-1}\Gamma\left(\frac{p-1}{2}\right)}{2\Gamma\left(\frac{p+n+2}{2}\right)}\int_{B_k(0,1)} \left(1 - \left\|y^{(2)}\right\|^2\right)^{\frac{n+p}{4}} dy^{(2)} \\
&= \sum_{i=2}^{n} c_{ii}\frac{\sqrt{\pi}^{n+k-1}\Gamma\left(\frac{p-1}{2}\right)}{2\Gamma\left(\frac{p+n+2}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathcal{B}\left(\frac{k}{2}, \frac{n+p+4}{4}\right).
\end{aligned}
$$

We collect the above calculations to arrive at

$$
\begin{aligned}
J = \frac{J_1+J_3}{2} &= \frac{\sqrt{\pi}^{n+k-1}}{2\Gamma\left(\frac{p+n+2}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathcal{B}\left(\frac{k}{2},\frac{n+p+4}{4}\right)\left(c_{11}\Gamma\left(\frac{p+1}{2}\right)+\frac{1}{2}\sum_{i=2}^{n}c_{ii}\Gamma\left(\frac{p-1}{2}\right)\right)\\
&= \frac{\sqrt{\pi}^{n+k-1}\Gamma\left(\frac{p-1}{2}\right)}{2\Gamma\left(\frac{p+n+2}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathcal{B}\left(\frac{k}{2},\frac{n+p+4}{4}\right)\left(c_{11}\frac{p-1}{2}+\sum_{i=2}^{n}c_{ii}\right)\\
&= \frac{\sqrt{\pi}^{n+k-1}\Gamma\left(\frac{p-1}{2}\right)}{4\Gamma\left(\frac{p+n+2}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathcal{B}\left(\frac{k}{2},\frac{n+p+4}{4}\right)\left(c_{11}(p-2)+\operatorname{tr}(C)\right).
\end{aligned}
$$

Above, we again appeal to the same property of Gamma functions as in (3.48). We sum up our calculations and upon dividing $J$ by $I$ to obtain the following

$$
\begin{aligned}
\gamma_0 = \frac{I}{J} &= \frac{\frac{\sqrt{\pi}^{n+k-1}\Gamma\left(\frac{p-1}{2}\right)}{4\Gamma\left(\frac{p+n+2}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathcal{B}\left(\frac{k}{2},\frac{n+p+4}{4}\right)\left(c_{11}(p-2)+\operatorname{tr}(C)\right)}{\frac{\sqrt{\pi}^{k+n-1}\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{n+p}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathcal{B}\left(\frac{k}{2},\frac{n+p+2}{4}\right)}\\
&= \frac{\Gamma\left(\frac{n+p}{2}\right)}{4\Gamma\left(\frac{n+p+2}{2}\right)}\frac{\mathcal{B}\left(\frac{k}{2},\frac{n+p+4}{4}\right)}{\mathcal{B}\left(\frac{k}{2},\frac{n+p+2}{4}\right)}\left(c_{11}(p-2)+\operatorname{tr}(C)\right)\\
&= \frac{1}{2(n+p)}\frac{\mathcal{B}\left(\frac{k}{2},\frac{n+p+4}{4}\right)}{\mathcal{B}\left(\frac{k}{2},\frac{n+p+2}{4}\right)}\left(c_{11}(p-2)+\operatorname{tr}(C)\right).
\end{aligned}
$$

In order to finish this part of the proof, we express the constants $c_{11}$ and $\operatorname{tr}(C)$ in terms of matrix $A$ and the vector $\xi$. Recall that $C = R^TAR$ and $R^T\xi = e_1$, which implies that

$$
c_{11} = \langle Ce_1, e_1\rangle = \langle CR^T\xi, R^T\xi\rangle = \langle R(R^TAR)R^T\xi,\xi\rangle = \langle A\xi,\xi\rangle
$$

and due to the orthogonality of $R$ there holds $\operatorname{tr}(C) = \operatorname{tr}(R^TAR) = \operatorname{tr}(A)$. Therefore, we conclude that

$$
\gamma_0 = \frac{1}{2(n+p)}\frac{\mathcal{B}\left(\frac{k}{2},\frac{n+p+4}{4}\right)}{\mathcal{B}\left(\frac{k}{2},\frac{n+p+2}{4}\right)}\left(\langle A\xi,\xi\rangle(p-2)+\operatorname{tr}(A)\right).
$$

Then, upon substituting $\xi$ with $\xi/|\xi|$ we arrive at the assertion (3.39).

We are now in a position to show the second assertion of the lemma, namely the asymptotic formula (3.40) for $\mu_p(\varepsilon, u)$.

Let $\varepsilon > 0$ be such that $\overline{B(x,\varepsilon)} \subset \Omega$. Consider function $q(y)$ as in (3.38) with

$$
q(x) = u(x), \quad \xi = \nabla_{V_1}u(x), \quad A = \nabla^2_{V_1}u(x), \quad \eta = 2\nabla_{V_2}u(x).
$$

Notice that with this notation

$$
\Delta^N_{p,\mathbb{G}}u(x) = \operatorname{tr}(A) + (p-2)\frac{\langle A\xi,\xi\rangle}{|\xi|^2}.
$$

Set $u_\varepsilon(z) = u(x\delta_\varepsilon(z))$ and $q_\varepsilon(z) = q(x\delta_\varepsilon(z))$. Since $u \in C^2(\Omega)$ it holds that for all $t > 0$ there is $\varepsilon = \varepsilon(t) > 0$ such that for every $z \in \overline{B}$ and all $\varepsilon \in (0, \varepsilon(t))$ it holds

$$
|u_\varepsilon(z) - q_\varepsilon(z)| < t\varepsilon^2.
$$

Furthermore, by parts (4) and (5) of Theorem 3.11 we have $\mu_p(\varepsilon, q \pm t\varepsilon^2)(x) = \mu_p(\varepsilon, q)(x) \pm t\varepsilon^2$. These observations together with Corollary 3.12 and part (3) of Theorem 3.11 allow us to obtain the following estimates:

$$
\frac{\mu_p(\varepsilon,q)-u(x)}{\varepsilon^2} - t \le \frac{\mu_p(\varepsilon,u)-u(x)}{\varepsilon^2} \le \frac{\mu_p(\varepsilon,q)-u(x)}{\varepsilon^2} + t.
$$

By applying (3.39) we obtain

$$C(p,n,k)\Delta^N_{p,\mathbb{G}}u(x)-t \le \liminf_{\varepsilon\to 0}\frac{\mu_p(\varepsilon,q)-u(x)}{\varepsilon^2} \le \limsup_{\varepsilon\to 0}\frac{\mu_p(\varepsilon,q)-u(x)}{\varepsilon^2} \le C(p,n,k)\Delta^N_{p,\mathbb{G}}u(x)+t,$$

where $C(p,n,k) := \frac{1}{2(n+p)}\frac{\mathcal{B}\left(\frac{k}{2},\frac{n+p+4}{4}\right)}{\mathcal{B}\left(\frac{k}{2},\frac{n+p+2}{4}\right)}$.

### 3.5.2 Case 2: $p=\infty$

The main difference between this case and the proof presented in Chapter 3.4.2 is hidden in the auxiliary results to which we refer. While in Chapter 3.4.2 we refer to the work of Ferrari–Liu–Manfredi (Lemma 3.1 and 3.2 in [FLM14]) concerning the Heisenberg group, here we have to use a more refined result by Ferrari–Pinamonti (Lemma 1.5 and 1.6 in [FP15]) valid in general Carnot group.

Recall that for $p=\infty$ there holds

$$\begin{aligned}\gamma_\varepsilon &= \frac{\mu_\infty(\varepsilon,q)-q(x)}{\varepsilon^2}\\ &= \frac{1}{2\varepsilon}\left(\min_{y\in\overline{B}}\left[\langle\xi,y^{(1)}\rangle+\varepsilon\left(\langle\eta,y^{(2)}\rangle+\frac{1}{2}\langle Ay^{(1)},y^{(1)}\rangle\right)\right]\right.\\ &\left.+\max_{y\in\overline{B}}\left[\langle\xi,y^{(1)}\rangle+\varepsilon\left(\langle\eta,y^{(2)}\rangle+\frac{1}{2}\langle Ay^{(1)},y^{(1)}\rangle\right)\right]\right).\end{aligned} \tag{3.49}$$

In order to show that there exists the limit of $\gamma_\varepsilon$ we define a function $g:\mathbb{G}\to\mathbb{R}$ with $g(y)=\langle\xi,y^{(1)}\rangle+\langle\eta,y^{(2)}\rangle+\frac{1}{2}\langle Ay^{(1)},y^{(1)}\rangle$. Observe further, that by $\delta_\varepsilon(y)=:z$ there holds

$$\min_{y\in\overline{B}}\left[\langle\xi,y^{(1)}\rangle+\varepsilon\left(\langle\eta,y^{(2)}\rangle+\frac{1}{2}\langle Ay^{(1)},y^{(1)}\rangle\right)\right]=\frac{1}{\varepsilon}\min_{z\in\overline{B(0,\varepsilon)}}g(z),$$

and

$$\max_{y\in\overline{B}}\left[\langle\xi,y^{(1)}\rangle+\varepsilon\left(\langle\eta,y^{(2)}\rangle+\frac{1}{2}\langle Ay^{(1)},y^{(1)}\rangle\right)\right]=\frac{1}{\varepsilon}\max_{z\in\overline{B(0,\varepsilon)}}g(z).$$

Furthermore, notice that $\nabla_{V_1}g(0)=\xi\neq 0$.

Before we apply Lemma 1.5 and 1.6 in [FP15] let us comment on the differences between these results and Lemma 3.1 and 3.2 in [FLM14] which we applied in the proof of analogous case in Chapter 3.4.2. Lemma 3.1 in [FLM14] and Lemma 1.5 in [FP15] assert existence of points $P_{\varepsilon,M}$ and $P_{\varepsilon,m}$ (see below) and their proofs are the same. The main difference lies in the asymptotic results: Lemma 3.2 in [FLM14] is rather straightforward (the main tool used in the proof is the method of Lagrange multipliers), while the proof of Lemma 1.6 in [FP15] is much more technically involved, which is due to a complicated geometry of general Carnot groups. We apply Lemma 1.5 and 1.6 in [FP15] to obtain, that for all small enough $\varepsilon$, there exist points $P_{\varepsilon,M}=(y^{(1)}_{\varepsilon,M},y^{(2)}_{\varepsilon,M})$ and $P_{\varepsilon,m}=(y^{(1)}_{\varepsilon,m},y^{(2)}_{\varepsilon,m})$ in $\partial B(0,\varepsilon)$ with the following properties:

$$\max_{\overline{B(0,\varepsilon)}}g=g(P_{\varepsilon,M}),\quad \min_{\overline{B(0,\varepsilon)}}g=g(P_{\varepsilon,m}).$$

Moreover,

$$\lim_{\varepsilon\to 0}\frac{y^{(1)}_{\varepsilon,M}}{\varepsilon}=\frac{\xi}{|\xi|},\quad \lim_{\varepsilon\to 0}\frac{y^{(1)}_{\varepsilon,m}}{\varepsilon}=-\frac{\xi}{|\xi|}. \tag{3.50}$$

We use these to estimate (3.49) in the following way

$$\frac{1}{2\varepsilon^2}\left(g(P_{\varepsilon,m})+g(-P_{\varepsilon,m})\right)\le\frac{1}{2\varepsilon^2}\left(\min_{z\in\overline{B(0,\varepsilon)}}g(z)+\max_{z\in\overline{B(0,\varepsilon)}}g(z)\right)\le\frac{1}{2\varepsilon^2}\left(g(P_{\varepsilon,M})+g(-P_{\varepsilon,M})\right). \tag{3.51}$$

Compute

$$\frac{1}{2\varepsilon^2}(g(P_{\varepsilon,M})+g(-P_{\varepsilon,M})) = \frac{1}{4\varepsilon^2}\left(\langle Ay^{(1)}_{\varepsilon,M}, y^{(1)}_{\varepsilon,M}\rangle + \langle A - y^{(1)}_{\varepsilon,M}, -y^{(1)}_{\varepsilon,M}\rangle\right)$$
$$= \frac{1}{2}\left\langle A\frac{y^{(1)}_{\varepsilon,M}}{\varepsilon}, \frac{y^{(1)}_{\varepsilon,M}}{\varepsilon}\right\rangle \xrightarrow{\varepsilon\to 0} \frac{1}{2}\frac{\langle A\xi,\xi\rangle}{|\xi|^2}.$$

We treat the left-hand side of (3.51) similarly to conclude that

$$\mu_\infty(\varepsilon, q) = q(x) + \frac{\varepsilon^2}{2}\frac{\langle A\xi,\xi\rangle}{|\xi|^2} + o(\varepsilon^2).$$

### 3.5.3 Case 3: $p = 1$

Recall, that for $p = 1$ the unique number $\gamma_\varepsilon$ is defined with the following equation (cf. (3.41)):

$$|\{z\in B : \langle\xi, z^{(1)}\rangle + \varepsilon\left(\frac{1}{2}\langle Az^{(1)}, z^{(1)}\rangle + \langle\eta, z^{(2)}\rangle\right) < \varepsilon\gamma_\varepsilon\}|$$
$$= |\{z\in B : \langle\xi, z^{(1)}\rangle + \varepsilon\left(\frac{1}{2}\langle Az^{(1)}, z^{(1)}\rangle + \langle\eta, z^{(2)}\rangle\right) > \varepsilon\gamma_\varepsilon\}|.$$

Let us apply the change of variables as described in the paragraph following formula (3.42) (for the sake of simplicity we still use the variable $z$) and divide both inequalities by $\varepsilon$ to arrive at

$$|\{z\in B : \frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz^{(1)}, z^{(1)}\rangle + \langle\eta, z^{(2)}\rangle < \gamma_\varepsilon\}| = |\{z\in B : \frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz^{(1)}, z^{(1)}\rangle + \langle\eta, z^{(2)}\rangle > \gamma_\varepsilon\}|. \tag{3.52}$$

As previously we assume that $|\xi| = 1$ and denote $C = R^TAR$, where $R$ denotes the rotation matrix as defined in the discussion following (3.42). Equation (3.52) means that for each fixed $\varepsilon > 0$, $\gamma_\varepsilon$ is the median $\mu_1(1,h) =: \mu_1(h)$ of the function $h : \overline{B} \to \mathbb{R}$ defined with the following formula

$$h(z) := \frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz^{(1)}, z^{(1)}\rangle + \langle\eta, z^{(2)}\rangle.$$

Denote by $c' := \left\|\frac{1}{2}\langle Cz^{(1)}, z^{(1)}\right\|_{L^\infty(B)} < \infty$. Similarly to the reasoning in the proof in Chapter 3.4.3 we observe that by monotonicity of $\mu_1$ and property (4) in Theorem 3.11, we obtain the following estimates

$$\begin{aligned}\gamma_\varepsilon &= \mu_1\left(\frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz^{(1)}, z^{(1)}\rangle + \langle\eta, z^{(2)}\rangle\right)\\ &\le \mu_1\left(\frac{z_1}{\varepsilon} + \langle\eta, z^{(2)}\rangle\right) + c',\end{aligned} \tag{3.53}$$

and

$$\begin{aligned}\gamma_\varepsilon &= \mu_1\left(\frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz^{(1)}, z^{(1)}\rangle + \langle\eta, z^{(2)}\rangle\right)\\ &\ge \mu_1\left(\frac{z_1}{\varepsilon} + \langle\eta, z^{(2)}\rangle\right) - c'.\end{aligned} \tag{3.54}$$

As in the proof for $p = 1$ in Chapter 3.4.3 it holds that for all $\varepsilon > 0$ we have

$$|\{z\in B : \frac{z_1}{\varepsilon} + \langle\eta, z^{(2)}\rangle < 0\}| = |\{z\in B : \frac{z_1}{\varepsilon} + \langle\eta, z^{(2)}\rangle > 0\}|,$$

and so $\mu_1\left(\frac{z_1}{\varepsilon} + \langle\eta, z^{(2)}\rangle\right) = 0$. By estimates (3.53) and (3.54) we get that $-c' \le \gamma_\varepsilon \le c'$. Hence all $\gamma_\varepsilon$ are bounded, and after passing to a subsequence, there exists $\gamma_0 := \lim_{\varepsilon\to 0}\gamma_\varepsilon$.

We apply to both sides of (3.52) the following change of variables

$$(z_1, z_2, \dots, z_n, z^{(2)}) \mapsto (\varepsilon z_1, z_2, z_3, \dots, z_n, z^{(2)}) =: \varepsilon z_1 e_1 + \tilde{z},$$

where $\tilde{z} := (0, z_2, \dots, z_n, z^{(2)})$. The Jacobian of this transformation is constant, hence it cancels out on both sides and (3.52) becomes

$$\begin{aligned} &|\{z \in \mathbb{R}^m : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1, \quad z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1 e_1 + \tilde{z}^{(1)}), (\varepsilon z_1 e_1 + \tilde{z}^{(1)})\rangle + \langle \eta, z^{(2)}\rangle\right) < \gamma_\varepsilon\}| \\ &= |\{z \in \mathbb{R}^m : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1, \quad z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1 e_1 + \tilde{z}^{(1)}), (\varepsilon z_1 e_1 + \tilde{z}^{(1)})\rangle + \langle \eta, z^{(2)}\rangle\right) > \gamma_\varepsilon\}|. \end{aligned} \tag{3.55}$$

Let us denote by $\tilde{B} := \{(z_2, \dots, z_n, z^{(2)}) \in \mathbb{R}^{m-1} : |(0, z_2, \dots, z_n, z^{(2)})|_{\mathbb{G}} < 1\}$ and consider a function $F : \{z \in \mathbb{R}^m : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1\} \to \mathbb{R}$ defined by

$$F(z) := z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1 e_1 + \tilde{z}^{(1)}), (\varepsilon z_1 e_1 + \tilde{z}^{(1)})\rangle + \langle \eta, z^{(2)}\rangle\right).$$

For small $\varepsilon$, we are going to represent the intersection of the boundaries of sets in (3.55), i.e., the surface $\{F(z) = \gamma_\varepsilon : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1\}$, as the graph of a function of the form $\tilde{z} \to g_\varepsilon(\tilde{z}) e_1 + \tilde{z}$ where $g_\varepsilon : \tilde{B} \to \mathbb{R}$ and $e_1 = (1, 0, 0, \dots, 0) \in \mathbb{R}^{n+k}$.

Let us observe, that the derivative $F_{z_1}$ can be estimated from below:

$$F_{z_1}(z) = 1 + \varepsilon^2 c_{11} z_1 + \varepsilon(c_{12} z_2 + c_{13} z_3 \dots + c_{1n} z_n) > \frac{1}{2}$$

for $\varepsilon$ sufficiently small. This follows from $|\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1$ and the fact that

$$-\varepsilon \sum_{i=1}^{n} |c_{1i}| \le \varepsilon^2 c_{11} z_1 + \varepsilon(c_{12} z_2 + c_{13} z_3 \dots + c_{1n} z_n) \le \varepsilon \sum_{i=1}^{n} |c_{1i}|.$$

Hence for a fixed $\tilde{z} \in \tilde{B}$ the function $z_1 \to F(z_1 e_1 + \tilde{z})$ is monotone increasing and therefore has an inverse $h_{\varepsilon,\tilde{z}}(t)$. It follows that $F(h_{\varepsilon,\tilde{z}}(t) e_1 + \tilde{z}) = t$ and $g_\varepsilon(\tilde{z}) = h_{\varepsilon,\tilde{z}}(\gamma_\varepsilon)$ is a point in the intersection of the boundaries of sets in (3.55). Furthermore, let us observe that, possibly after passing to a subsequence, the following limit exists for all $\tilde{z} \in \tilde{B}$

$$g_\varepsilon(\tilde{z}) \to \gamma_0 - \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle - \langle \eta, z^{(2)}\rangle \quad \text{as } \varepsilon \to 0^+. \tag{3.56}$$

Indeed, for all $\tilde{z} \in \tilde{B}$ the equation $F(g_\varepsilon(\tilde{z}) e_1 + \tilde{z}) = \gamma_\varepsilon$ equivalently reads:

$$g_\varepsilon(\tilde{z}) + \frac{1}{2}\langle C(\varepsilon g_\varepsilon(\tilde{z}) e_1 + \tilde{z}^{(1)}), (\varepsilon g_\varepsilon(\tilde{z}) e_1 + \tilde{z}^{(1)})\rangle + \langle \eta, z^{(2)}\rangle = \gamma_\varepsilon.$$

From this we get that

$$g_\varepsilon(\tilde{z}) + \frac{1}{2}\left(\varepsilon^2 c_{11} g_\varepsilon^2(\tilde{z}) + 2\varepsilon \sum_{i=2}^{n} c_{1i} g_\varepsilon(\tilde{z}) z_i + \langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle\right) + \langle \eta, z^{(2)}\rangle = \gamma_\varepsilon,$$

which for fixed $\tilde{z}$ and $c_{11} \neq 0$ is the following quadratic equation in $g_\varepsilon(\tilde{z})$:

$$g_\varepsilon^2(\tilde{z}) \frac{\varepsilon^2 c_{11}}{2} + g_\varepsilon(\tilde{z})\left(1 + 2\varepsilon \sum_{i=2}^{n} c_{1i} z_i\right) + \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle + \langle \eta, z^{(2)}\rangle - \gamma_\varepsilon = 0.$$

Therefore, $g_\varepsilon(\tilde{z})$ has to be either equal to

$$g_\varepsilon(\tilde{z}) = \frac{-1 - 2\varepsilon \sum_{i=2}^{n} c_{1i} z_i + \sqrt{\left(1 + 2\varepsilon \sum_{i=2}^{n} c_{1i} z_i\right)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle + \langle \eta, z^{(2)}\rangle - \gamma_\varepsilon\right)}}{\varepsilon^2 c_{11}},$$

or equal to

$$g_\varepsilon(\tilde{z}) = \frac{-1 - 2\varepsilon\sum_{i=2}^n c_{1i}z_i - \sqrt{\left(1 + 2\varepsilon\sum_{i=2}^n c_{1i}z_i\right)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle + \langle \eta, z^{(2)}\rangle - \gamma_\varepsilon\right)}}{\varepsilon^2 c_{11}}.$$

Likewise for $\mathbb{G} = \mathbb{H}_1$ and $p = 1$ we observe, that the second solution is of order $\varepsilon^{-2}$ and therefore does not lie in the set $\{z : |\varepsilon g_\varepsilon(\tilde{z})e_1 + \tilde{z}|_{\mathbb{G}} < 1\}$ for $\varepsilon \to 0^+$. We consider the first solution, which after cancellation reads

$$g_\varepsilon(\tilde{z}) = \frac{2\left(\gamma_\varepsilon - \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle - \langle \eta, z^{(2)}\rangle\right)}{\sqrt{\left(1 + 2\varepsilon\sum_{i=2}^n c_{1i}z_i\right)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle + \langle \eta, z^{(2)}\rangle - \gamma_\varepsilon\right)} + 1 + 2\varepsilon\sum_{i=2}^n c_{1i}z_i}.$$

If $c_{11} = 0$ then

$$g_\varepsilon(\tilde{z})\left(1 + 2\varepsilon\sum_{i=2}^n c_{1i}z_i\right) = \gamma_\varepsilon - \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle - \langle \eta, z^{(2)}\rangle.$$

Therefore, we conclude (3.56). Thus, we can represent the measures of the sets appearing in (3.55) as integrals, and obtain the following

$$\int_{\tilde{B}} \left[\min\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\} + G_\varepsilon(\tilde{z})\right] d\tilde{z} = \int_{\tilde{B}} \left[G_\varepsilon(\tilde{z}) - \max\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\}\right] d\tilde{z}, \tag{3.57}$$

where

$$G_\varepsilon(\tilde{z}) := \frac{1}{\varepsilon}\sqrt{\sqrt{1 - (z_{n+1}^2 + \ldots + z_{n+k}^2)} - (z_2^2 + \ldots + z_n^2)}.$$

The function $G_\varepsilon$ is the non-negative solution $z_1$ to the equation $|\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} = 1$ describing the boundary of $B$. Observe, that (3.57) is equivalent to

$$\int_{\tilde{B}} \left[\min\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\} + \max\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\}\right] d\tilde{z} = 0. \tag{3.58}$$

Applying the dominated convergence theorem to the case $\varepsilon \to 0^+$ in (3.58) gives the following

$$\int_{\tilde{B}} \left(\gamma_0 - \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle - \langle \eta, z^{(2)}\rangle\right) d\tilde{z} = 0. \tag{3.59}$$

The symmetry of $\tilde{B}$ shows that $\int_{\tilde{B}}\langle \eta, z^{(2)}\rangle = 0$ and so (3.59) becomes

$$\gamma_0 = \frac{1}{2}\fint_{\tilde{B}} \langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle d\tilde{z}.$$

Due to symmetries of $\tilde{B}$ the right-hand side can be written as

$$\gamma_0 = \frac{1}{2}\sum_{i=2}^n c_{ii}\fint_{\tilde{B}} z_i^2 d\tilde{z}. \tag{3.60}$$

Observe, that the calculation of the above integrals is essentially covered in (3.44) and that due to symmetries of $\tilde{B}$ they do not depend on the choice of $i$. Recall, that $\tilde{z}^{(1)} = (z_2, \ldots, z_n)$ and $\tilde{z}^{(2)} = (z_{n+1}, \ldots, z_{n+k})$. Let us go ahead the computations in case $i = 2$:

$$\begin{aligned}
\int_{\tilde{B}} z_2^2 d\tilde{z} &= \int\limits_{B_k(0,1)} \int\limits_{B_{n-1}(0,\sqrt[4]{1-\|\tilde{z}^{(2)}\|^2})} z_2^2 \, d\tilde{z}^{(1)} d\tilde{z}^{(2)} \\
&= \int\limits_{B_k(0,1)} \int\limits_{B_{n-1}(0,1)} \left(1 - \|\tilde{z}^{(2)}\|^2\right)^{\frac{n-1}{4}} y_2^2 \left(1 - \|\tilde{z}^{(2)}\|^2\right)^{\frac{1}{2}} d\tilde{y}^{(1)} \, d\tilde{z}^{(2)} \\
&= \int\limits_{B_k(0,1)} \left(1 - \|\tilde{z}^{(2)}\|^2\right)^{\frac{n+1}{4}} d\tilde{z}^{(2)} \int_{B_{n-1}(0,1)} y_2^2 \, d\tilde{y}^{(1)},
\end{aligned} \tag{3.61}$$

where we applied the change of variables formula $z_2 = y_2 \sqrt[4]{1-\left\|\tilde{z}^{(2)}\right\|^2}, \ldots, z_n = y_n \sqrt[4]{1-\left\|\tilde{z}^{(2)}\right\|^2}$ in the inner integral. We apply Lemma 3.16 to calculate the second integral of (3.61):

$$\int_{B_{n-1}(0,1)} y_2^2 \, d\tilde{y}^{(1)} = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)^{n-2}}{\Gamma\left(\frac{n+3}{2}\right)}. \tag{3.62}$$

Let us proceed with the first integral in (3.61):

$$\begin{aligned}
\int\limits_{B_k(0,1)} \left(1-\left\|\tilde{z}^{(2)}\right\|^2\right)^{\frac{n+1}{4}} d\tilde{z}^{(2)} &= \int_0^1 \int_{\partial B_k(0,r)} (1-r^2)^{\frac{n+1}{4}} dS^{k-1}(\tilde{z}^{(2)}) dr \\
&= |\partial B_k(0,1)| \int_0^1 (1-r^2)^{\frac{n+1}{4}} r^{k-1} dr \\
&= \frac{k\pi^{\frac{k}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} \int_0^1 (1-t)^{\frac{n+1}{4}} t^{\frac{k-1}{2}} \frac{1}{2} t^{-\frac{1}{2}} dt \\
&= \frac{k\pi^{\frac{k}{2}}}{2\Gamma\left(\frac{n+2}{2}\right)} \mathcal{B}\left(\frac{k}{2}, \frac{n+5}{4}\right),
\end{aligned} \tag{3.63}$$

where we used the change of variables $t := r^2$. Notice, that in order to calculate the measure of $\tilde{B}$ we need to compute the following integral

$$|\tilde{B}| = \int\limits_{B_k(0,1)} \left(1-\left\|\tilde{z}^{(2)}\right\|^2\right)^{\frac{n-1}{4}} d\tilde{z}^{(2)} \int_{B_{n-1}(0,1)} 1 \, d\tilde{y}^{(1)}, \tag{3.64}$$

which is analogous to (3.62) and (3.63). Therefore, we conclude that

$$\int_{B_{n-1}(0,1)} 1 \, d\tilde{y}^{(1)} = |B_{n-1}(0,1)| = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \tag{3.65}$$

$$\int\limits_{B_k(0,1)} \left(1-\left\|\tilde{z}^{(2)}\right\|^2\right)^{\frac{n-1}{4}} d\tilde{z}^{(2)} = \frac{k\pi^{\frac{k}{2}}}{2\Gamma\left(\frac{n+2}{2}\right)} \mathcal{B}\left(\frac{k}{2}, \frac{n+3}{4}\right). \tag{3.66}$$

We sum up observations (3.61)–(3.63) and (3.64)–(3.66) to rewrite (3.60) in the following way

$$\gamma_0 = \frac{1}{2} \frac{k\pi^{\frac{k}{2}}}{2\Gamma\left(\frac{n+2}{2}\right)} \mathcal{B}\left(\frac{k}{2}, \frac{n+5}{4}\right) \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)^{n-2}}{\Gamma\left(\frac{n+3}{2}\right)} \frac{2\Gamma\left(\frac{n+2}{2}\right)}{k\pi^{\frac{k}{2}} \mathcal{B}\left(\frac{k}{2}, \frac{n+3}{4}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}} \sum_{i=2}^n c_{ii},$$

which upon simplification reads

$$\gamma_0 = \frac{\mathcal{B}\left(\frac{k}{2}, \frac{n+5}{4}\right)}{2(n+1)\mathcal{B}\left(\frac{k}{2}, \frac{n+3}{4}\right)} \sum_{i=2}^n c_{ii}.$$

Therefore,

$$\gamma_0 = \frac{1}{2(n+1)} \frac{\mathcal{B}\left(\frac{k}{2}, \frac{n+5}{4}\right)}{\mathcal{B}\left(\frac{k}{2}, \frac{n+3}{4}\right)} \left(\operatorname{tr}(A) - \frac{\langle A\xi, \xi\rangle}{|\xi|^2}\right),$$

which follows from the same argument used in the case $1 < p < \infty$, and the same reasoning allows us to conclude (3.39) and (3.40) for $p = 1$ as well. Thus, the proof of Lemma 3.22 is completed for all $1 \leq p \leq \infty$. □

# Chapter 4

# AMV harmonic functions on metric measure spaces

## 4.1 Introduction

This chapter is based on results obtained in [AKS20]. The results presented here are obtained in the setting of locally doubling metric measure spaces, which is more general than those studied in Chapters 2 and 3. For the courtesy of interested reader at the end of this chapter we briefly describe those results from [AKS20], which for the sake of consistency are not discussed entirely.

Let us define the central object of this chapter. Let $X = (X, d, \mu)$ be a metric measure space and function $u \in L^1_{loc}(X)$. The *r-laplacian* of $u$ is defined as follows

$$\Delta_r^\mu u(x) = \Delta_r u(x) = \frac{u_{B(x,r)} - u(x)}{r^2}, \quad x \in X, \tag{4.1}$$

where $u_{B(x,r)}$ stands for the mean-value of $u$ over a ball $B(x, r)$.

In this chapter we consider the notion of *strongly asymptotically mean value harmonic* functions, often abbreviated to (strongly) amv-harmonic functions, arising from assuming that the following limit exists almost uniformly

$$\lim_{r \to 0} \Delta_r u = 0,$$

see Definition 4.3 below. Recall from Chapter 2, that classical mean value property states that, in a Euclidean domain $\Omega$, a harmonic function $u$ satisfies $\Delta_r u(x) = 0$ for all $0 < r < \operatorname{dist}(x, \partial\Omega)$, cf. Definition 2.1 and Definition 4.7.

It turns out that harmonic functions rarely enjoy the mean value property outside the Euclidean setting, see the discussion in Chapter 2.6 summed up by an observation, that the space of strongly harmonic functions in $\mathbb{R}^2$ with respect to $l^p$-norm is finite dimensional for $p \neq 2$. Instead of considering the mean value property as in Definition 2.1, we believe that it is better to study functions which satisfy an *asymptotic mean value property*, where the pointwise limit $r \to 0$ in (4.1) vanishes. For example, harmonic functions in the smooth Riemannian manifolds have asymptotic mean value property, whereas the mean value property for harmonic functions on manifolds is known to hold on the so-called harmonic manifolds. The Lichnerowicz conjecture, proven for manifold dimensions 2-5, characterizes harmonic manifolds as either flat or rank-one symmetric, see Example 4 in [AGG19] and references therein. The converse statement, namely when the asymptotic mean value property implies that a function satisfies the appropriate Laplace equation is known as the Blaschke–Privaloff–Zaremba (BPZ) theorem, and will be discussed in more detail below. Apart from the classical setting, the $r$-laplacian also arises in approximation problems of Riemannian manifolds by graphs [BIK13], and the mean value property plays a role in geometric group theory in Kleiner's proof of Gromov's polynomial growth theorem [Kle10].

In the setting of Carnot groups the $r$-Laplacian and its relations to the subelliptic harmonic functions have been studied, for instance, in [AW20; FLM14; FP15]. Furthermore, Theorem 1 in [CO20] indirectly relates the amv-harmonicity to functions with bounded variations. Namely, the result characterizes $C^1$-minimal surfaces $S$ by observing that a certain piecewise constant function $f_S$ is amv-harmonic in the sense that $\Delta_r f_S \to 0$, as $r \to 0$ on $S$. Moreover, the proof of [CO20, Theorem 1] uses the relation between the amv-harmonic operator and a nondegenerate 1-Laplacian: $\operatorname{div}(\nabla/\sqrt{1+|\nabla|^2})$.

In the setting of Heisenberg groups we obtain a BPZ-type result, whereby *pointwise* vanishing of the limit $r \to 0$ in (4.1) for a continuous function implies harmonicity and thus strong amv-harmonicity, see Chapter 4.3.3.

In metric spaces with a doubling measure we consider strongly amv-harmonic functions and prove that they are Hölder continuous for any exponent below 1, see Theorem 4.18. This result is in fact true for a larger class of functions with finite $L^\infty$ amv-norm, which we introduce below. Moreover, using the method of refined averaging, we obtain an auxiliary regularity result for strongly harmonic functions in Theorem 4.8. For a complete discussion we refer the reader to Chapter 4.3, where we also study the finite dimensionality of the space of strongly harmonic functions with polynomial growth.

In Chapter 4.4 we discuss Hajłasz–Sobolev functions in amv-harmonic class with their blow-ups. We show that at almost every point such blow-up satisfy the global mean value property (Theorem 4.41), which in general is very rare, see the discussion above. Theorem 4.41 can be seen as an infinitesimal connection between amv- and strong harmonicity, and may serve as an obstruction to having many amv-harmonic functions on metric spaces that are too irregular.

As a toy model, we study amv-harmonicity in weighted Euclidean spaces, where it becomes evident that the connection between weak amv-harmonicity and energy-minimizers breaks down in the presence of weights. This is related to the failure of the $r$-laplacian to be asymptotically self-adjoint. Nevertheless, in the weighted Euclidean spaces, we obtained more concrete PDE description of the amv-harmonic functions, see operator (4.23), Theorem 4.45 and Chapter 4.5 for full discussion of the results.

Let us now describe the results from [AKS20], which we are not included in this Chapter.

First of all, a weaker version of amv-harmonicity is considered, namely we say that a function $u \in L^2(X)$ is *weakly amv-harmonic* if $\lim_{r\to 0} \int_X \varphi \Delta_r u d\mu = 0$ for every compactly supported Lipschitz function $\varphi$.

Moreover, the results of [AKS20] are obtained for the setting of RCD spaces. The notion of the $RCD$ spaces (*Riemannian curvature dimension spaces*) grows from the synthetic approach to curvature bounds and the idea of introducing the unifying notion of a curvature in metric measure spaces. The origins of $RCD$ spaces go back to works by Otto, Villani, Sturm, Ambrosio, Gigli, Savare to mention just few names and in recent 5-10 years, the area of $RCD$ and $CD$ spaces has become one of the most rapidly developing areas of analysis and geometry on metric measure spaces. It combines techniques of Ricci curvature and the Riemannian geometry (with the Bochner identity as one of the cornerstones), heat semigroups with functional analysis, measure theory and the optimal transportation theory. The precise definition of the $RCD$ spaces requires introducing, among others, the entropy functional and the Wasserstein distance and will not be used in our work. Instead, for the definition and further properties of the $RCD$ spaces we refer to extensive literature on the subject, e.g. [Amb18; AGS14b; AGS14a; Gig15; LV09; Stu06a; Stu06b; Vil16]). For our needs let us emphasize that one of the key features of the $RCD$ spaces is that the natural Sobolev space $W^{1,2}$ is a Hilbert space (*infinitesimally Hilbertian*), which equivalently can be expressed in terms of the linearity if the harmonic heat flow and the fact that the Cheeger differential is a quadratic form, see e.g. [AGS14b; AGS14a].

Let us denote by $\theta^N$ the *Bishop–Gromov density* defined with

$$\theta_r^N(x) := \frac{\mu(B_r(x))}{\omega_N r^N}, \quad \theta^N(x) := \lim_{r\to 0} \theta_r^N(x).$$

We say that an $RCD(K,N)$-space $(X,d,\mu)$ is *non-collapsed*, if $\theta^N(x) \leq 1$ for $\mu$-almost every

$x \in X$. We say that $X$ has *vanishing mm-boundary*, if the signed Radon measures $\frac{1-\theta_r^N}{r} d\mu$ for $0 < r \leq 1$ are uniformly bounded in the total variation norm, and converge weakly to zero as $r \to 0$.

In [KS93], the authors defined a Sobolev space of functions with values in a complete metric space $X$ by considering the so-called *Korevaar–Schoen energy* for $u \in L^2_{loc}(X)$:

$$E^2_{KS}(u) := \sup_{\varphi \in C_c(X), 0<\varphi\leq 1} \limsup_{r\to 0} \int_X \varphi(x) \frac{1}{2} \fint_{B_r(x)} \left| \frac{u(y)-u(x)}{r} \right|^2 d\mu(y) d\mu.$$

In this setting it is proved in [AKS20] that harmonic functions on non-collapsed RCD spaces with vanishing metric measure boundary are weakly amv-harmonic. Moreover, the relation between the Korevaar-Schoen energy and the $r$-laplacian is obtained and the connection between weakly amv-harmonic functions and local minimizers of the Korevaar-Schoen energy is attained.

In the next chapter we introduce preliminary notions and definitions used throughout this chapter.

## 4.2 Preliminaries

Given a subset $F \subset X$ of a metric space and $r > 0$, we denote

$$N_r(F) = \{x \in X : \operatorname{dist}(x, F) < r\} \text{ and } \overline{N}_r(F) = \{x \in X : \operatorname{dist}(x, F) \leq r\}$$

the open and closed $r$-neighbourhood of $F$ (note that $\overline{N}_r(F)$ need not be the closure of $N_r(F)$ unless $X$ is a length space). For $x \in X$, we denote by $B(x,r) := N_r(\{x\})$ and $\overline{B}(x,r) := \overline{N}_r(\{x\})$, respectively, an open and closed ball centered at $x$ with radius $r$. The Lipschitz constant of a map $f : (X, d_x) \to (Y, d_Y)$ between metric spaces is

$$\operatorname{LIP}(f) := \sup_{x \neq y} \frac{d_Y(f(x), f(x))}{d_X(x,y)}.$$

A measure $\mu$ on a separable metric space $X$ is called *locally doubling* if, for every compact $K \subset X$, there exists $r_K > 0$ and a constant $C_K > 0$, such that $\overline{N}_{r_K}(K)$ is compact and

$$\mu(B(x, 2r)) \leq C_K \mu(B(x,r)) \tag{4.2}$$

for every $x \in K$ and $0 < r \leq r_K$. If $\mu$ is locally doubling, for every compact $K \subset X$ there exists a constant $C_K > 0$ for which

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C \left(\frac{r}{R}\right)^Q, \quad y \in B(x,R),\ 0 < r \leq R \leq r_K, \tag{4.3}$$

where $Q = \log_2 C_K$. If the constant $C_K = C_\mu$ can be chosen independently of the set $K \subset X$, and $r_K = \infty$, we say that $\mu$ is *doubling*, and the number $Q = \log_2 C_\mu$ is called the *doubling exponent* of $\mu$.

The following definition is due to Buckley, see [Buc99, Section 1], and is stronger than the doubling condition.

**Definition 4.1.** Let $(X, d, \mu)$ be a metric measure space with a doubling measure $\mu$. We say that $X$ satisfies the *$\alpha$-annular decay property* with some $\alpha \in (0, 1]$ if there exists $A \geq 1$ such that for all $x \in X$, $r > 0$ and $\varepsilon \in (0, 1)$ it holds that

$$\mu\left(B(x,r) \setminus B(x, r(1-\varepsilon))\right) \leq A\varepsilon^\alpha \mu(B(x,r)). \tag{4.4}$$

If $\alpha = 1$, then we say that $X$ satisfies the *strong annular decay property.*

**Example 4.2.** The Euclidean space satisfies strong annular decay property. Metric measure spaces with strong annular decay property include geodesic metric spaces with uniform measures and Heisenberg groups $\mathbb{H}^n$ equipped with a left-invariant Haar measures. By [Buc99, Corollary 2.2], a length space with a doubling measure has the $\alpha$-annular decay property for some $\alpha \in (0,1]$ with $\alpha$ depending only on a doubling constant of the measure. In fact, it is enough for the metric measure space to be the so-called $(\alpha,\beta)$-chain space to conclude that it has the $\delta$-annular decay property, see Theorem 2.1 in [Buc99].

From now on a metric measure space $X = (X, d, \mu)$ is a separable metric space $(X, d)$ equipped with a measure $\mu$ that is finite and nontrivial on balls, i.e. $0 < \mu(B) < \infty$ for all balls $B \subset X$.

Now we are in a position to define the central object of this chapter.

**Definition 4.3.** A function $u \in L^1_{loc}(X)$ is *strongly amv-harmonic*, if

$$\lim_{r\to 0} \|\Delta_r u\|_{L^\infty(K)} = 0$$

for any compact set $K \subset X$. Here $\Delta_r u$ denotes the $r$-laplacian of $u$, see (4.1).

In general metric measure spaces there is no natural limit operator of $\Delta_r$ as $r \to 0$. Therefore, it is highly non-obvious what should be a domain of such an limit operator. We define the space of functions with finite amv-norm.

**Definition 4.4.** Let $(X, d, \mu)$ be a metric measure space and $p \in [1, \infty]$. We set

$$\mathrm{AMV}^p(X) := \{u \in L^p(X) : \ \|u\|_{\mathrm{AMV}^p} < \infty\},$$

where

$$\|u\|_{\mathrm{AMV}^p} := \limsup_{r\to 0} \|\Delta_r u\|_{L^p(X)}$$

is the *amv-norm* of $u$. Moreover, we define the class of functions with locally finite amv-norm: $\mathrm{AMV}^p_{loc}(X)$ consist of functions $u \in L^p_{loc}(X)$ for which $\limsup_{r\to 0} \|\Delta_r u\|_{L^p(K)}$ for every compact set $K \subset X$.

**Remark 4.5.** Observe, that any strongly amv-function has locally finite amv-norm, but the converse is not necessarily true. Let us consider a domain $X = \Omega \subset \mathbb{R}^n$. In Proposition 4.47 we see that functions with locally finite amv-norm coincide with the space $W^{2,p}_{loc}(\Omega)$, while by the Blaschke–Privaloff–Zaremba we know, that if for a continuous function $u : \Omega \to \mathbb{R}$ its $r$-laplacian converges pointwise to 0, which holds true for strongly amv-harmonic functions, then the function is harmonic, hence analytic.

### 4.2.1 Doubling measures and averaging operators

Let $(X, d, \mu)$ be a metric measure space and $r > 0$. Given a locally integrable function $u \in L^1_{loc}(X)$, we denote by

$$A^\mu_r u(x) = \fint_{B(x,r)} u \, d\mu, \quad x \in X,$$

the $r$-average function of $u$. Whenever the measure $\mu$ is clear from the context, we will omit writing the measure in the superscript. Note that

$$A_r u(x) = u_{B(x,r)};$$

we will use the two notations interchangeably, depending on whether we want to view the average as a number, or an operator on a function space. Indeed, the function $A_r u : X \to \mathbb{R}$ is measurable, and $A_r$ defines a bounded linear operator $A_r : L^1(X) \to L^1(X)$ if and only if $a_r \in L^\infty(X)$, where

$$a_r(x) = \int_{B(x,r)} \frac{d\mu(y)}{\mu(B(y,r))}, \quad x \in X.$$

Moreover, in this case the operator norm satisfies $\|A_r\|_{L^1\to L^1}= \|a_r\|_{L^\infty}$, see [Ald19, thm 3.3]. This is true in particular when $\mu$ is a doubling measure. On the other hand, it is true that by the Lebesgue differentiation theorem

$$u(x) = \lim_{r\to 0} A_r u(x) \text{ for almost every } x \in X,$$

if $\mu$ is infinitesimally doubling, cf. [Hei+15, Remark 3.4.29].

If $X$ is doubling as a metric space, then there exists $C > 0$ so that $\|A_r\|_{L^p\to L^p}\leq C$ for every $r > 0$ and every $1 \leq p < \infty$, cf. [Ald19, thm 3.5]. However, $A_r$ is not a self-adjoint operator; the formal adjoint $A_r^*$ of $A_r$ is given by

$$(A_r^\mu)^* u(x) = A_r^* u(x) := \int_{B(x,r)} \frac{u(y)d\mu(y)}{\mu(B(y,r))}, \quad x \in X,$$

for $u \in L^1_{loc}(X)$. Indeed, a direct computation using the Fubini theorem yields that

$$\int_X vA_r u d\mu = \int_X uA_r^* v d\mu, \quad u \in L^p(X),\ v \in L^q(X),$$

where $1/p + 1/q = 1$.

We may write the $r$-laplacian using the averaging operator as

$$\Delta_r u = \frac{A_r u - u}{r^2}, \quad u \in L^1_{loc}(X).$$

We denote by

$$\Delta_r^* u := \frac{A_r^* u - u}{r^2}, \quad u \in L^1_{loc}(X),$$

the formal adjoint of the $r$-laplacian. Note that if $A_r : L^p(X) \to L^p(X)$ is bounded, then $\Delta_r : L^p(X) \to L^p(X)$ and $\Delta_r^* : L^q(X) \to L^q(X)$ are both bounded, where $1/p + 1/q = 1$.

**Remark 4.6.** While most results are formulated for metric measure spaces, the results encompass the case of an open set $\Omega \subset X$ in the introduction. Indeed, an open subset $\Omega \subset X$ of a metric measure space can be regarded as a metric measure space $\Omega = (\Omega, d|_\Omega, \mu|_\Omega)$. In particular, *if $X$ is locally doubling, then $\Omega$ is locally doubling.*

## 4.3 Refined averaging and strongly harmonic functions

In Chapter 2 we broadly studied strongly harmonic functions in the weighted Euclidean case. In this chapter we intend to refine regularity of such functions on metric measure spaces by showing their local Lipschitz regularity assuming merely the doubling property of the underlying measure, see Theorem 4.8. We also prove a dimension bound on the space of strongly harmonic functions with polynomial growth in the spirit of the celebrated result of Colding–Minicozzi [CM97b] confirming Yau's conjecture, see Proposition 4.22. Our approach emphasizes the role of the averaging operators.

Let us rephrase Definition 2.1 equivalently using the notion of $r$-laplacian, cf. Chapter 2.1.

**Definition 4.7.** Let $X = (X, d, \mu)$ be a metric measure space. We say that a function $u \in L^1_{loc}(X)$ is *strongly harmonic* (or has the *mean value property*) if

$$\Delta_r u = 0 \text{ on } K$$

for any compact set $K \subset X$ and $r < r_K := \sup\{\rho > 0 : \overline{N}_r(K) \text{ is compact}\}$.

Throughout the chapter we abbreviate the mean value of a function $u \in L^1_{loc}(X)$ over a ball $B(x,r)$ as follows:

$$u_r(x) := A_r u(x) = \fint_{B(x,r)} u(y)dy.$$

If $X$ is a complete doubling metric measure space with doubling exponent $Q$, and $\Omega \subset X$ domain, then recall that by $\mathcal{H}(\Omega)$ we denote the space of strongly harmonic functions on $\Omega$. Note that, if $u \in \mathcal{H}(\Omega)$, then by the very definition it holds that

$$u(x) = A_r u(x), \quad x \in \Omega,\ r < \operatorname{dist}(x, X \setminus \Omega).$$

### 4.3.1 Local Lipschitz continuity of strongly harmonic functions

In this chapter we show that the mean value property yields higher regularity than obtained in [AGG19]. If a measure $\mu$ has the $\alpha$-annular decay property (see Definition 4.1), then [AGG19, Theorem 4.2] shows that strongly harmonic functions are $\alpha$-Hölder continuous. Below, we will prove that in fact strongly harmonic functions are Lipschitz continuous even when the doubling measure does not satisfy the annular decay condition.

**Theorem 4.8.** *Let $\Omega \subset X$ be an open subset of a complete locally doubling metric measure space $X = (X, d, \mu)$, and $u \in L^1_{loc}(\Omega)$ a strongly harmonic function on $\Omega$. Then $u$ is locally Lipschitz and satisfies the bound*

$$\operatorname{LIP}(u|_{B(x_0,r)}) \leq \frac{C}{r} \inf_{c\in\mathbb{R}} \fint_{B(x_0,3r)} |u-c|d\mu \tag{4.5}$$

*whenever $\bar{B}(x_0, 3r) \subset \Omega$.*

The idea of the proof of Theorem 4.8 is to consider a refined averaging process, wherein we average over the radius as well as the space variable. Given a function $u \in L^1_{loc}(X)$ we define

$$A^r u(x) := \frac{2}{r}\int_{r/2}^{r} u_t(x)dt = \frac{2}{r}\int_{r/2}^{r} \left( \fint_{B(x,t)} u(y)dy \right) dt, \quad x \in X. \tag{4.6}$$

For $x \in X$ and $r \leq R$ we introduce the following notation

$$A_{r,R}(x) = \bar{B}(x,R) \setminus B(x,r)$$

for a closed annulus centered at $x$, with inner radius equal to $r$ and outer to $R$. We use the convention that $B(x,r) = \emptyset$ for $r \leq 0$ and $A_{r,R}(x) = \emptyset$ if $r > R$. The following elementary lemma will play a crucial role in proving that $A^r u$ is locally Lipschitz.

**Lemma 4.9.** *Let $f \in L^1_{loc}(\Omega)$ be a nonnegative function and $x \in \Omega$. Let $0 \leq r \leq R < \infty$, and $-\infty < d_1 \leq d_2 < \infty$ Then*

$$\int_r^R \int_{A_{t+d_1,t+d_2}(x)} f d\mu\ dt \leq (d_2 - d_1) \int_{A_{r+d_1,R+d_2}(x)} f d\mu.$$

*Proof.* Let us fix $x \in \Omega$ and define function $g : \mathbb{R} \to \mathbb{R}$ as follows: $g(t) = 0$ for $t \leq 0$ and

$$g(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ \int_{B(x,t)} f d\mu & \text{for } t > 0. \end{cases}$$

Notice, that $g$ is a nondecreasing function.

Let us fix $t \in \mathbb{R}$. If $R + d_1 \leq r + d_2$, then $R - r \leq d_2 - d_1$ and the following estimate follows trivially

$$\int_r^R \int_{A_{t+d_1,t+d_2}(x)} f d\mu dt \leq \int_r^R \int_{A_{r+d_1,R+d_2}(x)} f d\mu dt \leq (d_2 - d_1) \int_{A_{r+d_1,R+d_2}(x)} f d\mu.$$

Conversely, if $R+d_1 > r+d_2$ then the following estimate holds true

$$\int_r^R \int_{A_{t+d_1,t+d_2}(x)} f d\mu\, dt = \int_r^R [g(t+d_2) - g(t+d_1)]dt = \int_{r+d_2}^{R+d_2} gdt - \int_{r+d_1}^{R+d_1} gdt$$
$$= \int_{R+d_1}^{R+d_2} gdt - \int_{r+d_1}^{r+d_2} gdt \leq (d_2-d_1)g(R+d_2) - (d_2-d_1)g(r+d_1)$$
$$=(d_2-d_1)\int_{A_{r+d_1,R+d_2}(x)} f d\mu,$$

which ends the proof. □

Now we are in a position to prove Lipschitz regularity of $A^r u$.

**Proposition 4.10.** *Let $(X,d,\mu)$ be a locally doubling metric measure space, $u \in L^1_{loc}(X)$. Let $K \subset X$ be compact and $r_K > 0$ such that (4.2) holds for $r < r_K$. Then, for any $r < r_K$, the function $A^r u$ is Lipschitz on $K$ and, for any $c \in \mathbb{R}$, satisfies the Hajlasz type estimate*

$$|A^r u(x) - A^r u(y)| \leq \frac{Cd(x,y)}{r}\left(\fint_{B(x,2r)} |u-c|d\mu + \fint_{B(y,2r)} |u-c|d\mu\right), \tag{4.7}$$

*for any pair of points $x,y \in K$ with distance $d(x,y) < r$. The constant $C$ depends only on the doubling constant of the measure $\mu$ on $K$. In particular*

$$\mathrm{LIP}(A^r u|_{B(x_0,r)}) \leq \frac{C}{r}\fint_{B(x_0,3r)} |u-c|d\mu$$

*whenever $\bar{B}(x_0,3r) \subset X$ is compact.*

*Proof of Proposition 4.10.* We begin with the prove of the second part of the hypothesis, i.e. the Lipschitz estimate assuming the first part (4.7). Let $K = B(x_0,r)$, $c \in \mathbb{R}$ and assume that $x,y \in B(x_0,r)$ with $d(x,y) < r$. Then (4.7) directly yields

$$|u(x) - u(y)| \leq \frac{Cd(x,y)}{r}\left(\fint_{B(x,2r)} |u-c|d\mu + \fint_{B(y,2r)} |u-c|d\mu\right) \leq \frac{Cd(x,y)}{r}\fint_{B(x_0,3r)} |u-c|d\mu.$$

If $d(x,y) \geq r$, then $d(x,x_0) + d(y,x_0) \leq 2d(x,y)$, and thus

$$\begin{aligned}|A^r u(x) - A^r u(y)| &\leq |A^r u(x) - A^r u(x_0)| + |A^r u(y) - A^r u(x_0)| \\ &\leq \frac{C(d(x,x_0)+d(y,x_0))}{r}\fint_{B(x_0,3r)} |u-c|d\mu \\ &\leq \frac{Cd(x,y)}{r}\fint_{B(x_0,3r)} |u-c|d\mu.\end{aligned}$$

Now it suffices to prove (4.7). Let $K$ and $r$ be as in the claim and denote by $C_K$ the doubling constant of $\mu$ in $K$. Given $x,y \in K$ with $d := d(x,y) \leq r$ and $r/2 \leq t < r$, we have that for the symmetric difference of two balls it holds

$$B(x,t) \triangle B(y,t) \subset A_{t-d,t+d}(x) \subset N_{2r}(K). \tag{4.8}$$

Indeed, let us take a point $z \in B(x,t) \triangle B(y,t)$. Then either (1) $d(x,z) < t$ and $d(y,z) \geq t$ or (2) $d(x,z) \geq t$ and $d(y,z) < t$. In case (1) $d(x,z) < t \leq t+d$ and $d(x,z) \geq d(y,z) - d(x,y) \geq t-d$, hence $z \in A_{t-d,t+d}(x)$. In case (2) $d(x,z) \geq t \geq t-d$ and $d(x,z) \leq d(y,z) + d(x,y) \leq t+d$. The second inclusion is trivial.

Let us fix $c \in \mathbb{R}$ and follow the reasoning in the proof of [AGG19][Propostion 4.1]:

$$|u_t(x) - u_t(y)| = \left| \fint_{B(x,t)} (u-c)d\mu - \fint_{B(y,t)} (u-c)d\mu \right|$$
$$\leq \frac{\mu(B(x,t) \triangle B(y,t))}{\mu(B(x,t))\mu(B(y,t))} \int_{B(y,t)} |u-c|d\mu + \frac{1}{\mu(B(x,t))} \int_{B(x,t)\triangle B(y,t)} |u-c|d\mu.$$

Let us apply (4.8) to the right-hand side

$$|u_t(x) - u_t(y)| \leq \frac{\mu(A_{t-d,t+d}(x))}{\mu(B(x,t))\mu(B(y,t))} \int_{B(y,t)} |u-c|d\mu + \frac{1}{\mu(B(x,t))} \int_{A_{t-d,t+d}(x)} |u-c|d\mu.$$

Use the doubling property of $\mu$ on $K$ and the assumption $\frac{r}{2} \leq t < r$ to obtain $\mu(B(x,t)) \geq \frac{1}{C_K}\mu(B(x,r))$ and $\mu(B(y,t)) \geq \frac{1}{C_K}\mu(B(y,r))$. This together with monotonicity allows us to conclude

$$|u_t(x) - u_t(y)| \leq \frac{C_K^2 \mu(A_{t-d,t+d}(x))}{\mu(B(x,r))\mu(B(y,r))} \int_{B(y,r)} |u-c|d\mu + \frac{C_K}{\mu(B(x,r))} \int_{A_{t-d,t+d}(x)} |u-c|d\mu. \quad (4.9)$$

Notice, that by Lemma 4.9 for $f \equiv 1$, $d_2 = d$ and $d_1 = -d$ there holds

$$\int_{r/2}^{r} \mu(A_{t-d,t+d}(x))dt \leq 2d\mu(A_{r/2-d,r+d}(x)).$$

Let us integrate both sides of (4.9) with respect to $t \in (r/2, r)$ and apply the above observation

$$\int_{r/2}^{r} |u_t(x) - u_t(y)|dt \leq \frac{2dC_K^2 \mu(A_{r/2-d,r+d}(x))}{\mu(B(x,r))\mu(B(y,r))} \int\limits_{B(y,r)} |u-c|d\mu + \frac{2dC_K}{\mu(B(x,r))} \int\limits_{A_{r/2-d,r+d}(x)} |u-c|d\mu$$
$$\leq 2dC_K^4 \left( \fint_{B(y,2r)} |u-c|d\mu + \fint_{B(x,2r)} |u-c|d\mu \right),$$

where in the last inequality we once again appeal to the doubling property (4.3) of $\mu$, inclusion $A_{r/2-d,r+d}(x) \subset B(x,2r)$ and the monotonicity of integral. We sum up the above observations

$$|A^r u(x) - A^r u(y)| \leq \frac{2}{r} \int_{r/2}^{r} |u_t(x) - u_t(y)|dt \leq \frac{4C_K^4 d(x,y)}{r} \left( \fint_{B(y,2r)} |u-c|d\mu + \fint_{B(x,2r)} |u-c|d\mu \right),$$

proving assertion (4.7). □

*Proof of Theorem 4.8.* By considering the metric measure space $(\Omega, d, \mu|_\Omega)$ we may assume that $\Omega = X$, cf. Remark 4.6. Since $u$ is strongly harmonic for every compact $K \subset X$, there exists $r_K > 0$ so that $u = u_r$ on $K$ for all $r < r_K$. In particular,

$$u = A^r u \text{ on } K$$

whenever $r < r_K$. The Lipschitz continuity of $u$ and the estimate (4.5) then follows by Proposition 4.10. □

**Remark 4.11.** When $X$ is complete and $\mu$ is globally doubling, strongly harmonic functions satisfy $u = u_R$ on $X$, for *any* $R > 0$. Consequently (4.5) yields

$$\mathrm{LIP}(u|_{B(x,R)}) \leq \frac{C}{R} \fint_{B(x,3R)} |u - u(p)|d\mu, \quad p \in X.$$

Remark 4.11 together with Harnack's inequality [AGG19][Lemma 4.1] imply, in particular, that there are no non-constant strongly harmonic functions of sublinear growth. A function $u$ is said to have *sublinear growth*, if

$$\limsup_{R\to\infty} \frac{1}{R} \sup_{B(x,R)} |u| = 0$$

for some, and hence any $p \in X$.

The following observation can be considered as a counterpart of Cheng's result for harmonic functions with sublinear growth on complete manifolds with nonnegative Ricci curvature, see Corollary 1.5 in [Li06], and it is also related to the celebrated Phragmen-Lindelöf theorem.

**Corollary 4.12.** *Let $(X, d, \mu)$ be a doubling metric measure space. If $u$ is strongly harmonic and has sublinear growth, then it is constant.*

In the next chapter we discuss regularity of amv-harmonic functions.

### 4.3.2 Regularity of amv-harmonic functions

In this chapter we prove Hajłasz–Sobolev and Hölder regularity of functions with finite amv-norm. Let us define local fractional Hajłasz–Sobolev spaces.

**Definition 4.13.** Let $(X, d, \mu)$ be a metric measure space, and $1 < p \leq \infty$, $0 < \alpha \leq 1$. The local fractional Hajłasz–Sobolev space $M^{\alpha,p}_{loc}(X)$ consists of Borel functions $u \in L^p_{loc}(X)$ with the following property: there exists a null set $N \subset X$ and, for every compact $K \subset X$, a non-negative function $g_K \in L^p_{loc}(X)$ and $r_K > 0$ with $\overline{N}_{r_K}(K)$ compact, and

$$|u(x) - u(y)| \leq d(x, y)^\alpha [g_K(x) + g_K(y)], \quad x, y \in K \setminus N,\ d(x, y) < r_K.$$

To our best knowledge the fractional Hajłasz–Sobolev functions were firstly defined on Euclidean sets by Hu [Hu03] and then on metric spaces by Yang [Yan03]. The main motivation is to study a counterpart of Sobolev spaces on fractals. These spaces help to investigate the geometry of fractals from inside the set and enable to study analysis on fractals. For example, Hu showed that there exists $\alpha > 1$ such that $M^{\alpha,2}(S)$ is dense in $C(S)$, where $S$ is the Sierpiński gasket in $\mathbb{R}^n$.

We recall the *fractional sharp maximal function*, see [HK98, pg. 606], as follows. Let $0 < \alpha < \infty$, $R > 0$ and $u \in L^1_{loc}(X)$. Then

$$\mathcal{M}^{\#}_{\alpha,R} u(x) := \sup_{0<r<R} r^{-\alpha} \fint_{B(x,r)} |u - u_{B(x,r)}| d\mu, \quad x \in X.$$

We denote by $\mathcal{M}^{\#}_R u := \mathcal{M}^{\#}_{0,R} u$ and $\mathcal{M}^{\#}_\alpha u := \mathcal{M}^{\#}_{\alpha,\infty} u$. Moreover, we denote by $\mathcal{M}$ the Hardy–Littlewood maximal function and by $\mathcal{M}_R$ the restricted maximal function

$$\mathcal{M}u(x) := \sup_{r>0} \fint_{B(x,r)} |u|\, d\mu, \quad \mathcal{M}_R u(x) := \sup_{0<r<R} \fint_{B(x,r)} |u|\, d\mu.$$

Throughout this chapter $X = (X, d, \mu)$ denotes a locally compact and doubling metric measure space with doubling exponent $Q$. We begin by considering the refined average $A^r : L^1_{loc}(X) \to \mathrm{LIP}_{loc}(X)$ defined in (4.6). We employ an iterative argument, which is based on the following observation.

**Proposition 4.14.** *Let $\Omega \subset X$ be a domain, and $u \in \mathrm{AMV}^p(\Omega)$. Then $u \in M^{1/2,p}_{loc}(\Omega)$. Moreover, if $u \in M^{\alpha,p}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$, then $u \in M^{\alpha',p}_{loc}(\Omega)$, where*

$$\alpha' = \frac{2 - 1/p}{3 - \alpha - 1/p} > \alpha.$$

*Proof.* Let $K \subset \Omega$ be compact, and define

$$R_K = \frac{1}{6}\min\{\mathrm{dist}(K, X \setminus \Omega)^2, r_K^2, 1\},$$

where $r_K$ is given by the locally doubling condition (4.2). Let $x, y \in K$ satisfy $d(x,y) < R_K$. For any $r_K > r > d(x,y)$ there holds

$$\begin{aligned}|u(x) - u(y)| \leq & |u(x) - A^r u(x)| + |u(y) - A^r u(y)| + |A^r u(x) - A^r u(y)| \\ \leq & 2r \int_0^r (|\Delta_t u(x)| + |\Delta_t u(y)|) dt + \frac{Cd(x,y)}{r} \fint_{B(x,3r)} |u - u_{B(x,3r)}| d\mu,\end{aligned} \tag{4.10}$$

where the third term is estimated using Proposition 4.10 with $c = u_{B(x,3r)}$ and the first two terms are treated in the following way

$$\begin{aligned}|u(x) - A^r u(x)| \leq & \frac{2}{r} \int_{\frac{r}{2}}^r \left| u(x) - \fint_{B(x,t)} u(y) d\mu(y) \right| dt = 2r \int_{\frac{r}{2}}^r \frac{t^2}{r^2} |\Delta_t u(x)|\, dt \\ \leq & 2r \int_{\frac{r}{2}}^r |\Delta_t u(x)|\, dt \leq 2r \int_0^r |\Delta_t u(x)|\, dt.\end{aligned}$$

By choosing $r = d(x,y)^{1/2} > d(x,y)$ we obtain

$$|u(x) - u(y)| \leq d(x,y)^{1/2}[g(x) + g(y)],$$

where

$$g(x) = 2\int_0^{r_K} |\Delta_t u(x)| dt + C\mathcal{M}^{\#}_{3r_K} u(x).$$

Moreover, suppose that $u \in M^{\alpha,p}_{loc}(\Omega)$, and let $g_K$ be the Hajłasz gradient and $\tilde{r}_K$ the scale in Definition 4.13. Define

$$R_K = \frac{1}{6}\min\{\tilde{r}_K, \mathrm{dist}(K, X \setminus \Omega)^{3-\alpha-1/p}, r_K^2\}$$

From (4.10) we obtain that as long as $6r < \tilde{r}_K$ and $d(x,y) < R_K$, then

$$|u(x) - u(y)| \leq Cr^{2-1/p} \left( \int_0^r (|\Delta_t u(x)|^p + |\Delta_t u(y)|^p) dt \right)^{1/p} + C\frac{d(x,y)}{r^{1-\alpha}}[A_{6r} g_K(x) + A_{6r} g_K(y)]. \tag{4.11}$$

Indeed, applying the Hölder inequality to the first term in the right-hand side of (4.10) we obtain, up to a multiplicative constant, that

$$r\int_0^r |\Delta_t u(x)|\, dt \leq r \left( \int_0^r |\Delta u(x)|^p \right)^{\frac{1}{p}} r^{\frac{p-1}{p}} = r^{2-\frac{1}{p}} \left( \int_0^r |\Delta u(x)|^p \right)^{\frac{1}{p}}.$$

Whereas the second term in (4.10) is estimated using the Hajłasz inequality, up to a multiplicative constant, in the following way

$$
\begin{aligned}
&\frac{d(x,y)}{r} \fint_{B(x,3r)} |u-u_{B(x,3r)}| d\mu \\
&\qquad = \frac{d(x,y)}{r} \fint_{B(x,3r)} \left| u(w) - \fint_{B(x,3r)} u(z) d\mu(z) \right| d\mu(w) \\
&\qquad \le \frac{d(x,y)}{r} \fint_{B(x,3r)} \fint_{B(x,3r)} |u(w)-u(z)| \, d\mu(z) d\mu(w) \\
&\qquad \le \frac{d(x,y)}{r} \fint_{B(x,3r)} \fint_{B(x,3r)} d(w,z)^\alpha (g_K(w)+g_K(z)) d\mu(z) d\mu(w) \\
&\qquad \le \frac{d(x,y)}{r} (6r)^\alpha \fint_{B(x,3r)} \fint_{B(x,3r)} \left(\frac{d(w,z)}{6r}\right)^\alpha (g_K(w)+g_K(z)) d\mu(z) d\mu(w) \\
&\qquad \le C \frac{d(x,y)}{r^{1-\alpha}} [A_{6r} g_K(x) + A_{6r} g_K(y)].
\end{aligned}
$$

We choose $r$ in (4.11) such that

$$
r^{2-1/p} = \frac{d(x,y)}{r^{1-\alpha}},
$$

i.e.

$$
r = d(x,y)^{1/(3-\alpha-1/p)} < r_K^{1/(3-\alpha-1/p)} < \tilde{r}_K,
$$

to obtain

$$
|u(x)-u(y)| \le C d(x,y)^{\alpha'} [g(x)+g(y)]
$$

where

$$
\alpha' = \frac{2-1/p}{3-\alpha-1/p}
$$

and

$$
g(x) = C \left( \int_0^{\tilde{r}_K} |\Delta_t u(x)|^p dt \right)^{1/p} + C \mathcal{M}_{\tilde{r}_K} g_K(x).
$$

□

We iterate Proposition 4.14 to improve regularity of functions from $\mathrm{AMV}^p_{loc}(\Omega)$.

**Theorem 4.15.** *Let $\Omega \subset X$ be an open subset of a complete locally doubling metric measure space $X = (X, d, \mu)$, and let $u \in \mathrm{AMV}^p_{loc}(\Omega)$. Then $u \in M^{\alpha,p}_{loc}(\Omega)$ for every $0 < \alpha < 1$.*

*Proof.* Define $\alpha_0 = 1/2$ and

$$
\alpha_{n+1} = \frac{2-1/p}{3-\alpha_n-1/p}, \quad n \ge 0.
$$

We see that $\alpha_n$ is an increasing sequence and converges to 1. By Proposition 4.14 we have that $u \in M^{\alpha_k,p}_{loc}(\Omega)$ for every $k$. The claim follows. □

Theorem 4.15 is not quantitative, because it does not give an explicit bound on the fractional Hajłasz–Sobolev gradient in terms of the amv-norm of the function $u$. We apply the regularization $A^r u$ to function $u \in \mathrm{AMV}^p_{loc}(X)$ to prove the following result which more explicitly describes the Hajłasz gradient of $u$.

**Proposition 4.16.** *Suppose that $X$ is a locally doubling metric measure space, function $u \in \mathrm{AMV}^p_{loc}(X)$ and $x_0 \in X$. For each $k \in \mathbb{N}$ there exists $r_k > 0$ and $g_k \in L^p(B(x_0, r_k))$ such that $\bar{B}(x_0, r_k^{k/(k+1)}) =: \bar{B}_k$ is compact and*

$$|u(x) - u(y)| \leq d(x,y)^{k/(k+1)}[g_k(x) + g_k(y)], \quad x, y \in B_k \setminus E,$$

*where $E$ is a null set.*

*Proof.* Let $x, y \in B(x_0, r_0)$ where $B(x_0, 3r_0^{1/2}) \subset \Omega$. For any $r < r_0^{1/2}$ we have

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - A^r u(x)| + |u(y) - A^r u(y)| + |A^r u(x) - A^r u(y)| \\ &\leq \frac{2}{r}\int_0^r t^2[|\Delta_t u|(x) + |\Delta_t u|(y)]dt + |A^r u(x) - A^r u(y)| \\ &\leq Cr \int_0^r [|\Delta_t u|(x) + |\Delta_t u|(y)]dt + \frac{Cd(x,y)}{r} \fint_{B(x,3r)} |u - c| d\mu; \end{aligned} \tag{4.12}$$

cf. Proposition 4.10. Choosing $r = d(x,y)^{1/2}$ we obtain

$$|u(x) - u(y)| \leq d(x,y)^{1/2}[g_1(x) + g_1(y)], \quad x, y \in B(x_0, r_0),$$

where

$$g_1(x) := C\int_0^{r_0^{1/2}} |\Delta_t u|(x)dt + C\mathcal{M}^{\#}_{r_0^{1/2}} u(x).$$

We have that $g_1 \in L^p(B(x_0, r_0))$, since

$$\int_{B(x_0,r_0)} g_1^p d\mu \leq C r_0^{(p-1)/2} \int_0^{r_0^{1/2}} \int_{B(x_0,r_0)} |\Delta_t u|^p(x) d\mu(x)\, dt + C\int_{B(x_0,r_0)} (\mathcal{M}^{\#}_{r_0^{1/2}} u(x))^p d\mu(x) < \infty.$$

To iterate this process, suppose the claim in the proposition holds for $k \in \mathbb{N}$. Let $0 < r_{k+1} < r_k$ be such that $B(x_0, 3r_{k+1}^{(k+1)/(k+2)}) \subset \Omega$. For $x, y \in B(x_0, r_k)$ we get, using (4.12) with $c = u(x)$, that

$$\begin{aligned} |u(x) - u(y)| &\leq Cr \int_0^{r_0^{1/2}} [|\Delta_t u|(x) + |\Delta_t u(y)|]dt + \frac{Cd(x,y)}{r} \fint_{B(x,3r)} r^{k/(k+1)}(g_k(x) + g_k(y))d\mu(y) \\ &\leq Cr \int_0^{r_0^{1/2}} [|\Delta_t u|(x) + |\Delta_t u(y)|]dt + \frac{Cd(x,y)}{r^{1/(k+1)}} \mathcal{M}_{r_k^{\frac{k+1}{k+2}}} g_k(x). \end{aligned}$$

Here, $\mathcal{M}_{r_k^{\frac{k+1}{k+2}}} g_k(x)$ denotes the restricted Hardy–Littlewood maximal function for radii $0 < r < r_k^{\frac{k+1}{k+2}}$. Choosing $r = d(x,y)^{(k+1)/(k+2)}$ we obtain

$$|u(x) - u(y)| \leq Cd(x,y)^{(k+1)/(k+2)}[g_{k+1}(x) + g_{k+1}(y)], \quad x, y \in B(x_0, r_k),$$

where

$$g_{k+1}(x) := \int_0^{r_0^{1/2}} |\Delta_t u|(x)dt + \mathcal{M}_{r_0^{1/2}} g_k(x) \in L^p(B(x_0, r_k)).$$

□

Let us recall the fractional Morrey embedding theorem, see part i) and iii) in [Yan03][Corollary 1.4].

**Proposition 4.17.** *Let $X$ be locally compact doubling metric measure space. Suppose, that $B \subset X$ is a ball such that on $B$ there holds (4.3) with doubling exponent $Q$. Suppose that $0 < \alpha \leq 1$, $0 < p < \infty$ and $\alpha p > Q$. Then, there holds the following embedding*

$$M^{\alpha,p}(B) \subset C^{\alpha - \frac{Q}{p}}(B).$$

We apply the Morrey embedding theorem for fractional Hajłasz–Sobolev spaces and Theorem 4.15 to prove Hölder regularity of strongly amv-harmonic functions.

**Theorem 4.18.** *Let $\Omega \subset X$ be a domain in a doubling metric measure space with doubling exponent $Q$. If $p > Q$ and $u \in \mathrm{AMV}^p_{loc}(\Omega)$, then $u$ is locally $\alpha$-Hölder continuous for every $\alpha < 1 - Q/p$. Moreover, any strongly amv-harmonic function on $\Omega$ is locally $\alpha$-Hölder continuous for any $\alpha \in (0,1)$.*

*Proof.* Let us fix a ball $B \subset \Omega$, an exponent $\alpha < 1 - \frac{Q}{p}$ and function $u \in \mathrm{AMV}^p_{loc}(\Omega)$. Then, by Theorem 4.15 we get that $u \in M^{\beta,p}(B)$ for any $\beta \in (0,1)$. We choose $\beta$ so that $\alpha < \beta - \frac{Q}{p}$ and $\beta p > Q$. Apply Proposition 4.17 to obtain that $u$ is $\beta - \frac{Q}{p}$-Hölder continuous on $B$ and hence $\alpha$-Hölder continuous on $B$.

In order to prove that a strongly amv-harmonic function $u$ is Hölder continuous we only need to observe, that $u \in \mathrm{AMV}^p_{loc}(\Omega)$ for every $p \in (1,\infty)$ and use the first part of the hypothesis. □

### 4.3.3 Improving the regularity: the Blaschke-Privaloff-Zaremba theorem for the amv-harmonic functions on the Heisenberg group

In this chapter we are going to present, that strongly amv-harmonic functions beyond the Euclidean setting may posses higher regularity, than Hölder continuity proven in the previous chapter. Namely, we discuss the so-called Blaschke-Privaloff-Zaremba theorem (the BPZ theorem, for short). In its classical version in the setting of Euclidean spaces, see e.g. [Llo15, Theorem 2.1.5], the BPZ theorem asserts that given an open set in $\mathbb{R}^n$ a continuous pointwise amv-harmonic function solves locally the Laplace equation. Thus, the pointwise nullity of the amv-harmonic operator $\lim_{r\to 0^+} \Delta_r$ improves the regularity of amv-harmonic functions to being analytic. Below we show that this is also the case of amv-harmonic functions in Heisenberg group $\mathbb{H}_1$. For the convenience of the reader we briefly recall the setting of the Heisenberg group, cf. Example 3.5.

Our model for $\mathbb{H}^1$ is the group $(\mathbb{R}^3, \circ)$ where the group law is given by

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)).$$

By using this group law, one introduces a frame of left-invariant vector fields which agree with the standard basis at the origin:

$$X_1 := \frac{\partial}{\partial x_1} - \frac{1}{2} x_2 \frac{\partial}{\partial x_3}, \quad X_2 := \frac{\partial}{\partial x_2} + \frac{1}{2} x_1 \frac{\partial}{\partial x_3}, \quad X_3 := \frac{\partial}{\partial x_3}.$$

The *Korányi–Reimann distance* is a metric defined by

$$d_{\mathbb{H}^1}(x,y) := \|y^{-1}x\|_{\mathbb{H}^1}, \quad \text{where } \|(x_1,x_2,x_3)\|_{\mathbb{H}^1} = \sqrt[4]{(x_1^2 + x_2^2)^2 + x_3^2}.$$

Let $p_0 \in \mathbb{H}_1$ and $R > 0$. An open ball in $\mathbb{H}_1$ centered at $p_0$ with radius $R$ with respect to metric $d_{\mathbb{H}^1}$ is defined as follows: $B(p_0,R) := \{p \in \mathbb{H}_1 \, : \, \|p^{-1}p_0\|_{\mathbb{H}^1} < R\}$. The subelliptic Laplace operator $\Delta_{\mathbb{H}_1}$ on the Heisenberg group is defined as $\Delta_{\mathbb{H}_1} u := X_1^2 u + X_2^2 u$ and in the local coordinates $(x_1, x_2, x_3)$ reads $\Delta_{\mathbb{H}_1} u = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{4}(x_1^2 + x_2^2)\frac{\partial^2}{\partial x_3^2} - x_2 \frac{\partial^2}{\partial x_1 \partial x_3} + x_1 \frac{\partial^2}{\partial x_2 \partial x_3}$. Solutions to the subelliptic Laplace equation are $C^2$ due to results e.g. in [Cap97] and [MM07]. The Dirichlet problem on Korányi–Reimann balls for the continuous boundary data has the classical $C^2$-solution for harmonic subelliptic equation in $\mathbb{H}_1$, see [GV85]. The subelliptic Laplacian on $\mathbb{H}_n$ is hypoelliptic, which improves the regularity of harmonic functions to being real analytic. Hovewer, it is shown in [HH87], that for $\mathbb{H}_n$ for $n \geq 2$ balls in the Carnot-Carathéodory distance are not regular at the characteristic points. Therefore, due to the approach we take in the proof of Theorem 4.19 below, we will restrict our discussion to the case of $\mathbb{H}_1$ and balls with respect to the *Korányi–Reimann* distance.

**Theorem 4.19** (The Blaschke-Privaloff-Zaremba theorem in $\mathbb{H}_1$)**.** *Let $\Omega \subset \mathbb{H}_1$ be a domain in the first Heisenberg group $\mathbb{H}_1$ equipped with metric $d_{\mathbb{H}^1}$ and let $f : \Omega \to \mathbb{R}$ be a continuous amv-harmonic function in $\Omega$. Then $f$ is a sub-elliptic harmonic function, i.e. for all $x \in \Omega$ it holds that*

$$\lim_{r\to 0^+} \Delta_r f(x) = \frac{1}{3\pi}\Delta_{\mathbb{H}_1} f(x) = 0.$$

The constant $\frac{1}{3\pi}$ is computed in [FLM14, Lemma 3.3]. Notice, that in [FLM14] the authors choose different vector fields $X$ and $Y$, but their proof and hence the constant is independent on that choice.

*Proof.* In the proof we follow the original idea of Privaloff developed for the setting of $\mathbb{R}^n$, see [Pri25, Theorem II]. Let $p_0 \in \Omega$ and $B = B(p_0, R)$ be a ball centered at $p_0$ with radius $R > 0$ such that $B \subset \Omega$. Theorem in [GV85] allows us to infer that the sub-elliptic Dirichlet problem on $B$ with the boundary data $f$ has the unique solution, denoted by $u$, such that $u \in C(\overline{B})$ and $u = f|_{\partial B}$. Set $\phi = f - u$. Then $\phi \in C(\overline{B})$ and $\phi|_{\partial B} \equiv 0$. The assertion will be proven if we show that $\phi \equiv 0$ in $B$. We argue by contradiction. Namely, suppose that there exists $q \in B$ such that $\phi(q) \neq 0$ and without the loss of generality we assume that $\phi(q) < 0$. Let us define the following function on $B$

$$F(p) = \phi(p) + \frac{\phi(q)}{2}\left(\frac{\|(p^{-1}p_0)_H\|^2_{\mathbb{R}^2} - R^2}{R^2}\right),$$

where $\|(p^{-1}p_0)_H\|_{\mathbb{R}^2}$ stands for the Euclidean length in $\mathbb{R}^2$ of the horizontal part of point $p^{-1}p_0 \in \Omega$. It follows that $F \in C(\overline{B})$, $F|_{\partial B} \geq 0$ and that $F(q) < 0$. Hence, there is $q_m \in \overline{B}$ such that $F(q_m) = \min_{\overline{B}} F$ (in fact, $q_m \in B$). Moreover, $\Delta_r F(q_m) \geq 0$ for all $r \leq R$ and by direct computations we verify that

$$\Delta_r F(q_m) = \Delta_r \phi(q_m) + \frac{\phi(q)}{2}\Delta_r\left(\frac{\|(p^{-1}p_0)_H\|^2_{\mathbb{R}^2} - R^2}{R^2}\right)(q_m).$$

Therefore, upon applying the definition of $\Delta_r$, we arrive at the following estimate

$$0 \leq \Delta_r F(q_m) = \Delta_r f(q_m) - \Delta_r u(q_m) + \frac{\phi(q)}{2}\Delta_r\left(\frac{\|(p^{-1}p_0)_H\|^2_{\mathbb{R}^2} - R^2}{R^2}\right)(q_m).$$

Let us denote the coordinates of $p$ and $p_0$ as follows: $p = (x, y, t)$ and $p_0 = (x_0, y_0, t_0)$. Then

$$\frac{\phi(q)}{2}\Delta_{\mathbb{H}_1}\left(\frac{\|(p^{-1}p_0)_H\|^2_{\mathbb{R}^2} - R^2}{R^2}\right)(q_m) = \frac{\phi(q)}{2R^2}\Delta_{\mathbb{H}_1}\left((x-x_0)^2 + (y-y_0)^2\right)(q_m).$$

Recall, that

$$\Delta_{\mathbb{H}_1} = \frac{\partial^2}{\partial_x^2} + \frac{\partial^2}{\partial_y^2} + \frac{1}{4}(x^2+y^2)\frac{\partial^2}{\partial t^2} - y\frac{\partial^2}{\partial x \partial t} + x\frac{\partial^2}{\partial y \partial t}.$$

Therefore, in our case the sub-Laplacian reduces to the Laplacian in $\mathbb{R}^2$

$$\Delta_{\mathbb{H}_1}\left((x-x_0)^2 + (y-y_0)^2\right)(q_m) = \Delta\left((x-x_0)^2 + (y-y_0)^2\right)(q_m) = 4.$$

Since $f$ is assumed to be strongly amv-harmonic and $u \in C^2(B)$, the definition of amv-harmonic functions together with [FLM14, Lemma 3.3] imply that upon $r \to 0^+$ it holds that $\Delta_r f(q_m) \to 0$ and $\Delta_r u(q_m) \to 0$. Moreover, by applying [FLM14, Lemma 3.3] again we obtain that

$$\lim_{r\to 0^+} \frac{\phi(q)}{2}\Delta_r\left(\frac{\|(p^{-1}p_0)_H\|^2_{\mathbb{R}^2} - R^2}{R^2}\right)(q_m) = \frac{1}{3\pi}\frac{\phi(q)}{2}\Delta_{\mathbb{H}_1}\left(\frac{\|(p^{-1}p_0)_H\|^2_{\mathbb{R}^2} - R^2}{R^2}\right)(q_m) = \frac{2\phi(q)}{3\pi R^2} < 0,$$

since above we assume that $\phi(q) < 0$. In a consequence, we get that $0 \leq \frac{2\phi(q)}{3\pi R^2} < 0$ contradicting our assumption. The proof of the theorem is completed. □

In the next chapter we prove, that the dimension of the space of strongly harmonic functions of polynomial growth is finite.

### 4.3.4 Mean value-harmonic functions of polynomial growth

Colding and Minicozzi proved in [CM97a; CM97b] a conjecture of Yau on the finite dimensionality of the space of harmonic functions in the sense of solutions to the Beltrami–Laplace equation with polynomial growth of degree $m$ by showing that, in a Riemannian $n$-manifold $M$ of non-negative Ricci curvature holds the following bound

$$\dim \mathcal{H}^m(M) \le C(n) m^{n-1}.$$

This result has been extended to Alexandrov and RCD-spaces, see [Hua11; HKX16]. An argument of Li [Li97] uses the doubling property and the mean value inequality of subharmonic functions to obtain the estimate

$$\dim \mathcal{H}^m(M) \le C(n) m^{Q} \tag{4.13}$$

for manifolds with a measure satisfying (4.3) with doubling exponent $Q$ and a uniform Poincaré inequality. By the latter we mean that there exists a constant $C > 0$ such that for every function $u \in W^{1,2}_{loc}(M)$, point $x \in M$ and radius $r > 0$ there holds $\int_{B(x,r)} |u - u_{B(x,r)}|^2 \le Cr^2 \int_{B(x,r)} |\nabla u|^2$. The estimate (4.13) remains valid in the context of strongly harmonic functions on doubling spaces.

In fact a modification of the same argument improves the bound (4.13) if $\mu$ satisfies an annular decay property, cf. Definition 4.1. We follow the strategy in [Li97], see Lemmas 4.24 and 4.25 below, and present the modifications needed for our result.

**Definition 4.20.** A function $u \in \mathcal{H}(X)$ is said to have *growth rate at most* $m$ for $m > 0$ if there exists $p \in X$ and $C > 0$ such that for all $x \in X$ there holds

$$|u(x)| \le C(1 + d_p(x))^m,$$

where $d_p : X \to \mathbb{R}$ is the distance function $x \mapsto d(p, x)$. We denote by $\mathcal{H}^m(X)$ the space of $u \in \mathcal{H}(X)$ with growth rate at most $m$.

**Remark 4.21.** This definition is independent on the choice of point $p \in X$. Indeed, suppose that $u \in \mathcal{H}^m(X)$ for some $p \in X$ and take any $q \in X$. Then, the following estimate holds true

$$\begin{aligned} |u(x)| &\le C(1 + d(p,x))^m \le C(1 + d(p,q) + d(q,x))^m \\ &\le C(1 + d(p,q))^m \left(1 + \frac{d(q,x)}{1 + d(p,q)}\right)^m \le C(1 + d(p,q))^m (1 + d(q,x))^m. \end{aligned}$$

Therefore, $u \in \mathcal{H}^m(X)$ as well for the choice of point $q$ and a constant $C' = C(1 + d(p,q))^m$.

Notice, that for every $m > 0$ space $\mathcal{H}^m(X)$ is nonempty, because it contains constant functions. In case of $X$ being a harmonic Riemannian manifold the class $\mathcal{H}^m(X)$ consist of harmonic polynomials of degree at most $m$. Moreover, if $X$ is a Carnot group, then $\mathcal{H}^m(X)$ contains spherical harmonic polynomials of degree at most $m$, see [AW20].

**Proposition 4.22.** *Let $(X, d, \mu)$ be a complete doubling metric measure space with doubling exponent $Q := \log_2 C_\mu > 1$, and suppose $\mu$ has $\alpha$-annular decay* (4.4)*. Then, for any $m > 0$, we have that*

$$\dim \mathcal{H}^m(X) \le C m^{Q-\alpha},$$

*where the constant $C = C(Q, \alpha)$ depends only on $Q$ and $\alpha$.*

Doubling measures on length spaces always satisfy an annular decay property for some $\alpha$, see the discussion following Definition 4.1. Thus, Proposition 4.22 implies the following corollary.

**Corollary 4.23.** *Let $(X, d, \mu)$ be a complete geodesic doubling metric measure space with $Q > 1$. Then there exists $\delta > 0$, depending only on a doubling exponent $Q$, so that*

$$\dim \mathcal{H}^m(X) \le C(Q) m^{Q-\delta}.$$

Let $(X, d, \mu)$ be a complete doubling metric measure space, where $\mu$ has $\alpha$-annular decay. Given $R > 0$, we define a bi-linear form

$$A_R(u, v) := \int_{B(p,R)} uv d\mu,$$

for $u, v \in \mathcal{H}(X)$. Note that $A_R$ is symmetric and positive semidefinite. It follows from the proof of [Hua11][Lemma 3.4] that, for any finite dimensional vector subspace $V \subset \mathcal{H}(X)$, there exists a radius $R_0 > 0$ so that $A_R$ is an inner product on $V$ for every $R \geq R_0$.

In order to prove Proposition 4.22 we need the following auxiliary results.

**Lemma 4.24.** *Let $V$ be a $k$-dimensional linear subspace of $\mathcal{H}^m(X)$. For any $p \in X$, $\beta > 1$, $\delta > 0$, $R_0 > 0$ there exists $R > R_0$ such that if $u_1, \ldots, u_k$ is an orthonormal basis for $V$ with respect to the inner product $A_{\beta R}$, then*

$$\int_{B(p,R)} (u_1^2 + \ldots + u_k^2)\, d\mu \geq \frac{k}{\beta^{2m+Q+\delta}}.$$

*Proof.* The proof of [Li97, Lemma 2] for manifolds carries over to the setting of metric measure spaces under our assumptions, because it is based only on linear tools and the measure growth condition (4.3). See also [Hua11, Lemma 3.7], where the lemma is proven in the setting of Alexandrov spaces, and [HKX16, Lemma 5.2] for the formulation of the lemma in the $RCD^*(0, N)$ spaces. □

**Lemma 4.25.** *Let $V$ be a $k$-dimensional linear subspace of $\mathcal{H}^m(X)$. Then, there exists a constant $C = C(Q)$ such that for any base $u_1, \ldots, u_k$ of $V$, any $p \in X$, $R > 0$ and any $\varepsilon \in (0, \frac{1}{2})$ it holds that*

$$\int_{B(p,R)} (u_1^2 + \ldots + u_k^2)\, d\mu \leq \frac{C}{\varepsilon^{Q-\alpha}} \sup_{\sum_{i=1}^k a_i^2 = 1} \int_{B(p,(1+\varepsilon)R)} |a_1 u_1 + \ldots + a_k u_k|^2 d\mu.$$

*Proof.* We follow closely the proof of [HKX16][Lemma 5.3]. Fix $q \in B(p, r)$ and define $V_q := \{u \in V : u(q) = 0\}$. The subspace $V_q \subset V$ is of $\operatorname{codim} V_q \leq 1$ since, if $u, v \notin V_q$, then $u - \frac{u(q)}{v(q)} v \in V_q$.

There exists an orthogonal change of variables $A$ on $V$ such that $A(u_i) := v_i$ for all $i = 1, \ldots, k$ with $v_i \in V_q$ for $i = 2, \ldots, k$.

We recall the relevant part of [AGG19][Proposition 3.1]: if $f \in \mathcal{H}(X)$ and $F : f(X) \to \mathbb{R}$ is convex, then $F \circ f$ is *subharmonic*, i.e. for all $x \in X$ and $r > 0$ there holds

$$F(f(x)) \geq \fint_{B(x,r)} F(f(y)) d\mu(y).$$

We apply this result for $f = v_1$ and $F(s) = s^2$ to obtain that

$$\begin{aligned}\sum_{i=1}^k u_i^2(q) = \sum_{i=1}^k v_i^2(q) = v_1^2(q) &\leq \fint_{B(q,(1+\varepsilon)R - d_p(q))} v_1^2(z)\, d\mu(z) \\ &\leq \sup_{\sum_{i=1}^k a_i^2 = 1} \frac{1}{\mu(B(q, (1+\varepsilon)R - d_p(q)))} \int_{B(p,(1+\varepsilon)R)} \left| \sum_{i=1}^k a_i u_i(z) \right|^2 d\mu(z). \end{aligned} \tag{4.14}$$

We apply (4.3) to obtain that

$$\frac{\mu(B(q, (1+\varepsilon)R - d_p(q)))}{\mu(B(p, (1+\varepsilon)R))} \geq \frac{1}{C_\mu^2} \left( \frac{(1+\varepsilon)R - d_p(q)}{(1+\varepsilon)R} \right)^Q.$$

Therefore,

$$\frac{1}{\mu(B(q, (1+\varepsilon)R - d_p(q)))} \leq \frac{C_\mu^2}{\mu(B(p, (1+\varepsilon)R))} \left( \frac{(1+\varepsilon)R}{(1+\varepsilon)R - d_p(q)} \right)^Q.$$

Hence, upon integrating (4.14), we arrive at

$$\begin{aligned}
&\sum_{j=1}^{k} A_R(u_j,u_j)\\
\le&\frac{C}{\mu(B(p,R))}\left(\int_{B(p,R)}\left(1+\varepsilon-\frac{d_p(q)}{R}\right)^{-Q}d\mu(q)\right)\\
&\cdot\sup_{\sum_{i=1}^k a_i^2=1}\int_{B(p,(1+\varepsilon)R)}|a_1u_1+\ldots+a_ku_k|^2d\mu.
\end{aligned}\tag{4.15}$$

Denote

$$f:[0,1]\to\mathbb{R},\quad f(t)=(1+\varepsilon-t)^{-Q},$$

and note that the claim follows directly from (4.15) and the estimate

$$\fint_{B(p,R)} f(d_p/R)d\mu\le\frac{C}{\varepsilon^{Q-\alpha}}.\tag{4.16}$$

To obtain (4.16), note that $f$ is smooth, the derivative $f'(t)=Q(1+\varepsilon-t)^{-Q-1}$ is positive, hence $f$ is increasing and thus

$$\begin{aligned}
\int_{B(p,R)} f(d_p/R)d\mu&=\int_0^\infty \mu(B(p,R)\cap\{q\in X: f\circ(d_p/R)(q)\ge\lambda\})d\lambda\\
&=\int_{-\infty}^1 f'(s)\mu(B(p,R)\cap\{d_p\ge sR\})ds
\end{aligned}$$

after a change of variables $f(s)=\lambda$. The $\alpha$-annular decay implies

$$\mu(B(p,R)\cap\{d_p\ge sR\})=\mu(B(p,R)\setminus B(p,sR))\le C(1-s)^\alpha\mu(B(p,R))$$

and therefore

$$\begin{aligned}
\fint_{B(p,R)} f(d_p/t)d\mu\le&C\int_{-\infty}^1 f'(s)(1-s)^\alpha ds\le CQ\int_{-\infty}^1(1+\varepsilon-s)^{-Q-1+\alpha}ds\\
=&\frac{CQ}{Q-\alpha}\frac{1}{\varepsilon^{Q-\alpha}},
\end{aligned}$$

establishing (4.16). □

*Proof of Proposition 4.22.* Let $R$ be large enough, $\beta=1+\varepsilon$ and $\varepsilon=1/(2m)$. Let $\{u_1,\ldots,u_k\}$ be an orthonormal basis with respect to $A_{\beta R}$. Combining the estimates in Lemmas 4.24 and 4.25 we obtain

$$\frac{k}{(1+\varepsilon)^{2m+Q+\delta}}\le\sum_{j=1}^k A_R(u_j,u_j)\le\frac{C(Q)}{\varepsilon^{Q-\alpha}},$$

since

$$\sup_{a_1,\ldots,a_k:\sum_{i=1}^k a_i^2=1}\int_{B(p,(1+\varepsilon)R)}|a_1u_1+\ldots+a_ku_k|^2d\mu=1.$$

Thus

$$k\le C(Q)(1+1/(2m))^{2m+Q+\delta}(2m)^{Q-\alpha}\le Cm^{Q-\alpha}$$

after letting $\delta\to 0$. □

## 4.4 Blow-ups of Hajłasz–Sobolev functions with finite AMV-norm

In this chapter we study blow-ups of functions with finite amv-norm. A blow-up of a metric measure space and a function around a point is a pointed Gromov–Hausdorff limit of a rescaling, see Chapter 4.4.3. Blow ups are a tool of analysis on metric spaces used to study local behaviour and geometry of a metric measure space and functions on that space. One of the essential reasons to study blow ups is that the class of doubling metric measure spaces supporting a Poincaré inequality is closed under Gromov–Hausdorff convergence, see [Hei+15][Chapter 11].

The main result of this chapter is Theorem 4.41 which says, that a blow up of a Hajłasz–Sobolev function $u \in \mathrm{AMV}^p_{loc}(\Omega)$ is strongly harmonic on the tangent space. An immediate consequence is that tangent functions to a strongly amv-harmonic function are strongly harmonic.

We begin with a review of pointed measured Gromov–Hausdorff convergence of spaces and functions, which we will abbreviate to pmGH-convergence. In the literature there are several variants of pmGH-convergence. Here we follow the presentation of [Kei03], and refer the interested reader to [Hei+15] for more extensive discussion and the relations between various notions.

### 4.4.1 Pointed measured Gromov–Hausdorff convergence

In this chapter we announce notions which are fundamental for our further studies: the Hausdorff convergence of sets and pointed Gromov–Hausdorff convergence of metric measure spaces and functions. Moreover, we present basic properties of these objects as semicontinuity of Hausdorff convergence, compactness of proper metric measure spaces and the relation of Gromov–Hausdorff convergence of functions to weak convergence.

We begin with introducing the notion of Hausdorff convergence of closed sets in metric spaces and recalling the weak convergence of measures.

**Definition 4.26.** Let $F_m, F \subset Z$ be closed sets in a metric space $Z$. We say that $F_n$ Hausdorff-converges to $F$ and denote it by $F_m \to F$, if

$$\lim_{m\to\infty} \sup_{z\in F_m\cap B(q,R)} \mathrm{dist}_Z(z,F) = 0 \quad \text{and} \quad \lim_{m\to\infty} \sup_{z\in F\cap B(q,R)} \mathrm{dist}_Z(z,F_m) = 0$$

for every $q \in Z$ and $R > 0$.

**Definition 4.27.** If $\nu_m, \nu$ are Radon measures on $Z$, we say that $\nu_m$ converges to $\nu$ weakly, denoted $\nu_m \rightharpoonup \nu$, if

$$\lim_{m\to\infty} \int_Z \varphi d\nu_m = \int_Z \varphi d\nu$$

for every continuous function $\varphi : Z \to \mathbb{R}$ with bounded support.

Let us prove the following semicontinuity-type result.

**Lemma 4.28.** *Let $\nu_m$ be a sequence of measures on $Z$ converging weakly to a measure $\nu$. Suppose that $F_m$ is a sequence of compact sets Hausdorff-converging to a compact set $F$. Then for any fixed $\varepsilon > 0$ there holds*

$$\limsup_{m\to\infty} \nu_m(F_m) \le \nu(F) \le \liminf_{m\to\infty} \nu_m(N_\varepsilon(F_m)).$$

*Proof.* Let us fix $\varepsilon > 0$. By Definition 4.26 there exists $m_0 \in \mathbb{N}$ such that

$$F \subset N_\varepsilon(F_m) \quad \text{and} \quad F_m \subset N_\varepsilon(F)$$

whenever $m \ge m_0$. By [Hei+15][Remark 11.4.1] we obtain

$$\limsup_{m\to\infty} \nu_m(F_m) \le \limsup_{m\to\infty} \nu(\bar{N}_\varepsilon(F)) \le \nu(\bar{N}_\varepsilon(F)).$$

Taking infimum over $\varepsilon > 0$ we obtain the first inequality.

Similarly, to conclude the second inequality we use [Hei+15][Remark 11.4.1] and estimate

$$\nu(F) \leq \nu(N_\varepsilon(F)) \leq \liminf_{m\to\infty} \nu_m(N_\varepsilon(F)) \leq \liminf_{m\to\infty} \nu_m(N_{2\varepsilon}(F_m)),$$

which ends the proof. □

For the next definition, we recall that a pointed metric measure space $(X, d, \mu, p)$ consists of a metric measure space $(X, d, \mu)$ and a distinguished point $p \in X$. We consider only proper spaces here.

**Definition 4.29.** A sequence $X_m = (X_m, d_m, \mu_m, p_m)$ of pointed proper metric measure spaces *pmGH-converges* to a pointed proper metric measure space $X = (X, d, \mu, p)$, denoted $X_m \xrightarrow{GH} X$, if there exists a pointed proper metric space $(Z, q)$ and isometric embeddings $\iota_m : X_m \to Z$ and $\iota : X \to Z$ so that

(1) $\iota_m(p_m) = \iota(p) = q$, and $\iota_m(X_m)$ Hausdorff-converges to $\iota(X)$;

(2) there holds the weak convergence of pushforwards $\iota_{m*}\mu_m \rightharpoonup \iota_*\mu$.

We also define Gromov–Hausdorff convergence for sequences of functions. Since we consider pointed measured spaces, we nevertheless include them (see also Definition 4.34).

**Definition 4.30.** Let $u_m : X_m \to \mathbb{R}$ and $u : X \to \mathbb{R}$ be functions on pointed proper metric measure spaces. We say that $u_m$ Gromov–Hausdorff converges to $u$, denoted $u_m \xrightarrow{GH} u$, if there are isometric embeddings $\iota_m : X_m \to Z$ and $\iota : X \to Z$ satisfying conditions (1) and (2) in Definition 4.29 and

(3) $u_m(z_m) \to u(z)$ whenever $z_m \in X_m$, $z \in X$, and $\iota_m(z_m) \to \iota(z)$.

The embeddings $\iota_m : X_m \to Z$ and $\iota : X \to Z$ satisfying (1) and (2) (resp. (3)) in Definition 4.29 are said to *realize* the convergence $X_m \xrightarrow{GH} X$ (resp. $u_m \xrightarrow{GH} u$).

Two central properties of Gromov–Hausdorff convergence are its compactness properties, see Proposition 4.32 below, and the stability of properties which are central in metric geometry and analysis. For our purposes the stability of length spaces and the doubling property of the measure is important. For a detailed discussion see [Hei+15][Section 11] and [Kei03].

**Definition 4.31.** Let $(X, d, p)$ be a pointed metric space. We denote by $N_X(\varepsilon, R)$ the maximal number of disjoint closed balls of radius $\varepsilon$ inside $B(p, R)$. We say that a sequence of pointed metric spaces $(X_m, d_m, p_m)$ is *totally bounded* if

$$\sup_m N_{X_m}(\varepsilon, R) < \infty$$

for every choice $\varepsilon, R > 0$.

The following compactness property is proved in [Kei03][Proposition 5.1.9].

**Proposition 4.32.** *Let $X_m = (X_m, d_m, \mu_m, p_m)$ be a totally bounded sequence of proper metric measure spaces, satisfying*

$$\sup_m \mu_m(B(p_m, R)) < \infty \quad \textit{for every } r > 0. \tag{4.17}$$

*Then there exists a subsequence and a pointed proper metric measure space $X = (X, d, \mu, p)$ so that $X_m \xrightarrow{GH} X$.*

Another compactness result for sequences of functions on pointed metric spaces can be proved using Proposition 4.32 and a diagonal argument, as in the proof of the Arzela–Ascoli theorem. Under a different notion of convergence (which is equivalent to ours under the hypotheses there), Proposition 4.33 appears in [Kei04]. For the additional statement (2), see [Hei+15][Section 11] and [Kei03].

To state the result, let $(X_m) = (X_m, d_m, \mu_m, p_m)$ be a sequence of pointed proper metric measure spaces. We say that a sequence $(f_m)$ of functions $f_m : X_m \to \mathbb{R}$ for $m = 1, 2, \ldots$ is *equicontinuous* if, for every $\varepsilon, R > 0$, there exists $\delta > 0$ such that if $x_m, y_m \in B(p_m, R)$ satisfy $d(x_m, y_m) < \delta$, then $|f_m(x_m) - f_m(y_m)| < \varepsilon$.

**Proposition 4.33.** *Let $(X_m) = (X_m, d_m, \mu_m, p_m)$ be a totally bounded sequence of pointed proper metric measure spaces satisfying (4.17). If a sequence of functions $(f_m)$, where $f_m : X_m \to \mathbb{R}$ for $m = 1, 2, \ldots$ is an equicontinuous sequence of functions, for which*

$$\sup_m |f_m(p_m)| < \infty,$$

*then there exists a subsequence of $(f_m)$ and a continuous function $f : X \to \mathbb{R}$, defined on a proper pointed metric measure space $X$, for which $f_m \xrightarrow{GH} f$. Moreover,*

*(1) if each $f_m$ is L-Lipschitz, then $f$ is L-Lipschitz;*

*(2) if each $X_m$ is a doubling length space with doubling constant $\leq C$, then $X$ is a length space with doubling constant $\leq C^2$.*

To study tangents of Hajłasz–Sobolev functions, we also consider a notion of weak convergence for functions. The following definition is a slight modification of the weak convergence in [Eri+20].

**Definition 4.34.** Let $(X_m) = (X_m, d_m, \mu_m, p_m)$ and $X = (X, d, \mu, p)$ be pointed proper metric measure spaces. A sequence $(u_m)$ of functions $u_m \in L^1_{loc}(X_m)$ converges *weakly* to $u \in L^1_{loc}(X)$, denoted $u_m \overset{GH}{\rightharpoonup} u$, if there exist isometric embeddings $\iota_m : X_m \to Z$, $\iota : X \to Z$ satisfying (1) and (2) in Definition 4.29, and for which

(3') $\iota_{m*}((u_m)_+ d\mu_m) \rightharpoonup \iota_*(u_+ d\mu)$ and $\iota_{m*}((u_m)_- d\mu_m) \rightharpoonup \iota_*(u_- d\mu)$

Here, for a function $f : Z \to \mathbb{R}$ we denote

$$f_+ = \max\{f, 0\} \text{ and } f_- = -\min\{f, 0\}.$$

The Gromov-Hausdorff convergence of functions is analogous to the uniform convergence on compact sets and indeed coincides with this notion if $X_m = X = Z$ for all $m$. The weak convergence of functions as in Definition 4.34 corresponds to weak convergence of signed measures. In keeping with these analogies, we indeed have the natural implication between the two notions.

**Lemma 4.35.** *Let metric measure spaces $(X_m)$ and $X$ be as in Definition 4.34. Suppose further, that a sequence $(u_m)$ of functions $u_m \in L^1_{loc}(X_m)$ Gromov–Hausdorff converges to a continuous function $u : X \to \mathbb{R}$. Then $(u_m)$ converges to $u$ weakly.*

*More precisely, if $\iota_m : X_m \to Z$, $\iota : X \to Z$ realize the convergence $u_m \xrightarrow{GH} u$, then*

$$\lim_{m\to\infty} \int_{X_m} (\varphi \circ \iota_m)(u_m)_\pm d\mu_m = \int_X (\varphi \circ \iota) u_\pm \, d\mu.$$

*for any boundedly supported continuous $\varphi : Z \to \mathbb{R}$.*

*Proof.* It is easy to see that if $u_m \xrightarrow{GH} u$, then $(u_m)_\pm \xrightarrow{GH} u_\pm$, and the embeddings realizing the first convergence also realize the latter convergence. Thus we may assume that $u_m$ and $u$ are non-negative, and $\iota_m : X_m \to Z$, $\iota : X \to Z$ realize the convergence $u_m \xrightarrow{GH} u$. It suffices to show that $\iota_{m*}(u_m d\mu) \rightharpoonup \iota_*(u d\mu)$.

Let $\tilde{u} : Z \to \mathbb{R}$ be a continuous extension of $u \circ \iota^{-1} : \iota(X) \to \mathbb{R}$, and set

$$\tilde{u}_m = u \circ \iota_m^{-1}|_{\iota_m(X_m)}.$$

Given any continuous $\varphi : Z \to \mathbb{R}$ with bounded (thus compact) support, we have

$$\int_{X_m} (\varphi \circ \iota_m) u_m d\mu_m - \int_X (\varphi \circ \iota) u d\mu = \int_Z \varphi(\tilde{u}_m - \tilde{u}) \iota_{m*}(d\mu_m) + \int_Z \varphi \tilde{u} \iota_{m*}(d\mu_m) - \int_Z \varphi \tilde{u} \iota_*(d\mu).$$

Since $\iota_{m*}(d\mu_m) \rightharpoonup \iota_*(d\mu)$, it suffices to prove that

$$\int_Z \varphi\ (\tilde{u}_m - \tilde{u})\ \iota_{m*}(d\mu_m) \to 0.$$

If $B \subset Z$ is a closed ball containing the support of $\varphi$, we obtain

$$\begin{aligned}\limsup_{m\to\infty} \left| \int_Z \varphi\ (\tilde{u}_m - \tilde{u})\ \iota_{m*}(d\mu_m) \right| &\leq \limsup_{m\to\infty} \|\varphi\|_{L^\infty(B)} \mu_m(\iota_m^{-1} B) \|\tilde{u}_m - \tilde{u}\|_{L^\infty(\iota_m(X_m)\cap B)} \\ &\leq \|\varphi\|_{L^\infty(B)} \mu(\iota^{-1} B) \limsup_{m\to\infty} \|\tilde{u}_m - \tilde{u}\|_{L^\infty(\iota_m(X_m)\cap B)}.\end{aligned}$$

Suppose $\limsup_{m\to\infty} \|\tilde{u}_m - \tilde{u}\|_{L^\infty(\iota_m(X_m)\cap B)} > \varepsilon_0$ for some $\varepsilon_0 > 0$. Then, there is a sequence $(x_m)$ such that $x_m \in X_m$ with $z_m := \iota_m(x_m) \in B$ and

$$|\tilde{u}_m(z_m) - \tilde{u}(z_m)| = |u_m(x_m) - \tilde{u}(\iota_m(x_m))| > \varepsilon_0$$

for $m$ large enough. Since $B$ is compact, a subsequence satisfies $z_m \to z \in \iota(X)$ for some $z$. By the Gromov–Haudorff convergence $u_m \xrightarrow{GH} u$ and the continuity of $\tilde{u}$, we obtain

$$\lim_{m\to\infty} |u_m(x_m) - \tilde{u}(\iota_m(x_m))| = |u(\iota^{-1}(z)) - \tilde{u}(z)| = 0,$$

which is a contradiction. This completes the proof. □

### 4.4.2 Gromov–Hausdorff convergence and averaging operators

In this chapter we prove preliminary results which we will use in the proof of the main result of Chapter 4.4, i.e. Theorem 4.41. We focus on characterizing limit of average operators applied to a Gromov–Hausdorff convergent sequence of functions. This characterization is attained using the following result.

**Proposition 4.36.** *Let $X_m = (X_m, d_m, \mu_m, p_m)$ and $X = (X, d, \mu, p)$ be proper locally doubling length spaces. Suppose the sequence $u_m \in L^1_{loc}(X_m)$ converges weakly to $u \in L^1_{loc}(X)$, and let $\iota_m : X_m \to Z$, $\iota : X \to Z$ realize this convergence. If $z_m \in X_m$ and $z \in X$ are such that $\iota_m(z_m) \to \iota(z)$, then*

$$\lim_{m\to\infty} \int_{B(z_m,r)} u_m d\mu_m = \int_{B(z,r)} u d\mu$$

*for any $r > 0$.*

*Proof.* By considering $u_{m\pm}$, we may assume that the functions $u_m$ and $u$ are non-negative. We note that

$$\iota_m(B(z_m, r)) \to \iota(B(z, r)) \tag{4.18}$$

for every $r > 0$ in the sense of Hausdorff-convergence. Indeed, given $q \in Z$, $R > 0$ and arbitrary $\varepsilon > 0$ the convergence $\iota_m(X_m) \to \iota(X)$ and $\iota_m(z_m) \to \iota(z)$ imply that, for large enough $m$

$$\begin{aligned}&B(q, R) \cap \iota_m(B(z_m, r)) \\ &\quad = B(p, R) \cap \iota_m(X_m) \cap B(\iota_m(z_m), r) \subset N_\varepsilon(\iota(X)) \cap B(\iota(z), r + \varepsilon) = N_\varepsilon(\iota(B(z, r)))\end{aligned}$$

and

$$B(q,R)\cap\iota(B(z,r))$$
$$= B(p,R)\cap\iota(X)\cap B(\iota(z),r)\subset N_\varepsilon(\iota_m(X_m))\cap B(\iota_m(z_m),r+\varepsilon)=N_\varepsilon(\iota(B(z_m,r))).$$

The convergence (4.18) follows. Note that the property of being a length space was used in the last equality, cf. [Hei+15][Lemma 11.3.10]. By Lemma 4.28 we have, for any $\varepsilon>0$, that

$$\limsup_{m\to\infty}\nu_m(\bar B(z_m,r))\le\nu(\bar B(z,r))\le\liminf_{m\to\infty}\nu_m(B(z_m,\varepsilon+r))$$

where $d\nu=ud\mu$ and $\nu_m=u_m d\mu_m$. Recall, that $A_{r,R}(x)=\bar B(x,R)\setminus B(x,r)$ for $0<r\le R$ and $x\in Z$. It holds that

$$\begin{aligned}\liminf_{m\to\infty}\nu_m(B(z_m,\varepsilon+r)) &= \liminf_{m\to\infty}[\nu_m(\bar B(z_m,r))+\nu_m(A_{r,r+\varepsilon}(z_m))]\\ &\le \liminf_{m\to\infty}\nu_m(B(z_m,r))+\limsup_{m\to\infty}\nu_m(A_{r,r+\varepsilon}(z_m)).\end{aligned}$$

The argument used to establish (4.18) also yields that

$$A_{r,r+\varepsilon}(z_m)\to A_{r,r+\varepsilon}(z)$$

in the sense of Hausdorff-convergence. Applying Lemma 4.28 once more we obtain

$$\limsup_{m\to\infty}\nu_m(\bar B(z_m,r))\le\nu(\bar B(z,r))\le\liminf_{m\to\infty}\nu_m(B(z_m,r))+\nu(A_{r,r+\varepsilon}(z)).$$

Since $\mu$ is a locally doubling measure on a length space it has an annular decay property (see the discussion after Definition 4.1). Therefore,

$$\lim_{\varepsilon\to0}\mu(A_{r,r+\varepsilon}(z))=0$$

which, by the absolute continuity of $\nu$ with respect to $\mu$, implies

$$\lim_{\varepsilon\to0}\nu(A_{r,r+\varepsilon}(z))=0.$$

We have obtained

$$\limsup_{m\to\infty}\nu_m(\bar B(z_m,r))\le\nu(\bar B(z,r))\le\liminf_{m\to\infty}\nu_m(\bar B(z_m,r)),$$

which completes the proof. □

Proposition 4.36 has the following immediate corollary.

**Corollary 4.37.** *Let $(X_m)=(X_m,d_m,\mu_m,p_m)$ and $X=(X,d,\mu,p)$ be proper locally doubling length spaces and $(u_m)$ be a sequence of functions $u_m\in L^1_{loc}(X_m)$ converging weakly to $u\in L^1_{loc}(X)$. Then*

*(a)* $A_r^{\mu_m}u_m\xrightarrow{GH}A_r^{\mu}u$, *and*

*(b)* $(A_r^{\mu_m})^*u_m\xrightarrow{GH}(A_r^{\mu})^*u$

*for each $r>0$. In particular, if $u$ is continuous, then*

$$\Delta_r^{\mu_m}u_m\overset{GH}{\rightharpoonup}\Delta_r^{\mu}u\quad\textit{and}\quad(\Delta_r^{\mu_m})^*u_m\overset{GH}{\rightharpoonup}(\Delta_r^{\mu})^*u.$$

*Proof.* The first claim follows directly from Proposition 4.36. Note that, if $\iota_m : X_m \to Z$ and $\iota : X \to Z$ realize the convergence $X_m \xrightarrow{GH} X$, Proposition 4.36 implies in particular that

$$\mu_m(B(z_m, r)) \to \mu(B(z, r)) \quad \text{whenever } \iota_m(z_m) \to \iota(z).$$

It follows that, for any $r > 0$, the sequence $f_m \in L^1_{loc}(X_m)$,

$$f_m(z) = \frac{u_m(z)}{\mu_m(B(z, r))}$$

weakly converges to

$$f(z) := \frac{u(z)}{\mu(B(z, r))}.$$

This yields part (b) of the assertion. □

### 4.4.3 Blow-ups of Hajłasz–Sobolev functions with finite amv-norm

In this chapter we prove Theorem 4.41. We begin with defining the main object considered in this chapter, i.e. a blow up of a metric measure space.

Let us consider a proper locally doubling length space $X = (X, d, \mu)$. Given a point $x \in X$ and $r > 0$, the pointed metric measure space

$$X_r = (X, d_r, \mu_r, x), \quad d_r := \frac{d}{r}, \quad \mu_r := \frac{1}{\mu(B(x, r))}\mu$$

is called a *rescaling* of $X$ at $x$ by $r$.

Let $(r_m)$ be a sequence of positive numbers converging to zero, and denote by $X_m := X_{r_m}$. A pointed measured Gromov–Hausdorff limit $X_\infty$ of $X_m$ is called a *tangent space of $X$ at $x$ subordinate to $(r_m)$.*

Similarly, if $f : X \to \mathbb{R}$ is a function, $x \in X$ and $r > 0$, the function

$$f_r := \frac{f - f(x)}{r} : X_r \to \mathbb{R}$$

is a rescaling of $f$ at $x$ by $r$. A Gromov–Hausdorff limit $f_\infty : X_\infty \to \mathbb{R}$ of $f_m := f_{r_m}$ is called a *tangent of $f$ at $x$ subordinate to $(r_m)$*. If the convergence $f_m \to f_\infty$ is weak (cf. Definition 4.34), we say that $f_\infty$ is an *approximate tangent of $f$ at $x$, subordinate to $(r_m)$.*

It is worth remarking that, in general, tangents are highly non-unique – different sequences can produce different limits. However, any sequence of rescalings is totally bounded and satisfies (4.17). Moreover, $f_m(x) = 0$ for $m$, and thus Proposition 4.33 implies the existence of tangents of Lipschitz functions at any point.

**Proposition 4.38.** *Let $X$ be a proper locally doubling metric measure space, and $f : X \to \mathbb{R}$ an $L$-Lipschitz function. Fix a sequence $(r_m)$ of positive numbers converging to zero. Then, for any $x \in X$, there exists a subsequence of the rescalings $f_m : X_m \to \mathbb{R}$ at $x$, and a tangent function $f_\infty : X_\infty \to \mathbb{R}$ such that $f_m \xrightarrow{GH} f_\infty$.*

In particular, any tangent space $X_\infty$ is doubling, and any tangent $f_\infty : X_\infty \to \mathbb{R}$ is $L$-Lipschitz. Next, we present a variant for Hajłasz–Sobolev functions.

**Proposition 4.39.** *Let $X$ be a proper locally doubling metric measure space, $p > 1$, and $u \in M^{1,p}(X)$. Given a sequence $(r_m)$, for $\mu$-almost every point $x \in X$, there is a subsequence of the rescalings $u_m : X_m \to \mathbb{R}$ and a Lipschitz function $u_\infty : X_\infty \to \mathbb{R}$, so that $u_m \overset{GH}{\rightharpoonup} u_\infty$.*

It can be shown that the Lipschitz constant of $u_\infty$ satisfies

$$\frac{1}{C} g(x) \le \mathrm{LIP}(u_\infty) \le C g(x)$$

for a constant depending only on the local doubling constant of $\mu$ near $x$; cf. [Eri+20].

*Proof of Proposition 4.39.* We follow the ideas of [Eri+20]. Let $g \in L^p(X)$ be a Hajłasz upper gradient of $u$, and set

$$E_n = \{x \in X : g(x) > n\}, \quad n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} \mu(E_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{E_n} g d\mu = 0.$$

Thus there exists a null set $N \subset X$ for which every $x \in X \setminus N$ has the property that

$$\lim_{r\to 0} \frac{\mu(B(x,r)\cap E_n)}{\mu(B(x,r))} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap E_n} g(y) d\mu(y) = 0 \tag{4.19}$$

for some $n \in \mathbb{N}$. We fix $x \in X \setminus N$ and $n \in \mathbb{N}$ satisfying (4.19).

Note that $u|_{X\setminus E_n}$ is $2n$-Lipschitz and let $\tilde{u}_n : X \to \mathbb{R}$ be a $2n$-Lipschitz extension of $u|_{X\setminus E_n}$. By Proposition 4.38 there is a subsequence of the rescalings $(\tilde{u}_n)_m : X_m \to \mathbb{R}$ and a Lipschitz function $\tilde{u} : X_\infty \to \mathbb{R}$ so that $(\tilde{u}_n)_m \xrightarrow{GH} \tilde{u}$.

We show that, for this subsequence, the rescalings $u_m : X_m \to \mathbb{R}$ converges weakly to $\tilde{u}$. (This is different from claiming that $(\tilde{u}_n)_m \overset{GH}{\rightharpoonup} \tilde{u}$, which follows from Lemma 4.35.)

Let $Z$ be a proper metric space and $\iota_m : X_m \to Z$ isometric embeddings realizing the convergence $(\tilde{u}_n)_m \to \tilde{u}$. Given $\varphi \in C_b(Z)$, fix a large number $R > 0$ so that $\operatorname{spt}\varphi \subset B(\iota_m(x), R)$ for all $m \in \mathbb{N}$. Then $\operatorname{spt}(\varphi \circ \iota_m) \subset B_R^{X_m}(x) = B(x, r_m R)$. We have

$$\begin{aligned}
&\left| \int_{X_m} \varphi\circ\iota_m \; u_m d\mu_m - \int_{X_\infty} \varphi\circ\iota_\infty \; \tilde{u} d\mu_\infty \right| \\
&= \left| \int_X \varphi\circ\iota_m \; (\tilde{u}_n)_m d\mu_m - \int_{X_\infty} \varphi\circ\iota_\infty \; \tilde{u} d\mu_\infty + \int_{E_n} \varphi\circ\iota_m \; [u_m - (\tilde{u}_n)_m] d\mu_m \right| \\
&\le \left| \int_X \varphi\circ\iota_m \; (\tilde{u}_n)_m d\mu_m - \int_{X_\infty} \varphi\circ\iota_\infty \; \tilde{u} d\mu_\infty \right| + \int_{E_n} |\varphi\circ\iota_m \; u_m| d\mu_m + \int_{E_n} |\varphi\circ\iota_m \; (\tilde{u}_n)_m| d\mu_m
\end{aligned}$$

The first term converges to zero since $(\tilde{u}_n)_m \xrightarrow{GH} \tilde{u}$. We may estimate the second term by

$$\int_{E_n} |\varphi\circ\iota_m| \; |u_m| d\mu_m \le \frac{\|\varphi\|_\infty}{\mu(B(x,r_m))} \int_{E_n\cap B(x,r_m R)} (g(x) + g(y)) d\mu(y)$$

which, by (4.19) converges to zero as $m \to \infty$. Similarly,

$$\int_{E_n} |\varphi\circ\iota_m| \; |(\tilde{u}_n)_m| d\mu_m \le 2n\|\varphi\|_\infty \frac{\mu(E_n \cap B(x, r_m R))}{\mu(B(x,r_m))}$$

converges to zero as $m \to \infty$. This completes the proof. □

By a suitable cut-off argument, we obtain the following corollary whose proof we omit.

**Corollary 4.40.** *Let $\Omega \subset X$ be a domain in a proper locally doubling metric measure space, and let $u \in M^{1,p}_{loc}(\Omega)$. Given a sequence $r_m \downarrow 0$, for $\mu$-almost every $x \in \Omega$, there is a subsequence of the rescalings $u_m : X_m \to \mathbb{R}$ at $x$, and Lipschitz function $u_\infty : X_\infty \to \mathbb{R}$, so that $u_m \overset{GH}{\rightharpoonup} u_\infty$.*

We are now ready to prove that having finite amv-norm forces tangent maps to be strongly harmonic.

**Theorem 4.41.** *Let $\Omega \subset X$ be an open subset of a proper locally doubling measured length space $X = (X, d, \mu)$, and let $1 < p < \infty$. Suppose $u \in M^{1,p}_{loc}(\Omega) \cap \mathrm{AMV}^p_{loc}(\Omega)$ and $(r_k)$ is a positive sequence converging to zero. Then for $\mu$-almost every $x \in \Omega$, any approximate tangent map $u_\infty : X_\infty \to \mathbb{R}$ at $x$, subordinate to a subsequence of $(r_k)$, is strongly harmonic.*

*Proof.* By Corollary 4.40, $u$ has approximate tangent maps, subordinate to $(r_m)$, for $\mu$-almost every $x \in X$. For any $k \in \mathbb{N}$, $r \in \mathbb{Q}_+$, and compact $K \subset \Omega$, we have

$$\int_K \liminf_{m\to\infty} \fint_{B(x,kr_m)} |\Delta_{r_m r} u| d\mu \, d\mu(x) \le \liminf_{m\to\infty} \int_K \fint_{B(x,kr_m)} |\Delta_{r_m r} u| d\mu \; d\mu(x) \le C \limsup_{\rho\to 0} \int_K |\Delta_\rho u| d\mu,$$

cf. [Ald19, Theorem 3.3]. Thus, almost every $x_0 \in X$ has the following property: for every subsequence of $(r_m)$ there exists a further subsequence (not relabeled), for which

$$\lim_{m\to\infty} \fint_{B(x_0,kr_m)} |\Delta_{r_m r} u| d\mu < \infty \quad \text{for every } k \in \mathbb{N} \text{ and } r \in \mathbb{Q}_+. \tag{4.20}$$

Let $x_0 \in \Omega$ be a point where (4.20) and the claim of Corollary 4.40 holds for $(r_m)$. Furthermore, let $(X_\infty, d_\infty, \mu_\infty, x_\infty)$ be a pointed measured Gromov–Hausdorff limit of a sequence

$$X_m = (X, d_m, \mu_m, x_0) = \left( X, \frac{d}{r_n}, \frac{\mu}{\mu(B(x_0, r_m))}, x_0 \right),$$

and $u_\infty : X_\infty \to \mathbb{R}$ a weak limit of the sequence

$$u_m := \frac{u - u(x_0)}{r_m} : X_m \to \mathbb{R},$$

for a subsequence of $(r_m)$. We pass to a further subsequence (again not relabeled) for which (4.20) holds. Note that

$$A_r^{\mu_m} u_m(z) - u_m(z) = \fint_{B(z,r_m r)} \frac{u(y) - u(z)}{r_m} d\mu(y) = r_m r^2 \Delta_{r_m r} u(z)$$

for any $z \in X_m$ and $r > 0$. Let $\iota_m : X_m \to Z$ and $\iota : X_\infty \to Z$ realize the convergence $u_m \overset{GH}{\to} u_\infty$. Corollary 4.37 implies that

$$\begin{aligned}\int_{X_\infty} \varphi \circ \iota_\infty(z) \left[ A_r^{\mu_\infty} u_\infty(z) - u_\infty(z) \right] d\mu_\infty(z) &= \lim_{m\to\infty} \int_{X_m} \varphi \circ \iota_m(z) \left[ A_r^{\mu_m} u_\infty(z) - u_m(z) \right] d\mu_m(z) \\ &= \lim_{m\to\infty} r_m r^2 \int_{X_m} \varphi \circ \iota_m(z) \Delta_{r_m r} u(z) d\mu_m(z)\end{aligned}$$

for every compactly supported $\varphi \in C(Z)$. Since

$$\begin{aligned}\left| \int_{X_m} \varphi \circ \iota_m(z) \Delta_{r_m r} u(z) d\mu_m(z) \right| &\le \frac{\|\varphi \circ \iota_m\|_\infty}{\mu(B(x_0, r_m))} \int_{B(x_0,kr_m)} |\Delta_{r_m r} u(z)| d\mu(z) \\ &\le C_{k,\varphi} \fint_{B(x_0,kr_m)} |\Delta_{r_m r} u| d\mu < \infty\end{aligned}$$

it follows that

$$\int_{X_\infty} \varphi \circ \iota_\infty(z) \left[ (u_\infty)_{B_\infty(z,r)} - u_\infty(z) \right] d\mu_\infty(z) = 0.$$

Since $\varphi$ is arbitrary, this establishes the claim for all rational $r > 0$. The claim follows for all $r > 0$ by continuity, since $X_\infty$ is a length space and $\mu_\infty$ a doubling measure. □

## 4.5 Weighted Euclidean spaces. Elliptic PDEs and amv-harmonic functions

This chapter is mostly devoted to characterization of limits of $r$-laplacian in the weighted Euclidean setting. We find the explicit PDE for the $L^p$-limit of the averaging operator $\Delta_r$ assuming the $W^{1,\infty}_{loc}$-regularity for a positive weight, see Theorem 4.45. We discuss differences between weighted and unweighted $r$-laplacian, which is best seen by (4.22) in Lemma 4.44.

The difference between mean value and amv-harmonic functions is perhaps best seen by considering the Euclidean space $\mathbb{R}^n$ with an arbitrary norm $\|\cdot\|$. In Chapter 2.6.2 and 2.6.3, summarized in (2.42) and Remark 2.26 we showed that strongly harmonic functions on $(\mathbb{R}^2, \|\cdot\|_p)$ form a finite dimensional space, when $p \neq 2$. In the presence of a weight, strongly harmonic functions on weighted Euclidean domains satisfy system (2.1) of PDE's which, in the case of smooth weights, is infinite. In contrast, amv-harmonicity is characterized by an elliptic second order non-divergence form PDE, as presented below.

We begin with recalling the setting of weighted Euclidean spaces introduced in Chapter 2. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and denote by $\mathcal{H}^n$ the corresponding Hausdorff $n$-measure, which is a constant multiple of the Lebesgue measure. Given an open domain $\Omega \subset \mathbb{R}^n$, consider the metric measure space $(\Omega, \|\cdot\|, \mathcal{H}^n|_\Omega)$ and denote

$$|A| := \mathcal{H}^n|_\Omega(A), \ \int_A \varphi dy := \int_A \varphi d\mathcal{H}^n|_\Omega, \ A_r := A_r^{\mathcal{H}^n|_\Omega} \text{ and } \Delta_r := \Delta_r^{\mathcal{H}^n|_\Omega}$$

for Borel sets $A \subset \mathbb{R}^n$ and integrable functions $\varphi : A \to \mathbb{R}$. The unit ball $B(0,1) := B^{\|\cdot\|}(0,1)$ is an open, symmetric convex set. Let $M \in \mathbb{R}^{n\times n}$ denote the *matrix of second moments* of the unit ball $B(0,1) \subset \mathbb{R}^n$ of $\|\cdot\|$, given by

$$M = (m_{ij}), \quad m_{ij} := \fint_{B(0,1)} y_i y_j d\mathcal{H}^n(y), \quad 1 \leq i,j \leq n.$$

**Remark 4.42.** Notice, that entries $m_{ij}$ correspond to coefficients $A_\alpha$ for $|\alpha| = 2$ defined in lines of Theorem 2.2. Therefore, the matrix $M$ is symmetric and positive definite, see (2.22). Moreover, ball $B(x,r)$ is symmetric for every $x \in \mathbb{R}^n$ and $r > 0$, hence we have that

$$\fint_{B(x,r)} (y-x)_i dy = r \fint_{B(0,1)} z_i dz = 0, \quad i = 1, \ldots, n$$

for $x \in \mathbb{R}^n$ and $r > 0$.

Suppose that $\Omega \subset \mathbb{R}^n$ is a domain, $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and a weight $w : \Omega \to \mathbb{R}$ is positive and locally Lipschitz. Let us consider the *weighted* metric measure space

$$\Omega_w = (\Omega, \|\cdot\|, \mu),$$

where $\mu := w\mathcal{H}^n|_\Omega$. We use the notation (cf. Chapter 4.2.1)

$$A_r^w := A_r^\mu, \ \Delta_r^w := \Delta_r^\mu.$$

The following elementary facts follow from the assumptions on the weight function $w$.

**Remark 4.43.** Since $w$ is continuous and positive on $\Omega$, the measure $w\mathcal{H}^n|_\Omega$ is locally doubling. Moreover,

(1) $A_r^w f \to f$ locally uniformly in $\Omega$ as $r \to 0$, whenever $f : \Omega \to \mathbb{R}$ is continuous;

(2) For each $p \in [1,\infty]$, $L^p_{loc}(\Omega) = L^p_{loc}(\Omega_w)$ as sets, and $L^p$-convergence on compact subsets of $\Omega$ with respect to $\mathcal{H}^n|_\Omega$ and $w\mathcal{H}^n|_\Omega$ agree.

The next lemma provides two different representations for $\Delta_r^w$ in terms of $\Delta_r$ and will prove useful in further parts of this chapter.

**Lemma 4.44.** *Suppose $f \in L^1_{loc}(\Omega)$. Then*

$$\Delta_r^w f = \frac{1}{A_r w}(\Delta_r(fw) - f\Delta_r w), \tag{4.21}$$

*and*

$$\Delta_r^w f = \Delta_r f + \frac{1}{A_r w}\langle f, w\rangle_r, \tag{4.22}$$

*where*

$$\langle f, g\rangle_r(x) := \fint_{B(x,r)} \frac{f(y) - A_r f(x)}{r}\frac{g(y) - A_r g(x)}{r} dy,$$

*for* $f, g \in L^1_{loc}(\Omega)$ *such that* $fg \in L^1_{loc}(\Omega)$.

*Proof.* The first claim of the lemma is a direct consequence of the pointwise identity

$$(f(y) - f(x))w(y) = f(y)w(y) - f(x)w(x) + f(x)(w(x) - w(y)), \quad x, y \in \Omega.$$

We integrate both sides with respect to $y \in B(x,r)$ and divide by $r^2 \int_{B(x,r)} w(y)dy$, which equals to $r^2 |B(x,r)| A_r w(x)$, to arrive at the following identity

$$\Delta_r^w f(x) = \frac{1}{r^2 A_r w(x)} \left( \fint_{B(x,r)} (f(y)w(y) - f(x)w(x))dy - f(x) \fint_{B(x,r)} (w(y) - w(x))dy \right).$$

This proves (4.21). To see the second assertion (4.22) we fix $r > 0$, $x \in \Omega$ and compute

$$\begin{aligned}
\frac{\langle f, w\rangle_r(x)}{A_r w(x)} =& \frac{1}{A_r w(x)} \fint_{B(x,r)} \frac{(f(y) - A_r f(x))(w(y) - A_r w(x))}{r^2} dy \\
=& \frac{1}{A_r w(x)} \fint_{B(x,r)} \frac{f(y)w(y) - A_r w(x) f(y) - A_r f(x) w(y) + A_r f(x) A_r w(x)}{r^2} dy \\
=& \frac{1}{r^2 A_r w(x)} \left[ \fint_{B(x,r)} f(y)w(y)dy - A_r w(x) \fint_{B(x,r)} f(y)dy \right. \\
& \left. - A_r f(x) \fint_{B(x,r)} w(y)dy + A_r f(x) A_r w(x) \right] \\
=& \frac{1}{r^2 A_r w(x)} \left[ \fint_{B(x,r)} f(y)w(y)dy - A_r f(x) A_r w(x) \right] \\
=& \frac{A_r^w f(x) - A_r f(x)}{r^2} \\
=& \frac{A_r^w f(x) - f(x) - (A_r f(x) - f(x))}{r^2} = \Delta_r^w f(x) - \Delta_r f(x).
\end{aligned}$$

□

Consider the unbounded operator $L_w : L^p(\Omega_w) \to D^*(\mathbb{R}^n)$ defined as

$$L_w u := \frac{1}{2}\mathrm{div}(M\nabla u) + \frac{1}{w}\langle \nabla w, M\nabla u\rangle = \frac{1}{2}\mathrm{div}(M\nabla u) + \langle \nabla \ln w, M\nabla u\rangle. \tag{4.23}$$

This can be interpreted as a distribution for any $u \in L^p(\Omega_w)$ but makes sense as an $L^p_{loc}$-function for $u \in W^{2,p}_{loc}(\Omega)$. The main result of this chapter is the following result linking limit of $\Delta_r^w$ to the operator $L_w$.

**Theorem 4.45.** *Let $w$ be a locally Lipschitz positive weight on $\Omega$ and $p \in (1, \infty)$. Then $W^{1,p}(\Omega) \cap \mathrm{AMV}^p_{loc}(\Omega_w) = W^{2,p}_{loc}(\Omega)$. Moreover, for every $u \in W^{2,p}_{loc}(\Omega_w)$ we have that $\Delta_r^w u \to L_w u$ in $L^p_{loc}(\Omega)$ as $r \to 0$.*

In order to prove Theorem 4.45 we firstly consider the case of a constant weight. We begin with proving that under the $C^2$-regularity assumption strong amv-harmonic functions solve the second order elliptic PDE whose coefficients depend only on the geometry (i.e. on matrix $M$) of the unit ball in the underlying metric, see Lemma 4.46.

**Lemma 4.46.** *If $u \in C^2(\Omega)$, then*

$$\Delta_r u \to \frac{1}{2}\mathrm{div}(M\nabla u)$$

*locally uniformly in $\Omega$, as $r \to 0$.*

*Proof.* Let $x \in \Omega$ and $r < \mathrm{dist}(x, \partial\Omega)$. For every fixed $z \in B(0,1)$ the Taylor expansion of $u$ yields

$$u(x+rz) - u(x) = r\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x) z_i + \frac{1}{2}r^2 \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) z_i z_j + E_r(x,z),$$

where $E_r(x,z)$ is the Taylor remainder and $E_r(x,z)/r^2 \to 0$ locally uniformly in $\Omega \times \bar{B}(0,1)$, as $r \to 0$. Thus, upon dividing by $r^2$ and taking the mean integral with respect to $z \in B(0,1)$ we have

$$\Delta_r u(x) = \fint_{B(0,1)} \frac{u(x+rz)-u(x)}{r^2} dz = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \fint_{B(0,1)} z_i z_j dz + \fint_{B(0,1)} \frac{E_r(x,z)}{r^2} dz,$$

where the integral of the first order term over $B(0,1)$ vanishes by Remark 4.42. Thus,

$$\lim_{r\to 0} \Delta_r u(x) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) m_{ij} = \frac{1}{2}\mathrm{div}(M\nabla u),$$

and the claim follows. □

In the next result we weaken the regularity assumption from $C^2$ to $W^{2,p}_{loc}$ and prove the assertion corresponding to Theorem 4.45 in Proposition 4.47. In particular, we identify the amv-harmonic functions with locally finite amv-norms with functions in the Sobolev space $W^{2,p}_{loc}$. In the proof of Proposition 4.47 we use the notion of $W^{2,p}$ extension domain. Recall, that for $p \in [1,\infty]$ and $k \in \mathbb{N}$ a domain $\Omega \subset \mathbb{R}^n$ is called a $W^{k,p}$ extension domain, if there exists a continuous linear extension operator $T_\Omega : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$. Furthermore, recall that any Lipschitz domain is an extension domain.

**Proposition 4.47.** *Let $p \in (1,\infty)$. Then $\mathrm{AMV}^p_{loc}(\Omega) = W^{2,p}_{loc}(\Omega)$. Moreover, for each $u \in W^{2,p}_{loc}(\Omega)$, we have that*

$$\Delta_r u \to \frac{1}{2}\mathrm{div}(M\nabla u) \tag{4.24}$$

*in $L^p_{loc}(\Omega)$, as $r \to 0$.*

*Proof.* Assume $u \in W^{2,p}_{loc}(\Omega)$ and set

$$R(x,y) := u(y) - u(x) - \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x)(y-x)_i, \quad x,y \in \Omega.$$

Let us fix a $W^{2,p}$-extension domain $\Omega'$ compactly contained in $\Omega$. Denote by $B^E(x,r)$ an Euclidean ball and by $\partial B^E(0,1)$ an Euclidean unit sphere. We apply [BIK13, Theorem 2.5 and (21)] to arrive at

$$\lim_{r\to 0} \int_{\Omega'} \fint_{B^E(x,r)} \left| \frac{R(x,y)}{r^2} \right|^p dy dx = c(n,p) \int_{\Omega'} \int_{\partial B^E(0,1)} \left| 2\sum_{i\le j} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) e_i e_j \right|^p de\, dx. \tag{4.25}$$

By Remark 4.42 we see that

$$\fint_{B(x,r)} \frac{R(x,y)}{r^2} dy = \Delta_r u(x), \quad \bar{B}(x,r) \subset \Omega. \tag{4.26}$$

Together with (4.25) and the fact that $\|\cdot\|$ is comparable to the Euclidean norm, (4.26) implies that

$$\limsup_{r\to 0}\int_{\Omega'}|\Delta_r u|^p dx \leq \limsup_{r\to 0}\int_{\Omega'}\fint_{B(x,r)}\left|\frac{R(x,y)}{r^2}\right|^p dydx \leq c\lim_{r\to 0}\int_{\Omega'}\fint_{B^E(x,r)}\left|\frac{R(x,y)}{r^2}\right|^p dydx < \infty. \tag{4.27}$$

Notice, that we showed (4.27) for any $W^{2,p}$-extension domain $\Omega'$ compactly contained in $\Omega$. We want to use this observation to prove that $u \in \mathrm{AMV}^p_{loc}(\Omega)$. In order to do that, we fix a compact $K \subset \Omega$ and for each point $x \in K$ find its neighbourhood $\Omega'_x$ which is a $W^{2,p}$-extension domain and is a compact subset of $\Omega$. By compactness of $K$ we find a finite cover $\Omega'_1, \dots, \Omega'_N$ and estimate using (4.27) in the following way

$$\limsup_{r\to 0}\int_K |\Delta_r u|^p\, dx \leq \limsup_{r\to 0}\sum_{i=1}^N \int_{\Omega'_i}|\Delta_r u|^p\, dx < \infty.$$

Hence, $u \in \mathrm{AMV}^p_{loc}(\Omega)$.

Conversely, suppose $u \in \mathrm{AMV}^p_{loc}(\Omega)$. Then, for any positive sequence $(r_m)$ converging to zero there is a further subsequence and a function $g \in L^p_{loc}(\Omega)$ such that $\Delta_{r_m}u \rightharpoonup g$ weakly in $L^p_{loc}(\Omega)$ as $m\to\infty$. In particular, for any $\varphi \in C^2_c(\Omega)$, we have

$$\int_\Omega \varphi g dx = \lim_{m\to\infty}\int_\Omega \varphi\Delta_{r_m}u dx = \lim_{m\to\infty}\int_\Omega u\Delta_{r_m}\varphi dx = \frac{1}{2}\int_\Omega u\mathrm{div}(M\nabla\varphi)dx,$$

since $\Delta_r\varphi \to \frac{1}{2}\mathrm{div}(M\nabla\varphi)$ locally uniformly in $\Omega$ and we used the following Green-type identity, see [MT19][Theorem 5.3]:

$$\int_X \varphi\Delta_r u - u\Delta_r\varphi d\mu = \frac{1}{r^2}\int_X u(x)\int_{B(x,r)}\varphi(y)\left(\frac{1}{\mu(B(y,r))} - \frac{1}{\mu(B(x,r))}\right)d\mu(y)d\mu(x).$$

Notice, that in our case $\mu(B(y,r)) = \mu(B(x,r))$ is constant, hence the right-hand side is equal to zero. This shows that $g$ is a unique limit and agrees with the distribution $g = \frac{1}{2}\mathrm{div}(M\nabla u)$ on $\Omega$, because $\int_\Omega \frac{1}{2}\mathrm{div}(M\nabla u)\varphi = \int_\Omega \frac{1}{2}\mathrm{div}(M\nabla\varphi)u$. Since $\mathrm{div}(M\nabla u) \in L^p_{loc}(\Omega)$ we have that $u \in W^{2,p}_{loc}(\Omega)$, see e.g. Theorem 6.29 in [Gru09] applied for the differential operator $P = \Delta$.

It remains to prove the convergence in (4.24). For this assume $u \in W^{2,p}_{loc}(\Omega)$. Denote by $\nabla^2 u$ the matrix of second weak partial derivatives of $u$ and by $Q(x,s)$ a concentric cube centered at $x$ with side length $s$. Then, by [BHS02, Theorem 3.3] (applied with $m = 2$, $S = B(x,r)$ according to the notation in [BHS02]) we observe that there exist constant $C' = C'(n)$ and $\sigma \geq 1$ such that

$$|\Delta_r u(x)| = \frac{1}{r^2}\left|u(x) - \fint_{B(x,r)}u(y)dy\right| \leq \frac{C'(n)}{r^2}\int_{Q(x,\sigma r)}\frac{\left|\nabla^2 u\right|(y)}{|x-y|^{n-2}}dy.$$

We apply [BHS02, Lemma 3.4] with $u = \left|\nabla^2 u\right|$, $\mu_1 = \mu_2 = 2$, according to the notation in [BHS02], in the above observation to obtain that there exists a constant $C = C(n)$ such that

$$|\Delta_r u(x)| \leq \frac{C'(n)}{r^2}\int_{Q_{\sigma r}(x)}\frac{|\nabla^2 u|(y)}{|x-y|^{n-2}}dy \leq C\mathcal{M}_{\sqrt{n}\sigma r}|\nabla^2 u|(x), \quad B(x,\sigma r)\subset\Omega. \tag{4.28}$$

Let $\Omega' \subset \Omega$ be compactly contained and let $(u_m)$ be a sequence of smooth functions converging to $u$ in $W^{2,p}(\Omega')$ as $m\to\infty$. In particular, $|\nabla^2(u-u_m)| \to 0$ and $\mathrm{div}(M\nabla(u-u_m)) \to 0$ in $L^p(\Omega')$

as $m \to \infty$. By Lemma 4.46 and (4.28) we obtain

$$
\begin{aligned}
&\limsup_{r\to\infty} \|\Delta_r u - \frac{1}{2}\operatorname{div}(M\nabla u)\|_{L^p(\Omega')} \\
&\le \limsup_{r\to 0} \left( \|\Delta_r(u-u_m)\|_{L^p(\Omega')} + \|\Delta_r u_m - \frac{1}{2}\operatorname{div}(M\nabla u_m)\|_{L^p(\Omega')} + \frac{1}{2}\|\operatorname{div}(M\nabla(u-u_m))\|_{L^p(\Omega')} \right) \\
&\le C \limsup_{r\to 0} \|\mathcal{M}_{\sqrt{n}\sigma r}|\nabla^2(u-u_m)|\|_{L^p(\Omega')} + \frac{1}{2}\|\operatorname{div}(M\nabla(u-u_m))\|_{L^p(\Omega')} \\
&\le C \left\|\nabla^2(u-u_m)\right\|_{L^p(\Omega')} + \frac{1}{2}\|\operatorname{div}(M\nabla(u-u_m))\|_{L^p(\Omega')}.
\end{aligned}
$$

Upon letting $m \to \infty$ the claim follows. □

The last auxiliary result which is needed in the proof of Theorem 4.45 in order to reduce to the unweighted case is presented below.

**Lemma 4.48.** *If $f, g \in W^{1,1}_{loc}(\Omega)$ and in addition $g \in L^n_{loc}(\Omega)$, then*

$$\langle f, g\rangle_r \overset{r\to 0}{\longrightarrow} \langle M\nabla f, \nabla g\rangle$$

*pointwise almost everywhere in $\Omega$.*

*Proof.* The Sobolev embedding implies that $f \in L^{n/(n-1)}_{loc}(\Omega)$. Thus $fg \in L^1_{loc}(\Omega)$ and $\langle f, g\rangle_r$ is finite. Sobolev functions in $W^{1,1}_{loc}$ satisfy the following approximation by tangent planes for almost every $x \in \Omega$, cf. [EG92][Theorem 2 in Section 6.1.2]

$$\lim_{r\to 0} \left( \fint_{B(x,1)} \left| \frac{f(x+rz) - f(x) - \nabla f(x)\cdot(rz)}{r} \right|^{n/(n-1)} dz \right)^{(n-1)/n} = 0.$$

It follows that

$$\lim_{r\to 0} \left( \fint_{B(x,1)} \left| \frac{f(x+rz) - A_r f(x) - \nabla f(x)\cdot(rz)}{r} \right|^{n/(n-1)} dz \right)^{(n-1)/n} = 0, \tag{4.29}$$

since

$$
\begin{aligned}
&\left( \fint_{B(x,1)} \left| \frac{f(x+rz) - A_r f(x) - \nabla f(x)\cdot(rz)}{r} \right|^{n/(n-1)} dz \right)^{(n-1)/n} \\
\le &\left( \fint_{B(x,1)} \left| \frac{f(x+rz) - f(x) - \nabla f(x)\cdot(rz)}{r} \right|^{n/(n-1)} dz \right)^{(n-1)/n} + \left| \frac{A_r f(x) - f(x)}{r} \right| \\
\le 2 &\left( \fint_{B(x,1)} \left| \frac{f(x+rz) - f(x) - \nabla f(x)\cdot(rz)}{r} \right|^{n/(n-1)} dz \right)^{(n-1)/n}.
\end{aligned}
$$

For the next calculations we use the shorthand

$$R^f_r(x,z) := \frac{f(x+rz) - A_r f(x) - \nabla f(x)\cdot(rz)}{r}, \quad \bar{B}(x,r) \subset \Omega.$$

The definition of the matrix $M$ yields

$$\langle M\nabla f(x), \nabla g(x)\rangle = \fint_{B(x,1)} \sum_{i,j=1}^n \partial_i f(x)\partial g_j(x) z_i z_j dz = \fint_{B(x,1)} (\nabla f(x)\cdot z)(\nabla g\cdot z) dz$$

which may be expanded to

$$\begin{aligned}\langle M\nabla f(x),\nabla g(x)\rangle &= \fint_{B(x,1)} \frac{(f(x+rz)-A_rf(x))}{r}(\nabla g(x)\cdot z)dz - \fint_{B(x,1)} R_r^f(x,z)(\nabla g(x)\cdot z)dz\\ &= \fint_{B(x,1)} \frac{f(x+rz)-A_rf(x)}{r}\frac{g(x+rz)-A_rg(x)}{r}dz\\ &\quad - \fint_{B(x,1)}[R_r^g(x,z)(\nabla f(x)\cdot z)+R_r^f(x,z)(\nabla g(x)\cdot z)]dz.\end{aligned}$$

Thus

$$\langle f,g\rangle_r(x)-\langle M\nabla f(x),\nabla g(x)\rangle = \fint_{B(x,1)}[R_r^g(x,z)(\nabla f(x)\cdot z)+R_r^f(x,z)(\nabla g(x)\cdot z)]dz$$

tends to zero as $r\to 0$ for almost every $x\in\Omega$, by (4.29). □

We are now in a position to discuss the proof of Theorem 4.45.

*Proof of Theorem 4.45.* We first observe that, if $u\in W^{1,p}_{loc}(\Omega)$, then

$$\begin{aligned}|\langle u,w\rangle_r(x)|&\le \fint_{B(x,r)}\frac{|u-A_ru(x)|}{r}\frac{|w-A_rw(x)|}{r}dy\\ &\le \|\nabla w\|_{L^\infty(B(x,r))}\fint_{B(x,r)}\frac{|u-A_ru(x)|}{r}dy \le C\|\nabla w\|_{L^\infty(B(x,r))}\mathcal{M}_r|\nabla u|(x),\end{aligned}$$

which holds due to the fact that $w$ is locally Lipschitz and by the boundedness of the maximal function. From this we conclude, that for each compactly contained $\Omega'\subset\Omega$ there exists a function $g\in L^p(\Omega')$ such that

$$|\langle u,w\rangle_r|\le g \tag{4.30}$$

almost everywhere in $\Omega'$. We apply Lemma 4.48 in (4.30) to obtain that

$$\langle u,w\rangle_r\to\langle M\nabla u,\nabla w\rangle \quad \text{in}\ \ L^p_{loc}(\Omega)$$

Now suppose that $u\in W^{2,p}_{loc}(\Omega)$. Then $\langle u,w\rangle_r\to\langle M\nabla u,\nabla w\rangle$ and $\Delta_r u\to\frac{1}{2}\mathrm{div}(M\nabla u)$ in $L^p_{loc}(\Omega)$. Thus (4.22) in Lemma 4.44 implies that

$$\Delta_r^w u = \Delta_r u+\frac{\langle u,w\rangle_r}{A_rw}\xrightarrow{r\to 0}\frac{1}{2}\mathrm{div}(M\nabla u)+\frac{1}{w}\langle M\nabla u,\nabla w\rangle = L_wu$$

in $L^p_{loc}(\Omega)$. This also implies that $u\in \mathrm{AMV}^p_{loc}(\Omega_w)$.

Conversely, suppose that $u\in W^{1,p}_{loc}(\Omega)\cap\mathrm{AMV}^p_{loc}(\Omega_w)$. Then, by (4.22) in Lemma 4.44 and (4.30) we obtain for every compactly contained $\Omega'\subset\Omega$ that there exists a function $g\in L^p(\Omega')$ such that

$$|\Delta_r u|\le|\Delta_r^w u|+\frac{1}{A_rw}g \quad \text{on}\ \ \Omega'.$$

Thus $u\in\mathrm{AMV}^p_{loc}(\Omega)$ which, by Proposition 4.47, implies that $u\in W^{2,p}_{loc}(\Omega)$. □

In the last part of this chapter we briefly discuss the remaining results in [AKS20] obtained in the setting of metric measure space $\Omega_w$.

Assuming higher regularity of the weight $w\in W^{2,\infty}_{loc}(\Omega)$ the authors can describe more precisely the class $\mathrm{AMV}^p_{loc}(\Omega_w)$ refining Theorem 4.45 in the following way.

**Theorem 4.49.** *Suppose that the weight $w\in W^{2,\infty}(\Omega)$ and $p\in(1,\infty)$. Then $\mathrm{AMV}^p_{loc}(\Omega_w) = W^{2,p}_{loc}(\Omega)$. Moreover, for every $u\in W^{2,p}_{loc}(\Omega)$ there holds $\Delta_r^w u\to L_wu$ in $L^p_{loc}(\Omega)$ as $r\to 0$.*

Additionally, the authors study in [AKS20] weakly harmonic functions. We say, that a function $u \in L^2(X)$ is weakly amv-harmonic on a metric measure space $(X, d, \mu)$ if $\lim_{r\to 0} \int_X \varphi \Delta_r u d\mu = 0$ for every compactly supported Lipschitz function $\varphi$. The following characterization of both weakly and strongly amv-harmonic functions is attained in [AKS20].

**Theorem 4.50.** *Let $u \in W^{1,2}_{loc}(\Omega)$. Then, the following conditions are equivalent:*

1. *$u$ is a weak solution to $L_w u = 0$ in $\Omega$,*
2. *$u$ is weakly amv-harmonic in $\Omega_w$,*
3. *$u \in \mathrm{AMV}^2_{loc}(\Omega)$, and $\Delta^w_r u \to 0$ in $L^2_{loc}(\Omega_w)$ as $r \to 0$.*

*Moreover, if $w \in C^\infty(\Omega)$, then all above conditions are equivalent to the following: $u$ is strongly amv-harmonic in $\Omega_w$.*

# Bibliography


[AS64] M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series 55, U.S. Government Printing Office, Washington, D.C., 1964.

[AGG19] T. Adamowicz, M. Gaczkowski, P. Górka, "Harmonic functions on metric measure spaces", *Rev. Math. Complut.* 32.1 (2019), pp. 141–186.

[Ada+20] T. Adamowicz, A. Kijowski, A. Pinamonti, B. Warhurst, "Variational approach to the asymptotic mean-value property for the $p$-Laplacian on Carnot groups", *Nonlinear Anal.* 198 (2020), p. 22.

[AKS20] T. Adamowicz, A. Kijowski, T. Soultanis, "Asymptotically mean value harmonic function in doubling metric measure spaces", *arXiv:2005.13902* (2020).

[AW20] T. Adamowicz, B. Warhurst, "Mean value property and harmonicity on Carnot-Carathéodory groups", *Potential Anal.* 52 (2020), pp. 495–525.

[Ald19] J. M. Aldaz, "Boundedness of averaging operators on geometrically doubling metric spaces", *Ann. Acad. Sci. Fenn. Math.* 44.1 (2019), pp. 497–503.

[Amb18] L. Ambrosio, *Calculus, heat flow and curvature-dimension bounds in metric measure spaces*, Proceedings of the International Congress of Mathematician–Rio de Janeiro 2018. Vol. I. Plenary lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 301–340.

[AGS14a] L. Ambrosio, N. Gigli, G. Savaré, "Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below", *Invent. Math.* 195.2 (2014), pp. 289–391.

[AGS14b] L. Ambrosio, N. Gigli, G. Savaré, "Metric measure spaces with Riemannian Ricci curvature bounded from below", *Duke Math. J.* 163.7 (2014), pp. 1405–1490.

[AB12] D. Andrica, V. Bulgarean, *Some remarks on the group of isometries of a metric space.* Vol. 68, Nonlinear Analysis. Springer Optimization and Its Applicationss, Springer, New York, 2012.

[AL16a] A. Arroyo, J. G. Llorente, "On the asymptotic mean value property for planar $p$-harmonic functions", *Proc. Amer. Math. Soc.* 144.9 (2016), pp. 3859–3868.

[AL16b] A. Arroyo, J. G. Llorente, "On the Dirichlet problem for solutions of a restricted nonlinear mean value property", *Differential Integral Equations* 29.1-2 (2016), pp. 151–166.

[AL18] A. Arroyo, J. G. Llorente, "A priori Hölder and Lipschitz regularity for generalized $p$-harmonious functions in metric measure spaces", *Nonlinear Anal.* 168 (2018), pp. 32–49.

[BZ80] C. A. Berenstein, L. Zalcman, "Pompeiu's problem on symmetric spaces", *Comment. Math. Helv.* 55.4 (1980), pp. 593–621.

[BM19] D. Berti, R. Magnanini, "Asymptotics for the resolvent equation associated to the game-theoretic $p$-laplacian", *Appl. Anal.* 10 (2019), pp. 1827–1842.

[Bie06] T. Bieske, “Equivalence of weak and viscosity solutions to the $p$-Laplace equation in the Heisenberg group”, *Ann. Acad. Sci. Fenn. Math.* 31.2 (2006), pp. 363–379.

[BB11] A. Björn, J. Björn, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics 17, European Math. Soc., Zurich, 2011.

[BHS02] B. Bojarski, P. Hajlasz, P. Strzelecki, “Improved $C^{k,\lambda}$ approximation of higher order Sobolev functions in norm and capacity”, *Indiana Univ. Math. J.* 51.3 (2002), pp. 507–540.

[BIK13] B. Bojarski, L. Ihnatsyeva, J. Kinnunen, “How to recognize polynomials in higher order Sobolev spaces”, *Math. Scand.* 112.2 (2013), pp. 161–181.

[BLU07] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[Bos65] A. K. Bose, “Functions satisfying a weighted average property”, *Trans. Amer. Math. Soc.* 118 (1965), pp. 472–487.

[Bos66] A. K. Bose, “Functions satisfying a weighted average property II”, *Trans. Amer. Math. Soc.* 124 (1966), pp. 540–551.

[Bos68] A. K. Bose, “Generalized eigenfunctions of the Laplace operator and weighted average property”, *Proc. Amer. Math. Soc.* 19 (1968), pp. 55–62.

[Buc99] S. Buckley, “Is the maximal function of a Lipschitz function continuous?”, *Ann. Acad. Sci. Fenn. Math.* 24.2 (1999), pp. 519–528.

[BIK19] D. Burago, S. Ivanov, Y. Kurylev, “Spectral stability of metric-measure Laplacians”, *Isr. J. Math.* 232 (2019), pp. 125–158.

[CT76] P. C., G. Talenti, “Elliptic (second-order) partial differential equations with measurable coefficients and approximating integral equations”, *Advances in Math.* 19 (1976), pp. 48–105.

[Cap97] L. Capogna, “Regularity of quasi-linear equations in the Heisenberg group”, *Comm. Pure Appl. Math.* 50.9 (1997), pp. 867–889.

[CDG96] L. Capogna, D. Danielli, N. Garofalo, “Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations”, *Amer. J. Math.* 118.6 (1996), pp. 1153–1196.

[Cap+07] L. Capogna, D. Danielli, S. Pauls, J. Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, vol. 259, Progress in Mathematics, Birkhäuser Verlag, Basel, 2007.

[Che99] J. Cheeger, “Differentiability of Lipschitz functions on metric measure spaces”, *Geom. Funct. Anal.* 9.3 (1999), pp. 428–517.

[CM97a] T. H. Colding, W. P. I. Minicozzi, “Harmonic Functions on Manifolds”, *Annals of Math. (2)* 146.3 (1997), pp. 725–747.

[CM97b] T. H. Colding, W. P. I. Minicozzi, “Harmonic functions with polynomial growth”, *J. Differential Geom.* 46.1 (1997), pp. 1–77.

[CO20] A. Córdoba, J. Ocáriz, “A note on generalized laplacians and minimal surfaces”, *Bull. Lond. Math. Soc.* 52 (2020), pp. 153–157.

[CIL92] M. Crandall, H. Ishii, P.-L. Lions, “User’s guide to viscosity solutions of second order partial differential equations”, *Bull. Amer. Math. Soc. (N.S.)* 27.1 (1992), pp. 1–67.

[DR92a] E. Damek, F. Ricci, “A class of nonsymetric harmonic Riemannian spaces”, *Bull. Amer. Math. Soc.* 27 (1992), pp. 813–829.

[DR92b] E. Damek, F. Ricci, “Harmonic analysis on solvable extensions of H-type groups”, *J. Geom. Anal.* 2.3 (1992), pp. 213–248.

[Del58] J. Delsarte, “Note sur une propriété nouvelle des fonctions harmoniques”, *C. R. Acad. Sci. Paris* 246 (1958), pp. 1358–1360.

[Edw22] J. Edwards, *A Treatise on the Integral Calculus*, vol. II, Chelsea Publishing Company, New York, 1922.

[Eri+20] S. Eriksson-Bique, J. T. Gill, P. Lahti, N. Shanmugalingam, “Asymptotic behavior of BV functions and sets of finite perimeter in metric measure spaces”, *arXiv:1810.05310* (2020).

[Eva98] L. C. Evans, *Partial differential equations*, vol. 19, Graduate Studies in Mathematics, American Mathematical Society, Providence, 1998.

[EG92] L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[FLM14] F. Ferrari, Q. Liu, J. J. Manfredi, “On the characterization of $p$-harmonic functions on the Heisenberg group by mean value properties”, *Discrete Contin. Dyn. Syst.* 34.7 (2014), pp. 2779–2793.

[FP15] F. Ferrari, A. Pinamonti, “Characterization by asymptotic mean formulas of $q$-harmonic functions in Carnot groups”, *Potential Anal.* 42.1 (2015), pp. 203–227.

[Fla61] L. Flatto, “Functions with a mean value property”, *J. Math. Mech.* 10 (1961), pp. 11–18.

[Fla63] L. Flatto, “Functions with a mean value property II”, *Amer. J. Math.* 85 (1963), pp. 248–270.

[Fla65] L. Flatto, “The converse of Gauss’s theorem for harmonic functions”, *J. Differential Equations* 1 (1965), pp. 438–490.

[FL62] A. Friedman, W. Littman, “Functions satisfying the mean value property”, *Trans. Amer. Math. Soc.* 102 (1962), pp. 167–180.

[GG09] M. Gaczkowski, P. Górka, “Harmonic Functions on Metric Measure Spaces: Convergence and Compactness”, *Potential Anal.* 31 (2009), pp. 203–214.

[Gau40] C. F. Gauss, “Algemeine Lehrsätze in Beziehung auf die im verkehrtem Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abbstossungs-Kräfte”, *Werke* 5 (1840), Band, Göttingen, 1877.

[GV85] B. Gaveau, J. Vauthier, “The Dirichlet problem for the subelliptic Laplacian on the Heisenberg group. II”, *Canad. J. Math.* 37.4 (1985), pp. 760–766.

[Gig15] N. Gigli, “On the differential structure of metric measure spaces and applications”, *Mem. Amer. Math. Soc.* 236.1113 (2015), pp. vi+91.

[GT01] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[Gro96] M. Gromov, *Carnot-Carathéodory spaces seen from within*, vol. 144, Subriemannian Geometry, Progress in Mathematics, Birkhäuser, Basel, 1996, pp. 79–323.

[Gru09] G. Grubb, *Distributions and operators*, Graduate Texts in Mathematics 252, Springer, New York, 2009.

[Haj96] P. Hajłasz, “Sobolev spaces on an arbitrary metric space”, *Potential Anal.* 5.4 (1996), pp. 403–415.

[HK98] P. Hajłasz, J. Kinnunen, “Hölder quasicontinuity of Sobolev functions on metric spaces”, *Rev. Mat. Iberoamericana* 14.3 (1998), pp. 601–622.

[HK00] P. Hajłasz, P. Koskela, *Sobolev Met Poincaré*, Mem. Amer. Math. Soc. 145, 2000.

[HH87] W. Hansen, H. Hueber, “The Dirichlet problem for sub-Laplacians on nilpotent Lie groups-geometric criteria for regularity”, *Math. Ann.* 275.4 (1987), pp. 537–547.

[HN93] W. Hansen, N. Nadirashvili, "A converse to the mean value theorem for harmonic functions", *Acta Math.* 171.2 (1993), pp. 139–163.

[HR11] D. Hartenstine, M. Rudd, "Asymptotic statistical characterizations of p-harmonic functions of two variables", *Rocky Mountain J. Math.* 41.2 (2011), pp. 493–504.

[HR13] D. Hartenstine, M. Rudd, "Statistical functional equations and $p$-harmonious functions", *Adv. Nonlinear Stud.* 13 (2013), pp. 191–207.

[Heb06] J. Heber, "On harmonic and asymptotically harmonic homogeneous spaces", *Geom. Funct. Anal.* 16.4 (2006), pp. 869–890.

[Hei+15] J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev spaces on metric measure spaces. An approach based on upper gradients*, New Mathematical Monographs 27, Cambridge University Press, Cambridge, 2015.

[Hu03] J. Hu, "A note on Hajłasz-Sobolev spaces on fractals", *J. Math. Anal. Appl.* 280.1 (2003), pp. 91–101.

[Hua11] B. Hua, "Harmonic functions of polynomial growth on singular spaces with nonnegative Ricci curvature", *Proc. Amer. Math. Soc.* 139.6 (2011), pp. 2191–2205.

[HKX16] B. Hua, M. Kell, C. Xia, "Harmonic functions on metric measure spaces", *arXiv:1308.-3607* (2016).

[IMW17] M. Ishiwata, R. Magnanini, H. Wadade, "A natural approach to the asymptotic mean value property for the $p$-Laplacian", *Calc. Var. Partial Differential Equations* 56.4 (2017), Art. 97, 22 pp.

[Iwa12] K. Iwasaki, "Cubic harmonics and Bernoulli numbers", *J. Combin. Theory Ser.* 119.6 (2012), pp. 1216–1234.

[Joh55] F. John, *Plane waves and spherical means applied to partial differential equations*, Interscience Publishers, New York-London, 1955.

[JLM01] P. Juutinen, P. Lindqvist, J. Manfredi, "On the equivalence of viscosity solutions and weak solutions for a quasi-linear elliptic equation", *SIAM J. Math. Anal.* 33.3 (2001), pp. 699–717.

[KMP12] B. Kawohl, J. J. Manfredi, M. Parviainen, "Solutions of nonlinear PDEs in the sense of averages", *J. Math. Pures Appl.* 97.2 (2012), pp. 173–188.

[Kei03] S. Keith, "Modulus and the Poincaré inequality on metric measure spaces", *Math. Z.* 245.2 (2003), pp. 255–292.

[Kei04] S. Keith, "A differentiable structure for metric measure spaces", *Adv. Math.* 183.2 (2004), pp. 271–315.

[Kle10] B. Kleiner, "A new proof of Gromov's theorem on groups of polynomial growth", *J. Amer. Math. Soc.* 23.3 (2010), pp. 815–829.

[Koe06] P. Koebe, "Herleitung der partiellen Differentialgleichungen der Potentialfunktion aus deren Integraleigenschaft", *Sitzungsber. Berlin. Math. Gessellschaft* 5 (1906), pp. 39–42.

[Koi04] S. Koike, *A beginner's guide to the theory of viscosity solutions*, MSJ Memoirs 13, Mathematical Society of Japan, Tokyo, 2004.

[KS93] N. Korevaar, R. Schoen, "Sobolev spaces and harmonic maps for metric space targets", *Comm. Anal. Geom.* 1.3–4 (1993), pp. 561–659.

[LMR20] M. Lewicka, J. J. Manfredi, D. Ricciotti, "Random walks and random tug of war in the Heisenberg group", *Math. Ann.* 77.1–2 (2020), pp. 797–846.

[Li97] P. Li, "Harmonic sections of polynomial growth", *Math. Res. Lett.* 4.1 (1997), pp. 35–44.

[Li06] P. Li, *Harmonic functions and applications to complete manifolds*, XIV Escola de Geometria Diferencial, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2006.

[Lic44] A. Lichnerowicz, "Sur les espaces riemanniens complètement harmoniques", *Bull. Soc. Math.* 72 (1944), pp. 146–168.

[LM16] P. Lindqvist, J. J. Manfredi, "On the mean value property for the $p$-Laplace equation in the plane", *Proc. Amer. Math. Soc.* 144.1 (2016), pp. 144–149.

[LY13] H. Liu, X. Yang, "Asymptotic mean value formula for sub-$p$-harmonic functions on the Heisenberg group", *J. Funct. Anal.* 264.9 (2013), pp. 2177–2196.

[Llo15] J. G. Llorente, "Mean value properties and unique continuation", *Commun. Pure Appl. Anal.* 14.1 (2015), pp. 185–199.

[LV09] J. Lott, C. Villani, "Ricci curvature for metric-measure spaces via optimal transport", *Ann. of Math. (2)* 169.3 (2009), pp. 903–991.

[Łys18a] G. Łysik, *Private communication* (2018).

[Łys18b] G. Łysik, "A characterization of real analytic functions", *Ann. Acad. Sci. Fenn. Math.* 43.1 (2018), pp. 475–482.

[MM07] J. J. Manfredi, G. Mingione, "Regularity results for quasilinear elliptic equations in the Heisenberg group", *Math. Ann.* 339.3 (2007), pp. 485–544.

[MPR10] J. J. Manfredi, M. Parviainen, J. Rossi, "An asymptotic mean value characterization for $p$-harmonic functions", *Proc. Amer. Math. Soc.* 138.3 (2010), pp. 881–889.

[MPR13] J. J. Manfredi, M. Parvianen, J. D. Rossi, "On the definition and properties of $p$-harmonious functions", *Ann. Sc. Norm. Super. Pisa Cl. Sc.* 11 (2013), pp. 215–241.

[MPR12] J. Manfredi, M. Parviainen, J. Rossi, "On the definition and properties of $p$-harmonious functions", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 11.2 (2012), pp. 215–241.

[MT19] A. Minne, D. Tewodrose, "Asymptotic Mean Value Laplacian in Metric Measure Spaces", *arXiv:1912.00259* (2019).

[Mon02] R. Montgomery, *A tour of Subriemannian geometries, their geodesics and applications*, vol. 91, Mathematical Surveys and Monographs, American Mathematical Society, Providence, 2002.

[NV94] I. Netuka, J. Vesely, "Mean value property and harmonic functions, Classical and modern potential theory and applications (Chateau de Bonas, 1993)", *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* 430.Kluwer Acad. Publ., Dordrecht (1994), pp. 359–398.

[Nik05] Y. Nikolayevsky, "Two theorems on harmonic manifolds", *Comment. Math. Helv.* 80.1 (2005), pp. 29–50.

[Per+09] Y. Peres, O. Schramm, S. Sheffield, D. B. Wilson, "Tug-of-war and the infinity Laplacian", *J. Amer. Math. Soc.* 22.1 (2009), pp. 167–210.

[PS08] Y. Peres, S. Sheffield, "Tug-of-war with noise: a game-theoretic view of the $p$-Laplacian", *Duke Math. J.* 145.1 (2008), pp. 91–120.

[PS15] N. Peyerimhoff, E. Samiou, "Integral geometric properties of non-compact harmonic spaces", *J. Geom. Anal.* 25.1 (2015), pp. 122–148.

[PW89] M. A. Picardello, W. Woess, "A converse to the mean value property on homogeneous trees", *Trans. Amer. Math. Soc.* 311.1 (1989), pp. 209–225.

[Pri25] I. Privaloff, "Sur les fonctions harmoniques", *Rec. Math. Moscou (Mat. Sbornik)* 32 (1925), pp. 464–471.

[Sch14a] R. Schneider, *Convex bodies: the Brunn-Minkowski theory. Second expanded edition.* Vol. 151, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2014.

[Sch14b] D. Schymura, "An upper bound on the volume of the symmetric difference of a body and a congruent copy", *Adv. Geom.* 14.2 (2014), pp. 287–298.

[Sha00] N. Shanmugalingam, "Newtonian spaces: an extension of Sobolev spaces to metric measure spaces", *Rev. Mat. Iberoamericana* 16.2 (2000), pp. 243–279.

[Stu06a] K.-T. Sturm, "On the geometry of metric measure spaces. I", *Acta Math.* 196.1 (2006), pp. 65–131.

[Stu06b] K.-T. Sturm, "On the geometry of metric measure spaces. II", *Acta Math.* 196.1 (2006), pp. 133–177.

[Sza90] Z. I. Szabó, "The Lichnerowicz conjecture on harmonic manifolds", *J. Diff. Geom.* 31 (1990), pp. 1–28.

[Vil16] C. Villani, "Synthetic theory of Ricci curvature bounds", *Jpn. J. Math.* 11.2 (2016), pp. 219–263.

[Wal49] A. C. Walker, "On Lichnerowicz's conjecture for harmonic 4-spaces", *J. London Math. Soc.* 24 (1949), pp. 1–24.

[Wil50] T. J. Willmore, "Mean-value theorems in harmonic Riemannian spaces", *J. London Math. Soc.* 25 (1950), pp. 54–57.

[Yan03] D. Yang, "New characterizations of Hajłasz-Sobolev spaces on metric spaces", *Sci. China Ser. A* 46.5 (2003), pp. 675–689.

[Zal73] L. Zalcman, "Mean values and differential equations", *Israel J. Math* 14 (1973), pp. 339–352.

[Zuc02] F. Zucca, "The mean value property for harmonic functions on graphs and trees", *Ann. Mat. Pura Appl. (4)* 181.1 (2002), pp. 105–130.