Report on the doctoral thesis *On the geometry of metric spaces defined by group actions: from circle rotations to super-expanders* by Damian Sawicki

Background

Sawicki's thesis studies *warped cones*. Warped cones were introduced by Roe in [11] in the context of foliations to provide a sort of coarse geometric model for the ideas underlying Alain Connes' foliation C^*-algebra [2]. To give a vague idea of an overly-generalized version (for the sake of simplicity), let X be a compact topological space, equipped with an equivalence relation $\sim$. The key examples occur when X is a foliated manifold and $\sim$ the relation of being in the same leaf, or X could be equipped with a group action and $\sim$ could be the relation of being in the same orbit. In Connes' noncommutative geometry program, one builds a C^*-algebra out of $(X, \sim)$ in which it is 'hard' to tell apart related points (but maybe not impossible! – one does not want to pass to the classical quotient $X/\sim$ as this would lose all information about points in the same equivalence class). The idea of the warped cone is to build a metric space where related points are 'hard' to tell apart: in coarse geometry, this means that they are nearby. Making this precise in a non-trivial way requires one to choose generators for the relation in some sense (some form of infinitessimal generators determined by a Riemannian metric in the case of a foliation, honest generators of the group for a group action).

Now, Roe's original motivations for this construction in [11] were index-theoretic. However, around ten years later, Roe [13] used the warped cone idea for group actions to build interesting examples of purely coarse geometric interest; it is really with this later paper that the ideas in Sawicki's thesis start. I think motivated largely by the work of Sawicki and his advisor Piotr Nowak, there has been an explosion of interest in the coarse geometric properties of warped cones[1] in recent years. The paradigm for many of these results, starting with Roe's paper [13], is that one can relate dynamical

[1]Interestingly, the index theoretic warped cone ideas from Roe's earlier paper [11] have also recently been substantially developed in the PhD thesis of Christopher Wulff [22] – it seems warped cones are an idea whose time has come!

properties of an action $\Gamma \circlearrowright X$ of a finitely generated group Γ on a compact space X to coarse geometric properties of the associated warped cone $\mathcal{O}_\Gamma X$. This paradigm was established earlier in the (significantly simpler) case of *box spaces* [12, Chapter 11] by Roe and also Guentner: roughly sequences of finite quotients of a fixed residually finite infinite group; many warped cone results are developments of earlier known results for box spaces. This paradigm allows one to build many interesting (counter)examples in coarse geometry, and throws up a variety of related questions: for example *rigidity* questions of the form 'how much of the action $\Gamma \circlearrowright X$ does the coarse geometry of the warped cone $\mathcal{O}_\Gamma X$ remember?'.

Sawicki has made much progress in understanding some of these issues: indeed, he has written six papers on these topics during his thesis work, two of them already published or accepted in strong international journals [14, 16, 18, 15, 19, 17]. Sawicki starts his thesis giving a thorough overview of background and basic facts, and then proceeds to the main results, as well as discussing many illuminating examples. Without attempting to be completely exhaustive, I will discuss some of this below.

Generalizations of the results of Roe, and related results

The main results of Roe in the paper [13] discussed above relate amenability of an action $\Gamma \circlearrowright X$ to property A of the associated warped cone, and a-T-menability of the acting group to coarse embeddability of the warped cone $\mathcal{O}_\Gamma X$ into Hilbert space. Roe was able to show that if X is (say) a compact manifold and Γ acts amenably by Lipschitz homeomorphisms, then $\mathcal{O}_\Gamma X$ has property A, and moreover that the converse holds when X has an invariant probability measure and the action is free (the converse result is not stated like this, but the same proof works in this level of generality). Sawicki proves among other things (Corollary 3.24 in his thesis, from his joint work with Wu) that if the action is free and by Lipschitz homeomorphisms on (say) a manifold, then the action is amenable if and only if $\mathcal{O}_\Gamma X$ has property A.

Relatedly Roe proved that if Γ acts freely and there is a finite invariant measure, then coarse embeddability of $\mathcal{O}_\Gamma X$ into Hilbert space implies a-T-menability of the action. Sawicki again generalizes this in several ways: in particular, there are versions for coarse embeddings into, and actions on, more general Banach spaces, a subject of much current interest. One particularly striking result (a special case of Corollary 3.15, again from joint work with Wu) states that if Γ acs by isometries on a manifold X, then Γ has the

Haagerup property if and only if $\mathcal{O}_\Gamma X$ fibred coarsely embeds into Hilbert space in the sense of [1].

I find the result removing the assumption of invariant probability measure from Roe's work quite satisfactory; and the proof requires a genuinely new idea over the original version. Fibered coarse embeddability was introduced in [1] after Roe's work [13]; with hindsight, it is fairly clear that it 'should' be the right property of $\mathcal{O}_\Gamma X$ to relate to the Haagerup property of the action $\Gamma \circlearrowright X$, but the details are not obvious (that coarse embeddability itself is not the 'right' property in this regard is made particularly stark by the example in Sawicki's Corollary 6.23). The generalizations to other Banach spaces and the use of 'linearizability' seem particularly nice to me here.

There are other results in Sawicki's thesis that use or fit into this paradigm. One example are results (partly due to Wu and Zacharias) relating the asymptotic dimension of $\mathcal{O}_\Gamma X$ to certain equivariant dimensions for the action $\Gamma \circlearrowright X$; this is used to get some purely dynamical consequences via a clever argument of Yamauchi [23], in particular answering some questions of mine about estimates on, and precise values of, these dynamical dimensions in quite a satisfactory way.

Rigidity

As mentioned already, it is natural to ask how much information about an action $\Gamma \circlearrowright X$ is remembered by the associated warped cone $\mathcal{O}_\Gamma X$. Sawicki has several interesting results here, partly coming from his joint work with Kielak. One particularly nice result is Theorem 5.2, which states that if Γ and Λ act freely and isometrically on compact manifolds M and N of dimensions m and n respectively, then if $\mathcal{O}_\Gamma M$ and $\mathcal{O}_\Lambda N$ are quasi-isometric, then $\Gamma \times \mathbb{Z}^m$ is quasi-isometric to $\Gamma \times \mathbb{Z}^n$ (morally, the factors of $\mathbb{Z}^m$ arises as an m-manifold looks 'more and more like' $\mathbb{Z}^m$ as one blows up the metric). This should be compared to results of Fisher, Nguyen, and van Limbeek [5] of around the same time, who can even get conjugacy of the actions $\Gamma \circlearrowright M$ and $\Lambda \circlearrowright N$ under suitable assumptions (that in particular guarantee the free abelian factors can be ignored); and also to results of Delabie and Khukhro [3] on box spaces. Such results can be used to show that the class of warped cones is coarse geometrically rather rich. Note moreover that other results show that the theory of warped cones in some sense properly subsumes the theory of box spaces (compare Section 6.3 and Theorem 5.14).

Equally interesting are various *non*-rigidity results from Section 6.2, which give examples showing that the free abelian factors in the results above are really necessary in some sense. These results also mesh well with the work of Fisher et. al. mentioned above.

(Super)-expansion and embeddings

Inspired to a large extent by work of Yu [24] and Kasparov and Yu [7, 8] there has been substantial interest in determining which metric spaces coarsely embed into 'nice' classes of Banach spaces, as already mentioned briefly above. One of the first known examples that do not coarsely embed come from box spaces of groups with variants on property (T) (or at least lacking the Haagerup property): the strongest such non-embeddability examples come from Lafforgue's work on various Banach space versions of property (T) [9] and its subsequent developments due to several authors. Sawicki (from his joint work with Nowak) shows an analogue of Lafforgue's box spaces results for warped cone in Theorem 8.1 from his thesis: the input here is an action $\Gamma \circlearrowright X$ with a particularly strong form of spectral gap; this is interesting as it gives a much more general class of non-embeddable examples, noting that Sawicki's results on profinite completions show that the theory of warped cones properly subsumes that of box spaces in some sense.

Another development of these ideas, building on work of Vigolo [20], shows that warped cones can be used to build families of super-expanders, roughly expander graphs that are expanders with respect to a very large family of Banach spaces. Previously, the only known examples of super-expanders come from Lafforgue's work [9] and its developments, and a more combinatorial construction of Mendel and Naor [10]; both are deep and difficult. One should expect Sawicki's examples to be very rich: indeed, in their already cited work, Fisher et. al. show that the class if super-expanders arising in this way is very coarse geometrically rich [5], in particular containing continuum many different examples in some sense.

The coarse Baum-Connes conjecture

The last part of Sawicki's results that I want to discuss specifically cover counterexamples to the coarse Baum-Connes conjecture. This is an important conjecture that relates manifold topology, coarse geometry, and index theory; positive results on the conjecture such as Yu's [24] are responsible for some of the strongest results on problems like the Novikov conjecture and the non-existence of positive scalar curvature metrics. Nonetheless, the conjecture is known to be false [6] for examples arising from so-called 'ghost projections' arising from expander graphs.

Now, Drutu and Nowak [4] have constructed analogues of these ghost projections from warped cones $\mathcal{O}_\Gamma X$ arising from actions $\Gamma \curvearrowright X$ with spectral gap. They conjectured that their warped cones should also be counterexamples to the coarse Baum-Connes conjecture (Roe also made similar speculations around the time [13] was written). Sawicki was able to prove this conjecture. The method of proof is based on an unpublished proof of Higson for the expander case (as exposited later by Yu and myself [21]); nonetheless to develop this method to warped cones, Sawicki had to overcome substantial technical difficulties (even to show that the projections of Drutu an Nowak lie in the right C^*-algebra!). I must admit that these technical details were not obvious to me at the time Drutu and Nowak made their conjecture, and I am impressed that Sawicki was able to overcome them, especially in a subject (C^*-algebra K-theory) that lies at some distance from the rest of the material in his thesis.

Summary

The thesis is in general well-written, and the results are correct as far as I have been able to tell. Sawicki includes substantial background and historical material, evidencing that he has internalized much associated theory, and has a good knowledge of the place that his own work takes in the broader subject. Although all centered on the examples of warped cones, the thesis touches on a large range of mathematics: coarse geometry, group representations, dynamical systems, Banach space geometry, operator algebras, K-theory and so on. It is probably fair to say that many of the results are in some sense technical improvements and developments of material already appearing in the literature (say in the much simpler box space case), but the improvements are often highly non-trivial and require substantial new ideas.

In summary, I am very impressed by this thesis, and unhesitatingly give it my highest recommendation. In particular, I judge it to be 'outstanding' in the sense of being in the top 20% of PhD theses that I have reviewed.

Sincerely,

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