Ranks of tensors, related varieties and rank additivity property for small cases

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Oświadczam, że niniejsza rozprawa została napisana przeze mnie samodzielnie.

Filip Rupniewski

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(data i podpis)

Niniejsza rozprawa jest gotowa do oceny przez recenzentów.

dr hab. Jarosław Buczyński

....
(data i podpis)

To my family.

Contents

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Abstract

The thesis is concerned with ranks of tensors. Topics of central interest are tensor rank, tensor border rank, cactus rank, border cactus rank and related varieties, i.e. secant variety and cactus variety.

In the case of tensor rank and tensor border rank, we analyze the problem of additivity with respect to the direct sum of two independent tensors (of order 3) $p' \in A' \otimes B' \otimes C'$, $p'' \in A'' \otimes B'' \otimes C''$. Namely, we study if the (border) rank of their direct sum is equal to the sum of their individual (border) ranks. For the tensors of order 2 (matrices), tensor rank equals the rank of the corresponding matrix and the additivity holds. In the case of tensors of a bigger order, a positive answer to the problem was previously known as Strassen's conjecture (1973). It was disproved by Shitov (2019). However, his proof was not constructive, and still, an explicit counterexample is not known.

In this thesis, we prove that the additivity of tensor rank holds for some small three-way tensors. For instance, if the tensor p'' is concise and its rank is less or equal dimension of A'' plus 2, then the additivity holds. It is the case also if $p'' \in A'' \otimes (B'' \otimes \mathbb{k}^1 + \mathbb{k}^2 \times C'')$. When we restrict our base field to real or complex numbers, the sufficient condition for rank additivity is that dimensions of both B'' and C'' are equal 3. For $p' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$, $p'' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ or $p' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$, $p'' \in \mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ the additivity also holds. If the base field is $\mathbb{C}$ and the rank of p'' is smaller than 7, it holds as well. As a consequence, the pair of 2×2 matrix multiplication tensors has a rank additivity property. It gives a negative answer to the question of the existence of a faster algorithm for the multiplication of two pairs of 2×2 matrices. The optimal method is to multiply the first pair and then the second one independently.

In addition, we also treat some cases of the additivity of the border rank of tensors. In particular, we show that it holds if the direct sum tensor is contained in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$.

Tensors of a given border rank form a secant variety. Cactus variety is its generalization. It is defined with linear spans of arbitrary finite schemes of bounded length, while secant variety definition uses isolated reduced points only. In particular, any secant variety is always contained in the respective cactus variety, and, except in a few initial cases (when the length is small), the inclusion is strict. It is known that lots of natural criteria on membership in secant varieties are actually only tests for membership in cactus varieties. In this thesis, we propose a pioneering technique for distinguishing actual secant variety from the cactus variety. Our method works in the case of the cactus variety defined for Veronese variety $\nu_d(\mathbb{P}^n)$. We present an algorithm for deciding whether a point in the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}^n))$ belongs to the secant variety $\sigma_{14}(\nu_d(\mathbb{P}^n))$ for $6 \leq d, 6 \leq n$. We obtain similar results for the Grassmann cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}^n))$.

For a tensor $p \in \mathbb{C}^k \otimes \mathbb{C}^l \otimes \mathbb{C}^m$ (border) rank of p equals (border) rank of the image of the linear map $(\mathbb{C}^k)^* \to \mathbb{C}^l \otimes \mathbb{C}^m$ induced by p. We extensively use this tool, known as the slice technique, when studying the additivity of (border) rank. We present counterexamples for the slice techniques in the case of cactus rank and border cactus rank. In some sense, the counterexamples which we provide are the smallest possible.

keywords: tensor rank, additivity of tensor rank, Strassen's conjecture, slices of tensor, secant variety, border rank, cactus variety, cactus rank, Hilbert scheme, apolarity

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Streszczenie

Rozprawa dotyczy rang tensorów. Głównymi tematami pracy są ranga tensorowa, brzegowa ranga tensorowa, ranga kaktusowa oraz związane z nimi rozmaitości, tj. rozmaitość siecznych i rozmaitość kaktusowa.

W przypadku pierwszych dwóch rodzajów rang analizujemy problem addytywności ze względu na sumę prostą dla dwóch niezależnych tensorów (rzędu 3) $p' \in A' \otimes B' \otimes C'$, $p'' \in A'' \otimes B'' \otimes C''$. Badamy, czy ranga (brzegowa) ich sumy prostej jest równa sumie poszczególnych rang (brzegowych). Dla tensorów rzędu 2 (macierzy) ranga tensora jest równa rzędowi odpowiadającej macierzy oraz zachodzi addytywności rangi. W przypadku tensorów większego rzędu pozytywna odpowiedź na zagadnienie addytywność rangi tensorowej była znana jako hipoteza Strassena (1973). Została ona obalona przez Shitova (2019). Jednak jego dowód nie jest konstruktywny i wciąż jeszcze żaden konkretny kontrprzykład nie jest znany.

W pracy doktorskiej dowodzimy, że dla pewnych małych tensorów rzędu 3 addytywność rangi tensorowej zachodzi. Dzieje się tak na przykład, gdy tensor p'' jest treściwy oraz jego ranga jest mniejsza lub równa wymiarowi przestrzeni A'' powiększonemu o 2. Zachodzi ona również gdy $p'' \in A'' \otimes (B'' \otimes \mathbb{k}^1 + \mathbb{k}^2 \times C'')$. Jeżeli ograniczymy ciało bazowe do liczb rzeczywistych lub zespolonych, warunkiem wystarczającym na addytywność jest, żeby wymiary obu przestrzeni B'' oraz C'' były równe 3. W przypadku gdy $p' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ i $p'' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ lub $p' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ i $p'' \in \mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ również zachodzi addytywność. Jest tak też, gdy ciałem bazowym są liczby zespolone oraz ranga p'' jest mniejsza niż 7. Stąd, para tensorów mnożenia macierzy 2×2 ma własność addytywności rangi. Daje to negatywną odpowiedź na pytanie o istnienie szybszego algorytmu mnożenia dwóch par macierzy 2×2. Optymalnym sposobem jest niezależnie od siebie pomnożyć pierwszą parę, a następnie drugą.

Badamy również przypadki addytywności rangi brzegowej wspomnianych tensorów. W szczególności pokazujemy, że zachodzi ona, gdy suma prosta tensorów jest zawarta w $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$.

Tensory ustalonej rangi brzegowej tworzą rozmaitość siecznych. Jej uogólnieniem jest rozmaitość kaktusowa. Definiuje się ją przy pomocy przestrzeni liniowych rozpiętych przez dowolne skończone schematy ograniczonej długości, podczas gdy rozmaitość siecznych jest zdefiniowana przy pomocy tylko izolowanych zredukowanych punktów. W szczególności każda rozmaitość siecznych jest zawsze

zawarta w odpowiadającej jej rozmaitości kaktusowej. Poza kilkoma początkowymi przykładami (gdy długość jest mała) zawieranie jest ścisłe. Dużo naturalnych kryteriów na bycie punktem rozmaitości siecznych sprawdza jedynie przynależność do rozmaitości kaktusowej. W rozprawie prezentujemy technikę, która jako pierwsza pozwala na odróżnianie rozmaitości siecznych od rozmaitości kaktusowej. Nasza metoda działa w przypadku rozmaitości kaktusowej zdefiniowanej dla rozmaitości Veronese $\nu_d(\mathbb{P}^n)$. Podajemy algorytm stwierdzający, czy punkt rozmaitości kaktusowej $\kappa_{14}(\nu_d(\mathbb{P}^n))$ należy do rozmaitości siecznych $\sigma_{14}(\nu_d(\mathbb{P}^n))$ dla $6 \leq d$ i $6 \leq n$. Przedstawiamy także podobny rezultat dla rozmaitości kaktusowej Grassmanna $\kappa_{8,3}(\nu_d(\mathbb{P}^n))$.

Narzędziem, którego wielokrotnie używamy w części dotyczącej addytywności rangi (brzegowej) jest tzw. technika plastrów. Mówi ona, że ranga (brzegowa) tensora $p \in \mathbb{C}^k \otimes \mathbb{C}^l \otimes \mathbb{C}^m$ jest równa randze (brzegowej) obrazu przekształcenia liniowego $(\mathbb{C}^k)^* \to \mathbb{C}^l \otimes \mathbb{C}^m$ zadanego przez p. Podajemy przykłady świadczące o tym, że technika plastrów w przypadku rangi kaktusowej i brzegowej rangi kaktusowej nie działa. W pewnym sensie nasze kontrprzykłady są najmniejszymi możliwymi do uzyskania.

Słowa kluczowe: ranga tensorowa, addytywność rangi tensorowej, hipoteza Strassena, plastry tensorów, rozmaitość siecznych, ranga brzegowa, rozmaitość kaktusowa, ranga kaktusowa, schemat Hilberta, abiegunowość

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Chapter 1

Introduction

1.1 Preface

Tensors appear in many different branches of sciences. The classical question related to tensors is how complicated they are, i.e. what is their rank, border rank, cactus rank. When do different notions of ranks coincide? Is the rank of a sum of two given tensors equal the sum of their ranks?

In the present thesis we analyze the relation of these questions with Commutative Algebra and Algebraic Geometry and give answers for the mentioned questions in some particular cases.

1.2 Additive decomposition

1.2.1 Integers as a sum of powers of integers

Some natural numbers are squares (e.g. 4), some are not, but can be presented as a sum of squares ($5 = 1^2 + 2^2$). There are also such that cannot be presented even using 3 (or less) squares of natural numbers. For example

$$7 = 1^2 + 1^2 + 1^2 + 2^2,$$

$$2023 = 1^2 + 2^2 + 13^2 + 43^2$$

and one can check that there is no presentation using fewer squares. We can ask:

Question. *Is it possible to present every non-negative integer as a sum of 4 squares of non-negative integers?*

This question intrigued humanity already in ancient times. Claude Gaspard Bachet de Méziriac observes that Diophantus of Alexandria (AD 200-284) appears to assume that any number is a sum of up to four squares [Hea10, Arithmetica Book IV Problem 29]. However, the proof was not known, so the claim became known as Bachet's conjecture, after his translation of Diophantus from 1621.

We had to wait until 1770 for the proof that the answer for the question is affirmative. Since then, the conjecture is known as a famous Lagrange's Four Square Theorem.

Theorem 1.2.1.1 (Lagrange's Four Square Theorem)**.** *Any non-negative integer can be expressed as the sum of four integer perfect squares.*

In the same year in which Joseph-Luis Lagrange solved the Bachet's conjecture, the number theorist Edward Waring stated, with no proof [War91] that:

- every integer is a cube or the sum of at most 9 cubes;
- every integer is also the square of a square, or the sum of up to 19 such,
- and so forth.

Waring looked for a generalization of the question. He was trying to show two things. First, that any positive integer may be represented as the sum of other integers raised to a specific power. Second, that given a power there is always a maximum number of summands needed for such minimal presentation.

We had to wait next 139 years for a proof of the following statement.

Theorem 1.2.1.2 (Hilbert, 1909)**.** *For any positive integer d, there exist the smallest number $g(d)$ such that every non-negative integer n can be written as:*

$$n = a_1^d + a_2^d + ... + a_{g(d)}^d, \text{ for } a_i \in \mathbb{Z}_{\geq 0}.$$

Using the notion of $g(d)$ from the statement, we can translate Lagrange's Four Square Theorem to $g(2) \leq 4$, and Waring's statement to $g(3) \leq 9$ and $g(4) \leq 19$.

1.2.2 Waring decomposition of homogeneous polynomials

The last subsection was devoted to the question about presenting positive integers as a sum of powers. We can ask if the similar problem can be stated and solved in the case of rings. Let us focus on the polynomial ring and try to describe a given homogeneous polynomial as a linear combination of powers of linear forms.

Throughout the article, we use the following notation. For any graded ring P (for example polynomial ring $\mathbb{k}[x_1, x_2, ..., x_n]$), by P_i we denote the homogeneous part of degree i, and

$$P_{\leq i} = \bigoplus_{j \leq i} P_j.$$

Definition 1.2.2.1. The *Waring rank* of a homogeneous polynomial $F \in \mathbb{k}[x_1, x_2, ..., x_n]_d$ of degree d in n variables, is the smallest number r such that F is a linear combination of r d-th powers of linear forms. The presentation as a such sum is called a *minimal decomposition*).

$$\begin{aligned} R_{\nu_d(\mathbb{P}(\mathbb{k}^n))}(F) := \min\{r \mid & \text{there exist linear forms } L_1, L_2 ..., L_r \in \mathbb{k}[x_1, x_2, ..., x_n]_1 \\ & \text{and scalars } a_1, a_2, ..., a_r \in \mathbb{k} \text{ such that} \\ & F = a_1 L_1^d + a_2 L_2^d + ... + a_r L_r^d\} \end{aligned}$$

We will briefly say *rank* for *Waring rank* and write R instead of $R_{\nu_d(\mathbb{P}(\mathbb{k}^n))}$, if there is no risk of confusion with other notions of ranks. For the definitions of the map ν_d and rank with respect to any projective variety see Definitions 1.3.0.2, 1.3.0.3.

The decomposition as in Definition 1.2.2.1 is particularly important in the process of blind identification of underdetermined mixtures, i.e. linear mixtures of independent random variables (the so-called sources) when the number of sources exceeds the dimension of the observation space. For more details see [CGLM08] and the references therein.

Notice, that in Definition 1.2.2.1 one can set all a_i equal to 1, if the base field is algebraically closed. Another fact is, that the rank $R(F)$ of a homogeneous polynomial F of degree d and the rank $R(\lambda F)$ of a multiplication of F by a nonzero scalar λ, are equal.

If we are interested in the rank of a polynomial F of degree d, it is natural to look at the $[F] \in \mathbb{P}(\Bbbk[x_1, x_2, ..., x_n]_d)$ in place of $F \in \Bbbk[x_1, x_2, ..., x_n]_d$. Here, we use $\mathbb{P}(V)$ to denote the projectivization of the vector space V, i.e the quotient of $V \setminus 0$ by the equivalence relation: $v \sim w$ if and only if there exists a number $\lambda \neq 0$ such that $v = \lambda w$. We will also use two other notations for a projective space. The projectivization of a N-dimensional vector space will be denoted by $\mathbb{P}(\Bbbk^N)$, or $\mathbb{P}^{N-1}$ if the base field $\Bbbk$ is clear from the context.

Let us look at the set of the polynomials of a given rank.

Definition 1.2.2.2. The set of polynomials in n variables of degree d and rank r will be denoted by:

$$\mathring{\sigma}_r(\nu_d(\mathbb{P}(\Bbbk[x_1, x_2, ..., x_n]_1))) = \{[F] \in \mathbb{P}(\Bbbk[x_1, x_2, ..., x_n]_d) \mid R(F) = r\}$$

It is important to observe that the set $\mathring{\sigma}_r$ is not always closed, as the following example demonstrates.

Example 1.2.2.3. *The Waring rank of $xy^2 \in \mathbb{C}[x, y]_3$ is greater than 2, but $[xy^2]$ is contained in the closure of $\mathring{\sigma}_2(\nu_d(\mathbb{P}(\mathbb{C}[x, y]_1)))$*

$R(xy^2)$ obviously is not 1. To see it is not 2, assume there exists $\{a, b, c, d\}$ such that

$$xy^2 = (ax + by)^3 + (cx + dy)^3.$$

Then

$$xy^2 = (a^3 + c^3)x^3 + 3(a^2b + c^2d)x^2y + 3(ab^2 + cd^2)xy^2 + (b^3 + d^3)y^3.$$

Thus, $\{a, b, c, d\}$ satisfy the following system of equations

$$\begin{cases} a^3 + c^3 = 0 \\ a^2b + c^2d = 0 \\ 3(ab^2 + cd^2) = 1 \\ b^3 + d^3 = 0 \end{cases} \tag{1.2.2.4}$$

However, one can check that 1.2.2.4 has no solution. We obtained a contradiction.

To prove that $[xy^2]$ is contained in the closure of $\mathring{\sigma}_2(\nu_d(\mathbb{P}(\mathbb{C}[x, y]_1)))$ let us look at the limit $\lim_{t\to 0}(\frac{y}{\sqrt[3]{3t}})^3 + (\frac{tx-y}{\sqrt[3]{3t}})^3$. Indeed $(\frac{y}{\sqrt[3]{3t}})^3 + (\frac{tx-y}{\sqrt[3]{3t}})^3 = \frac{1}{3}t^2x^3 - tx^2y + xy^2$, so it converges to xy^2 when t goes to 0.

We say that the polynomial $xy^2 \in \mathbb{C}[x, y]_3$ has a *border rank* two. In general, for an algebraically closed field $\Bbbk$, the polynomial $p \in \Bbbk[x_1, x_2, \ldots, x_n]_d$ has border rank at most r, if and only if it is a limit of polynomials of rank at most r. The border rank of p is denoted by $\underline{R}_{\nu_d(\mathbb{P}(\Bbbk^{n-1}))}(p)$ or $\underline{R}(p)$ if there is no risk of confusion.

1.2.3 Additive decomposition of tensors

For every quadratic homogeneous $f \in \Bbbk[x_1, x_2, .., x_n]_2$ over a base field of characteristic different than two, one can find a symmetric matrix M such that $(x_1, x_2, ..., x_n)M(x_1, x_2, ..., x_n)^T = f(x_1, x_2, ..., x_n)$. This correspondence is unique [AIK89, Ch. 2 Sect. 3.9]. Standard methods of the linear algebra states how to obtain a basis $\mathcal{B} = \{\tilde{x_1}, \tilde{x_2}, \ldots, \tilde{x_n}\}$ of $\Bbbk[x_1, x_2, .., x_n]_1$ such that writing M in $\mathcal{B}$ we obtain the diagonal matrix $\widetilde{M}$ [AIK89, Ch. 2 Sect. 3.7a]. Thus, we recover the minimal decomposition of f by calculating $(\tilde{x_1}, \tilde{x_2}, \ldots, \tilde{x_n})\widetilde{M}(\tilde{x_1}, \tilde{x_2}, \ldots, \tilde{x_n})^T$.

Example 1.2.3.1. *Given $f = xy \in \mathbb{C}[x, y]$, we can present it as $(x, y)M(x, y)^T$ where $M = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$. The matrix M in a basis $\mathcal{B} = \{x + y, x - y\}$ is $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$ so f is of Waring rank 2 and $\frac{1}{4}((x + y)^2 - (x - y)^2)$ is the minimal presentation.*

Notice that rank of a homogeneous polynomial $f \in \mathbb{C}[x_1, x_2, ..., x_n]$ of degree 2 is the same as the rank of corresponding matrix (in arbitrary basis). We may go one step deeper and look at the generalization of rank to multi-dimensional matrices, called tensors. Before that, lets look at the one of the definitions of the rank of matrix.

Definition 1.2.3.2. Matrix $M \in \Bbbk^{k \times l}$ is of rank r if and only if there exist $v_1, v_2, ..., v_r \in \Bbbk^k$ and $u_1, u_2, ..., u_r \in \Bbbk^l$ such that $M = v_1u_1^T + v_2u_2^T + ... + v_nu_n^T$.

This definition generalizes to tensors. For simplicity, here we define it only in case of three-way tensors.

Definition 1.2.3.3. Tensor $p \in \Bbbk^k \otimes \Bbbk^l \otimes \Bbbk^m$ is of rank r if and only if there exist $a_1, a_2, ..., a_r \in \Bbbk^k$, $b_1, b_2, ..., b_r \in \Bbbk^l$ and $c_1, c_2, ..., c_r \in \Bbbk^l$ such that $p = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + ... + a_r \otimes b_r \otimes c_r$ and such a presentation is minimal with respect to the number of summands. Tensors of rank 1 are called *simple tensors*.

The tensor decomposition, also known as *Canonical Polyadic Decomposition* and *CANDECOMP/PARAFAC (CP)* tensor decomposition, can be considered to be higher order generalizations of the matrix singular value decomposition (SVD). In the analogy to the analyzing complicated data coming from physical world, the rank should correspond to the number of simple ingredients affecting our complicated state.

In the last twenty years, interest in the subject has expanded to other fields. Examples include signal processing [DLDM97], numerical linear algebra [DLDMV00], computer vision [VT02], numerical analysis [HKT05], data mining [SPY06], graph analysis [KBK05], neuroscience [Arn06], and more. More about tensor decomposition one can read in the survey [KB09]. For more motivations to study tensor rank see for instance [Com02], [Lan12], [CGO14] and references therein.

Tensors and tensor rank appear also in physics. In quantum information theory, a tensor $p = \sum_{i_i=1}^{k}\sum_{i_l=1}^{l}\sum_{i_3=1}^{m} p_{i_i,i_2,i_3} e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \in \mathbb{C}^k \otimes \mathbb{C}^l \otimes \mathbb{C}^m$ (where e_i are elements of orthonormal basis of $\mathbb{C}^{max\{k,l,m\}}$) corresponds to a pure state of a quantum system composed out of 3 subsystems of k, l and m levels respectively. In Dirac notation used by physicists $p = \sum_{i_i=1}^{k}\sum_{i_l=1}^{l}\sum_{i_3=1}^{m} p_{i_i,i_2,i_3}|i_1 i_2 i_3\rangle$. The quantity $p^2_{i_i,i_2,i_3}$ represents the probability of the state to be measured in the corresponding base state and the sum $\sum_{i_i=1}^{k}\sum_{i_l=1}^{l}\sum_{i_3=1}^{m} p^2_{i_i,i_2,i_3}$ has to equal 1. In quantum mechanics, the rank of a tensor is a measure of degree of entanglement. Simple tensor is called separable state, while other tensors are entangled states. Separable states correspond to a product probability tensor $p(X, Y, Z = i_1, i_2, i_3) = p(X = i_1)p(Y = i_2)p(Z = i_3)$. Degree of entanglement is interpreted as a degree of quantum correlation between subsystems. For detailed introduction to problems of pure states entanglement and connection with variants of tensor rank, see [BFZ20].

A simple criterion of the complexity of a given tensor, in particular a matrix, is its rank. The computation of the matrix rank is usually obtained by applying the Gaussian elimination process. A classical result says that it is computable in a polynomial time [Far88, p.12]. In contrast to matrix rank, there is no effective algorithm calculating the rank of a given tensor. Hastad [Has90] proved that the tensor rank is NP-hard to compute.

1.2.4 Matrix multiplication

Let us recall the standard way to calculate the product of two 2×2 matrices:

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\ a_{2,1}b_{1,2} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} \end{bmatrix}$$

We have used 8 multiplications and 4 additions. Since additions are computationally cheaper than multiplications, it is natural to ask if there is another algorithm, which uses possibly more additions but fewer multiplications. In 1969, Strassen presented an algorithm for the multiplication of two 2×2 matrices, using 18 additions, but only 7 multiplications.

Theorem 1.2.4.1 (Strassen's algorithm for multiplication of two 2×2 matrices, [Str69]). *Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two 2×2 matrices and $C = AB = (c_{i,j})$*

be their product. Then calculating 7 products (of numbers):

$$\begin{aligned} I :=&(a_{1,1}+a_{2,2})(b_{1,1}+b_{2,2});\\ II :=&(a_{2,1}+a_{2,2})b_{1,1};\\ III :=&a_{1,1}(b_{1,2}-b_{2,2});\\ IV :=&a_{2,2}(-b_{1,1}+b_{2,1});\\ V :=&(a_{1,1}+a_{1,2})b_{2,2};\\ VI :=&(-a_{1,1}+a_{2,1})(b_{1,1}+b_{1,2});\\ VII :=&(a_{1,2}-a_{2,2})(b_{2,1}+b_{2,2}); \end{aligned}$$

we can present $(c_{i,j})$ using just their sums:

$$\begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} = \begin{bmatrix} I+IV-V+VII & II+IV \\ III+V & I+III-II+VI \end{bmatrix}$$

Since the multiplication of two $n \times n$ matrices can be made in blocks, there is a generalization of Strassen's algorithm to multiplication of bigger matrices. As a consequence, matrix multiplication can be performed by using on the order of $n^{log_2(7)} \approx n^{2.81}$ arithmetic operations, in contrary to the standard algorithm which uses the order of n^3. The natural question is: what is the smallest possible exponent?

In [BCS97] authors proved that looking for the answer, we do not need to worry about the number of additions. To be more precise — the exponent of the order of the required arithmetic operations equals the exponent of the order of the required multiplications. More recent results decreased the exponent close to 2. From 1990 until 2010 the smallest known exponent was 2.375477 [CW87], given by the Coppersmith–Winograd algorithm. The state of the art is 2.3728639 [LG14]. The famous conjecture in algebraic complexity theory states that the number is exactly 2 [Lan08, Subsect. 3.9]. Roughly speaking, it says that as matrices get large, it becomes as easy to multiply them as to add them. For a more detailed description see [Lan17].

The matrix multiplication is a bilinear map $\mu_{i,j,k}\colon \mathcal{M}^{i\times j} \times \mathcal{M}^{j\times k} \to \mathcal{M}^{i\times k}$, where $\mathcal{M}^{l\times m}$ is the linear space of $l \times m$ matrices with coefficients in a field $\Bbbk$. In particular, $\mathcal{M}^{l\times m} \simeq \Bbbk^{l\cdot m}$, where $\simeq$ denotes an isomorphism of vector spaces. We can interpret $\mu_{i,j,k}$ as a three-way tensor

$$\mu_{i,j,k} \in (\mathcal{M}^{i\times j})^* \otimes (\mathcal{M}^{j\times k})^* \otimes \mathcal{M}^{i\times k}.$$

Example 1.2.4.2. *The 2×2 matrix multiplication tensor is $\mu_{2,2,2} = (a_{1,1}\otimes b_{1,1} + a_{1,2}\otimes b_{2,1})\otimes c_{1,1} + (a_{1,1}\otimes b_{1,2} + a_{1,2}\otimes b_{2,2})\otimes c_{1,2} + (a_{2,1}\otimes b_{1,2} + a_{2,2}\otimes b_{2,1})\otimes c_{2,1} + (a_{2,1}\otimes b_{1,2} + a_{2,2}\otimes b_{2,2})\otimes c_{2,2}$.*

Using Theorem 1.2.4.1 we can write $\mu_{2,2,2} = (a_{1,1}+a_{2,2})\otimes(b_{1,1}+b_{2,2})\otimes(c_{1,1}+c_{2,2}) + (a_{2,1}+a_{2,2})\otimes b_{1,1}\otimes(c_{1,2}-c_{2,2}) + a_{1,1}\otimes(b_{1,2}-b_{2,2})\otimes(c_{2,1}+c_{2,2}) + a_{2,2}\otimes(-b_{1,1}+b_{2,1})\otimes(c_{1,1}+c_{1,2}) + (a_{1,1}+a_{1,2})\otimes b_{2,2}\otimes(-c_{1,1}+c_{1,2}) + (-a_{1,1}+a_{2,1})\otimes(b_{1,1}+b_{1,2})\otimes c_{2,2} + (a_{1,2}-a_{2,2})\otimes(b_{2,1}+b_{2,2})\otimes c_{1,1}$

The question "what is the minimal number of multiplications required to calculate the product of two matrices M, N, for any $M \in \mathcal{M}^{i\times j}$ and $N \in \mathcal{M}^{j\times k}$?" is the same question as "what is the *tensor rank* of $\mu_{i,j,k}$?".

1.2.5 Strassen additivity problem

One of our main interest is the *additivity* of the tensor rank. Given arbitrary four matrices $M' \in \mathcal{M}^{i'\times j'}$, $N' \in \mathcal{M}^{j'\times k'}$, $M'' \in \mathcal{M}^{i''\times j''}$, $N'' \in \mathcal{M}^{j''\times k''}$, suppose we want to calculate both products $M'N'$ and $M''N''$ simultaneously. What is the minimal number of multiplications needed to obtain the result? Is it equal to the sum of the ranks $R(\mu_{i',j',k'})+R(\mu_{i'',j'',k''})$? More generally, the same question can be asked for arbitrary tensors. For two tensors in independent vector spaces, is the rank of their sum equal to the sum of their ranks? A positive answer to this question was widely known as Strassen's Conjecture [Str73, p. 194, §4, Vermutung 3], [Lan12, Sect. 5.7], until it was disproved by Shitov [Shi19].

Definition 1.2.5.1. Assume $A = A' \oplus A''$, $B = B' \oplus B''$, and $C = C' \oplus C''$, where all $A, \dots, C''$ are finite dimensional vector spaces over a field $\Bbbk$. Pick $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ and let $p = p' + p''$, which we will write as $p = p' \oplus p''$. We say that the pair p', p'' has a *rank additivity property* if the following equality holds

$$R(p) = R(p') + R(p''). \tag{1.2.5.2}$$

Problem 1.2.5.3 (Strassen's additivity problem)**.** *Given p', p'' as in the definition 1.2.5.1 decide if they poses rank additivity property.*

Theorem 1.2.5.4 (Strassen's additivity does not always hold, [Shi19])**.** *There exist $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$, where $A' = A'' = \dots = C' = C'' = \mathbb{C}^n$ and $n \geq 450$ such that*

$$R(p' \oplus p'') < R(p') + R(p'')? \tag{1.2.5.5}$$

Shitov did not gave a constructive proof, so there is still work needed to find an explicit example of a pair without rank additivity property.

In the article [BPR20] we address several cases of Problem 1.2.5.3 and its generalisations. It is known that if one of the factor vector spaces is small, then the additivity of the tensor rank holds.

Theorem 1.2.5.6 ([JT86])**.** *Using notation from Definition 1.2.5.1, if one of the vector space A', A'', B', B'', C', C'' over an arbitrary field $\Bbbk$ has dimension bounded by 2, then*

$$R(p' \oplus p'') = R(p') + R(p'').$$

See [JT86] for the original proof and Section 2.2.2 for a discussion of more recent approaches.

One of our results includes the next case, that is if say $\dim B'' = \dim C'' = 3$, then (1.2.5.2) holds. The following theorem summarizes our main results, i.e. Corollary 3.1.3.9, Theorems 3.1.4.1–3.1.4.3 and Corollary 3.1.4.13.

Theorem 1.2.5.7. *Let $\Bbbk$ be any base field and let A', A'', B', B'', C', C'' be vector spaces over $\Bbbk$ of dimensions $\mathbf{a}', \mathbf{a}'', \ldots, \mathbf{c}''$ respectively. Assume $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ and let*

$$p = p' \oplus p'' \in (A' \oplus A'') \otimes (B' \oplus B'') \otimes (C' \oplus C'').$$

If at least one of the following conditions holds, then the additivity of the rank holds for p, that is $R(p) = R(p') + R(p'')$:

(i) $R(p'') \le \mathbf{a}''+2$ and p'' is not contained in $\tilde{A}'' \otimes B'' \otimes C''$ for any linear subspace $\tilde{A}'' \subsetneqq A''$ (this part of the statement is valid for any base field $\Bbbk$).

(ii) $\Bbbk = \mathbb{R}$ (real numbers) or $\Bbbk$ is an algebraically closed field of characteristic $\neq 2$ and $R(p'') \le 6$,

(iii) $\Bbbk = \mathbb{C}$ or $\Bbbk = \mathbb{R}$ (complex or real numbers) and $p'' \in A'' \otimes \Bbbk^3 \otimes \Bbbk^3$

(iv) $p'' \in A'' \otimes (B'' \otimes \Bbbk^1 + \Bbbk^2 \otimes C'')$ (this part of the statement is valid for any base field $\Bbbk$).

(v) $\Bbbk = \mathbb{C}$ and the pair $((\mathbf{a}', \mathbf{b}', \mathbf{c}'), (\mathbf{a}'', \mathbf{b}'', \mathbf{c}''))$ equals either $((4,4,3),(4,4,3))$ or $((4,4,3),(4,3,4))$,

(vi) $\Bbbk = \mathbb{C}$ and both tensors have ranks less or equal 7. In particular, $R(\mu_{2,2,2} \oplus \mu_{2,2,2}) = R(\mu_{2,2,2}) + R(\mu_{2,2,2})$.

Analogous statements hold if we exchange the roles of A, B, C and/or of $'$ and $''$.

Remark 1.2.5.8. Although most of our arguments are characteristic free, we partially rely on some earlier results which often are proven only over the base fields of the complex or the real numbers, or other special fields. Specifically, we use upper bounds on the maximal rank of small tensors, such as [BH13] or [SMS10]. See Section 3.1.4 for a more detailed discussion. In particular, the consequence of the proof of Theorem 3.1.4.3 is that if (over any base field $\Bbbk$) there are p' and p'' such that $R(p'') \le 6$ and $R(p' \oplus p'') < R(p') + R(p'')$, then $p'' \in \Bbbk^3 \otimes \Bbbk^3 \otimes \Bbbk^3$ and $R(p'') = 6$. In [BH13] it is shown that if $\Bbbk = \mathbb{Z}_2$ (the field with two elements), then tensors $p'' \in \Bbbk^3 \otimes \Bbbk^3 \otimes \Bbbk^3$ with $R(p'') = 6$ exist. Thus, the minimal case in which the counterexample to additivity can occur is $p' \oplus p'' \in \mathbb{Z}_2^{3+3} \otimes \mathbb{Z}_2^{3+3} \otimes \mathbb{Z}_2^{3+3}$, $R(p_1) = R(p_2) = 6$. Over the base field $\mathbb{C}$ it is $p' \oplus p'' \in \mathbb{C}^{4+4} \otimes \mathbb{C}^{4+4} \otimes \mathbb{C}^{4+4}$ such that $R(p') = 8$, $R(p'') = 7$ (see Remark 3.1.4.15).

1.2.6 Strassen additivity problem for the border rank

Next we turn our attention to the *border rank*. Roughly speaking, over the complex numbers, a tensor p has border rank at most r, if and only if it is a limit of tensors of rank at most r. The border rank of p is denoted by $\underline{R}(p)$. One can pose the analogue of Problem 1.2.5.3 for the border rank. For which tensors

$p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ is the border rank additive, that is $\underline{R}(p' \oplus p'') = \underline{R}(p') + \underline{R}(p'')$?

In general, the answer is negative; in fact there exist examples for which $\underline{R}(p' \oplus p'') < \underline{R}(p') + \underline{R}(p'')$. Schönhage [Sch81] proposed a family of counterexamples amongst which the smallest is

$$\underline{R}(\mu_{2,1,3}) = 6, \quad \underline{R}(\mu_{1,2,1}) = 2, \quad \underline{R}(\mu_{2,1,3} \oplus \mu_{1,2,1}) = 7,$$

see also [Lan12, Sect. 11.2.2].

An interesting question is what is the smallest counterexample to the additivity of the border rank? The smallest example of Schönhage lives in $\mathbb{C}^{2+2} \otimes \mathbb{C}^{3+2} \otimes \mathbb{C}^{6+1}$, that is it requires using a seven dimensional vector space. In [BPR20] we show that if all three spaces A, B, C have dimensions at most 4, then it is impossible to find a counterexample to the additivity of the border rank.

Theorem 1.2.6.1 ([BPR20, Thm 1.3])**.** *Suppose $A', A'', B', B'', C', C''$ are vector spaces over $\mathbb{C}$ and $A = A' \oplus A''$, $B = B' \oplus B''$, and $C = C' \oplus C''$. If $\dim A, \dim B, \dim C \leq 4$, then for any $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ the additivity of the border rank holds:*

$$\underline{R}(p' \oplus p'') = \underline{R}(p') + \underline{R}(p'').$$

We prove the theorem in Section 3.2 as Corollary 3.2.0.2, Propositions 3.2.1.1 and 3.2.2.1, which in fact cover a wider variety of cases.

1.3 Geometry of secants

There is a definition which generalizes Definitions 1.2.2.1 and 1.2.3.3. To state it we need to observe that the set of simple tensors is naturally isomorphic to the Cartesian product of projective spaces. The image of the embedding in the tensor space is called the *Segre variety*.

Definition 1.3.0.1. For $A_1, A_2, \ldots A_d$ vector spaces over $\Bbbk$, the *Segre variety* is defined as the image of the map, called *Segre embedding*:

$$\begin{aligned} Seg : \mathbb{P}A_1 \times \mathbb{P}A_2 \times \cdots \times \mathbb{P}A_d &\to \mathbb{P}(A_1 \otimes A_2 \otimes \cdots \otimes A_d) \\ ([a_1], [a_2], \ldots, [a_d]) &\mapsto [a_1 \otimes a_2 \otimes \cdots \otimes a_d]. \end{aligned}$$

If there is no risk of confusion we will denote the image by

$$\mathrm{Seg} = \mathrm{Seg}_{A_1, A_2, \ldots A_d} := \mathbb{P}A_1 \times \mathbb{P}A_2 \times \cdots \times \mathbb{P}A_d \subset \mathbb{P}(A_1 \otimes A_2 \otimes \cdots \otimes A_d).$$

Definition 1.3.0.2. For $A = \Bbbk^{n+1}$ and a positive integer d, consider the map

$$\nu_d : \mathbb{P}A \to \mathbb{P}(\operatorname{Sym}^d A)$$
$$[a] \mapsto [a^d].$$

Here, $\operatorname{Sym}^d \Bbbk^{n+1}$ denotes the space of polynomials of order d in $n+1$ variables. The *Veronese variety* is defined as the image of this map, called *Veronese embedding*.

Definition 1.3.0.3. For a projective variety $X \subset \mathbb{P}W \simeq \mathbb{P}(\Bbbk^{N+1})$ and $[p] \in \mathbb{P}W$ we define *X-rank* of p to be the minimal number r such that there exist $\{s_1, s_2, ..., s_r\} \subseteq X$ such that $[p]$ lies in the projective span $\langle s_1, s_2, ..., s_r\rangle$. To ease the notation we will say that X-rank of a point $p \in Z$ is the X-rank of a projective class of point $[p] \in \mathbb{P}W$. This number will be denoted by $R_X(p)$. Compare to Definition 1.2.2.1.

For the notion of the rank of a linear subspace, see Definition 2.1.0.1.

Example 1.3.0.4. *Taking Veronese and Segre variety (see Definitions 1.3.0.1, 1.3.0.2) as X in Definition 1.3.0.3 we recover Waring rank (see Definition 1.2.2.1) and tensor rank (see Definition 1.2.3.3), correspondingly.*

A central task in many problems is to test tensor membership in a given set (e.g., if a tensor has rank r). Some of these sets are defined as the zero sets of collections of polynomials, i.e. as algebraic varieties. However in general, the set of tensors of rank at most r is neither open nor closed. One of the very few exceptions is the case of matrices, that is tensors in $A \otimes B$. The sets which are not algebraic varieties, we can expand to varieties by taking Zariski closure. For example, the set of tensors of border rank at most r is the Zariski closure of the set of tensors of rank at most r.

Definition 1.3.0.5. For a algebraically closed base field $\Bbbk$ and a projective variety $X \subset \mathbb{P}W \simeq \mathbb{P}(\Bbbk^{N+1})$ the r'th secant variety of X is the closure of the union of all linear subspaces spanned by r points of X:

$$\sigma_r(X) := \bigcup \overline{\{\langle s_1, s_2, \ldots, s_r\rangle : s_i \in X\}} \subset \mathbb{P}W$$

where the overline $\overline{\{\cdot\}}$ denotes the closure in the Zariski topology.

Example 1.3.0.6. *When $X = \operatorname{Seg}_{A,B,C}$, then $\sigma_r(X) \subset \mathbb{P}A \otimes \mathbb{P}B \otimes \mathbb{P}C$ is the set of tensors of border rank at most r.*

Example 1.3.0.7. *When $X = \nu_d(\Bbbk[x_1, x_2, ..., x_n]_1)$, then $\sigma_r(X) \subset \Bbbk[x_1, x_2, ..., x_n]_d$ is the set of classes of homogeneous polynomials of border rank at most r.*

In the definition of r'th secant variety over $\mathbb{C}$, the resulting set coincides with the Euclidean closure. This is a classically studied algebraic variety [Pal06], [Zak93], [Ådl87] and leads to a definition of border rank of a point.

Definition 1.3.0.8. For a projective variety $X \subset \mathbb{P}W \simeq \mathbb{P}(\Bbbk^{N+1})$ and for $[p] \in \mathbb{P}W$ define $\underline{R}_X(p)$, *the X border rank* of $[p]$, to be the minimal number r such that $[p]$ belongs to $\sigma_r(X)$. We say that $p \in Z$ has border rank r if $[p] \in \mathbb{P}W$ has border rank r. We follow the standard convention that $\underline{R}_X(p) = 0$ if and only if $p = 0$. We will drop X from the subscript, if the variety we work with is clear from the context.

For the notion of the border rank of a linear subspace, see Definitions 2.1.2.1 and 2.4.1.1.

There are many related variants of the notion of X-rank. To mention only one of them, the *X-cactus rank* of a point $p \in Z$ is the minimal number r such that there exists a finite subscheme $R \subset X$ of length r such that $[p]$ is in scheme theoretic linear span of R. It follows from the definition that the X-cactus rank is less or equal than X-rank. For more details, including definition of the border cactus rank and the analogue of secant variety for the cactus rank (i.e. cactus variety), see Subsection 2.4.2.

1.3.1 Secant variety of Veronese variety

Secant varieties of a non-degenerate variety $X \subseteq \mathbb{P}W \simeq \mathbb{P}(\Bbbk^{N+1})$ eventually fill the projective space $\mathbb{P}W$. To see this, it is enough to make the following two observations. One can check that, if $\sigma_m(X) = \sigma_{m+1}(X)$, then $\sigma_m(X) = \sigma_{m+a}(X)$ for every $a \in \mathbb{N}$ [CGO14, Exer. 2.5]. In this case also $\sigma_m(X)$ has to be a linear subspace of $\mathbb{P}W$. It follows that if X is non-degenerate, namely is contained in a space isomorphic to $\mathbb{P}(\Bbbk^N)$, then we have a filtration:

$$X \subsetneq \sigma_2(X) \subsetneq \sigma_3(X) \subsetneq \sigma_4(X) \subsetneq \ldots \subsetneq \sigma_m(X) = \mathbb{P}W.$$

Given a class of point $[p]$ and a non-degenerate variety, we would like to check to which secant variety $[p]$ belongs. As every variety, $\sigma_r(X)$ is given by some polynomial equations, so if we know them, we can check if $[p] \in \sigma_r(X)$. Unfortunately, these equations are hard to compute and are unknown in general.

The paper [LO13] presents many methods of obtaining equations vanishing on the secant variety in the setting of vector bundles. However, the equations given in this way are equations of a bigger variety, the cactus variety $\kappa_r(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$, see [Gał17] for the proof, and the discussion in [Lan17, Sect. 10.2]. In fact, we are not aware of any explicit equation of the secant variety $\sigma_r(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ which does not vanish on the respective cactus variety. Moreover, the cactus varieties fill up the projective spaces quicker than the secant varieties, see [BJMR17].

An interesting question is for which values of r, d, n, the secant variety $\sigma_r(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ coincides with the corresponding cactus variety. This problem is related to the geometry of Hilbert schemes. If $\mathcal{H}ilb_r^{Gor}(\mathbb{P}(\mathbb{C}^{n+1}))$, i.e. the open locus of $\mathcal{H}ilb_r(\mathbb{P}(\mathbb{C}^{n+1}))$ of Gorenstein subschemes of the Hilbert scheme of r points in $\mathbb{P}(\mathbb{C}^{n+1})$, is irreducible, then $\sigma_r(\nu_d(\mathbb{P}(\mathbb{C}^{n+1}))) = \kappa_r(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for any d (see

[BB14, Prop. 2.2]). The paper [CJN15, Thm A, B] shows that for $r < 14$, the scheme $\mathcal{H}ilb_r^{Gor}(\mathbb{P}(\mathbb{C}^{n+1}))$ is irreducible and that $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}(\mathbb{C}^{n+1}))$ is reducible if and only if $n \geq 6$. That is why we focus on studying $\kappa_{14}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for $n \geq 6$ in Section 4.2.

The notion of the Waring rank of a single homogeneous polynomial can be generalized to the rank of a subspace spanned by several homogeneous polynomials of the same degree. This leads to a generalization of secant varieties to the notion of a Grassmann secant variety $\sigma_{r,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1}))) \subseteq \mathrm{Gr}(k, \mathbb{P}(S^d\mathbb{k}^{n+1})$ for positive integers r, k, d, n (see Section 2.4 for the definitions). We have $\sigma_{r,1}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1}))) = \sigma_r(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ as defined above. For a Grassmann secant variety, there is an analogous notion of a Grassmann cactus variety $\kappa_{r,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ (see Section 2.4).

As in the case $k = 1$, it is interesting to investigate for which values of r, k, d, n the Grassmann secant variety $\sigma_{r,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ coincides with the Grassmann cactus variety $\kappa_{r,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$. However, in this case it is not enough to focus only on the locus of Gorenstein schemes. The reduction to the case of Gorenstein schemes was possible because of [BB14, Prop. 2.2] which is not true in the case of a vector space. In particular, Theorem 1.3.1.3 describe two irreducible components of $\kappa_{8,3}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$, while if we consider only Gorenstein schemes then (as it was mentioned before) from [CJN15, Thm A, B] follows that $\mathcal{H}ilb_8^{Gor}(\mathbb{P}(\mathbb{C}^{n+1}))$ is irreducible.

The paper [CEVV09] shows that for $r \leq 7$ and any n, the scheme $\mathcal{H}ilb_r(\mathbb{P}(\mathbb{C}^{n+1}))$ is irreducible. Moreover, $\mathcal{H}ilb_8(\mathbb{P}(\mathbb{C}^{n+1}))$ is reducible if and only if $n \geq 4$. As a consequence, $\sigma_{r,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1}))) = \kappa_{r,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for $r \leq 7$, and any n, k. Furthermore, $\sigma_{8,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1}))) = \kappa_{8,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for $n \leq 3$, and any k. That is why we focus on studying $\kappa_{8,3}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for $n \geq 4$ in Section 4.3. See Remark 4.3.0.1 for the reason why we consider $\kappa_{8,3}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ among all other $\kappa_{8,k}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ varieties with $k \geq 2$.

We solve the problem of identification of points of the Grassmann secant variety inside the Grassmann cactus variety in the minimal cases where these varieties differ. It turns out, that in the case of $\sigma_{14}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for $d \geq 5$ and $n = 6$ the closure of the set-theoretic difference between the cactus variety and the secant variety, consists exactly of polynomials divisible by a large power of a linear form.

Theorem 1.3.1.1 ([GMR20, Thm 1.1.]). *Let $d \geq 5$ be an integer and $T = \mathbb{C}[x_0, x_1, \dots, x_6]$. Then the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}T_1))$ has two irreducible components: the secant variety $\sigma_{14}(\nu_d(\mathbb{P}T_1))$ and the other one, denoted by $\eta_{14}(\nu_d(\mathbb{P}T_1))$, where*

$$\eta_{14}(\nu_d(\mathbb{P}T_1)) = \{[L^{d-3}F] | [L] \in \mathbb{P}T_1, [F] \in \mathbb{P}T_3\}.$$

To become more familiar with the notation used in the statement of a Theorem 1.3.1.1 let us look at a simple example of its application. We obtain that the polynomial

$$x_0^3(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3) \in \mathbb{C}[x_0, x_1, ..., x_5] \qquad (1.3.1.2)$$

has a border cactus rank 14.

If n is greater than 6 and $d \geq 5$, then the description of $\eta_{14}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1}))$ is more complicated, for the detailed statement see Theorem 4.0.0.2. The proofs are in Section 4.2.

For $d \geq 6$, this allows us to design an algorithm for deciding whether a point in the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ is in the secant variety $\sigma_{14}(\nu_d\mathbb{P}(\mathbb{C}^{n+1}))$ (Theorem 4.0.0.4). In Section 4.3 we prove results analogous to those of Section 4.2 for the case of the Grassmann secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for $n \geq 4$. In particular we prove the following theorem and its generalization, i.e. Theorem 4.0.0.3. We use notation $\mathrm{Gr}(k, V)$ for the Grassmannian of k-dimensional subspaces of a linear space V.

Theorem 1.3.1.3 ([GMR20, Thm 1.2.]). *Let $d \geq 5$ be an integer and $T = \mathbb{C}[x_0, x_1, \ldots, x_4]$. Then the Grassmann cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}T_1))$ has two irreducible components: the Grassmann secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}T_1))$ and the other one, denoted by $\eta_{8,3}(\nu_d(\mathbb{P}T_1))$, where*

$$\eta_{8,3}(\nu_d(\mathbb{P}T_1)) = \{[L^{d-2}U] | [L] \in \mathbb{P}T_1, [U] \in \mathrm{Gr}(3, T_2)\}.$$

The above results of our article [GMR20] are the first ones that provides a procedure to distinguish points of a secant variety and the corresponding cactus variety for an embedding of a smooth variety. An algorithm is contained in Theorem 4.0.0.4, while its Grassmann version is included in Theorem 4.3.1.5. In particular, as a consequence we can demonstrate that the polynomial from (1.3.1.2) is contained not only in $\kappa_{14}(\nu_6(\mathbb{C}[x_0, x_1, ..., x_6]))$ but also in $\sigma_{14}(\nu_6(\mathbb{C}[x_0, x_1, ..., x_6]))$ (see Corollary 4.0.0.5).

On our way to establishing Theorems 1.3.1.1, 1.3.1.3, 4.0.0.2, and 4.0.0.3, we prove Theorems 1.3.1.4 and 1.3.1.5, which determine the cactus rank and border cactus rank of polynomials and subspaces divisible by a large power of a fixed linear form in a much more general situation. We denote such polynomials and subspaces by f^{hom,d_2} and W^{hom,d_2} correspondingly. One can think about them as a particular way of homogenizing of polynomial $f \in S_{\leq d_1}$ and subspace $W \subseteq S_{\leq d_1}$. The degree of f^{hom,d_2} is increased by d_2 with respect to degree of f. Every polynomial $g \in W^{hom,d_2}$ equals $\tilde{f}^{hom,d_2}$ for a certain $\tilde{f} \in W$. See Definition 2.5.0.1 for a precise definition.

Theorem 1.3.1.4 (Polynomial case, [GMR20, Thm 1.8.]). *Let $f \in S_{\leq d_1}, f = F_{d_1} + \cdots + F_0$, and $r = S^*/\operatorname{Ann}(f)$. Assume that F_{d_1} is not a power of a linear form. Then we have the following:*

(i) The cactus rank $\mathrm{cr}(f^{hom,d_2})$ of f^{hom,d_2} is not greater than r.

(ii) If $d_2 \geq d_1 - 1$, then the border cactus rank $\underline{\mathrm{cr}}(f^{hom,d_2})$ of f^{hom,d_2} equals r. In particular, $\mathrm{cr}(f^{hom,d_2}) = \underline{\mathrm{cr}}(f^{hom,d_2}) = r$.

Theorem 1.3.1.5 (Subspace case, [GMR20, Thm 1.9.]). *Let $W \subseteq S_{\leq d_1}$, and $r = \dim_{\Bbbk} S^*/\operatorname{Ann}(W)$. We have the following:*

(i) The cactus rank $\mathrm{cr}(W^{hom,d_2})$ *of* W^{hom,d_2} *is not greater than* r*.*

(ii) If $d_2 \geq d_1$*, then the border cactus rank* $\underline{\mathrm{cr}}(W^{hom,d_2})$ *of* W^{hom,d_2} *equals* r*. In particular,* $\mathrm{cr}(W^{hom,d_2}) = \underline{\mathrm{cr}}(W^{hom,d_2}) = r$*.*

Additionally, we show more or less the uniqueness of the border cactus decomposition (see Theorems 4.1.0.2, 4.1.0.3 for more precise statements).

As an example of application of Theorem 1.3.1.4, one can obtain that the cactus rank of the polynomial $x_1^2 x_2 x_0^2 + x_2 x_0^4 \in \mathbb{C}[x_0, x_1, x_2]$ is 6. For the calculations see Corollary 4.1.0.5 and its proof.

Chapter 2

Main tools, methods and first results

This chapter is devoted to introduce the language and tools needed in Chapters 3 and 4. However it also contains first minor research results from [BPR20], [GMR20] and unpublished ones. The reader can find here the explanation of the slice technique for the (border) rank and a counterexample for it in the case of the cactus rank. The chapter is divided on four parts: tensor rank (Sections 2.1, 2.2), border rank (Section 2.3), cactus rank together with Hilbert schemes (Sections 2.4 and 2.5) and characterization of the sets of cubics and subspaces with a certain Hilbert functions (Sections 2.6 and 2.7).

2.1 Ranks and slices

This section reviews the notions of rank, border rank, slices and conciseness. Readers that are familiar to these concepts may easily skip this section. The main things to remember from here are Notation 2.1.1.1 and Proposition 2.1.4.2.

Throughout this thesis let $A_1, A_2, \dots, A_d$, A, B, C, V and W be finite dimensional vector spaces over a field $\Bbbk$. By the bold lowercase letters $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_d, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{v}, \mathbf{w}$ we denote their dimensions. If P is a subset of V, we denote by $\langle P\rangle$ its linear span. We will use the same notation, i.e. $\langle P\rangle$ for a projective span if P is a subset of classes of points from a projective space $\mathbb{P}^N$. If $P = \{p_1, p_2, \dots, p_r\}$ is a finite subset, we will write $\langle p_1, p_2, \dots, p_r\rangle$ rather than $\langle\{p_1, p_2, \dots, p_r\}\rangle$ to simplify notation.

Definition 2.1.0.1. For a projective variety $X \subseteq \mathbb{P}^N$ and $\mathbb{P}^k \simeq \mathbb{P}(W)$ a projective linear subspace of $\mathbb{P}^N$, define $R_X(W)$ and $R_X(\mathbb{P}W)$, *the X-rank* of W and *the X-rank* of $\mathbb{P}(W)$, to be the minimal number r such that there exist r classes of points $\{[s_1], [s_2], \dots, [s_r]\} \subset X$ with $\mathbb{P}(W)$ contained in $\langle s_1, s_2, \dots, s_r\rangle$. For $[p] \in \mathbb{P}^N$, we write $R_X(p) := R_X(\langle p\rangle)$ obtaining the same definition as in Definition 1.3.0.3. We will drop X from the subscript, if the variety we work with is clear from the context.

Example 2.1.0.2. *In the particular case where $X = \nu_d(T_1)$, the Definition 2.1.0.1 can be stated as follows. For a k-dimensional linear subspace $W \subset T_d$,*

$$R_{\nu_d}(W) := \min\{r \in \mathbb{Z}_{>0} \mid \mathbb{P}W \subseteq \langle L_1^{[d]}, \dots, L_r^{[d]}\rangle \text{ for some } L_i \in T_1\}.$$

In the setting of Definition 2.1.0.1, if $X = \mathrm{Seg}_{A_1,A_2,\dots,A_d}$ and $W \subseteq A_1 \otimes A_2 \otimes \cdots \otimes A_d$ and $d = 1$, then $R(W) = R(\mathbb{P}W) = \dim W$. If $d = 2$ and $W = \langle p\rangle$ is 1-dimensional, then $R(W)$ is the rank of p viewed as a linear map $A_1^* \to A_2$. If $d = 3$ and $W = \langle p\rangle$ is 1-dimensional, then $R(W)$ is equal to $R(p)$ in the sense of Subsection 1.2.3.

More generally, for an arbitrary d, one can relate the rank $R(p)$ of d-way tensors with the rank $R(W)$ of certain linear subspaces in the space of $(d-1)$-way tensors. This relation is based on the *slice technique*, which we are going to review in Section 2.1.4.

2.1.1 Variety of simple tensors

We will intersect linear subspaces of the tensor space with the Segre variety. Using the language of algebraic geometry, such intersection may have a non-trivial scheme structure. In Chapters 2, 3 we just ignore the scheme structure and all our intersections are set theoretic. To avoid ambiguity of notation, we write $(\cdot)_{\mathrm{red}}$ to underline this issue, while the reader not originating from algebraic geometry should ignore the symbol $(\cdot)_{\mathrm{red}}$.

Notation 2.1.1.1. Given a linear subspace of a tensor space, $V \subseteq A_1 \otimes A_2 \otimes \cdots \otimes A_d$, we denote:

$$V_{\mathrm{Seg}} := (\mathbb{P}V \cap \mathrm{Seg}_{A_1,A_2,\ldots,A_d})_{\mathrm{red}}.$$

Thus, V_{Seg} is (up to projectivization) the set of rank one tensors in V.

In this setting, we have the following trivial rephrasing of the definition of rank:

Proposition 2.1.1.2 ([BPR20, Prop. 2.3.])**.** *Suppose $W \subseteq A_1 \otimes A_2 \otimes \cdots \otimes A_d$ is a linear subspace. Then $R(W)$ is equal to the minimal number r such that there exists a linear subspace $V \subseteq A_1 \otimes A_2 \otimes \cdots \otimes A_d$ of dimension r with $W \subseteq V$ and $\mathbb{P}V$ is linearly spanned by V_{Seg}. In particular,*

(i) $R(W) = \dim W$ if and only if

$$\mathbb{P}W = \langle W_{\mathrm{Seg}} \rangle.$$

(ii) Let U be the linear subspace such that $\mathbb{P}U := \langle W_{\mathrm{Seg}} \rangle$. Then $\dim U$ tensors from W can be used in the minimal decomposition of W, that is there exist $s_1, \ldots, s_{\dim U} \in W_{\mathrm{Seg}}$ such that $W \subset \langle s_1, \ldots, s_{R(W)} \rangle$ and s_i are simple tensors.

2.1.2 Secant varieties and border rank

For this subsection (and also in Section 3.2) we assume $\Bbbk = \mathbb{C}$. See Remark 2.1.2.2 for generalizations.

Analogously to Definition 1.3.0.8 we may give the border rank definition for linear subspaces. Fix $A_1, \ldots, A_d$ and an integer k. Denote by $\mathrm{Gr}(k, A_1 \otimes \cdots \otimes A_d)$ the Grassmannian of k-dimensional linear subspaces of the vector space $A_1 \otimes \cdots \otimes A_d$. Let $\sigma_{r,k}(\mathrm{Seg}) \subset \mathrm{Gr}(k, A_1 \otimes \cdots \otimes A_d)$ be the *Grassmann secant variety* of the Segre variety [BL13], [CC01], [CC08]:

$$\sigma_{r,k}(\mathrm{Seg}_{A_1,A_2,\ldots,A_d}) := \overline{\{W \in \mathrm{Gr}(k, A_1 \otimes \cdots \otimes A_d) \mid R(W) \leq r\}}.$$

Definition 2.1.2.1. For $W \subseteq A_1 \otimes A_2 \otimes \cdots \otimes A_d$, a linear subspace of dimension k, define $\underline{R}_{\mathrm{Seg}_{A_1,\ldots,A_d}}(W)$, *the border rank* of W, to be the minimal number r such that $W \in \sigma_{r,k}(\mathrm{Seg}_{A_1,\ldots,A_d})$. We will drop the subscript, if the variety we work with is known.

In particular, if $k = 1$, then Definition 2.1.2.1 coincides with Definition 1.3.0.8: $\underline{R}_{\mathrm{Seg}_{A_1,\dots,A_d}}(p) = \underline{R}_{\mathrm{Seg}_{A_1,\dots,A_d}}(\langle p\rangle)$. An important consequence of the definitions of border rank of a point or of a linear space is that it is a semicontinuous function

$$\underline{R}\colon \mathrm{Gr}(k, A_1 \otimes \cdots \otimes A_d) \to \mathbb{N}$$

for every k. Moreover, $\underline{R}(p) = 1$ if and only if $\langle p\rangle \in Seg$.

Remark 2.1.2.2. When treating the border rank and secant varieties in Section 3.2, we assume the base field is $\Bbbk = \mathbb{C}$. It is convenient to use this field, because in this case the Zariski closure is the same as the Euclidean closure. However, the result of [BJ17, Sect. 6, Prop. 6.11] imply (roughly) that anything that we can say about a secant variety over $\mathbb{C}$, we can also say about the same secant variety over any base field $\Bbbk$ of characteristic 0. In particular, the same results for border rank over an algebraically closed field $\Bbbk$ of characteristic 0 will be true.

To state it more precisely, we need some notation. For every natural number i let $A_{i,\mathbb{Q}}$ be a vector space over $\mathbb{Q}$, and $A_{i,\Bbbk} = A_{i,\mathbb{Q}} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\Bbbk$ be a corresponding vector space over the field extension $\mathbb{Q} \subset \Bbbk$. Furthermore, we denote by $X_{\mathbb{Q}} = \mathrm{Seg}_{A_{1,\mathbb{Q}},\dots,A_{n,\mathbb{Q}}} \subseteq \mathbb{P}_{\mathbb{Q}}(A_{1,\mathbb{Q}} \otimes \cdots \otimes A_{n,\mathbb{Q}})$ the embedded Segre variety and $X_{\Bbbk} = X_{\mathbb{Q}} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\Bbbk \subseteq \mathbb{P}_{\Bbbk}(A_{1,\Bbbk} \otimes \cdots \otimes A_{n,\Bbbk})$.

We have the following commutative diagram [BJ17, Sect. 6, Prop. 6.11] where the arrows pointing down are inclusions

$$\begin{array}{ccc} \sigma_r(X_{\Bbbk}) & = & \sigma_r(X_{\mathbb{Q}}) \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\Bbbk \\ \downarrow & & \downarrow \\ \mathbb{P}(\Bbbk^{N+1}) & = & \mathbb{P}(\mathbb{Q}^{N+1}) \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\Bbbk. \end{array}$$

Thus, we can translate results about secant varieties of $X_{\mathbb{C}}$ to a results for $\sigma_r(X_{\mathbb{Q}})$ and then to $\sigma_r(X_{\Bbbk})$ where $\Bbbk$ is any base field of characteristic 0.

If $\Bbbk$ is not algebraically closed, then the definition of border rank above might not generalise immediately, as there might be a difference between the closure in the Zariski topology or in some other topology, the latter being the Euclidean topology in the case $\Bbbk = \mathbb{R}$. Over real numbers, if we fix the degree d, then the set of polynomials of ranks equal r, i.e. $\mathring{\sigma}_r(\nu_d(\mathbb{P}(\mathbb{R}[x_0, x_1 \dots, x_n]_1)))$ (see Definition 1.2.2.2) is semialgebraic and its interior (with the Euclidean topology) can be non-empty for more than one value of r. For example $\mathring{\sigma}_r(\nu_3(\mathbb{P}(\mathbb{R}[x_0, x_1]_1)))$ has non-empty interior for r equal to 2 and 3 [Ble15, Thm 2.4].

In the following sections, up to Section 2.3 we restrict to the case of rank with respect to the Segre variety. In Subsection 2.4.1 one may find generalizations of definitions of this subsection, i.e. X-(border) rank and X-cactus (border) rank for arbitrary variety X.

2.1.3 Independence of the rank of the ambient space

As defined above, the notions of rank and border rank of a vector subspace $W \subseteq A_1 \otimes A_2 \otimes \cdots \otimes A_d$, or of a tensor $p \in A_1 \otimes A_2 \otimes \cdots \otimes A_d$, might seem to depend on the ambient spaces A_i. However, it is well known, that the rank is actually independent of the choice of the vector spaces. We first recall this result for tensors, then we apply the slice technique to show it in general.

Lemma 2.1.3.1 ([Lan12, Prop. 3.1.3.1] and [BL13, Cor. 2.2]). *Suppose $\Bbbk = \mathbb{C}$ and $p \in A'_1 \otimes A'_2 \otimes \cdots \otimes A'_d$ for some linear subspaces $A'_i \subset A_i$. Then $R(p)$ (respectively, $\underline{R}(p)$) measured as the rank (respectively, the border rank) in $A'_1 \otimes \cdots \otimes A'_d$ is equal to the rank (respectively, the border rank) measured in $A_1 \otimes \cdots \otimes A_d$.*

We also state a stronger fact about the rank from the same references: in the notation of Lemma 2.1.3.1, any minimal expression $W \subseteq \langle s_1, \ldots, s_{R(W)} \rangle$, for simple tensors s_i, must be contained in $A'_1 \otimes \cdots \otimes A'_d$. Here, we show that the difference in the length of the decompositions must be at least the difference of the respective dimensions. We stress that the lemma below does not depend on the base field, in particular, it does not require algebraic closedness.

Lemma 2.1.3.2 ([BPR20, Lem. 2.8]). *Suppose that $p \in A'_1 \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_d$, for a linear subspace $A'_1 \subset A_1$, and that we have an expression $p \in \langle s_1, \ldots, s_r \rangle$, where $s_i = a_i^1 \otimes a_i^2 \otimes \cdots \otimes a_i^d$ are simple tensors. Then:*

$$R(p) + \dim\langle a_1^1, \ldots, a_r^1 \rangle - \dim A' \leq r.$$

In particular, Lemma 2.1.3.2 implies the rank part of Lemma 2.1.3.1 for any base field $\Bbbk$, which on its own can also be seen by following the proof of [Lan12, Prop. 3.1.3.1] or [BL13, Cor. 2.2].

Proof. For simplicity of notation, we assume that $A'_1 \subset \langle a_1^1, \ldots, a_r^1 \rangle$ (by replacing A'_1 with a smaller subspace if needed) and that $A_1 = \langle a_1^1, \ldots, a_r^1 \rangle$ (by replacing A with a smaller subspace). Set $k = \dim A_1 - \dim A'_1$ and let us reorder the simple tensors s_i in such a way that the first k of the a_i^1's are linearly independent and $\langle A' \sqcup \{a_1^1, \ldots, a_k^1\} \rangle = A_1$.

Let $A''_1 = \langle a_1^1, \ldots, a_k^1 \rangle$ so that $A_1 = A'_1 \oplus A''_1$ and consider the quotient map $\pi \colon A_1 \to A_1/A''_1$. Then the composition $A'_1 \to A_1 \xrightarrow{\pi} A_1/A''_1 \simeq A'_1$ is an isomorphism, denoted by ϕ. By a minor abuse of notation, let π and ϕ also denote the induced maps $\pi \colon A_1 \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_d \to (A_1/A''_1) \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_d$ and $\phi \colon A'_1 \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_d \simeq A'_1 \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_d$. We have

$$\begin{aligned}\phi(p) = \pi(p) &\in \pi\left(\left\langle a_1^1 \otimes a_1^2 \otimes \cdots \otimes a_1^d, \ldots, a_r^1 \otimes a_r^2 \otimes \cdots \otimes a_r^d \right\rangle\right) \\ &= \left\langle \pi(a_1^1) \otimes a_1^2 \otimes \cdots \otimes a_1^d, \ldots, \pi(a_r^1) \otimes a_r^2 \otimes \cdots \otimes a_r^d \right\rangle \\ &= \left\langle \pi(a_{k+1}^1) \otimes a_{k+1}^2 \otimes \cdots \otimes a_{k+1}^d, \ldots, \pi(a_r^1) \otimes a_r^2 \otimes \cdots \otimes a_r^d \right\rangle.\end{aligned}$$

Using the inverse of the isomorphism ϕ, we get a presentation of p as a linear combination of $(r-k)$ simple tensors, that is $R(p) \leq r - k$ as claimed. ∎

2.1.4 Slice technique and conciseness

We define the notion of conciseness of tensors and we review a standard *slice technique* that replaces the calculation of rank of three way tensors with the calculation of rank of linear spaces of matrices.

A tensor $p \in A_1 \otimes A_2 \otimes \cdots \otimes A_d$ determines a linear map $p\colon A_1^* \to A_2 \otimes \cdots \otimes A_d$. If we choose a basis $\{a_1, a_2, \ldots, a_{\mathbf{a}}\}$ of A_1 we can write

$$p = \sum_{i=1}^{\mathbf{a}} a_i \otimes w_i,$$

where $w_1, \ldots, w_{\mathbf{a}} \in W := p({A_1}^*) \subset A_2 \otimes \cdots \otimes A_d$.

The elements $w_1, \ldots, w_{\mathbf{a}} \in W$ are called *slices* of p. The point is that W essentially uniquely (up to an action of $GL(A_1)$) determines p (cf. [BL13, Cor. 3.6]). Thus, the subspace W captures the geometric information about p, in particular its rank and border rank.

Lemma 2.1.4.1 ([BL13, Thm 2.5])**.** *Suppose $p \in A_1 \otimes A_2 \otimes \cdots \otimes A_d$ and $W = p(A_1^*)$ as above. Then $R(p) = R(W)$ and (if $\Bbbk = \mathbb{C}$) $\underline{R}(p) = \underline{R}(W)$.*

Clearly, we may also replace A_1 with any of the A_i to define slices as images $p(A_i^*)$ and obtain the analogue of the lemma. Now, we can prove the analogue of Lemmas 2.1.3.1 and 2.1.3.2 for higher dimensional subspaces of the tensor space.

Proposition 2.1.4.2 ([BPR20, Prop. 2.10.])**.** *Suppose $W \subset A_2' \otimes \cdots \otimes A_d'$ for some linear subspaces $A_2' \subset A_2, \ldots, A_d' \subset A_d$.*

(i) The numbers $R(W)$ and $\underline{R}(W)$ measured as the rank and border rank of W in $A_2' \otimes \cdots \otimes A_d'$ are equal to its rank and border rank calculated in $A_2 \otimes \cdots \otimes A_d$ (in the statement about border rank, we assume that $\Bbbk = \mathbb{C}$).

(ii) Moreover, if we have an expression $W \subset \langle s_1, \ldots, s_r \rangle$, where $s_i = a_i^2 \otimes a_i^3 \otimes \cdots \otimes a_i^d$ are simple tensors, then:

$$r \geq R(W) + \dim\langle a_1^2, \ldots, a_r^2 \rangle - \dim A_2'$$

Proof. Reduce to Lemmas 2.1.3.1 and 2.1.3.2 using Lemma 2.1.4.1. ∎

We conclude this section by recalling the following definition.

Definition 2.1.4.3. Let $p \in A_1 \otimes A_2 \otimes \cdots \otimes A_d$ be a tensor or let $W \subset A_1 \otimes A_2 \otimes \cdots \otimes A_d$ be a linear subspace. We say that p or W is A_1*-concise* if for all linear subspaces $V \subset A_1$, if $p \in V \otimes A_2 \otimes \cdots \otimes A_d$ (respectively, $W \subset V \otimes A_2 \otimes \cdots \otimes A_d$), then $V = A_1$. Analogously, we define A_i-concise tensors and spaces for $i = 2, \ldots, d$. We say p or W is *concise* if it is A_i-concise for all $i \in \{1, \ldots, n\}$.

Remark 2.1.4.4. Notice, that $p \in A_1 \otimes A_2 \otimes \cdots \otimes A_d$ is A_1-concise if and only if $p\colon A_1^* \to A_2 \otimes \cdots \otimes A_d$ is injective. In particular, from injectivity and Lemma 2.1.4.1 follows that rank of a A_1-concise tensor is greater or equal than the dimension of A_1.

2.2 Direct sum tensors and spaces of matrices

For simplicity of notation we restrict the presentation to the case of tensors in $A \otimes B \otimes C$ or linear subspaces of $B \otimes C$. We introduce the following notation which will be adopted throughout Sections 2.2, 2.3 and Chapter 3: Rank and border rank additivity problems.

Notation 2.2.0.1. Let $A', A'', B', B'', C', C''$ be vector spaces over $\Bbbk$ of dimensions, respectively, $\mathbf{a}', \mathbf{a}'', \mathbf{b}', \mathbf{b}'', \mathbf{c}', \mathbf{c}''$. Assume $A = A' \oplus A''$, $B = B' \oplus B''$, $C = C' \oplus C''$ and $\mathbf{a} = \dim A = \mathbf{a}' + \mathbf{a}''$, $\mathbf{b} = \dim B = \mathbf{b}' + \mathbf{b}''$ and $\mathbf{c} = \dim C = \mathbf{c}' + \mathbf{c}''$.

For the purpose of illustration, we will interpret the two-way tensors in $B \otimes C$ as matrices in $\mathcal{M}^{\mathbf{b} \times \mathbf{c}}$. This requires choosing bases of B and C, but (whenever possible) we will refrain from naming the bases explicitly. We will refer to an element of the space of matrices $\mathcal{M}^{\mathbf{b} \times \mathbf{c}} \simeq B \otimes C$ as a $(\mathbf{b}' + \mathbf{b}'', \mathbf{c}' + \mathbf{c}'')$ *partitioned matrix*. Every matrix $w \in \mathcal{M}^{\mathbf{b} \times \mathbf{c}}$ is a block matrix with four blocks of size $\mathbf{b}' \times \mathbf{c}'$, $\mathbf{b}' \times \mathbf{c}''$, $\mathbf{b}'' \times \mathbf{c}'$ and $\mathbf{b}'' \times \mathbf{c}''$ respectively.

Notation 2.2.0.2. As in Section 2.1.4, a tensor $p \in A \otimes B \otimes C$ is a linear map $p : A^* \to B \otimes C$; we denote by $W := p(A^*)$ the image of A^* in the space of matrices $B \otimes C$. Similarly, if $p = p' \oplus p'' \in (A' \oplus A'') \otimes (B' \oplus B'') \otimes (C' \oplus C'')$ is such that $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$, we set $W' := p'(A'^*) \subset B' \otimes C'$ and $W'' := p''(A''^*) \subset B'' \otimes C''$. In such situation, we will say that $p = p' \oplus p''$ is a *direct sum tensor*.

We have the following direct sum decomposition:

$$W = W' \oplus W'' \subset (B' \otimes C') \oplus (B'' \otimes C'')$$

and an induced matrix partition of type $(\mathbf{b}' + \mathbf{b}'', \mathbf{c}' + \mathbf{c}'')$ on every matrix $w \in W$ such that

$$w = \begin{pmatrix} w' & \underline{0} \\ \underline{0} & w'' \end{pmatrix},$$

where $w' \in W'$ and $w'' \in W''$, and the two $\underline{0}$'s denote zero matrices of size $\mathbf{b}' \times \mathbf{c}''$ and $\mathbf{b}'' \times \mathbf{c}'$ respectively.

Proposition 2.2.0.3 ([BPR20, Prop. 3.3.])**.** *Suppose that p, W, etc. are as in Notation 2.2.0.2. Then the additivity of the rank holds for p, that is $R(p) = R(p') + R(p'')$, if and only if the additivity of the rank holds for W, that is $R(W) = R(W') + R(W'')$.*

Proof. It is an immediate consequence of Lemma 2.1.4.1. ■

2.2.1 Projections and decompositions

The situation we consider here concerns the direct sums and their minimal decompositions. We fix $W' \subset B' \otimes C'$ and $W'' \subset B'' \otimes C''$ and we choose a minimal decomposition of $W' \oplus W''$, that is a linear subspace $V \subset B \otimes C$ such that $\dim V = R(W' \oplus W'')$, $\mathbb{P}V = \langle V_{\mathrm{Seg}} \rangle$ and $V \supset W' \oplus W''$. Such linear spaces W', W'' and V will be fixed for the rest of Sections 2.2 and 3.1.

In addition to Notations 2.1.1.1, 2.2.0.1 and 2.2.0.2 we need the following.

Notation 2.2.1.1. Under Notation 2.2.0.1, let $\pi_{C'}$ denote the projection

$$\pi_{C'} : C \to C'',$$

whose kernel is the space C'. With slight abuse of notation, we shall denote by $\pi_{C'}$ also the following projections

$$\pi_{C'} : B \otimes C \to B \otimes C'', \text{ or } \pi_{C'} : A \otimes B \otimes C \to A \otimes B \otimes C'',$$

with kernels, respectively, $B \otimes C'$ and $A \otimes B \otimes C'$. The target of the projection is regarded as a subspace of C, $B \otimes C$, or $A \otimes B \otimes C$, so that it is possible to compose such projections, for instance:

$$\pi_{C'}\pi_{B''} \colon B \otimes C \to B' \otimes C'', \text{ and } \pi_{C'}\pi_{B''} \colon A \otimes B \otimes C \to A \otimes B' \otimes C''.$$

We also let $E' \subset B'$ (resp. $E'' \subset B''$) be the minimal vector subspace such that $\pi_{C'}(V)$ (resp. $\pi_{C''}(V)$) is contained in $(E' \oplus B'') \otimes C''$ (resp. $(B' \oplus E'') \otimes C'$).

By swapping the roles of B and C, we define $F' \subset C'$ and $F'' \subset C''$ analogously. By the lowercase letters $\mathbf{e}', \mathbf{e}'', \mathbf{f}', \mathbf{f}''$ we denote the dimensions of the subspaces E', E'', F', F''.

If the differences $R(W') - \dim W'$ and $R(W'') - \dim W''$ (which we will informally call the *gaps*) are large, then the spaces E', E'', F', F'' could be large too, in particular they can coincide with B', B'', C', C'' respectively. In fact, these spaces measure "how far" a minimal decomposition V of a direct sum $W = W' \oplus W''$ is from being a direct sum of decompositions of W' and W''.

In particular, we will show in Proposition 3.1.1.5 and Corollary 3.1.3.9, that if $E'' = \{0\}$ or if both E'' and F'' are sufficiently small, then $R(W) = R(W') + R(W'')$. Then, as a consequence of Corollary 2.2.1.4, if one of the gaps is at most two (say, $R(W'') = \dim W'' + 2$), then the additivity of the rank holds, see Theorem 3.1.4.1.

Lemma 2.2.1.2 ([BPR20, Lem. 3.5])**.** *In Notation 2.2.1.1 as above, with $W = W' \oplus W'' \subset B \otimes C$, the following inequalities hold.*

$$R(W') + \mathbf{e}'' \le R(W) - \dim W'', \qquad R(W'') + \mathbf{e}' \le R(W) - \dim W',$$
$$R(W') + \mathbf{f}'' \le R(W) - \dim W'', \qquad R(W'') + \mathbf{f}' \le R(W) - \dim W'.$$

$$
\begin{array}{c}
\;\; \overbrace{\phantom{v_{1,1}\;\; v_{1,2}\;\; v_{1,3}}}^{C'} \;\; \overbrace{\phantom{v_{4,4}\;\; v_{4,5}\;\; v_{4,6}}}^{C''} \\
\begin{matrix} B' \left\{ \vphantom{\begin{matrix} a\\a\\a \end{matrix}} \right. \\ B'' \left\{ \vphantom{\begin{matrix} a\\a\\a \end{matrix}} \right. \end{matrix}
\begin{bmatrix}
v_{1,1} & v_{1,2} & v_{1,3} & & & \\
v_{2,1} & v_{2,2} & v_{2,3} & & & \\
v_{3,1} & v_{3,2} & v_{3,3} & v_{3,4} & & \\
 & & v_{4,3} & v_{4,4} & v_{4,5} & v_{4,6} \\
 & & v_{5,3} & v_{5,4} & v_{5,5} & v_{5,6} \\
 & & & v_{6,4} & v_{6,5} & v_{6,6}
\end{bmatrix}
\begin{matrix} \\ \\ \\ \left. \vphantom{\begin{matrix} a\\a \end{matrix}} \right\} E'' \\ \\ \end{matrix}
\end{array}
$$

Figure 2.1: A minimal decomposition of $W' \oplus W''$, that is a linear subspace $V \subset B \otimes C$ such that $\dim V = R(W' \oplus W'')$, $\mathbb{P}V = \langle V_{\mathrm{Seg}} \rangle$ and $V \supset W' \oplus W''$. We denote by $E'' \subset B''$ the minimal vector subspace such that $\pi_{C''}(V) \subset B \otimes C'$ is contained in $(B' \oplus E'') \otimes C'$. In the presented case $(\mathbf{b}', \mathbf{b}'', \mathbf{c}', \mathbf{c}'') = (3,3,3,3)$ (we use Notation 2.2.0.1).

Proof. We prove only the first inequality $R(W') + \mathbf{e}'' \leq R(W) - \dim W''$, the other follow in the same way by swapping B and C or $'$ and $''$. By Proposition 2.1.4.2(i) and (ii) we may assume W' is concise: $R(W')$ or $R(W)$ are not affected by choosing the minimal subspace of B' by (i), also the minimal decomposition V cannot involve anyone from outside of the minimal subspace by (ii).

Since V is spanned by rank one matrices and the projection $\pi_{C''}$ preserves the set of matrices of rank at most one, also the vector space $\pi_{C''}(V)$ is spanned by rank one matrices, say

$$\pi_{C''}(V) = \langle b_1 \otimes c_1, \ldots, b_r \otimes c_r \rangle$$

with $r = \dim \pi_{C''}(V)$. Moreover, $\pi_{C''}(V)$ contains W'. We claim that

$$B' \oplus E'' = \langle b_1, \ldots, b_r \rangle .$$

Indeed, the inclusion $B' \subset \langle b_1, \ldots, b_r \rangle$ follows from the conciseness of W', as $W' \subset V \cap B' \otimes C'$. Moreover, the inclusions $E'' \subset \langle b_1, \ldots, b_r \rangle$ and $B' \oplus E'' \supset \langle b_1, \ldots, b_r \rangle$ follow from the definition of E'', cf. Notation 2.2.1.1.

Thus, Proposition 2.1.4.2(ii) implies that

$$r = \dim \pi_{C''}(V) \geq R(W') + \underbrace{\dim \langle b_1, \ldots, b_r \rangle}_{\mathbf{b}' + \mathbf{e}''} - \mathbf{b}' = R(W') + \mathbf{e}''. \tag{2.2.1.3}$$

Since V contains W'' and $\pi_{C''}(W'') = \{0\}$, we have

$$r = \dim \pi_{C''}(V) \leq \dim V - \dim W'' = R(W) - \dim W''.$$

The claim follows from the above inequality together with (2.2.1.3). ∎

Rephrasing the inequalities of Lemma 2.2.1.2, we obtain the following.

Corollary 2.2.1.4 ([BPR20, Cor. 3.6.]). *If* $R(W) < R(W') + R(W'')$, *then*

$$\begin{aligned} \mathbf{e}' &< R(W') - \dim W', & \mathbf{f}' &< R(W') - \dim W', \\ \mathbf{e}'' &< R(W'') - \dim W'', & \mathbf{f}'' &< R(W'') - \dim W''. \end{aligned}$$

This immediately recovers a known case of additivity, when the gap is equal to 0 [Lan12, Prop. 10.3.3.3], that is if $R(W') = \dim W'$, then $R(W) = R(W') + R(W'')$ (because $\mathbf{e}' \geq 0$). Moreover, it implies that if one of the gaps is equal to 1 (say $R(W') = \dim W' + 1$), then either the additivity holds or both E' and F' are trivial vector spaces. In fact, the latter case is only possible if the former case holds too.

Lemma 2.2.1.5 ([BPR20, Lem. 3.7]). *With Notation 2.2.1.1, suppose* $E' = \{0\}$ *and* $F' = \{0\}$. *Then the additivity of the rank holds* $R(W) = R(W') + R(W'')$. *In particular, if* $R(W') \leq \dim W' + 1$, *then the additivity holds.*

Proof. Since $E' = \{0\}$ and $F' = \{0\}$, by the definition of E' and F' we must have the following inclusions:

$$\pi_{B''}(V) \subset B' \otimes C' \text{ and } \pi_{C''}(V) \subset B' \otimes C'.$$

Therefore $V \subset B' \otimes C' \oplus B'' \otimes C''$ and V is obtained from the union of decompositions of W' and W''.

The last statement follows from Corollary 2.2.1.4 ∎

Later in Proposition 3.1.1.5 we will show a stronger version of the above lemma, namely it is sufficient to assume that only one of E' or F' is zero. In Corollary 3.1.3.9 we prove a further generalization based on the results in the following subsection.

2.2.2 "Hook"-shaped spaces

It is known since [JT86], that the additivity of the tensor rank holds for tensors with one of the factors of dimension 2 (Theorem 1.2.5.6). Namely, using Notation 2.2.0.1 and 2.2.0.2, if $\mathbf{a}' \leq 2$ then $R(p' + p'') = R(p') + R(p'')$. The same claim is recalled in [LM17, Sect. 4] after Theorem 4.1. The brief comment says that if rank of p' can be calculated by the *substitution method*, then the additivity of the rank holds. Landsberg and Michałek implicitly suggest that if $\mathbf{a}' \leq 2$, then the rank of p' can be calculated by the substitution method, [LM17, Items (1)–(6) after Prop. 3.1]. This is indeed the case (at least over an algebraically closed base field $\Bbbk$), although rather demanding to verify, at least in the version of the algorithm presented in the cited article. In particular, to show that the substitution method can calculate the rank of $p' \in \Bbbk^2 \otimes B' \otimes C'$, one needs to use the normal forms of such tensors [Lan12, Sect. 10.3] and understand all the cases, and it is hard to agree that this method is so much simplier than the original approach of [JT86].

Instead, probably, the intention of the authors of [LM17] was slightly different, with a more direct application of [LM17, Prop. 3.1] (or Proposition 2.2.2.3 below). This has been carefully detailed and described in [Rup17, Prop. 3.2.12].Here we present a stronger statement about small "hook"-shaped spaces (Corollary 3.1.3.9). We stress that the argument for [Rup17, Prop. 3.2.12] and [BPR20, Prop. 3.17] requires the assumption of an algebraically closed base field $\Bbbk$, while the our proof of Corollary 3.1.3.9, as well as the original approach of [JT86] works over any base field.

Definition 2.2.2.1. For non-negative integers e, f, we say that a linear subspace $W \subset B \otimes C$ is (e, f)*-hook shaped*, if $W \subset \Bbbk^e \otimes C + B \otimes \Bbbk^f$ for some choices of linear subspaces $\Bbbk^e \subset B$ and $\Bbbk^f \subset C$.

The name "hook shaped" space comes from the fact that under an appropriate choice of basis, the only nonzero coordinates form a shape of a hook ⌜ situated in the upper left corner of the matrix, see Example 2.2.2.2. The integers (e, f) specify how wide the edges of the hook are. A similar name also appears in the context of Young diagrams, see for instance [BR87, Def. 2.3].

Example 2.2.2.2. *A* $(1, 2)$*-hook shaped subspace of* $\Bbbk^4 \otimes \Bbbk^4$ *has only the following possibly nonzero entries in some coordinates:*

$$\begin{bmatrix} * & * & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}.$$

The following observation is presented in [LM17, Prop. 3.1] and in [AFT11, Lem. B.1]. Here, we have phrased it in a coordinate free way.

Proposition 2.2.2.3 ([BPR20, Prop. 3.10])**.** *Let* $p \in A \otimes B \otimes C$, $R(p) = r > 0$, *and pick* $\alpha \in A^*$ *such that* $p(\alpha) \in B \otimes C$ *is nonzero. Consider two hyperplanes in* A*: the linear hyperplane* $\alpha^\perp = (\alpha = 0)$ *and the affine hyperplane* $(\alpha = 1)$*. For any* $a \in (\alpha = 1)$*, denote*

$$\tilde{p}_a := p - a \otimes p(\alpha) \in \alpha^\perp \otimes B \otimes C.$$

Then:

(i) there exists a choice of $a \in (\alpha = 1)$ *such that* $R(\tilde{p}_a) \leq r - 1$*,*

(ii) if in addition $R(p(\alpha)) = 1$*, then for any choice of* $a \in (\alpha = 1)$ *we have* $R(\tilde{p}_a) \geq r - 1$*.*

See [LM17, Prop. 3.1] for the proof (note the statement there is over the complex numbers only, but the proof is base field independent) or, alternatively, using Lemma 2.1.4.1 translate it into the following statement on linear spaces of tensors:

Proposition 2.2.2.4 ([BPR20, Prop. 3.11])**.** *Suppose $W \subset B \otimes C$ is a linear subspace, $R(W) = r$. Assume $w \in W$ is a nonzero element. Then:*

(i) there exists a choice of a complementary subspace $\widetilde{W} \subset W$ such that $\widetilde{W} \oplus \langle w \rangle = W$ and $R(\widetilde{W}) \leq r - 1$, and

(ii) if in addition $R(w) = 1$, then for any choice of the complementary subspace $\widetilde{W} \oplus \langle w \rangle = W$ we have $R(\widetilde{W}) \geq r - 1$.

Proposition 2.2.2.4 and the following Lemma 2.2.2.5 were crucial in the original proof that the additivity of the rank holds for vector spaces, one of which is $(1, 2)$-hook shaped (provided that the base field is algebraically closed). It is presented in [BPR20, Subsect. 3.2].

After introducing repletion and digestion with respect to a distinguished matrix (§3.1.2 and § 3.1.3), we present a stronger version of Proposition 2.2.2.4(i), i.e. Corollary 3.1.3.5. This approach also simplifies the original proof of additivity of the rank holds for vector spaces, one of which is $(1, 2)$-hook shaped and let us to prove its generalization to arbitrary fields, i.e. Corollary 3.1.3.9 (and Theorem 1.2.5.7(iv) from Chapter 1: Introduction).

The proof of the following lemma is a dimension count, see also [Rup17, Prop. 3.2.11].

Lemma 2.2.2.5 ([BPR20], Lemma 3.16)**.** *Suppose $\Bbbk$ is algebraically closed (of any characteristic) and $0 \neq p \in A \otimes B \otimes \Bbbk^2$ is concise and $\dim A \geq \dim B$. Then, there exists a rank one matrix in $p(A^*) \subset B \otimes \Bbbk^2$.*

Proof. Since $\dim A \geq \dim B$, the projectivization of the image $\mathbb{P}(p(A^*)) \subset \mathbb{P}(B \otimes \Bbbk^2)$ intersects the Segre variety $\mathbb{P}(B) \times \mathbb{P}(\Bbbk^2)$. Indeed, the corresponding dimensions are: $\dim(A) - 1$, $\dim(B) \cdot 2 - 1$, $\dim(B)$. Thus, $\dim(\mathbb{P}(p(A^*))) + \dim(Seg(\mathbb{P}(B) \times \mathbb{P}(\Bbbk^2))) - \dim(\mathbb{P}(B \otimes \Bbbk^2)) \geq 0$ and by [Har77, Thm I.7.2] the intersection is nonempty. Note, that here we use that the base field $\Bbbk$ is algebraically closed. ∎

Our proof of Lemma 2.2.2.5 does not work for non algebraically closed fields, since we rely on [Har77, Thm I.7.2]. In this thesis, we use Lemma 2.2.2.5 and the generalization of Proposition 2.2.2.4 (Corollary 3.1.3.5), to prove that rank additivity holds for a certain small dimensional spaces, see Corollary 3.1.4.13 (and Theorem 1.2.5.7(v)).

2.3 Border rank additivity tools

In Section 3.2 we focus on the border rank additivity. To prove it holds in the cases described there, we need tools, definitions and notations which we introduce in this subsection.

We commence with the observation that if the border rank additivity holds for tensors more degenerate than a given pair, then it has to hold for the original pair too. For the precise statement we will need a proper definition what degenerate tensor is.

Definition 2.3.0.1. Assume $p, q \in A \otimes B \otimes C$ are two tensors. We say that p is *more degenerate* than q if $p \in \overline{GL(A) \times GL(B) \times GL(C) \cdot q}$.

Example 2.3.0.2. *Any concise tensor in* $\mathbb{C}^1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ *is more degenerate than any concise tensor in* $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

Example 2.3.0.3. *Consider concise tensors in* $\mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}^2$. *According to [Lan12, Table 10.3.1], there are two orbits of the action of* $GL_3 \times GL_2 \times GL_2$ *of such tensors, both orbits of border rank* 3. *One orbit is "generic", the other is* more degenerate. *The latter is represented by:*

$$p = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_1 \otimes c_2 + a_3 \otimes b_2 \otimes c_1.$$

Lemma 2.3.0.4 ([BPR20, Lem. 5.6])**.** *Suppose* $p' \in A' \otimes B' \otimes C'$ *is an arbitrary tensor and* $p'', q'' \in A'' \otimes B'' \otimes C''$ *are such that* $\underline{R}(p'') = \underline{R}(q'')$ *and* p'' *is more degenerate than* q''. *If the additivity of the border rank holds for* $p' \oplus p''$ *then it also holds for* $p' \oplus q''$.

Proof. Since p'' is more degenerate than q'' also $p' \oplus p''$ is more degenerate than $p' \oplus q''$. Thus,

$$\underline{R}(p' \oplus q'') \geq \underline{R}(p' \oplus p'') = \underline{R}(p') + \underline{R}(p'') = \underline{R}(p') + \underline{R}(q'').$$

■

2.3.1 Strassen's equations of secant varieties

Often as a criterion to determine whether a tensor is or is not of a given border rank, we exploit defining equations of the corresponding secant varieties. We review here one type of equations, that is most important for the small cases we consider in the next chapter (see Section 3.2).

First assume $\mathbf{b} = \mathbf{c}$ and consider the space of square matrices $B \otimes C$. Let $f_{\mathbf{b}} : (B \otimes C)^{\times 3} \to B \otimes C$ be the map of matrices defined as follows:

$$f_{\mathbf{b}}(x, y, z) = x \operatorname{adj}(y) z - z \operatorname{adj}(y) x, \tag{2.3.1.1}$$

where $\operatorname{adj}(y)$ denotes the adjoint matrix of y.

As in Section 2.1.4 write

$$p = \sum_{i=1}^{\mathbf{a}} a_i \otimes w_i,$$

where $w_1, \ldots, w_{\mathbf{a}} \in W := p(A^*) \subset B \otimes C$ are $\mathbf{b} \times \mathbf{c}$ matrices and $\{a_1, \ldots, a_{\mathbf{a}}\}$ is a basis of A.

Proposition 2.3.1.2. *Assume that $p \in A \otimes B \otimes C$.*

(i) *[Str88] Suppose $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$. Then $\underline{R}(p) \leq 3$ if and only if $f_3(x, y, z) = \underline{0}$ for every $x, y, z \in W$.*

(ii) *[LM08] Suppose $\mathbf{a} = \mathbf{b} = \mathbf{c}$ and $\underline{R}(p) \leq \mathbf{a}$. Then $f_{\mathbf{a}}(x, y, z) = \underline{0}$, for every $x, y, z \in W$.*

See also [Fri13, Thm 3.2].

We also recall Ottaviani's derivation of Strassen's equations ([Ott07], see also [Lan12, Sect. 3.8.1]) for secant varieties of three factor Segre embeddings. Given a tensor $p : B^* \to A \otimes C$, consider the contraction operator

$$p_A^\wedge : A \otimes B^* \to \Lambda^2 A \otimes C,$$

obtained as composition of the map $\mathrm{Id}_A \otimes p : A \otimes B^* \to A^{\otimes 2} \otimes C$ with the natural projection $A^{\otimes 2} \otimes C \to \Lambda^2 A \otimes C$.

Proposition 2.3.1.3 ([Ott07, Thm 4.1])**.** *Assume $3 \leq \mathbf{a} \leq \mathbf{b}, \mathbf{c}$. If $\underline{R}(p) \leq r$, then* $\mathrm{rk}(p_A^\wedge) \leq r(\mathbf{a} - 1)$.

If $\mathbf{a} = 3$, we can slice p as follows (cf. Section 2.1.4): $p = \sum_{i=1}^{3} a_i \otimes w_i \in A \otimes B \otimes C$, with $w_i \in B \otimes C$. Then the matrix representation of $p_A^\wedge$ in block matrices is the following $(\mathbf{b} + \mathbf{b} + \mathbf{b}, \mathbf{c} + \mathbf{c} + \mathbf{c})$ partitioned matrix

$$M_3(w_1, w_2, w_3) := \begin{bmatrix} \underline{0} & w_3 & -w_2 \\ -w_3 & \underline{0} & w_1 \\ w_2 & -w_1 & \underline{0} \end{bmatrix}. \tag{2.3.1.4}$$

Proposition 2.3.1.5 ([Lan12, Prop. 7.6.4.4])**.** *If $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$, the degree nine equation*

$$\det(p_A^\wedge) = 0$$

defines the variety $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A \otimes B \otimes C)$.

If $\mathbf{a} = 4$ and $p = \sum_{i=1}^{4} a_i \otimes w_i \in A \otimes B \otimes C$, with $w_i \in B \otimes C$, then the matrix representation of $p_A^\wedge$ in block matrices is the following $(4 \cdot \mathbf{b}, 6 \cdot \mathbf{c})$ partitioned matrix

$$M_4(w_1, w_2, w_3, w_4) := \begin{bmatrix} \underline{0} & w_3 & -w_2 & w_4 & \underline{0} & \underline{0} \\ -w_3 & \underline{0} & w_1 & \underline{0} & -w_4 & \underline{0} \\ w_2 & -w_1 & \underline{0} & \underline{0} & \underline{0} & w_4 \\ \underline{0} & \underline{0} & \underline{0} & -w_1 & w_2 & -w_3 \end{bmatrix}. \tag{2.3.1.6}$$

2.4 Ranks and Apolarity Lemmas

In algebraic geometry we would like to translate geometric questions into algebraic problems. In the case of certain notions of rank, the main tool is the Apolarity Lemma in its many variants. It allows us to convert a question about a rank, into a question about an existence of some ideals. Firstly, it can be applied for establishing upper bounds for rank by constructing certain ideals. Secondly, it can provide lower bounds, by proving that ideals satisfying given properties, do not exist. Some of the many examples of applying Apolarity Lemmas in both directions are [Gał20, Thm 1.5], [BB15, Sect. 4], [GOV19] as well as Theorems 4.1.0.2, 4.1.0.3. For the applications of Border Apolarity Lemmas see [CHL19],[HMV20], and [Mań20].

In this section we focus on the ranks of polynomials (symmetric tensors). We review definitions of various types of ranks of tensors and the corresponding variants of Apolarity Lemma. In the previous sections of this chapter, we frequently used slice technique 2.1.4.1 to deal with tensor rank and border rank. One can wonder if slice technique works also for other notions of ranks of tensors. That is not the case. Multigraded Cactus Apolarity Lemma 2.4.2.4 let us prove that the analogue of the slice technique 2.1.4.1 does not apply to the cactus rank (Proposition 2.4.2.5). In Chapter 4, we provide the counterexample for the slice technique in the setting of border cactus rank (Proposition 4.3.1.7).

Subsections 2.4.1 and 2.4.2 introduce geometric objects which correspond to ranks of subspaces, namely Grassmann secant and Grassmann cactus varieties. The problem of decomposing many forms simultaneously as linear combinations of powers of the same set of linear forms originates from the work of Terracini [Ter15]. It was later studied by Bronowski [Bro33] and it is strictly connected to the notion of Grassmann secant variety.

We introduce the divided power rings denoted by $\Bbbk_{dp}[x_0, \ldots, x_n]$ (following Iarrobino-Kanev [IK99, Appendix A]), which allows to generalize Theorems 1.3.1.1, 1.3.1.3 and to state lemmas needed to prove it. We state definitions for any algebraically closed field $\Bbbk$ since this generality is needed in Theorems 1.3.1.4 and 1.3.1.5.

Fix a positive integer n and let $T^* := \Bbbk[\alpha_0, \ldots, \alpha_n] = \bigoplus_{0\leq j} T_j^*$ be a polynomial ring with the graded dual ring $T := \bigoplus_{0\leq j} T_j$ and let $x_i \in T_1$ be dual to α_i. For a multi-index $\mathbf{u}$, we denote by $\alpha^{\mathbf{u}} = \alpha_1^{u_1}\alpha_2^{u_2}\cdots\alpha_n^{u_n}$ the standard monomial basis of T_j^*, where $j = |\mathbf{u}| = u_1 + u_2 + \cdots + u_n$. Another notation is $x^{[\mathbf{u}]} := x_1^{[u_1]}x^{[u_2]}\cdots x_n^{[u_n]}$ for the basis of T_j dual to the basis $\{\alpha^{\mathbf{u}} : |\mathbf{u}| = j\}$. For every i, j and $\varphi \in T_i^*$, $f \in T_j$, $\psi \in T_{j-i}^*$, the contraction map $\lrcorner : T_i^* \times T_j \to T_{j-i}$ is defined as follows

$$(\varphi \lrcorner f)(\psi) := \begin{cases} 0 & \text{if } j < i \\ f(\varphi\psi) & \text{otherwise} \end{cases}.$$

We extend these maps to a contraction map $\lrcorner : T^* \times T \to T$ by linearity. On the bases we have

$$\alpha^{\mathbf{u}} \lrcorner x^{[\mathbf{v}]} := \begin{cases} x^{[\mathbf{v}-\mathbf{u}]} & \text{if } u_k \leq v_k \text{ for } k = 0, 1, \ldots, n \\ 0 & \text{in the other case} \end{cases}.$$

The multiplication of monomials is given by the equality $x^{[\mathbf{u}]}x^{[\mathbf{v}]} := \frac{(\mathbf{u}+\mathbf{v})!}{\mathbf{u}!\mathbf{v}!}x^{[\mathbf{u}+\mathbf{v}]}$, where $\mathbf{v}! := v_0!v_1!\cdots v_n!$. Extending by linearity, we obtain a structure of a $\Bbbk$-algebra on T.

Notice, that the usage of divided powers is necessary, since there is no assumption that $\Bbbk$ has characteristic zero. If $\operatorname{char}\Bbbk = 0$, divided powers have a simple form. Namely, $x^{[\mathbf{v}]} = \frac{x^{\mathbf{v}}}{\mathbf{v}!}$.

Another frequently used notation in this thesis is the following $S^* := \Bbbk[\alpha_1 \dots, \alpha_n] \subseteq T^*$ (we omit the variable α_0 from T^*). Then the graded dual ring $S := \Bbbk_{dp}[x_1, \dots, x_n]$ is naturally a subring of T.

Definition 2.4.0.1. Let k be a positive integer, $T^* := \Bbbk[\alpha_0, \alpha_1, \dots, \alpha_k]$ be a polynomial ring, and $T := \Bbbk_{dp}[x_0, x_1, \dots, x_k]$ be its graded dual. Given a finite dimensional linear subspace $V \subseteq T$ and a basis $(y_0, y_1, \dots, y_k)$ of T_1 (the homogeneous part of T of degree 1), we define $V|_{y_0=1}$ to be the dehomogenization of V with respect to the basis. The polynomial $f|_{y_0=1}$ is characterized analogously.

We denote by $\operatorname{Ann}(V)$ the ideal $\operatorname{Ann}(V) := \{\theta \in T^* \mid \theta \lrcorner V = 0\}$. The corresponding quotient ring $T^*/\operatorname{Ann}(V)$ will be defined as $\operatorname{Apolar}(V)$ and called the apolar algebra of V. If $V = \langle f \rangle$ we write $\operatorname{Ann}(f)$ instead of $\operatorname{Ann}(\langle f \rangle)$ and $\operatorname{Apolar}(f)$ instead of $\operatorname{Apolar}(\langle f \rangle)$.

Now we state the definitions of Hilbert function which will be used in the dissertation repeatedly.

Definition 2.4.0.2. Given a $\mathbb{Z}$-graded T^*-module M its *Hilbert function* is defined as

$$H(M, d) := \dim_{\Bbbk} M_d,$$

for all $d \in \mathbb{Z}$. For a finite local $\Bbbk$-algebra $(A, \mathfrak{m})$ the (local) *Hilbert function* of A is the Hilbert function of the associated graded ring $\operatorname{gr}_{\mathfrak{m}} A$ ([Eis95, Sect.5.1]), i.e.

$$H(A, d) := \dim_{\Bbbk} \mathfrak{m}^d/\mathfrak{m}^{d+1},$$

for all $d \in \mathbb{Z}_{\geq 0}$.

In the case of $\Bbbk = \mathbb{C}$, we will consider also $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$-graded rings. For $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$-graded rings, the polynomial ring $\operatorname{Sym}(A \oplus B \oplus C)^* := \mathbb{C}[\alpha_1, \alpha_2, \dots, \alpha_{\mathbf{a}}, \beta_1, \beta_2, \dots, \beta_{\mathbf{b}}, \gamma_1, \gamma_2, \dots, \gamma_{\mathbf{c}}]$ and its graded dual $\operatorname{Sym}(A \oplus B \oplus C) := \mathbb{C}_{dp}[x_1, x_2, \dots, x_{\mathbf{a}}, y_1, y_2, \dots, y_{\mathbf{b}}, z_1, z_2, \dots, z_{\mathbf{c}}]$ will be used. The grading is defined in a way that α_i has a degree $(1, 0, 0)$, β_i has a degree $(0, 1, 0)$ and γ_i has a degree $(0, 0, 1)$ for all i. We define the *Hilbert function* of a $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$-graded $\operatorname{Sym}(A \oplus B \oplus C)^*$-module N as

$$H(N, (d_1, d_2, d_3)) := \dim_{\mathbb{C}} N_{(d_1, d_2, d_3)},$$

for all $(d_1, d_2, d_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

In the case of $\mathbb{Z} \times \mathbb{Z}$-graded rings, we use analogous definitions of $\operatorname{Sym}(A \oplus B), \operatorname{Sym}(A \oplus B)^*$, degrees of α_i, β_i and Hilbert function $H(N, (d_1, d_2))$ of a $\mathbb{Z} \times \mathbb{Z}$-graded $\operatorname{Sym}(A \oplus B)^*$-module N.

We introduce the following notation of the functions $h_{s,n}$, $h_{s,(n_1,n_2)}$, $h_{s,(n_1,n_2,n_3)}$ and $(s, n+1)$-standard Hilbert function. The first three will be used in order to state Border Apolarity Lemma (Proposition 2.4.1.5) and for tensors (Proposition 2.4.1.7), while the fourth one is necessary for Weak Border Cactus Apolarity Lemma (Proposition 2.4.2.3).

Definition 2.4.0.3. For positive integers s, n, n_1, n_2, n_3, let $h_{s,n}\colon \mathbb{Z} \to \mathbb{Z}$ be given by

$$h_{s,n}(a) := \min\{\binom{a+n-1}{n-1}, s\}.$$

Functions $h_{s,(n_1,n_2,n_3)}\colon \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ and $h_{s,(n_1,n_2,n_3)}\colon \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ are defined analogously. In the case of $h_{s,(n_1,n_2,n_3)}\colon \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ we assign

$$h_{s,(n_1,n_2,n_3)}(a_1,a_2,a_3) := \min\{\binom{a_1+n_1-1}{n_1-1}\binom{a_2+n_2-1}{n_2-1}\binom{a_3+n_3-1}{n_3-1}, s\}.$$

A function $h\colon \mathbb{Z} \to \mathbb{Z}$ satisfying the following conditions will be called an $(s, n+1)$-*standard Hilbert function*:

(i) $h(d) \le h(d+1)$ for all d,
(ii) if $d \ge 0$ and $h(d) = h(d+1)$, then $h(e) = s$ for all $e \ge d$,
(iii) $0 \le h(d) \le h_{s,n+1}(d)$ for all d.

2.4.1 Rank and border rank

For any algebraically closed field and $X \subseteq \mathbb{P}^N$, we can give the definition of the X-border rank for a linear subspace of $\mathbb{C}^{N+1}$. It is a generalization of Definitions 1.3.0.3 and 2.1.2.1.

Definition 2.4.1.1. For a k-dimensional linear subspace V of $\mathbb{C}^{N+1}$, the X-rank of V is

$$R_X(V) := \min\{r \in \mathbb{Z}_{>0} \text{ such that } \mathbb{P}V \text{ is contained in projective span } \langle s_1, s_2, \dots, s_n\rangle, \text{ where } s_i \in X\}.$$

The (r,k)-th Grassmann secant variety of X is

$$\sigma_{r,k}(X) := \overline{\{[V] \in \mathrm{Gr}(k, \mathbb{C}^{N+1}) \mid R_X(V) \le r\}}.$$

The X-border rank of V is

$$\underline{R}_X(V) := \min\{r \in \mathbb{Z}_{>0} \mid [V] \in \sigma_{r,k}(X)\}.$$

We drop the subscript X if the variety we work with is clear from the context. From now on, in Sections 2.4, 2.5 we mainly focus on polynomials and symmetric rank, i.e. $\nu_d(\mathbb{P}T_1)$-rank.

If $k = 1$, namely if $V = \langle F \rangle$ for an element $F \in T_d$, we obtain the classical notions of rank and border rank of F and the secant variety $\sigma_r(\nu_d(\mathbb{P}T_1))$ (cf. Definitions 1.2.2.1, 1.3.0.3, 1.3.0.8).

In order to state Border Apolarity Lemma, we have to introduce multigraded Hilbert schemes. We will also use the following notation.

Notation 2.4.1.2. Let $\mathrm{Sym}(A \oplus B \oplus C)^*$ denotes a multigraded ring $\mathbb{C}[\alpha_1, \alpha_2, ..., \alpha_{\mathbf{a}}, \beta_1, \beta_2, ..., \beta_{\mathbf{b}}, \gamma_1, \gamma_2, ..., \gamma_{\mathbf{c}}]$ with $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$-grading such that α_i has a degree $(1, 0, 0)$, β_i has a degree $(0, 1, 0)$ and γ_i has a degree $(0, 0, 1)$ for all i. We define $\mathrm{Sym}(A \oplus B)^*$ analogously.

In our case we restrict to $\mathrm{Hilb}^{h_{r,n+1}}_{T^*}$, $\mathrm{Hilb}^{h_{r,(\mathbf{a},\mathbf{b})}}_{\mathrm{Sym}(A\oplus B)^*}$ and $\mathrm{Hilb}^{h_{r,(\mathbf{a},\mathbf{b},\mathbf{c})}}_{\mathrm{Sym}(A\oplus B\oplus C)^*}$, i.e. the multigraded Hilbert scheme associated to the polynomial rings T^* (with the standard $\mathbb{Z}$-grading), $\mathrm{Sym}(A \oplus B)^*$ and $\mathrm{Sym}(A \oplus B \oplus C)^*$, correspondingly. The functions $h_{r,n+1}$,$h_{r,(\mathbf{a},\mathbf{b})}$,$h_{r,(\mathbf{a},\mathbf{b},\mathbf{c})}$ are as in Definition 2.4.0.3. The aforementioned multigraded Hilbert schemes parameterize homogeneous ideals with fixed Hilbert functions $h_{r,n+1}$, $h_{r,(\mathbf{a},\mathbf{b})}$ and $h_{r,(\mathbf{a},\mathbf{b},\mathbf{c})}$ correspondingly. For more about the topic see [HS04].

We also need to define $\mathrm{Slip}_{r,\mathbb{P}T_1} \subseteq \mathrm{Hilb}^{h_{r,n+1}}_{T^*}$, but first let us recall the definition of the saturation of ideal.

Definition 2.4.1.3. Let I, J be ideals of a ring R. We define the ideal quotient

$$(I : J) := \{f \in R \mid f \cdot J \subseteq I\} \subseteq R.$$

We call the following ideal $\bigcup_{d=1}^{\infty}(I : J^d) \subseteq R$ the *J-saturation* of I or the saturation of I with respect to J. The ideal is *J-saturated* if is equal to its J-saturation. If J is an irrelevant ideal (for example $(\alpha_1, \alpha_2, \dots, \alpha_{\mathbf{a}})(\beta_1, \beta_2, \dots, \beta_{\mathbf{b}})(\gamma_1, \gamma_2, \dots, \gamma_{\mathbf{c}}) \subseteq \mathrm{Sym}(A \oplus B \oplus C)^*$ or $(\alpha_0, \alpha_1, \dots, \alpha_n) \subseteq T^*$), we omit J and just say *saturation* of I and *saturated* ideal.

Definition 2.4.1.4. Let $\mathrm{Slip}_{r,\mathbb{P}T_1}$ be the closure in $\mathrm{Hilb}^{h_{r,n+1}}_{T^*}$ of points corresponding to saturated ideals of r points. We define $\mathrm{Slip}_{r,\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C}$ as a closure in $\mathrm{Hilb}^{h_{r,(\mathbf{a},\mathbf{b},\mathbf{c})}}_{\mathrm{Sym}(A\oplus B\oplus C)^*}$ of points corresponding to M-saturated radical ideals, where $M = (\alpha_1, \alpha_2, \dots, \alpha_{\mathbf{a}})(\beta_1, \beta_2, \dots, \beta_{\mathbf{b}})(\gamma_1, \gamma_2, \dots, \gamma_{\mathbf{c}}) \subseteq \mathrm{Sym}(A \oplus B \oplus C)^*$.

Now we can state the Border Apolarity Lemma.

Proposition 2.4.1.5 (Border Apolarity Lemma)**.** *Let $V \subseteq T_d$ be a k-dimensional subspace. Then $\underline{R}(V) \leq r$ if and only if there exists an ideal $[I] \in \mathrm{Slip}_{r,\mathbb{P}T_1}$ such that*

$$I \subseteq \mathrm{Ann}(V).$$

Proposition 2.4.1.5 follows from the proof of [BB20, Thm 1.3] if we set $\mathcal{H} = \mathcal{H}ilb^{sm}_r(\mathbb{P}^n)$, i.e. the smoothable component of the Hilbert scheme of r points on $\mathbb{P}^n$. We rewrite it here.

Proof. We know that

$$\underline{R}(V) \leq r \iff [V] \in \sigma_{r,k}(\nu_d(\mathbb{P}^n)) \iff \exists_{[I]\in \mathrm{Slip}_{r,\mathbb{P}T_1}} I_d \subseteq V^{\perp}$$

where $V^{\perp}$ is the subspace of T_d^* of forms annihilating V. The latter equivalence follows from [BB20, Prop. 6.1]. We need to prove that

$$\exists_{[I]\in \mathrm{Slip}_{r,\mathbb{P}T_1}} I_d \subseteq V^{\perp} \iff \exists_{[I]\in \mathrm{Slip}_{r,\mathbb{P}T_1}} I \subseteq \mathrm{Ann}(V)$$

One implication is clear. We show the implication from the left to the right. Let $\phi \in I_e$ for $e \in \mathbb{Z}$, then $T_{d-e}^* \cdot \phi \in I_d \subset V^{\perp} \subset \mathrm{Ann}(V)$. Thus, $(T_{d-e}^* \cdot \phi) \lrcorner V = 0$ which implies $T_{d-e}^* \lrcorner (\phi \lrcorner V) = 0$. We obtain $\phi \lrcorner V = 0$. ■

There is also a version of the Border Apolarity Lemma, which may be applied to investigate a border rank of not necessarily symmetric tensors. We state it for tensors $p \in A \otimes B \otimes C$, while the analogous proposition is true also for $p \in A \otimes B$. We will use it only in these two cases. For a more general statement and the proof see [BB19, Thm 3.15].

In order to state Multigraded Border Apolarity Lemma, we have to introduce the following notation.

Notation 2.4.1.6. Let $\{x_1, x_2, \ldots, x_{\mathbf{a}}\}$, $\{y_1, y_2, \ldots, y_{\mathbf{b}}\}$, $\{z_1, z_2, \ldots, z_{\mathbf{c}}\}$ be bases of $\mathbb{C}$-vector spaces A, B, C correspondingly. We can look at a tensor $p \in A \otimes B \otimes C$ as a homogeneous polynomial of multi-degree $(1,1,1)$ in $\mathrm{Sym}(A \oplus B \oplus C) := \mathbb{C}_{dp}[x_1, x_2, \ldots, x_{\mathbf{a}}, y_1, y_2, \ldots, y_{\mathbf{b}}, z_1, z_2, \ldots, z_{\mathbf{c}}]$, where $\{x_i\}_i$,$\{y_j\}_j$,$\{z_k\}_k$ are of degrees $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ correspondingly.

Proposition 2.4.1.7 (Multigraded Border Apolarity Lemma, [BB19, Thm 3.15])**.** *Let us use Notations 2.4.1.2, 2.4.1.6. The border rank $\underline{R}(p) \leq r$ if and only if there exists an ideal $[I] \in \mathrm{Slip}_{r,\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C}$ such that*

$$I \subseteq \mathrm{Ann}(p),$$

where $\mathrm{Ann}(p) = \{\theta \in \mathrm{Sym}(A \oplus B \oplus C)^* \mid \theta \lrcorner p = 0\}$.

2.4.2 Cactus rank and border cactus rank

Given a subscheme $R \subseteq \mathbb{P}^N$ we denote by $\langle R \rangle$ the projective linear span in $\mathbb{P}^N$ of R, i.e. the smallest projective linear subspace of $\mathbb{P}^N$ containing R.

Definition 2.4.2.1. For $X \subseteq \mathbb{P}^N$ and a k-dimensional linear subspace V of $\mathbb{C}^{N+1}$, the X-cactus rank of V is

$$\mathrm{cr}_X(V) := \min\{r \in \mathbb{Z}_{>0} \mid \mathbb{P}V \subseteq \langle R \rangle \text{ for a zero-dimensional subscheme } R \subseteq X \text{ of length } r\}.$$

To state the following definitions we need the assumption of algebraically closed base field $\Bbbk$. The (r,k)-th Grassmann cactus variety of the d-th Veronese variety of X is

$$\kappa_{r,k}(X) := \overline{\{[W] \in \mathrm{Gr}(k, T_d) \mid \mathrm{cr}_X(W) \le r\}}.$$

The X-border cactus rank of V is

$$\underline{\mathrm{cr}}_X(V) := \min\{r \in \mathbb{Z}_{>0} \mid [V] \in \kappa_{r,k}(X)\}.$$

We will drop the subscript X if the variety we work with is clear from the context. For a point $(0,0,\ldots,0) \neq p \in \mathbb{C}^{N+1}$ let $W \subseteq \mathbb{C}^{N+1}$ be a linear span of p. We define the X-cactus rank and X-border cactus rank of p as a corresponding ranks of W. X-cactus rank and X-border cactus rank of $(0,0,\ldots,0) \in \mathbb{C}^{N+1}$ are set to be equal zero. We denote $\kappa_r(X) := \kappa_{r,1}(X)$.

We mainly focus on symmetric tensors, so we state next Propositions 2.4.2.2 and 2.4.2.3 for $X = \nu_d(\mathbb{P}T_1) \subseteq \mathbb{P}T_d$.

Proposition 2.4.2.2 (Cactus Apolarity Lemma)**.** *Let $V \subseteq T_d$ be a nonzero subspace and $I(R)$ be the saturated ideal of a subscheme $R \subseteq \mathbb{P}T_1$. Then*

$$I(R) \subseteq \mathrm{Ann}(V) \iff \mathbb{P}V \subseteq \langle \nu_d(R) \rangle.$$

Therefore, $\mathrm{cr}(V) \le r$ if and only if there exists a zero-dimensional subscheme $R \subseteq \mathbb{P}T_1$ of length r such that

$$I(R) \subseteq \mathrm{Ann}(V).$$

For a proof, see [Tei14, Thm 4.7]. Similar results are already stated in [BR13], [BB14] and [IK99].

In order to state a version of apolarity for border cactus rank, we need to consider all $(r, n+1)$-standard Hilbert functions (see Definition 2.4.0.3), instead of $h_{r,n+1}$ as in Border Apolarity Lemma 2.4.1.5.

Proposition 2.4.2.3 (Weak Border Cactus Apolarity Lemma, [BB20, Thm 1.1])**.** *Let $V \subseteq T_d$ be a nonzero subspace. If $\underline{\mathrm{cr}}(V) \le r$, then there exists a homogeneous ideal $I \subseteq \mathrm{Ann}(V) \subseteq T^*$ such that the Hilbert function of T^*/I is an $(r, n+1)$-standard Hilbert function.*

There is also a version of the Cactus Apolarity Lemma, which may be applied to investigate a cactus rank of not necessarily symmetric tensors. We state it for tensors $p \in A \otimes B \otimes C$, while the analogous proposition is true also for $p \in A \otimes B$. Multigraded Cactus Apolarity Lemma can be expressed also for toric varieties. However, there is a little connection with our dissertation, so we do not state it in this setting. For the statement in full generality and a proof see [Gał20, Thm 1.1, Rem. 4.8].

Proposition 2.4.2.4 (Multigraded Cactus Apolarity Lemma)**.** *Using Notations 2.4.1.2, 2.4.1.6 let* $M := (\alpha_1, \alpha_2, \ldots, \alpha_{\mathbf{a}})(\beta_1, \beta_2, \ldots, \beta_{\mathbf{b}})(\gamma_1, \gamma_2, \ldots, \gamma_{\mathbf{c}}) \subseteq \mathrm{Sym}(A \oplus B \oplus C)^*$ *and* $I(R) \subseteq \mathrm{Sym}(A \oplus B \oplus C)^*$ *be the* M*-saturated ideal of a subscheme* $R \subseteq \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$. *Then*

$$I(R) \subseteq \mathrm{Ann}(p) \iff [p] \in \langle \mathrm{Seg}(R) \rangle,$$

.

Therefore, $\mathrm{cr}(V) \leq r$ *for* $V \subseteq A \otimes B \otimes C$ *if and only if there exists an* M*-saturated ideal* $I \subseteq \mathrm{Sym}(A \oplus B \oplus C)^*$ *such that for sufficiently large* i, j, k *Hilbert function* $H(\mathrm{Sym}(A \oplus B \oplus C)^*/I, (i,j,k)) = r$ *and*

$$I \subseteq \mathrm{Ann}(V).$$

Here, by $\mathrm{Ann}(V)$ *we denote* $\{\theta \in \mathrm{Sym}(A \oplus B \oplus C)^* \mid \forall_{p \in V} \theta \lrcorner p = 0\}$.

Gesmundo, Oneto and Ventura in [GOV19, Ex. 2.23] show that the simultaneous cactus rank of a family of forms cannot be read as the cactus rank of tensor living in a bigger space. The example they provide is

$$p = x_1 \otimes y_1^2 y_2 + x_2 \otimes y_1^2 y_3 \in \mathbb{C}^2 \otimes \mathrm{Sym}^3(\mathbb{C}^3).$$

Then $4 \leq \mathrm{cr}(p)$, while $\mathrm{cr}(p((\mathbb{C}^2)^*)) \leq 3$.

With a help of Multigraded Cactus Apolarity Lemma 2.4.2.4 we give another counterexample to the analogue of the slice technique (Lemma 2.1.4.1) for the cactus rank (see Proposition 2.4.2.5), i.e.

$$x_1 \otimes (y_1 \otimes z_2 + y_2 \otimes z_1) + x_2 \otimes (y_1 \otimes z_3 + y_3 \otimes z_1) \in \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3.$$

Because of the symmetries inside the brackets, the natural step further is to check, if

$$x_1 \otimes y_1 y_2 + x_2 \otimes y_1 y_3 \in \mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^3)$$

is a counterexample too. Indeed, in Proposition 2.4.2.7 we prove it is. Notice, that the tensor we provide is simpler and is contained in a smaller dimensional space, than the one given by Gesmundo, Oneto and Ventura.

Proposition 2.4.2.5. *Let* $p = x_1 \otimes (y_1 \otimes z_2 + y_2 \otimes z_1) + x_2 \otimes (y_1 \otimes z_3 + y_3 \otimes z_1) \in \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. *Then* $4 \leq \mathrm{cr}(p)$, *while* $\mathrm{cr}(p(A^*)) \leq 3$, *where* A *is the first factor of* $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

Proof. To prove $4 \leq \mathrm{cr}(p)$ we will apply Multigraded Cactus Apolarity Lemma 2.4.2.4. Assume the contrary holds, $\mathrm{cr}(p) \leq 3$ and there exists a zero-dimensional subscheme $R \subseteq \mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\mathbb{C}^3)$ of length 3 such that

$$I := I(R) \subseteq \mathrm{Ann}(p). \tag{2.4.2.6}$$

Hilbert function of $\mathrm{Ann}(p)$ at all of the multi-degrees

$$\mathcal{D} := \{(0,1,0),(0,0,1),(1,0,1),(1,1,0)\}$$

equals 3. Thus, Hilbert functions of I at the same multi-degrees $\mathcal{D}$ has to be at least 3. Even more, it has to be equal 3, because the saturated ideal of a subscheme of length 3 has a Hilbert function bounded by 3. Let us take the ideal J generated by all generators of $\mathrm{Ann}(p)$ of degrees $\mathcal{D}$. Its saturation J^{sat} equals $(\beta_2,\beta_3,\gamma_2,\gamma_3)$, thus $J^{sat} \not\subseteq \mathrm{Ann}(p)$ and $I \not\subseteq \mathrm{Ann}(p)$ as well, since $J^{sat} \subseteq I$. We obtained a contradiction with (2.4.2.6).

To prove $\mathrm{cr}(p(A^*)) \leq 3$, we also use Multigraded Cactus Apolarity Lemma 2.4.2.4. Let $M = (\beta_1,\beta_2,\beta_3)(\gamma_1,\gamma_2,\gamma_3)$. It is enough to construct the ideal $L \subseteq \mathrm{Ann}(p(A^*))$ such that it is M-saturated and its Hilbert function equals 3 in every multi-degree except $(0,0)$. To construct L, we analyze the ideal $\mathrm{Ann}(p(A^*))$. It has generators in multi-degrees $(0,2),(1,1),(2,0)$ only and the following Hilbert function in multi-degree (i,j)

i \ j	0	1	2	3	…
0	1	3	0	0	…
1	3	2	0	0	…
2	0	0	0	0	…
3	0	0	0	0	…
⋮	⋮	⋮	⋮	⋮	⋱

Let us define the ideal

$$N := (\beta_3\gamma_3, \beta_2\gamma_3, \beta_3\gamma_2, \beta_2\gamma_2, \beta_3\gamma_1 - \beta_1\gamma_3, \beta_2\gamma_1 - \beta_1\gamma_2).$$

It is generated by all generators of $\mathrm{Ann}(p(A^*))$ of multi-degree $(1,1)$ except $\beta_1\gamma_1$. One can check that after taking M-saturation, we obtain the desired ideal

$$L := (\gamma_3^2, \gamma_2\gamma_3, \beta_3\gamma_3, \beta_2\gamma_3, \gamma_2^2, \beta_3\gamma_2, \beta_2\gamma_2, \beta_3\gamma_1 - \beta_1\gamma_3, \beta_2\gamma_1 - \beta_1\gamma_2, \beta_3^2, \beta_2\beta_3, \beta_2^2).$$

■

In a similar way, we prove the next proposition.

Proposition 2.4.2.7. *Let* $p = x_1 \otimes y_1y_2 + x_2 \otimes y_1y_3 \in \mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^3)$. *Then* $4 \leq \mathrm{cr}(p)$, *while* $\mathrm{cr}(p(A^*)) \leq 3$, *where* A *is the first factor of* $\mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^3)$.

Proof. To prove both inequalities we will apply Multigraded Cactus Apolarity Lemma 2.4.2.4. Assume the contrary to the first inequality holds, i.e. $\mathrm{cr}(p) \leq 3$. It follows from Multigraded Cactus Apolarity Lemma that there exists a zero-dimensional subscheme $R \subseteq \mathbb{P}(\mathbb{C}^2) \otimes \mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^3))$ of length 3 such that

$$I := I(R) \subseteq \mathrm{Ann}(p). \tag{2.4.2.8}$$

The Hilbert function of $\operatorname{Ann}(p)$ at all of the multi-degrees

$$\mathcal{D} := \{(0,1),(1,1)\}$$

equals 3. Thus, the Hilbert function of I at the same multi-degrees $\mathcal{D}$ has to be at least 3. Even more, it has to be equal 3, because the saturated ideal of a subscheme of length 3 has a Hilbert function bounded by 3. Let us take the ideal J generated by all generators of $\operatorname{Ann}(p)$ of multi-degrees $\mathcal{D}$. Its saturation J^{sat} equals (β_2, β_3), thus $J^{sat} \not\subseteq \operatorname{Ann}(p)$ and $I \not\subseteq \operatorname{Ann}(p)$ as well, since $J^{sat} \subseteq I$. We obtained a contradiction with (2.4.2.8).

To prove $\operatorname{cr}(p(A^*)) \leq 3$, we also use Multigraded Cactus Apolarity Lemma. It is enough to construct the ideal $L \subseteq \operatorname{Ann}(p(A^*))$ such that it is $(\beta_1, \beta_2, \beta_3)$-saturated and its Hilbert function equals 3 in every degree except 0. To construct L, we analyze the ideal $\operatorname{Ann}(p(A^*))$. It has generators in degree 2 only and the Hilbert function $1, 3, 2, 0, 0, 0, \ldots$.

Let us define the ideal

$$N := (\beta_2^2, \beta_2\beta_3, \beta_3^2).$$

It is generated by all generators of $\operatorname{Ann}(p(A^*))$ except β_1^2. It is $(\beta_1, \beta_2, \beta_3)$-saturated. One can check that we obtained the desired ideal

$$L := N.$$

■

Remark 2.4.2.9. A reader familiar with Segre-Veronese varieties can notice that there exist a shorter proof of Propositions 2.4.2.5 and 2.4.2.7. Indeed, let

$$\mathrm{SV}_{d_1,d_2,\ldots,d_n} : \mathbb{P}V_1^* \times \mathbb{P}V_2^* \ldots \mathbb{P}V_n^* \to \mathbb{P}(\mathrm{Sym}^{d_1}(V_1^*) \otimes \mathrm{Sym}^{d_1}(V_1^*) \otimes \ldots \otimes \mathrm{Sym}^{d_1}(V_1^*))$$

be a Segre-Veronese embedding given by

$$([e_1],[e_2],\ldots,[e_n]) \mapsto [e_1^{d_n} \otimes e_2^{d_n} \otimes \ldots \otimes e_n^{d_n}].$$

One can prove that for the natural inclusions

$$i : \mathrm{Sym}^2(\mathbb{C}^3) \to \mathbb{C}^3 \otimes \mathbb{C}^3,\ i(e_1e_2) := e_1 \otimes e_2 + e_2 \otimes e_1,$$

$$\tilde{i} : \mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^3) \to \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3,\ \tilde{i} = id_{\mathbb{C}^2} \otimes i$$

and $p \in \mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^3)$, we have the following inequalities

$$\mathrm{cr}_{\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2)}(\tilde{i}(p)) \leq \mathrm{cr}_{\mathrm{SV}_{1,2}(\mathbb{P}^1 \times \mathbb{P}^2)}(p)$$

and

$$\mathrm{cr}_{\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)}(i(p(A^*))) \leq \mathrm{cr}_{\nu_2(\mathbb{P}^2)}(p(A^*)).$$

Thus, to prove both Propositions 2.4.2.5 and 2.4.2.7 at once, it is enough to show that

$$\mathrm{cr}_{\nu_2(\mathbb{P}^2)}(\langle y_1y_2, y_1y_3\rangle) \leq 3$$

and

$$4 \leq \mathrm{cr}_{\mathrm{Seg}(\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2)}(x_1 \otimes (y_1 \otimes z_2 + y_2 \otimes z_1) + x_2 \otimes (y_1 \otimes z_3 + y_3 \otimes z_1)).$$

The tensor from 2.4.2.7 is the smallest possible counterexample among these contained in a space of the form $\mathbb{C}^n \times X$, where X is a smooth variety. One can prove that if X is smooth and the cactus rank of a pencil is 2, then the cactus rank of a tensor corresponding to the pencil is 2 as well. Indeed, let X be smooth variety, $p \in \mathbb{C}^n \times X$ and $R \subseteq X$ be a scheme of length 2 such that $W := p((\mathbb{C}^n)^*) \subseteq \langle R\rangle$. Any such R is isomorphic either to 2 distinct reduced points or a double point. Thus, W has a border rank at most 2. By slice technique for a border rank (Lemma 2.1.4.1), follows that $\underline{R}(W) = \underline{R}(p) \leq 2$. Hence, there exists a scheme $R' \subseteq \mathbb{C}^n \times X$ of length 2 such that $p \in \langle R'\rangle$. We obtained $\mathrm{cr}(P) \leq 2$.

If X is not smooth, we cannot argue in the same way, because there is no implication $\mathrm{cr}(W) \leq 2 \Rightarrow \underline{R}(W) \leq 2$ (see [Gał20, Beginning of Sec. 7.3]). However, we believe that it is the smallest possible counterexample also among these contained in a space of the form $\mathbb{C}^n \times X$, where X is an arbitrary variety.

2.5 A bound for the border rank of a particular subspace of polynomials of a fixed degree

In this section we present some algebraic results which will be needed in Chapter 4 and we prove the bound for the border rank of particular subspace of polynomials of a fixed degree. For a precise statement see Lemma 2.5.0.15. We will use the notation introduced at the beginning of the Section 2.4 and the following definition.

Definition 2.5.0.1. Let $d_1 \geq 1, d_2 \geq 0$ be integers. Given $f = F_{\deg f} + \ldots + F_0 \in S_{\leq d_1}$ where $F_i \in S_i$, define $f^{hom,d_2} \in T_{d_1+d_2}$ as

$$f^{hom,d_2} := \sum_{i=0}^{\deg f} F_i x_0^{[d_2+d_1-i]}.$$

For a linear subspace $W \subseteq S_{\leq d_1}$, define a linear subspace W^{hom,d_2} of $T_{d_1+d_2}$ as

$$W^{hom,d_2} := \{f^{hom,d_2} \mid f \in W\}.$$

Notice the difference between f^{hom,d_2} and the classical homogenization, which we use on a dual side (2.5.0.2). In our definition we use divided powers and the resulting polynomials are of a fixed degree $d_1 + d_2$.

Let a polynomial $\varphi = \sum_{i=0}^{\deg \phi} \Phi_i \in S^*$, where $\Phi_i \in S_i^*$. We denote by $\varphi^{hom} \subseteq T^*$ its homogenization with respect to α_0, i.e.

$$\varphi^{hom} := \sum_{i=0}^{\deg \varphi} \Phi_i \alpha_0^{\deg(\phi)-i}. \tag{2.5.0.2}$$

We will also use the notion of the homogenization of an ideal, we recall it after [CLO15, Sect. 8.4]. For an ideal $I \subseteq S^*$ its homogenization with respect to α_0 is denoted by $I^{hom} \subseteq T^*$, where

$$I^{hom} := \langle \varphi^{hom} \mid \varphi \in I \rangle.$$

The following lemma says that the homogenization in T^* of an ideal in S^* is saturated. This will enable us to use Cactus Apolarity Lemma 2.4.2.2 in the proofs of Theorems 1.3.1.4 and 1.3.1.5.

Lemma 2.5.0.3 ([GMR20, Lem. 3.1.]). *Let $I \subseteq S^*$ be an ideal. Then the homogenization $I^{hom} \subseteq T^*$ is saturated with respect to the irrelevant ideal $(\alpha_0, \dots, \alpha_n)$.*

Proof. It is enough to show that $(I^{hom} : \alpha_0) = I^{hom}$ (here $(I^{hom} : \alpha_0)$ denotes the ideal quotient, see Definition 2.4.1.3). Take $\theta \in (I^{hom} : \alpha_0)$. We shall show that $\theta \in I^{hom}$. Since $(I^{hom} : \alpha_0)$ is a homogeneous ideal, we may assume that θ is homogeneous. By the definition of ideal quotient, for some integer s, there are $\zeta_1, \dots, \zeta_s \in I$ and $\xi_1, \dots, \xi_s \in T^*$ such that

$$\theta\alpha_0 = \xi_1\zeta_1^{hom} + \dots + \xi_s\zeta_s^{hom}.$$

Hence $\theta_{|\alpha_0=1} \in I$, so $(\theta_{|\alpha_0=1})^{hom} \in I^{hom}$. Thus,

$$\theta = \alpha_0^{\deg\theta - \deg(\theta_{|\alpha_0=1})}(\theta_{|\alpha_0=1})^{hom} \in I^{hom},$$

as claimed. We used [CLO15, Prop. 8.2.7 (iii) and (iv)]. ∎

If $I \subseteq S^*$ is a homogeneous ideal, then there is a simple way to calculate the Hilbert function of T^*/I^{hom} from the Hilbert function of S^*/I, namely

$$H(T^*/I^{hom}, e) = \sum_{i=0}^{e} H(S^*/I, i) \text{ for } e \in \mathbb{Z}_{\geq 0}.$$

In particular, if the Hilbert polynomial of S^*/I is zero, then for large enough e we have $H(T^*/I^{hom}, e) = \dim_{\Bbbk} S^*/I$. Lemma 2.5.0.4 and Corollary 2.5.0.5 generalize this claim to inhomogeneous ideals.

In the following lemma we use the definitions of monomial orders $<$, leading terms $\mathrm{LT}_<$, leading monomials $\mathrm{LM}_<$, and Gröbner bases as in [CLO15, Ch. 2].

Lemma 2.5.0.4 ([GMR20, Lem. 3.2.]). *Let $I \subseteq S^*$ be an ideal. Let $<$ be any monomial order on S^* which respects the degree. Then for every non-negative integer e*

$$H(T^*/I^{hom}, e) = \#\{\mu \in S^* \mid \mu \text{ is a monomial, } \deg \mu \leq e \text{ and } \mu \notin \mathrm{LT}_<(I)\}.$$

Proof. Let $e \in \mathbb{Z}_{\geq 0}$ and consider sets

$$A_{\leq e} := \{\mu \in S^* \mid \mu \text{ is a monomial, } \deg \mu \leq e\}$$

and

$$B_e := \{\mu \in T^* \mid \mu \text{ is a monomial, } \deg \mu = e\}.$$

These sets are in bijection given by

$$A_{\leq e} \ni \mu \mapsto \alpha_0^{e-\deg \mu}\mu \in B_e$$

and

$$B_e \ni \mu \mapsto \mu|_{\alpha_0 = 1} \in A_{\leq e}.$$

Let $<_h$ be the monomial order on T^* defined by

$$\alpha_0^{a_0} \dots \alpha_n^{a_n} <_h \alpha_0^{b_0} \dots \alpha_n^{b_n} \Leftrightarrow \alpha_1^{a_1} \dots \alpha_n^{a_n} < \alpha_1^{b_1} \dots \alpha_n^{b_n} \text{ or}$$
$$\alpha_1^{a_1} \dots \alpha_n^{a_n} = \alpha_1^{b_1} \dots \alpha_n^{b_n} \text{ and } a_0 < b_0.$$

We have $H(T^*/I^{hom}, e) = \#\{\mu \in B_e \mid \mu \notin \mathrm{LT}_{<_h}(I^{hom})\}$ (see [Eis95, Thm 15.3]). Therefore, it is enough to show that for $\mu \in A_{\leq e}$ the following equivalence holds $\mu \in \mathrm{LT}_<(I)$ if and only if $\alpha_0^{e-\deg \mu}\mu \in \mathrm{LT}_{<_h}(I^{hom})$.

Assume that $\mu \in A_{\leq e} \cap \mathrm{LT}_<(I)$ and let $\theta \in I$ be such that $\mathrm{LM}_<(\theta) = \mu$. Then $\alpha_0^{e-\deg \mu}\theta^{hom} \in I^{hom}$ and $\mathrm{LM}_{<_h}(\alpha_0^{e-\deg \mu}\theta^{hom}) = \alpha_0^{e-\deg \mu}\mathrm{LM}_{<_h}(\theta^{hom}) = \alpha_0^{e-\deg \mu}\mu$. The latter equality follows from the following observation: $\mathrm{LM}_{<_h}(\theta^{hom}) = \mathrm{LM}_<(\theta)$ for $\theta \in S^*$.

Now suppose that $\alpha_0^{e-\deg \mu}\mu \in B_e \cap \mathrm{LT}_{<_h}(I^{hom})$. Let $G := \{\zeta_1, \dots, \zeta_k\}$ be a Gröbner basis for I with respect to $<$. Then $G^{hom} = \{\zeta_1^{hom}, \dots, \zeta_k^{hom}\}$ is a Gröbner basis for I^{hom} with respect to $<_h$ (see [CLO15, Thm 8.4.4]). Therefore, for some $j \in \{1, .., k\}$ the monomial $\mathrm{LM}_{<_h}(\zeta_j^{hom}) = \mathrm{LM}_<(\zeta_j)$ divides $\alpha_0^{e-\deg \mu}\mu$. Thus, $\mathrm{LM}_<(\zeta_j)$ divides μ, because $\zeta_j \in S^*$. ∎

The following corollary of Lemma 2.5.0.4 will be used extensively. It shows that the Hilbert polynomial of the subscheme defined by $\mathrm{Ann}(W)^{hom}$ is equal to $\dim_{\Bbbk} S^*/\mathrm{Ann}(W)$. Moreover, it provides an upper bound on the minimal degree from which the Hilbert function agrees with the Hilbert polynomial.

Corollary 2.5.0.5 ([GMR20, Cor. 3.3.]). *Let $W \subseteq S_{\leq d_1}$ be a linear subspace. Then for $e \geq d_1$*

$$H(T^*/\mathrm{Ann}(W)^{hom}, e) = \dim_{\Bbbk} S^*/\mathrm{Ann}(W).$$

Proof. All monomials of degree at least d_1+1 are in $\operatorname{Ann}(W)$. Therefore, for $e \geq d_1$ it follows from Lemma 2.5.0.4 that

$$H(T^*/\operatorname{Ann}(W)^{hom}, e) = H(T^*/\operatorname{Ann}(W)^{hom}, d_1)$$

is equal to the number of monomials in S^*, which do not belong to $\mathrm{LT}_<(\operatorname{Ann}(W))$. This number is the dimension of the quotient algebra $S^*/\operatorname{Ann}(W)$ as a $\Bbbk$-vector space ([Eis95, Thm 15.3]). ■

The following result is similar to Lemma 2.5.0.4. It compares the Hilbert functions of two related quotient algebras, one of S^* and one of T^*. We will use it in the proof of Part (iii) of Theorem 4.1.0.3.

Lemma 2.5.0.6 ([GMR20, Lem. 3.4.]). *Let $J \subseteq T^*$ be a homogeneous ideal and $\theta = \alpha_0^d + \rho$ be an element of J_d with ρ of degree smaller than d with respect to α_0. Consider the contraction $J^c = J \cap S^*$. Then for any integer e we have*

$$H(T^*/J, e) \leq H(S^*/J^c, e) + H(S^*/J^c, e-1) + \ldots + H(S^*/J^c, e-d+1).$$

Proof. Let $<$ be a graded lexicographic order on T^* with $\alpha_n < \alpha_{n-1} < \ldots < \alpha_0$ and consider its restriction $<'$ to S^*. It follows from [Eis95, Thm 15.3]) that $H(T^*/J, e)$ is the number of monomials of degree e, not in $\mathrm{LT}_<(J)$. Observe, that every monomial divisible by α_0^d is in $\mathrm{LT}_<(J)$. Therefore

$$H(T^*/J, e) = \sum_{i=0}^{d-1} \#\{\mu \in S^*_{e-i} \mid \mu \text{ is a monomial and } \alpha_0^i\mu \notin \mathrm{LT}_<(J)\}.$$

Fix $0 \leq i \leq d-1$ and let μ be a monomial of degree $e-i$ from S^*. If $\mu \in \mathrm{LT}_{<'}(J^c)$, then there is a homogeneous $\zeta \in J^c$ such that $\mathrm{LT}_{<'}(\zeta) = \mu$. Therefore, $\alpha_0^i\zeta \in J$ and $\mathrm{LT}_<(\alpha_0^i\zeta) = \alpha_0^i\mu$. Thus, for $i \in \{0, \ldots, d-1\}$

$$\#\{\mu \in S^*_{e-i} \mid \mu \text{ is a monomial and } \alpha_0^i\mu \notin \mathrm{LT}_<(J)\} \leq H(S^*/J^c, e-i).$$

■

In [BBKT15, Prop. 1.6] it was proven that the annihilator of a homogeneous degree d polynomial that is not a power of a linear form has a set of minimal generators of degrees at most d. The following lemma generalizes it to inhomogeneous polynomials.

Lemma 2.5.0.7 ([GMR20, Lem. 3.5.]). *Let $f = F_{d_1} + F_{d_1-1} + \ldots + F_0$ be a polynomial of degree $d_1 \geq 2$ in S where $F_i \in S_i$. Assume that F_{d_1} is not a power of a linear form. Then $\operatorname{Ann}(f)^{hom} \subseteq T^*$ has a set of minimal generators of degrees at most d_1.*

Proof. We have $\operatorname{Ann}(f) \supseteq S^*_{d_1+1}$, so we may choose a set of its generators of the form

$$\operatorname{Ann}(f) = (\{\alpha^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{Z}^n_{\geq 0} \text{ s.t. } |\mathbf{u}| = d_1 + 1\}) + (\zeta_1, \dots, \zeta_k) \text{ with } \deg(\zeta_i) \leq d_1.$$

Using Buchberger's algorithm for this set of generators and grevlex monomial order, we obtain a Gröbner basis of $\operatorname{Ann}(f)$ of the form

$$\{\alpha^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{Z}^n_{\geq 0} \text{ s.t. } |\mathbf{u}| = d_1 + 1\} \cup \{\zeta_1, \dots, \zeta_k\} \cup \{\xi_1, \dots, \xi_l\}. \tag{2.5.0.8}$$

We claim that $\deg \xi_i \leq d_1$. To see this, notice that in Buchberger's algorithm we remove from the S-polynomials multiples of monomials $\{\alpha^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{Z}^n_{\geq 0} \text{ s.t. } |\mathbf{u}| = d_1 + 1\}$.

Then $\operatorname{Ann}(f)^{hom}$ is generated by homogenizations of generators (2.5.0.8) [CLO15, Thm 8.4.4]. It is enough to show that we are able to replace the monomial generators of degree $d_1 + 1$ written above by some generators of degree not greater than d_1. Let $\mathbf{u} \in \mathbb{Z}^n_{\geq 0}$ with $|\mathbf{u}| = d_1 + 1$. Then in S^*, we can write $\alpha^{\mathbf{u}} = \sum_{i=1}^m \delta_i \gamma_i$ for some $\delta_i \in \operatorname{Ann}(F_{d_1})_{d_1}$ and $\gamma_i \in S^*_1$ [BBKT15, Prop. 1.6]. We have $\delta_i \in \operatorname{Ann}(f)$ for degree reasons. Therefore, $\alpha^{\mathbf{u}} \in ((\operatorname{Ann}(f)^{hom})_{\leq d_1})$ as an element of T^*. ■

For a homogeneous polynomial $F_{d_1} \in S_{d_1}$ of positive degree, $\operatorname{Ann}(F_{d_1} x_0^{[d_2]}) = (\alpha_0^{d_2+1}) + \operatorname{Ann}(F_{d_1})^{hom}$. In particular, $\operatorname{Ann}(F_{d_1} x_0^{[d_2]})_{\leq d_2} = (\operatorname{Ann}(F_{d_1})^{hom})_{\leq d_2}$. Lemma 2.5.0.9 generalizes it to an arbitrary polynomial. Part (i) was proven in [BR13, Lem. 2]. However, from the notation of the authors it is not clear that they use divided powers, but they are essential for the lemma to work (see Example 2.5.0.12). For this reason we present their proof with an explicit use of divided powers.

Recall the notation of f^{hom,d_2} from Definition 2.5.0.1.

Lemma 2.5.0.9 ([GMR20, Lem. 3.6.])**.** *Let $f = F_{d_1} + F_{d_1-1} + \ldots + F_0$ be a degree $d_1 \geq 1$ polynomial in S and $r = \dim_{\Bbbk} S^*/\operatorname{Ann}(f)$. Let d_2 be a non-negative integer. We have*

(i) $\operatorname{Ann}(f)^{hom} \subseteq \operatorname{Ann}(f^{hom,d_2})$.

(ii) $(\operatorname{Ann}(f)^{hom})_{\leq d_2} = \operatorname{Ann}(f^{hom,d_2})_{\leq d_2}$

(iii) If $d_2 = d_1 - 1$, then $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1)$ equals r or $r - 1$. Moreover, in the latter case $\operatorname{Ann}(f^{hom,d_2}) = (\alpha_0^{d_1} + \rho) + \operatorname{Ann}(f)^{hom}$, where $\rho \in T^*_{d_1}$ has degree smaller than d_1 with respect to α_0.

Proof. The proof of the lemma is based on the following calculation. Let $\Gamma =$

$\alpha_0^d\Theta_0 + \alpha_0^{d-1}\Theta_1 + \ldots + \Theta_d$, where $\Theta_i \in S_i^*$. We can rewrite $\Gamma \lrcorner f^{hom,d_2}$ as follows

$$\begin{aligned} \Gamma \lrcorner f^{hom,d_2} &= \sum_{e=0}^{d_1} \sum_{j=0}^{\min(d_1-e,d)} (\alpha_0^{d-j}\Theta_j) \lrcorner (x_0^{[d_1+d_2-(e+j)]} F_{e+j}) \\ &= \sum_{e=0}^{d_1} \sum_{j=0}^{\min(d_1-e,d)} (\alpha_0^{d-j} \lrcorner x_0^{[d_1+d_2-(e+j)]})(\Theta_j \lrcorner F_{e+j}) \\ &= \sum_{e=0}^{\min(d_1,d_1+d_2-d)} \sum_{j=0}^{\min(d_1-e,d)} x_0^{[d_1+d_2-d-e]}(\Theta_j \lrcorner F_{e+j}) \\ &= \sum_{e=0}^{\min(d_1,d_1+d_2-d)} x_0^{[d_1+d_2-d-e]} \sum_{j=0}^{\min(d_1-e,d)} \Theta_j \lrcorner F_{e+j}. \end{aligned} \tag{2.5.0.10}$$

(i) Let $\theta = \Theta_0 + \cdots + \Theta_d \in \operatorname{Ann}(f)$, where Θ_i is homogeneous of degree i. We show that $\theta^{hom} = \alpha_0^d\Theta_0 + \alpha_0^{d-1}\Theta_1 + \ldots + \Theta_d$ is in the annihilator of f^{hom,d_2}. We put $\Gamma = \theta^{hom}$ in Equation (2.5.0.10).
For every $e = 0, \ldots, \min(d_1, d_1 + d_2 - d)$ the sum $\sum_{j=0}^{\min(d_1-e,d)} \Theta_j \lrcorner F_{e+j}$ is zero since $\theta \lrcorner f = 0$. Hence $\Gamma \lrcorner f^{hom,d_2} = 0$, and the claim follows.

(ii) We have $\operatorname{Ann}(f)^{hom} \subseteq \operatorname{Ann}(f^{hom,d_2})$ by Part (i). Assume that $d \le d_2$ and let $\Gamma = \alpha_0^d\Theta_0 + \alpha_0^{d-1}\Theta_1 + \ldots + \Theta_d$, where $\Theta_i \in S_i^*$, be such that $\Gamma \lrcorner f^{hom,d_2} = 0$. We claim that $(\Gamma|_{\alpha_0=1}) \lrcorner f = 0$.
By Equation (2.5.0.10) we have

$$0 = \sum_{e=0}^{d_1} x_0^{[d_1+d_2-d-e]} \sum_{j=0}^{\min(d_1-e,d)} \Theta_j \lrcorner F_{e+j}.$$

Since the exponents at x_0 are pairwise different,

$$\sum_{j=0}^{\min(d_1-e,d)} \Theta_j \lrcorner F_{e+j} = 0 \text{ for every } d_1 \ge e \ge 0.$$

This implies that $(\Gamma|_{\alpha_0=1}) \lrcorner f = 0$.

(iii) We start with the following
Observation. Assume that $k \ge 0$, then for $\Gamma = \alpha_0^{d_1-1}\Theta_{1+k} + \alpha_0^{d_1-2}\Theta_{2+k} + \ldots + \Theta_{d_1+k}$. We have

$$\Gamma \lrcorner f^{hom,d_2} = 0 \Rightarrow \Gamma \in \operatorname{Ann}(f)^{hom}.$$

Indeed, Equation (2.5.0.10) with $d_2 = d_1 - 1, d = d_1 + k$ becomes

$$0 = \Gamma \lrcorner f^{hom,d_2} = \sum_{e=0}^{d_1-k-1} x_0^{[d_1-k-e-1]} \sum_{j=0}^{d_1-e} \Theta_j \lrcorner F_{e+j}.$$

Since the exponents at x_0 are pairwise different,

$$\sum_{j=0}^{d_1-e} \Theta_j \lrcorner F_{e+j} = 0 \text{ for every } d_1 - k - 1 \geq e \geq 0. \tag{2.5.0.11}$$

For $d_1 \geq e > d_1-k-1$ we have $\sum_{j=0}^{d_1-e} \Theta_j \lrcorner F_{e+j} = 0$ since $\Theta_j = 0$ for $j < k+1$. Together with Equation (2.5.0.11), it implies that $\Gamma|_{\alpha_0=1}$ annihilates f and thus, $\Gamma \in \operatorname{Ann}(f)^{hom}$ as claimed.
We proceed to the proof of Part (iii). We claim that $\operatorname{Ann}(f^{hom,d_2})$ has at most one minimal homogeneous generator of degree d_1 modulo generators of $(\operatorname{Ann}(f)^{hom})_{d_1}$. Indeed, by the above observation with $k = 0$, any such generator is (up to a scalar) of the form $\alpha_0^{d_1} + \rho$ where $\alpha_0^{d_1}$ does not divide any monomial in ρ. Given two such generators, say $\alpha_0^{d_1} + \rho$ and $\alpha_0^{d_1} + \rho'$ we have $\alpha_0^{d_1} + \rho = (\alpha_0^{d_1} + \rho') + (\rho - \rho')$. It follows from the above observation for $k = 0$ that $\rho - \rho'$ is in $(\operatorname{Ann}(f)^{hom})_{d_1}$ so the second new generator is not needed. Therefore, either

$$H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = H(T^*/\operatorname{Ann}(f)^{hom}, d_1) = r, \text{ or}$$
$$H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = H(T^*/\operatorname{Ann}(f)^{hom}, d_1) - 1 = r - 1.$$

Now we assume $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r - 1$. Then there exists a homogeneous generator of $\operatorname{Ann}(f^{hom,d_2})$ of the form $\alpha_0^{d_1} + \rho$, where $\alpha_0^{d_1}$ does not divide any monomial in ρ. It is enough to show that for any $k \geq 0$, if $\Gamma = \alpha_0^{d_1-1}\Theta_{1+k} + \alpha_0^{d_1-2}\Theta_{2+k} + \ldots + \Theta_{d_1+k}$ annihilates f^{hom,d_2}, then $\Gamma \in \operatorname{Ann}(f)^{hom}$. This is the observation from the beginning of the proof of Part (iii).

■

The following example shows that without the use of divided powers in homogenization $f \mapsto f^{hom,d_2}$ (see Definition 2.5.0.1), the statement of Lemma 2.5.0.9 (i) is false.

Example 2.5.0.12. *Let $f = x_1^3 + x_2 \in \mathbb{C}[x_1, x_2]$ and $G = x_1^3 + x_0^2 x_2 \in \mathbb{C}[x_0, x_1, x_2]$ be its standard homogenization. Then*

$$\operatorname{Ann}(f)^{hom} = (\alpha_2^2, \alpha_1\alpha_2, \alpha_1^3 - 6\alpha_0^2\alpha_2),$$

and

$$\operatorname{Ann}(G) = (\alpha_0^3, \alpha_2^2, \alpha_0\alpha_1, \alpha_1\alpha_2, \alpha_1^3 - 3\alpha_0^2\alpha_2).$$

The element $\alpha_1^3 - 6\alpha_0^2\alpha_2 \in \operatorname{Ann}(f)^{hom}$ does not annihilate G.

The following lemma is a generalization of Lemma 2.5.0.9. Here, we use a subspace $W \subseteq S_{\leq d_1}$ instead of a polynomial $f \in S_{\leq d_1}$.

Lemma 2.5.0.13 ([GMR20, Lem. 3.8.]). *Let $W \subseteq S_{\leq d_1}$ be a linear subspace with $d_1 \geq 1$ and fix a non-negative integer d_2. We have:*
(i) $\mathrm{Ann}(W)^{hom} \subseteq \mathrm{Ann}(W^{hom,d_2})$,
(ii) $(\mathrm{Ann}(W)^{hom})_{\leq d_2} = \mathrm{Ann}(W^{hom,d_2})_{\leq d_2}$.

Proof. (i) Let $f \in W$. Then $\mathrm{Ann}(W) \subseteq \mathrm{Ann}(f)$. Therefore,

$$\mathrm{Ann}(W)^{hom} \subseteq \mathrm{Ann}(f)^{hom} \subseteq \mathrm{Ann}\left(\sum_{i=0}^{\deg f} F_i x_0^{[d_2+d_1-i]}\right)$$

by Lemma 2.5.0.9 (i). Varying f, this shows that

$$\mathrm{Ann}(W)^{hom} \subseteq \bigcap_{H \in W^{hom,d_2}} \mathrm{Ann}(H) = \mathrm{Ann}(W^{hom,d_2}).$$

(ii) Let $\Theta \in \mathrm{Ann}(W^{hom,d_2})_{\leq d_2}$ be homogeneous and let $f \in W$. Then

$$\Theta \in \mathrm{Ann}\left(\sum_{i=0}^{\deg f} F_i x_0^{[d_2+d_1-i]}\right)_{\leq d_2}.$$

Since $d_2 \leq d_2 + d_1 - \deg f$, it follows from Lemma 2.5.0.9(ii) that $\Theta|_{\alpha_0=1} \in \mathrm{Ann}(f)$. We stress that applying Lemma 2.5.0.9(ii), we use $(\deg f, d_1 + d_2 - \deg f)$ instead of (d_1, d_2). Since f was arbitrary, we obtain

$$\Theta|_{\alpha_0=1} \in \bigcap_{f \in W} \mathrm{Ann}(f) = \mathrm{Ann}(W).$$

Therefore, $\Theta \in \mathrm{Ann}(W)^{hom}$.

■

As an application of the lemmas stated up to now, we can prove the following Lemma 2.5.0.15, which provides a sufficient condition for bounding the border rank of W^{hom,d_2} by $\dim_{\Bbbk} S^*/\mathrm{Ann}(W)$. To state the lemma we introduce the following Definition 2.5.0.14.

Definition 2.5.0.14. For $X = \mathbb{P}^n$ or $\mathbb{A}^n$ let $\mathcal{H}ilb_r(X)$ denote the Hilbert scheme of r points on X and $\mathcal{H}ilb_r^{sm}(X)$ denote the closure of the set of smooth schemes. Let $I \subseteq S^*$ be an ideal such that the dimension of S^*/I as a vector space over $\mathbb{C}$ is a finite number r. We say that a quotient algebra S^*/I is *smoothable* if the point corresponding to $\mathrm{Spec}(S^*/I)$ is contained in the smoothable component $\mathcal{H}ilb_r^{sm}(\mathbb{A}^n)$.

Lemma 2.5.0.15 ([GMR20, Lem. 3.9.]). *Let $d_1 \geq 1, d_2 \geq 0$ be integers and $W \subseteq S_{d_1}$ a linear subspace. Let $I := \mathrm{Ann}(W) \subseteq S^*$ and $r := \dim_{\Bbbk} S^*/\mathrm{Ann}(W)$. If $S^*/\mathrm{Ann}(W)$ is smoothable, then the border rank of W^{hom,d_2} is at most r.*

Proof. Let multigraded Hilbert scheme $\mathrm{Hilb}^{h_{r,n+1}}_{T^*}$ and $\mathrm{Slip}_{r,\mathbb{P}T_1}$ be defined as in Subsection 2.4.1. Observe, that $\mathrm{Slip}_{r,\mathbb{P}T_1}$ surjects onto $\mathcal{H}ilb^{sm}_r(\mathbb{P}^n)$ under the natural map

$$\mathrm{Hilb}^{h_{r,n+1}}_{T^*} \to \mathcal{H}ilb_r(\mathbb{P}^n)$$

given on closed points by $[I] \mapsto [\mathrm{Proj}\, T^*/I]$. Thus, there is an ideal $[J] \in \mathrm{Slip}_{r,\mathbb{P}T_1}$ with $J^{sat} = \mathrm{Ann}(W)^{hom}$ (we used Lemma 2.5.0.3). Since $\mathrm{Ann}(W)^{hom} \subseteq \mathrm{Ann}(W^{hom,d_2})$ by Lemma 2.5.0.13(i), we have $J \subseteq \mathrm{Ann}(W^{hom,d_2})$. Hence $[W^{hom,d_2}] \in \sigma_{r,\dim(W^{hom,d_2})}(\nu_d(\mathbb{P}T_1))$ by the Border Apolarity Lemma 2.4.1.5. ■

2.6 Characterization of the set of cubics with Hilbert function $(1,6,6,1)$

In this section we state Lemma 2.6.0.2, which provides a useful characterization of cubics f such that the Hilbert function of $\mathrm{Apolar}(f)$ is $(1,6,6,1)$ (see Definitions 2.4.0.1 and 2.4.0.2). This is inspired by [BJMR17, Ex. 8]. Then we establish Lemma 2.6.0.5 about topological properties of the set of such cubics. The mentioned lemma will be needed in the proofs of Theorems 1.3.1.1 and 4.0.0.2, which one can find in subsection 4.2.1.

Until the end of this chapter $\Bbbk := \mathbb{C}$. In this setting, the graded dual ring of a polynomial ring is isomorphic to a polynomial ring. We denote $S^* := \mathbb{C}[\alpha_1, \dots, \alpha_n]$, and $S := \mathbb{C}[x_1, \dots, x_n]$ is its graded dual. Given $f \in S$, we will write F_j for its homogeneous part of degree j. For $U \subseteq T_1$, let us denote $\mathrm{Sym}^d U := \{\langle y_1 y_2 \dots y_d\rangle \mid y_i \in U\} \subseteq T_d$. In this section we assume that $n \geq 6$.

Lemma 2.6.0.1 ([GMR20, Lem. 5.1.]). *Let $W \subseteq S$ be a linear subspace. Then*

$$H(\mathrm{Apolar}(W), k) = \mathrm{codim}_{S^*_k} E_k,$$

where $E_k = \{\theta_k \in S^*_k \mid$ there exists $\theta_{\geq k+1} \in S^*_{\geq k+1}$ such that $(\theta_k + \theta_{\geq k+1}) \lrcorner W = 0\}$.

Proof. Let $\overline{\mathfrak{m}}$ be the maximal ideal of $\mathrm{Apolar}(W)$.

$$\begin{aligned}
H(\mathrm{Apolar}(W), k) &= \dim_{\mathbb{C}} \overline{\mathfrak{m}}^k / \overline{\mathfrak{m}}^{k+1} \\
&= \mathrm{codim}_{S^*_{\geq k}} \mathrm{Ann}(W) \cap S^*_{\geq k} - \mathrm{codim}_{S^*_{\geq k+1}} \mathrm{Ann}(W) \cap S^*_{\geq k+1} \\
&= \mathrm{codim}_{S^*_{\geq k}} S^*_{\geq k+1} - \mathrm{codim}_{\mathrm{Ann}(W) \cap S^*_{\geq k}} \mathrm{Ann}(W) \cap S^*_{\geq k+1} \\
&= \dim_{\mathbb{C}} S^*_k - \dim_{\mathbb{C}} \frac{\mathrm{Ann}(W) \cap S^*_{\geq k}}{\mathrm{Ann}(W) \cap S^*_{\geq k+1}} \\
&= \dim_{\mathbb{C}} S^*_k - \dim_{\mathbb{C}} E_k
\end{aligned}$$

■

Lemma 2.6.0.2 ([GMR20, Lem. 5.2.]). *For $[f] \in \mathbb{P}S_{\leq 3}$ the following are equivalent:*

(a) $\mathrm{Apolar}(f)$ *has Hilbert function* $(1,6,6,1)$,
(b) *there exists* $[U] \in \mathrm{Gr}(6, S_1)$ *such that* $F_3 \in \mathrm{Sym}^3 U$, $F_2 \in U \cdot S_1$ *and* $H(\mathrm{Apolar}(F_3), 1) = 6$.

Proof. By Iarrobino's symmetric decomposition (see [CJN15, Thm 2.3 and the following remarks]), the algebra $\mathrm{Apolar}(f)$ has Hilbert function $(1, c+e, c, 1)$, where $(1, c, c, 1)$ is the Hilbert function of $\mathrm{Apolar}(F_3)$. We know from Lemma 2.6.0.1 that $c + e = \mathrm{codim}_{S_1^*} E_1$, where $E_1 = \{\theta_1 \in S_1^* |$ there exists $\theta_{\geq 2} \in S_{\geq 2}^*$ such that $(\theta_1 + \theta_{\geq 2}) \lrcorner f = 0\}$. We use the following computation

$$(\theta_3 + \theta_2 + \theta_1) \lrcorner (F_3 + F_2 + F_1 + F_0) = (\theta_1 \lrcorner F_3) + (\theta_1 \lrcorner F_2 + \theta_2 \lrcorner F_3) + (\theta_1 \lrcorner F_1 + \theta_2 \lrcorner F_2 + \theta_3 \lrcorner F_3). \tag{2.6.0.3}$$

Assume that $\mathrm{Apolar}(f)$ has Hilbert function $(1, 6, 6, 1)$, we will show that condition (b) is satisfied. Let U be $S_2^* \lrcorner F_3$, which is 6 dimensional, since the Hilbert function of $\mathrm{Apolar}(F_3)$ is $(1, 6, 6, 1)$, by the above discussion. It is enough to show that $F_2 \in U \cdot S_1$. Suppose that this does not hold. Up to a linear change of variables $U = \langle x_1, x_2, \dots, x_6 \rangle$. Let $V := \langle x_7, x_8, \dots, x_n \rangle$. By the classification of quadratic forms over $\mathbb{C}$, we may assume that $F_2 = x_n^2 + H + K$ where $H \in \mathrm{Sym}^2(\langle x_7, x_8, \dots, x_{n-1} \rangle)$ and $K \in S_1 \cdot U$. Then $\alpha_n \lrcorner F_2 \notin U$ and hence $\alpha_n \notin E_1$ by Equation (2.6.0.3). Thus, $\dim_{\mathbb{C}} E_1 \leq n - 7$. This contradicts Lemma 2.6.0.1 and the assumption that $H(\mathrm{Apolar}(f), 1) = 6$.

For the other direction, suppose that (b) holds, we will show that $\mathrm{Apolar}(f)$ has Hilbert function $(1, 6, 6, 1)$. It is enough to demonstrate that $\mathrm{codim}_{S_1^*} E_1 = 6$. By assumption $\dim_{\mathbb{C}} \mathrm{Ann}(F_3)_1 = n - 6$, so it suffices to show that $E_1 = \mathrm{Ann}(F_3)_1$. Assume that $\theta = \theta_3 + \theta_2 + \theta_1 \in S_{\geq 1}^* \in \mathrm{Ann}(f)$, then it follows from Equation (2.6.0.3) that $\theta_1 \in \mathrm{Ann}(F_3)_1$. Thus, $E_1 \subseteq \mathrm{Ann}(F_3)_1$. Let us take $\theta_1 \in \mathrm{Ann}(F_3)_1$. From the assumption $F_2 = \sum_i u_i h_i$ where $u_i \in U, h_i \in S_1$. Therefore,

$$\theta_1 \lrcorner F_2 = \sum_i u_i (\theta_1 \lrcorner h_i) \in U.$$

Since $(-) \lrcorner F_3 : S_2^* \to U$ is surjective, there exists $\theta_2 \in S_2^*$ such that $\theta_2 \lrcorner F_3 = -\theta_1 \lrcorner F_2$. By Equation (2.6.0.3) it is enough to observe that there exists $\theta_3 \in S_3^*$ such that $\theta_3 \lrcorner F_3 = -(\theta_1 \lrcorner F_1 + \theta_2 \lrcorner F_2)$

■

To state the next lemma, we need to introduce the notion of the Gorenstein scheme.

Definition 2.6.0.4. A zero-dimensional scheme of finite type over $\mathbb{C}$ is *Gorenstein* if it is equal to $\mathrm{Spec}(A)$, where A is a product of local algebras (A_i, m_i), and each socle $(0 : m_i)$ is a one-dimensional vector space. Let $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$, denote the open subset of the Hilbert scheme of r points on $\mathbb{A}^n$ consisting of Gorenstein subschemes, and let $\mathcal{H}ilb_r^{Gor,sm}(\mathbb{A}^n)$ denote its smoothable component.

Lemma 2.6.0.5 ([GMR20, Lem. 5.3.]). *The following subset of $\mathbb{P}S_{\leq 3}$ is irreducible, of dimension $13n+5$, and locally closed*

$$\mathcal{A} := \{[f] \in \mathbb{P}S_{\leq 3} \mid \operatorname{Apolar}(f) \text{ has Hilbert function } (1,6,6,1)\}.$$

Moreover, the set

$$\mathcal{B} := \{[f] \in \mathcal{A} \mid [\operatorname{Spec} \operatorname{Apolar}(f)] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)\}$$

is dense in $\mathcal{A}$.

Proof. Consider

$$\mathfrak{A} := \{([U],[f]) \in \operatorname{Gr}(6, S_1) \times \mathbb{P}S_{\leq 3} \mid [f] \in \mathbb{P}(\operatorname{Sym}^3 U \oplus (S_1 \cdot U) \oplus S_{\leq 1})\}.$$

and

$$\mathfrak{A}^0 := \{([U],[f]) \in \mathfrak{A} \mid H(\operatorname{Apolar}(F_3), 1) = 6\}.$$

We have a pullback diagram

$$\begin{array}{ccc} \mathfrak{A} & \longrightarrow & \operatorname{Fl}(1, 7n+42, S_{\leq 3}) \\ \downarrow & & \downarrow \\ \operatorname{Gr}(6, S_1) & \longrightarrow & \operatorname{Gr}(7n+42, S_{\leq 3}) \end{array}$$

where $\operatorname{Fl}(1, 7n+42, S_{\leq 3})$ is the flag variety parametrizing flags of subspaces $M \subseteq N \subseteq S_{\leq 3}$ with $\dim_{\mathbb{C}} M = 1$, $\dim_{\mathbb{C}} N = 7n+42$ and the lower horizontal map sends $[U]$ to $[\operatorname{Sym}^3 U \oplus (S_1 \cdot U) \oplus S_{\leq 1}]$.

The varieties $\mathfrak{A}$ and $\operatorname{Gr}(6, S_1)$ are projective. Moreover, the left vertical map is surjective and its fibers are irreducible varieties isomorphic to $\mathbb{P}^{7n+41}$. Since $\mathbb{P}^{7n+41}$ is irreducible, it follows from [Sha13, Thm 1.25-26] that $\mathfrak{A}$ is irreducible and of dimension $6(n-6) + 7n + 41 = 13n + 5$.

We will show that $\mathfrak{A}^0$ is open in $\mathfrak{A}$. Consider the subset

$$\mathfrak{B} := \{([U],[f]) \in \operatorname{Gr}(6, S_1) \times \mathbb{P}S_{\leq 3} \mid H(\operatorname{Apolar}(F_3), 1) \geq 6\}.$$

Observe, that $\mathfrak{A}^0 = \mathfrak{A} \cap \mathfrak{B}$. It is enough to show that $\mathfrak{B}$ is open in $\operatorname{Gr}(6, S_1) \times \mathbb{P}S_{\leq 3}$. Let

$$\mathfrak{C} := \{[f] \in \mathbb{P}S_{\leq 3} \mid H(\operatorname{Apolar}(F_3), 1) \geq 6\}.$$

It suffices to show that $\mathfrak{C}$ is open in $\mathbb{P}S_{\leq 3}$, which holds since its complement is given by catalecticant minors. We have established that $\mathfrak{A}^0 = \mathfrak{A} \cap \mathfrak{B}$ is open in $\mathfrak{A}$.

Let $\pi_2 : \operatorname{Gr}(6, S_1) \times \mathbb{P}S_{\leq 3} \to \mathbb{P}S_{\leq 3}$ be the projection map. By Lemma 2.6.0.2 follows $\mathcal{A} = \pi_2(\mathfrak{A}^0)$. Since $\pi_2|_{\mathfrak{A}^0} : \mathfrak{A}^0 \to \mathcal{A}$ has a finite fiber over every point, it follows from [Vak17, Thm 11.4.1] that $\mathcal{A}$ is irreducible and of dimension $13n+5$.

We know that $\mathcal{A} = \pi_2(\mathfrak{A}^0) = \pi_2(\mathfrak{A}) \cap \mathfrak{C}$ which is locally closed since $\pi_2(\mathfrak{A})$ is closed and $\mathfrak{C}$ is open. Therefore, we have a morphism $\mu : \mathcal{A} \to \mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^n)$ defined on closed points by $[f] \mapsto [\operatorname{Spec} S^*/\operatorname{Ann}(f)]$, see [GMR20, Thm 7.1].

By [CJN15], the scheme $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^n)$ has two irreducible components $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$ and $\mathcal{H}_{1661}$. We obtain $\mathcal{B} = \mu^{-1}(\mathcal{H}_{1661} \setminus \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n))$, so it is open in $\mathcal{A}$. Since $\mathcal{B}$ is non-empty and $\mathcal{A}$ is irreducible, it follows that $\mathcal{B}$ is dense in $\mathcal{A}$. ■

2.7 Characterization of the set of subspaces with Hilbert function $(1, 4, 3)$

In this section we state Lemma 2.7.0.1, which provides a useful characterization of subspaces W of a polynomial ring such that the Hilbert function of $\mathrm{Apolar}(W)$ is $(1, 4, 3)$. Then we establish Lemma 2.7.0.2 about topological properties of the set of such subspaces. The mentioned lemma will be needed in the proof of Theorems 1.3.1.3 and 4.0.0.3, which is located in in Subsection 4.3.1.

As in the previous section, here $S^* := \mathbb{C}[\alpha_1, \ldots, \alpha_n]$, and $S := \mathbb{C}[x_1, \ldots, x_n]$ is its graded dual. We assume that $n \geq 4$. Given an integer i, and a linear subspace $W \subseteq S$, we denote by W_i the image of the projection of W onto the i-th graded part.

Lemma 2.7.0.1 ([GMR20, Lem. 6.2.]). *For $[W] \in \mathrm{Gr}(3, S_{\leq 2})$ the following are equivalent:*

(a) $\mathrm{Apolar}(W)$ *has Hilbert function* $(1, 4, 3)$*,*

(b) $\mathrm{Apolar}(W_2)$ *has Hilbert function* $(1, 4, 3)$*.*

(c) $\dim_{\mathbb{C}} W_2 = 3$*,* $[W] \in \mathrm{Gr}(3, \mathrm{Sym}^2 U \oplus S_{\leq 1})$ *for some* $[U] \in \mathrm{Gr}(4, S_1)$ *and* $H(\mathrm{Apolar}(W_2), 1) = 4$*,*

Proof. Conditions (b) and (c) are equivalent. We shall show that Conditions (a) and (b) are equivalent. Observe, that $H(\mathrm{Apolar}(W), 2) = 3$ if and only if $\dim_{\mathbb{C}} W_2 = 3$ since $H(\mathrm{Apolar}(W), 2) = H(\mathrm{Apolar}(W_2), 2)$.

Therefore, we are left to show that $H(\mathrm{Apolar}(W), 1) = 4$ if and only if $H(\mathrm{Apolar}(W_2), 1) = 4$. By Lemma 2.6.0.1, we obtain $H(\mathrm{Apolar}(W), 1) = \mathrm{codim}_{S_1^*}(E_1)$, where

$$E_1 := \{\theta_1 \in S_1^* \mid \text{ there exists } \theta_{\geq 2} \in S_{\geq 2}^* \text{ such that } \theta_1 + \theta_{\geq 2} \in \mathrm{Ann}(W)\}.$$

We will show that $E_1 = \mathrm{Ann}(W_2)_1$.
Let $W = \langle Q_j + L_j + C_j \mid j \in \{1, 2, 3\},\ Q_j \in S_2, L_j \in S_1 \text{ and } C_j \in S_0\rangle$. Assume that $\theta_1 \in E_1$ and let $\theta_1 + \theta_{\geq 2} \in \mathrm{Ann}(W)$ for some $\theta_{\geq 2} \in S_{\geq 2}^*$. Then for $j \in \{1, 2, 3\}$

$$0 = (\theta_1 + \theta_{\geq 2}) \lrcorner (Q_j + L_j + C_j) = (\theta_1 \lrcorner Q_j) + (\theta_1 \lrcorner L_j + \theta_{\geq 2} \lrcorner Q_j),$$

so $\theta_1 \lrcorner Q_j = 0$ for $j \in \{1, 2, 3\}$.

Now suppose, that $\theta_1 \in \mathrm{Ann}(W_2)$. Since $\dim_{\mathbb{C}} W_2 = 3$, there is $\theta_2 \in S_2^*$ such that $\theta_2 \lrcorner Q_j = -\theta_1 \lrcorner L_j$ for $j \in \{1, 2, 3\}$. Then $\theta_1 + \theta_2 \in \mathrm{Ann}(W)$ so $\theta_1 \in E_1$. ■

Lemma 2.7.0.2 ([GMR20, Lem. 6.3.]). *The following subset is irreducible, of dimension $7n+8$, and locally closed*

$$\mathcal{A} := \{[W] \in \operatorname{Gr}(3, S_{\le 2}) | \operatorname{Apolar}(W) \textit{ has Hilbert function } (1,4,3)\}.$$

Moreover, the set

$$\mathcal{B} := \{[W] \in \mathcal{A} | [\operatorname{Spec}\operatorname{Apolar}(W)] \notin \mathcal{H}ilb_8^{sm}(\mathbb{A}^n)\}$$

is dense in $\mathcal{A}$.

Proof. Consider

$$\mathfrak{A} := \{([U],[W]) \in \operatorname{Gr}(4, S_1) \times \operatorname{Gr}(3, S_{\le 2}) \mid [W] \in \operatorname{Gr}(3, \operatorname{Sym}^2 U \oplus S_{\le 1})\}.$$

and

$$\mathfrak{A}^0 := \{([U],[W]) \in \mathfrak{A} \mid H(\operatorname{Apolar}(W_2), 1) = 4\}.$$

We have a pullback diagram

$$\begin{array}{ccc} \mathfrak{A} & \longrightarrow & \operatorname{Fl}(3, n+11, S_{\le 2}) \\ \downarrow & & \downarrow \\ \operatorname{Gr}(4, S_1) & \longrightarrow & \operatorname{Gr}(n+11, S_{\le 2}) \end{array}$$

where $\operatorname{Fl}(3, n+11, S_{\le 2})$ is the flag variety parametrizing flags of subspaces $M \subseteq N \subseteq S_{\le 2}$ with $\dim_{\mathbb{C}} M = 3$, $\dim_{\mathbb{C}} N = n+11$ and the lower horizontal map sends $[U]$ to $[\operatorname{Sym}^2 U \oplus S_{\le 1}]$.

The varieties $\mathfrak{A}$ and $\operatorname{Gr}(4, S_1)$ are projective. Moreover, the left vertical map is surjective and its fibers are irreducible and isomorphic to $\operatorname{Gr}(3, n+11)$. Since $\operatorname{Gr}(3, n+11)$ is irreducible, it follows from [Sha13, Thm 1.25-26] that $\mathfrak{A}$ is irreducible and of dimension $4(n-4) + 3(n+8) = 7n+8$.

We will show that $\mathfrak{A}^0$ is open in $\mathfrak{A}$. Consider the subset

$$\mathfrak{B} := \{([U],[W]) \in \operatorname{Gr}(4, S_1) \times \operatorname{Gr}(3, S_{\le 2}) \mid H(\operatorname{Apolar}(W_2), 1) \ge 4\}.$$

Observe, that $\mathfrak{A}^0 = \mathfrak{A} \cap \mathfrak{B}$. Therefore, it is enough to show that $\mathfrak{B}$ is open in $\operatorname{Gr}(4, S_1) \times \operatorname{Gr}(3, S_{\le 2})$. Let

$$\mathfrak{C} := \{[W] \in \operatorname{Gr}(3, S_{\le 2}) \mid H(\operatorname{Apolar}(W_2), 1) \ge 4\}.$$

It is enough to show that $\mathfrak{C}$ is open in $\operatorname{Gr}(3, S_{\le 2})$. Let

$$\mathfrak{D} := \{([U],[W]) \in \operatorname{Gr}(3, S_1) \times \operatorname{Gr}(3, S_{\le 2}) \mid [W] \in \operatorname{Gr}(3, \operatorname{Sym}^2(U) \oplus S_{\le 1})\},$$

and $\rho_2 : \operatorname{Gr}(3, S_1) \times \operatorname{Gr}(3, S_{\le 2}) \to \operatorname{Gr}(3, S_{\le 2})$ be the natural projection. Notice, that the complement of $\mathfrak{C}$ in $\operatorname{Gr}(3, S_{\le 2})$ is equal to $\rho_2(\mathfrak{D})$ which is closed since $\mathfrak{D}$ is projective. This concludes the proof that $\mathfrak{A}^0$ is open in $\mathfrak{A}$.

Let $\pi_2 : \mathrm{Gr}(4, S_1) \times \mathrm{Gr}(3, S_{\leq 2}) \to \mathrm{Gr}(3, S_{\leq 2})$ be the projection map. By Lemma 2.7.0.1, we have $A = \pi_2(\mathfrak{A}^0) \cap \mathfrak{F}$ where

$$\mathfrak{F} := \{[W] \in \mathrm{Gr}(3, S_{\leq 2}) \mid \dim_{\mathbb{C}} W_2 = 3\}.$$

Since $\pi_2|_{\mathfrak{A}^0} : \mathfrak{A}^0 \to \pi_2(\mathfrak{A}^0)$ has a finite fiber over a general point, it follows from [Vak17, Thm 11.4.1] that $\pi_2(\mathfrak{A}^0)$ is irreducible and of dimension $7n+8$. The subset $\mathfrak{F} \subseteq \mathrm{Gr}(3, S_{\leq 2})$ is open and $\pi_2(\mathfrak{A}^0) \cap \mathfrak{F}$ is non-empty, so $\mathcal{A} = \pi_2(\mathfrak{A}^0) \cap \mathfrak{F}$ is irreducible and of dimension $7n + 8$.

We know that $\mathcal{A} = \pi_2(\mathfrak{A}) \cap \mathfrak{C} \cap \mathfrak{F}$, so $\mathcal{A}$ is locally closed since $\pi_2(\mathfrak{A})$ is closed and $\mathfrak{C}$, $\mathfrak{F}$ are open. Therefore, we have a morphism $\mu : \mathcal{A} \to \mathcal{H}ilb_8(\mathbb{A}^n)$ given on closed points by $[W] \mapsto [\operatorname{Spec} S^* / \operatorname{Ann}(W)]$, see [GMR20, Thm 7.1]. By [CEVV09, Thm 1.1], the scheme $\mathcal{H}ilb_8(\mathbb{A}^n)$ has two irreducible components $\mathcal{H}ilb_8^{Gor,sm}(\mathbb{A}^n)$ and $\mathcal{H}_{143}$. We obtain $\mathcal{B} = \mu^{-1}(\mathcal{H}_{143} \setminus \mathcal{H}ilb_8^{sm}(\mathbb{A}^n))$, so it is open in $\mathcal{A}$. Since $\mathcal{B}$ is non-empty and $\mathcal{A}$ is irreducible, it follows that $\mathcal{B}$ is dense in $\mathcal{A}$. ■

Chapter 3

Rank and border rank additivity problems

In this chapter we address several cases of Problem 1.2.5.3. We recall, that if one of the vector spaces A', A'', B', B'', C', C'' is at most two dimensional, then the additivity of the tensor rank (1.2.5.2) holds (Theorem 1.2.5.6). We prove that it holds as well, if p' is arbitrary and $p'' \in A'' \otimes (B'' \otimes \Bbbk^1 + \Bbbk^2 \otimes C'')$ (see Corollary 3.1.3.9). To give the proof and other conditions for the additivity in more complicated setting, there is a necessity of the analysis of slices of $(p_1 + p_2)((A' \oplus A'')^*)$. It is located in Section 3.1. We distinguish seven types of matrices from a minimal decomposition and show that to prove the additivity of the tensor rank, one can get rid of two of those types. In other words, there is a smaller example, a pair $(\tilde{p_1}, \tilde{p_2})$ without those two types in its minimal decomposition. If the additivity property holds for $(\tilde{p_1}, \tilde{p_2})$, then it also holds for the original pair. This is the core observation, which let us prove one of the main results of the thesis, Theorem 1.2.5.7. In particular, over the base field $\mathbb{C}$, we solve the problem of additivity for a pair of tensors such that rank of each is at most 7 (Corollary 3.1.4.14). As a corollary, we obtain that a pair of 2×2 matrix multiplication tensor has the rank additivity property, i.e. $R(\mu_{2,2,2} \oplus \mu_{2,2,2}) = R(\mu_{2,2,2}) + R(\mu_{2,2,2})$.

In Section 3.2 we turn our attention to the additivity of the border rank. Since the known counterexamples to this version of the additivity are much smaller than in the case of the additivity of the tensor rank, our methods are more restricted to very small cases. We prove, that it is not possible to find an example of a pair of tensors (p', p'') without border rank additivity property such that $p' \oplus p'' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. We conclude the section with a brief discussion of possibility of occurrence of counterexample for border rank additivity property, when $A = B = C = \mathbb{C}^5$.

Although the chapter is mainly based on [BPR20] it also contains unpublished results. Sections 3.1.2 and 3.1.3 are generalizations of [BPR20, Sect. 4.2, 4.3]. They give more control over the hook structure of tensors. In result, we are able to prove Corollary 3.1.3.9, which generalize [BPR20, Prop. 3.17] to an arbitrary base field. The proof itself is also easier and shorter than the proof of [BPR20, Prop. 3.17]. Another benefit is the opening the door to proof of additivity in cases not covered by [BPR20] (see Corollary 3.1.4.13, Corollary 3.1.4.14). For the summary of all the cases in which we proved that rank additivity holds, see Theorem 1.2.5.7 from Chapter 1: Introduction.

3.1 Rank one matrices and additivity of the tensor rank

As long as we have a rank one matrix in the linear space W' or W'', we have a good starting point for an attempt to prove the additivity of the rank. Throughout this section we will make a formal statement out of this observation and prove, that if there is a rank one matrix in the linear spaces, then either the additivity holds or there exists a "smaller" example of failure of the additivity. In Section 3.1.4 we exploit several versions of this claim in order to prove that rank additivity holds in cases listed in Theorem 1.2.5.7.

Throughout this section we follow Notations 2.1.1.1 (denoting the rank one elements in a vector space by the subscript $\cdot_{Seg}$), 2.2.0.1 (introducing the vector spaces $A, A', \dots, C''$ and their dimensions $\mathbf{a}, \mathbf{a}', \dots, \mathbf{c}''$), and also 2.2.1.1 (which explains the conventions for projections $\pi_{A'}, \pi_{A''}, \dots, \pi_{C''}$ and vector spaces E', E'', F', F'', which measure how much the decomposition V of W sticks out from the direct sum $B' \otimes C' \oplus B'' \otimes C''$). In this chapter, the letter V will denote the decomposition of a subspace W, as defined at the beginning of Subsection 2.2.1. We will also frequently use Notation 2.2.0.2 and Proposition 2.2.0.3. Together they define a direct sum tensor $p = p' \oplus p''$ and let us translate the problem of additivity of rank for tensors to the additivity of rank for the corresponding vector spaces W, W', W''.

3.1.1 Combinatorial splitting of the decomposition

We carefully analyze the structure of the rank one matrices in V. We will distinguish seven types of such matrices.

Lemma 3.1.1.1 ([BPR20, Lem. 4.1.]). *Every element of $V_{\mathrm{Seg}} \subset \mathbb{P}(B \otimes C)$ lies in the projectivization of one of the following subspaces of $B \otimes C$:*

(i) $B' \otimes C'$, $B'' \otimes C''$, (Prime, Bis)
(ii) $E' \otimes (C' \oplus F'')$, $E'' \otimes (F' \oplus C'')$, (HL, HR)
$(B' \oplus E'') \otimes F'$, $(E' \oplus B'') \otimes F''$, (VL, VR)
(iii) $(E' \oplus E'') \otimes (F' \oplus F'')$. (Mix)

The spaces in (i) are purely contained in the original direct summands, hence, in some sense, they are the easiest to deal with (we will show how to "get rid" of them and construct a smaller example justifying a potential lack of additivity).[1] The spaces in (ii) stick out of the original summand, but only in one direction, either horizontal (HL, HR), or vertical (VL, VR)[2]. The space in (iii) is mixed and it sticks out in all directions. It is the most difficult to deal with and we expect, that the typical counterexamples to the additivity of the rank will have mostly (or only) such mixed matrices in their minimal decomposition. The mutual configuration and layout of those spaces in the case $(\mathbf{b}', \mathbf{b}'', \mathbf{c}', \mathbf{c}'') = (3, 3, 3, 3)$, $(\mathbf{e}', \mathbf{e}'', \mathbf{f}', \mathbf{f}'') = (1, 2, 1, 1)$ is illustrated in Figure 3.1. We use our usual convention, that bold lower case letters denote dimensions of the spaces denoted by capital letter.

Proof of Lemma 3.1.1.1. Let $b \otimes c \in V_{\mathrm{Seg}}$ be a matrix of rank one. Write $b = b' + b''$ and $c = c' + c''$, where $b' \in B', b'' \in B'', c' \in C'$ and $c'' \in C''$. We consider the image

[1] The word Bis comes from the Polish way of pronouncing the $''$ symbol.
[2] Here, the letters "H, V, L, R" stand for "horizontal, vertical, left, right" respectively.

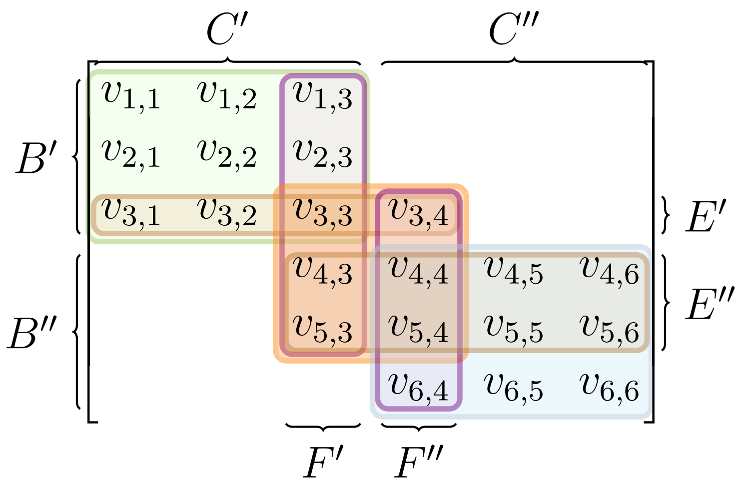

Figure 3.1: We use Notation 2.2.1.1. In the case $(\mathbf{b}', \mathbf{b}'', \mathbf{c}', \mathbf{c}'') = (3, 3, 3, 3)$, $(\mathbf{e}', \mathbf{e}'', \mathbf{f}', \mathbf{f}'') = (1, 2, 1, 1)$, choose a basis of E' and a completion to a basis of B' and, similarly, bases for $(E'', B''), (F', C'), (F'', C'')$. We can represent the elements of $V_{\mathrm{Seg}} \subset B \otimes C$ as matrices in one of the following subspaces: Prime (corresponding to the top-left green rectangle), Bis (bottom-right blue rectangle), VL (purple with entries $v_{1,3}, v_{2,3}, v_{3,3}, v_{4,3}, v_{5,3}$), VR (purple with entries $v_{3,4}, v_{4,4}, v_{5,4}, v_{6,4}$), HL (brown with entries $v_{3,1}, v_{3,2}, v_{3,3}, v_{3,4}$), HR (brown with entries $v_{4,3}, v_{4,4}, v_{4,5}, v_{4,6}, v_{5,3}, v_{5,4}, v_{5,5}, v_{5,6}$), and Mix (middle orange square with entries $v_{3,3}, v_{3,4}, v_{4,3}, v_{4,4}, v_{5,3}, v_{5,4}$).

of $b \otimes c$ via the four natural projections introduced in Notation 2.2.1.1:

$$\pi_{B'}(b \otimes c) = b'' \otimes c \in \qquad B'' \otimes (F' \oplus C''), \tag{3.1.1.2a}$$
$$\pi_{B''}(b \otimes c) = b' \otimes c \in \qquad B' \otimes (C' \oplus F''), \tag{3.1.1.2b}$$
$$\pi_{C'}(b \otimes c) = b \otimes c'' \in (E' \oplus B'') \otimes C'', \text{ and} \tag{3.1.1.2c}$$
$$\pi_{C''}(b \otimes c) = b \otimes c' \in (B' \oplus E'') \otimes C'. \tag{3.1.1.2d}$$

Notice, that b' and b'' cannot be simultaneously zero, since $b \neq 0$. Analogously, $(c', c'') \neq (0, 0)$.

Equations (3.1.1.2a)–(3.1.1.2d) prove, that the non-vanishing of one of b', b'', c', c'' induces a restriction on another one. For instance, if $b' \neq 0$, then by (3.1.1.2b) we must have $c'' \in F''$. Or, if $b'' \neq 0$, then (3.1.1.2a) forces $c' \in F'$, and so on. Altogether we obtain the following cases:

1. If $b', b'', c', c'' \neq 0$, then $b \otimes c \in (E' \oplus E'') \otimes (F' \oplus F'')$ (case Mix).
2. if $b', b'' \neq 0$ and $c' = 0$, then $b \otimes c = b \otimes c'' \in (E' \oplus B'') \otimes F''$ (case VR).
3. if $b', b'' \neq 0$ and $c'' = 0$, then $b \otimes c = b \otimes c' \in (B' \oplus E'') \otimes F'$ (case VL).
4. If $b' = 0$, then either $c' = 0$ and therefore $b \otimes c = b'' \otimes c'' \in B'' \otimes C''$ (case Bis), or $c' \neq 0$ and $b \otimes c = b'' \otimes c \in E'' \otimes (F' \oplus C'')$ (case HR).
5. If $b'' = 0$, then either $c'' = 0$ and thus $b \otimes c = b' \otimes c' \in B' \otimes C'$ (case Prime), or $c'' \neq 0$ and $b \otimes c = b' \otimes c \in E' \otimes (C' \oplus F'')$ (case HL).

This concludes the proof. ■

As in Lemma 3.1.1.1 every element of $V_{Seg} \subset \mathbb{P}(B \otimes C)$ lies in one of seven subspaces of $B \otimes C$. These subspaces may have nonempty intersection. We will now explain our convention with respect to choosing a basis of V consisting of elements of V_{Seg}.

Here and throughout the thesis, by $\sqcup$ we denote the disjoint union.

Notation 3.1.1.3. We choose a basis $\mathcal{B}$ of V in such a way that:

- $\mathcal{B}$ consist of rank one matrices only,
- $\mathcal{B} = \mathrm{Prime} \sqcup \mathrm{Bis} \sqcup \mathrm{HL} \sqcup \mathrm{HR} \sqcup \mathrm{VL} \sqcup \mathrm{VR} \sqcup \mathrm{Mix}$, where each of Prime, Bis, HL, HR, VL, VR, and Mix is a finite set of rank one matrices of the respective type as in Lemma 3.1.1.1 (for instance, $\mathrm{Prime} \subset B' \otimes C'$, $\mathrm{HL} \subset E' \otimes (C' \oplus F'')$, etc.).
- $\mathcal{B}$ has as many elements of Prime and Bis as possible, subject to the first two conditions,
- $\mathcal{B}$ has as many elements of HL, HR, VL and VR as possible, subject to all of the above conditions.

Let **prime** be the number of elements of Prime (equivalently, **prime** $= \dim \langle \mathrm{Prime} \rangle$) and analogously define **bis**, **hl**, **hr**, **vl**, **vr**, and **mix**. The choice of $\mathcal{B}$ need not be unique, but we fix one for the rest of the chapter. Instead, the numbers **prime**, **bis**, and **mix** are uniquely determined by V (there may be some non-uniqueness in dividing between **hl**, **hr**, **vl**, **vr**).

Thus, to each decomposition we associated a sequence of seven non-negative integers ($\mathbf{prime}, \dots, \mathbf{mix}$). We now study the inequalities between these integers and exploit them to get theorems about the additivity of the rank.

Proposition 3.1.1.4 ([BPR20, Prop. 4.3.]). *In Notations 2.2.1.1 and 3.1.1.3 the following inequalities hold:*

(i) $\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \min\left(\mathbf{mix}, \mathbf{e}'\mathbf{f}'\right) \geq R(W')$,
(ii) $\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \min\left(\mathbf{mix}, \mathbf{e}''\mathbf{f}''\right) \geq R(W'')$,
(iii) $\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \min\left(\mathbf{hr} + \mathbf{mix}, \mathbf{f}'(\mathbf{e}' + \mathbf{e}'')\right) \geq R(W') + \mathbf{e}''$,
(iv) $\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \min\left(\mathbf{vr} + \mathbf{mix}, \mathbf{e}'(\mathbf{f}' + \mathbf{f}'')\right) \geq R(W') + \mathbf{f}''$,
(v) $\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \min\left(\mathbf{hl} + \mathbf{mix}, \mathbf{f}''(\mathbf{e}' + \mathbf{e}'')\right) \geq R(W'') + \mathbf{e}'$,
(vi) $\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \min\left(\mathbf{vl} + \mathbf{mix}, \mathbf{e}''(\mathbf{f}' + \mathbf{f}'')\right) \geq R(W'') + \mathbf{f}'$.

Proof. To prove Inequality (i) we consider the composition of projections $\pi_{B''}\pi_{C''}$. The linear space $\pi_{B''}\pi_{C''}(V)$ is spanned by rank one matrices $\pi_{B''}\pi_{C''}(\mathcal{B})$ (where $\mathcal{B} = \mathrm{Prime} \sqcup \mathrm{Bis} \sqcup \dots \sqcup \mathrm{Mix}$ is as in Notation 3.1.1.3), and it contains W'. Thus $\dim(\pi_{B''}\pi_{C''}(V)) \geq R(W')$. But the only elements of the basis $\mathcal{B}$ that survive both projections (that is, they are not mapped to zero under the composition) are Prime, HL, VL, and Mix. Thus

$$\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{mix} \geq \dim(\pi_{B''}\pi_{C''}(V)) \geq R(W').$$

On the other hand, $\pi_{B''}\pi_{C''}(\mathrm{Mix}) \subset E' \otimes F'$, thus among $\pi_{B''}\pi_{C''}(\mathrm{Mix})$ we can choose at most $\mathbf{e}'\mathbf{f}'$ linearly independent matrices. Thus

$$\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{e}'\mathbf{f}' \geq \dim(\pi_{B''}\pi_{C''}(V)) \geq R(W').$$

The two inequalities prove (i).

To show Inequality (iii), we may assume that W' is concise as in the proof of Lemma 2.2.1.2. Moreover, as in that same proof (more precisely, Inequality (2.2.1.3)) we show that $\dim \pi_{C''}(V) \geq R(W') + \mathbf{e}''$. But $\pi_{C''}$ sends all matrices from Bis and VR to zero, thus

$$\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{hr} + \mathbf{mix} \geq \dim \pi_{C''}(V) \geq R(W') + \mathbf{e}''.$$

As in the proof of Part (i), we can also replace $\mathbf{hr} + \mathbf{mix}$ by $\mathbf{f}'(\mathbf{e}' + \mathbf{e}'')$, since $\pi_{C''}(\mathrm{HR} \cup \mathrm{Mix}) \subset (E' \oplus E'') \otimes F'$, concluding the proof of (iii).

The proofs of the remaining four inequalities are identical to one of the above, after swapping the roles of B and C or $'$ and $''$ (or swapping both pairs). ∎

Proposition 3.1.1.5 ([BPR20, Prop. 4.4.]). *If one among E', E'', F', F'' is zero, then $R(W) = R(W') + R(W'')$.*

Proof. Without loss of generality, we can assume that $E' = \{0\}$. Using the definitions of sets Prime, Bis, VR,... as in Notation 3.1.1.3, we see that $\mathrm{HL} = \mathrm{VR} = \mathrm{Mix} = \emptyset$, due to the order of choosing the elements of the basis $\mathcal{B}$: For instance, a potential candidate to became a member of HL, would be first elected to Prime, and similarly VR is consumed by Bis and Mix by HR. Thus:

$$R(W) = \dim(V_{Seg}) = \mathbf{prime} + \mathbf{bis} + \mathbf{hr} + \mathbf{vl}.$$

Proposition 3.1.1.4(i) and (ii) implies

$$R(W') + R(W'') \leq \mathbf{prime} + \mathbf{vl} + \mathbf{bis} + \mathbf{hr} = R(W),$$

while $R(W')+R(W'') \geq R(W)$ always holds. This shows the desired additivity. ∎

Corollary 3.1.1.6 ([BPR20, Cor. 4.5.])**.** *Assume that the additivity fails for W' and W'', that is, $d = R(W') + R(W'') - R(W' \oplus W'') > 0$. Then the following inequalities hold:*
(a) $\mathbf{mix} \geq d \geq 1$,
(b) $\mathbf{hl} + \mathbf{hr} + \mathbf{mix} \geq \mathbf{e}' + \mathbf{e}'' + d \geq 3$,
(c) $\mathbf{vl} + \mathbf{vr} + \mathbf{mix} \geq \mathbf{f}' + \mathbf{f}'' + d \geq 3$.

Proof. To prove (a) consider the inequalities (i) and (ii) from Proposition 3.1.1.4 and their sum:

$$\begin{aligned} \mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{mix} &\geq R(W'), \\ \mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \mathbf{mix} &\geq R(W''), \\ \mathbf{prime} + \mathbf{bis} + \mathbf{hl} + \mathbf{hr} + \mathbf{vl} + \mathbf{vr} + 2\mathbf{mix} &\geq R(W') + R(W''). \end{aligned} \tag{3.1.1.7}$$

The left-hand side of (3.1.1.7) is equal to $R(W) + \mathbf{mix}$, while its right-hand side is $R(W) + d$. Thus the desired claim.

Similarly, using inequalities (iii) and (v) of the same proposition we obtain (b), while (iv) and (vi) imply (c). Note, that $\mathbf{e}' + \mathbf{e}'' + d \geq 3$ and $\mathbf{f}' + \mathbf{f}'' + d \geq 3$ by Proposition 3.1.1.5. ∎

3.1.2 Replete pairs

This subsection is a generalization of [BPR20, Sect. 4.2.]. We distinguish a class of pairs W', W'' with particularly nice properties.

Definition 3.1.2.1. We say (W', W'') is *replete with respect to* $v \in$ Prime (or Bis), if $v \in W'$ (resp. $v \in W''$). Similarly, we say (W', W'') is *replete* if it is replete with respect to v for all $v \in \text{Prime} \sqcup \text{Bis}$.

Remark 3.1.2.2. Strictly speaking, the notion of *replete pair* depends also on the minimal decomposition V. But as always we consider a pair W' and W'' with a fixed decomposition $V = \langle V_{\text{Seg}} \rangle \supset W' \oplus W''$, so we refrain from mentioning V in the notation.

The first important observation is, that as long as we look for pairs that fail to satisfy the additivity, we are free to replenish any pair. More precisely, for any fixed W', W'', V and $v \in \mathrm{Prime}$ (or Bis) define the *repletion* of (W', W'') with respect to v as the pair $({}^{\Re(v)}W', {}^{\Re(v)}W'')$:

$$
\begin{aligned}
&{}^{\Re(v)}W' := W' + \langle v\rangle\,, \quad {}^{\Re(v)}W'' := W'', \qquad {}^{\Re(v)}W := {}^{\Re(v)}W' \oplus {}^{\Re(v)}W''.\\
&\text{or resp.}\\
&{}^{\Re(v)}W' := W', \qquad {}^{\Re(v)}W'' := W'' + \langle v\rangle\,, \quad {}^{\Re(v)}W := {}^{\Re(v)}W' \oplus {}^{\Re(v)}W''.
\end{aligned}
\tag{3.1.2.3}
$$

The result of consecutive repletion with respect to all elements of Prime and Bis will be denoted by $({}^{\Re}W', {}^{\Re}W'')$. This latter notion agrees with one introduced in [BPR20, Subsect. 4.2].

Proposition 3.1.2.4. *For any (W', W'') and $v \in \mathrm{Prime}$ (or Bis) we have:*

$$
\begin{aligned}
R(W') \le R({}^{\Re(v)}W') &\le R(W') + (\dim {}^{\Re(v)}W' - \dim W'),\\
R(W'') \le R({}^{\Re(v)}W'') &\le R(W'') + (\dim {}^{\Re(v)}W'' - \dim W''),\\
R({}^{\Re(v)}W) &= R(W).
\end{aligned}
$$

In particular, if the additivity of the rank fails for (W', W''), then it also fails for $({}^{\Re(v)}W', {}^{\Re(v)}W'')$. Moreover,

(i) V is a minimal decomposition of ${}^{\Re(v)}W$; in particular, the same distinguished basis $\mathrm{Prime} \sqcup \mathrm{Bis} \sqcup \cdots \sqcup \mathrm{Mix}$ works for both W and ${}^{\Re(v)}W$.

(ii) $({}^{\Re(v)}W', {}^{\Re(v)}W'')$ is a replete pair with respect to v.

(iii) The gaps $R({}^{\Re(v)}W') - \dim({}^{\Re(v)}W')$, $R({}^{\Re(v)}W'') - \dim({}^{\Re(v)}W'')$, and $R({}^{\Re(v)}W) - \dim({}^{\Re(v)}W)$, are at most (respectively) $R(W') - \dim(W')$, $R(W'') - \dim(W'')$, and $R(W) - \dim(W)$.

Proof. Since $W' \subset {}^{\Re(v)}W'$, the inequality $R(W') \le R({}^{\Re(v)}W')$ is clear. Moreover, ${}^{\Re(v)}W'$ is spanned by W' if $\dim {}^{\Re(v)}W' = \dim W'$, or by W' with additional matrix v in the other case. The matrix v is of rank one, so $R({}^{\Re(v)}W') \le R(W') + (\dim {}^{\Re(v)}W' - \dim W')$. The inequalities about $''$ and $R(W) \le R({}^{\Re(v)}W)$ follow similarly.

Further ${}^{\Re(v)}W \subset V$, thus V is a decomposition of ${}^{\Re(v)}W$. Therefore also $R({}^{\Re(v)}W) \le \dim V = R(W)$, showing $R({}^{\Re(v)}W) = R(W)$ and (i). Item (ii) follows from (i), while (iii) is a rephrasement of the initial inequalities. ∎

Moreover, if one of the inequalities of Lemma 2.2.1.2 is an equality, then the respective W' or W'' is not affected by the repletion.

Lemma 3.1.2.5. *If, say, $R(W') + \mathbf{e}'' = R(W) - \dim W''$, then for any $v \in \mathrm{Prime}$ (or Bis) we have $W'' = {}^{\Re(v)}W''$. The analogous statements hold for the other equalities coming from replacing $\le$ by $=$ in Lemma 2.2.1.2.*

Proof. By Lemma 2.2.1.2 applied to ${}^{\mathfrak{R}(v)}W = {}^{\mathfrak{R}(v)}W' \oplus {}^{\mathfrak{R}(v)}W''$ and by Proposition 3.1.2.4

$$\begin{aligned} R({}^{\mathfrak{R}(v)}W) - \mathbf{e}'' &\overset{2.2.1.2}{\geq} R({}^{\mathfrak{R}(v)}W') + \dim({}^{\mathfrak{R}(v)}W'') \\ &\overset{3.1.2.4}{\geq} R(W') + \dim W'' \\ &\overset{\text{assumptions of } 3.1.2.5}{=} R(W) - \mathbf{e}'' \overset{3.1.2.4}{=} R({}^{\mathfrak{R}(v)}W) - \mathbf{e}''. \end{aligned}$$

Therefore all inequalities are in fact equalities. In particular, $\dim({}^{\mathfrak{R}(v)}W'') = \dim W''$. The claim of the lemma follows from $W'' \subset {}^{\mathfrak{R}(v)}W''$. ■

As a corollary we can prove, that if $R(W'') \leq \dim W'' + 2$, then either rank additivity holds or W'' is equal to its repletion.

Corollary 3.1.2.6. *Assume $R(W'') \leq \dim W'' + 2$. Then either the additivity holds $R(W) = R(W') + R(W'')$ or:*

- $R(W'') = \dim W'' + 2$, *and*
- $R(W) = R(W') + R(W'') - 1$, *and*
- $\mathbf{e}'' = \mathbf{f}'' = 1$, *and*
- ${}^{\mathfrak{R}}W'' = W''$.

Proof. Assume, that the additivity does not hold. Then by Lemma 2.2.1.5 we must have $R(W'') = \dim W'' + 2$. By Proposition 3.1.1.5 follows $\mathbf{e}'' > 0$, $\mathbf{f}'' > 0$, while by Corollary 2.2.1.4 we obtain $\mathbf{e}'' < 2$ and $\mathbf{f}'' < 2$. Thus $\mathbf{e}'' = \mathbf{f}'' = 1$.

By Lemma 2.2.1.2 the inequality $R(W) \geq R(W') + 1 + \dim W''$ holds. The right hand side is equal to $R(W') + R(W'') - 1$ by the above discussion (the $\leq$ inequality follows from the failure of additivity).

The final claim ${}^{\mathfrak{R}}W'' = W''$ follows from Lemma 3.1.2.5. ■

Later, in Corollary 3.1.3.9 we will show that if the difference between rank and dimension of W'' is at most two, then rank additivity holds.

3.1.3 Digestion with respect to a rank one tensor

This subsection is a generalization of [BPR20, Sect. 4.2.]. For pairs which are replete with respect to $v \in \mathrm{Prime}$ (or Bis) it makes sense to consider the complement of $\langle v \rangle$ in W' (resp. $\langle v \rangle$ in W'').

Definition 3.1.3.1. Using Notation 3.1.1.3, let $v \in \mathrm{Prime} \sqcup \mathrm{Bis}$ and ${}^{\mathfrak{D}(v)}W'$, ${}^{\mathfrak{D}(v)}W''$ denote the following linear spaces:

$$\begin{cases} {}^{\mathfrak{D}(v)}W' := \langle \mathcal{B} \setminus \{v\} \rangle \cap W', & {}^{\mathfrak{D}(v)}W'' := W'', \qquad \text{if } v \in \mathrm{Prime}, \\ {}^{\mathfrak{D}(v)}W' := W', & {}^{\mathfrak{D}(v)}W'' := \langle \mathcal{B} \setminus \{v\} \rangle \cap W'', \quad \text{if } v \in \mathrm{Bis}. \end{cases}$$

We call the pair $({}^{\mathfrak{D}(v)}W', {}^{\mathfrak{D}(v)}W'')$ the *digested version of* (W', W'') *with respect to* v. Similarly, by $({}^{\mathfrak{D}}W', {}^{\mathfrak{D}}W'')$ we will denote the result of consecutive digestion with respect to all elements of Prime and Bis. This latter notion agrees with one introduced in [BPR20, Subsect. 4.3].

Lemma 3.1.3.2. *If* (W', W'') *is replete with respect to* $v \in \mathrm{Prime}$ *(or* Bis*), then* $W' = \langle v\rangle \oplus {}^{\mathfrak{D}(v)}W'$ *and* $W'' = {}^{\mathfrak{D}(v)}W''$. *(resp.* $W' = {}^{\mathfrak{D}(v)}W'$ *and* $W'' = \langle v\rangle \oplus {}^{\mathfrak{D}(v)}W''$*).*

Proof. We will prove only case when $v \in \mathrm{Prime}$. The case when $v \in \mathrm{Bis}$ is similar. Both $\langle v\rangle$ and ${}^{\mathfrak{D}(v)}W'$ are contained in W'. The intersection $\langle v\rangle \cap {}^{\mathfrak{D}(v)}W'$ is zero, since the seven sets $\mathrm{Prime}, \mathrm{Bis}, \mathrm{HR}, \mathrm{HL}, \mathrm{VL}, \mathrm{VR}, \mathrm{Mix}$ are disjoint and together they are linearly independent. Furthermore,

$$\operatorname{codim}({}^{\mathfrak{D}(v)}W' \subset W') \leq \\ \operatorname{codim}(\langle(\mathrm{Prime}\setminus\{v\}) \sqcup \mathrm{Bis} \sqcup \mathrm{HL} \sqcup \mathrm{HR} \sqcup \mathrm{VL} \sqcup \mathrm{VR} \sqcup \mathrm{Mix}\rangle \subset V) = 1.$$

Thus $\dim W' \leq \dim {}^{\mathfrak{D}(v)}W' + 1$, which concludes the proof. ■

These complements $({}^{\mathfrak{D}(v)}W', {}^{\mathfrak{D}(v)}W'')$ might replace the original pair (W', W'') replete with respect to $v \in \mathrm{Prime}$ (or Bis): as we will show in Lemma 3.1.3.3, if the additivity of the rank fails for (W', W''), it also fails for $({}^{\mathfrak{D}(v)}W', {}^{\mathfrak{D}(v)}W'')$. Moreover, $({}^{\mathfrak{D}(v)}W', {}^{\mathfrak{D}(v)}W'')$ does not involve $v \in \mathrm{Prime}$ (or Bis). The opposite implication is not true as Lemma 3.1.3.4 states.

Lemma 3.1.3.3. *Suppose* (W', W'') *is replete with respect to* $v \in \mathrm{Prime}$ *(or* Bis*), define* $S' := {}^{\mathfrak{D}(v)}W'$ *and* $S'' := {}^{\mathfrak{D}(v)}W''$ *and set* $S = S' \oplus S''$. *Then*

(i) $R(S) = R(W) - 1$ *and the space* $\langle(\mathrm{Prime}\setminus v), \mathrm{Bis}, \mathrm{HL}, \mathrm{HR}, \mathrm{VL}, \mathrm{VR}, \mathrm{Mix}\rangle$ *determines a minimal decomposition of* S.

(ii) If the additivity of the rank $R(S) = R(S') + R(S'')$ *holds for* S, *then it also holds for* W, *that is* $R(W) = R(W') + R(W'')$.

Proof. We will prove only case when $v \in \mathrm{Prime}$. The case when $v \in \mathrm{Bis}$ is similar. By Lemma 3.1.3.2 we have $W = S \oplus \langle v\rangle$, thus $R(W) \leq R(S) + 1$. On the other hand, $S \subset \langle(\mathrm{Prime}\setminus v), \mathrm{Bis}, \mathrm{HL}, \mathrm{HR}, \mathrm{VL}, \mathrm{VR}, \mathrm{Mix}\rangle$, hence $R(S) \leq R(W) - 1$. These two claims show the equality for $R(S)$ in (i) and that $\langle(\mathrm{Prime}\setminus v), \mathrm{Bis}, \mathrm{HL}, \mathrm{HR}, \mathrm{VL}, \mathrm{VR}, \mathrm{Mix}\rangle$ gives a minimal decomposition of S.

Finally, if $R(S) = R(S') + R(S'')$, then:

$$R(W) = R(S) + 1 = R(S') + R(S'') + 1 \geq R(W') + R(W''),$$

showing the statement (ii) for W. ■

Lemma 3.1.3.4. *Assume that there exist a counterexample to additivity of tensor rank over a base field* $\Bbbk$, *for example* $\Bbbk = \mathbb{C}$ *(see Theorem 1.2.5.4). Then, there exists an example of a pair* (W', W'') *of linear spaces over* $\Bbbk$ *such that:*

(i) (W', W'') *is replete with respect to* $v \in \mathrm{Prime}$,

(ii) *the additivity of the rank holds for* $W = W' \oplus W''$,

(iii) *the additivity of the rank does not hold for* $S = S' \oplus S''$, *where* $S' := {}^{\mathfrak{D}(v)}W'$, $S'' := {}^{\mathfrak{D}(v)}W''$.

Proof. Assume conversely, that additivity of the rank $R(S) = R(S') + R(S'')$ holds for S if and only if it holds for W, that is $R(W) = R(W') + R(W'')$.

Then take a pair (W', W'') such that $R(W' \oplus W'') < R(W') + R(W'')$ and a minimal basis $\mathcal{B}'$ of rank one matrices such that $W' \subseteq \langle \mathcal{B}' \rangle$. We construct $\tilde{W'}_1$ by adding an element $v \in \mathcal{B}'$ to W'. Let us observe, that $R(\tilde{W'}_1) = R(W')$. Indeed, $\tilde{W'}_1 \subseteq \langle \mathcal{B} \rangle$ implies $R(\tilde{W'}_1) \leq R(W')$ The opposite inequality follows from the fact, that $W' \subseteq \tilde{W'}_1$. If $R(\tilde{W'}_1 \oplus W'') = R(\tilde{W'}_1) + R(W'')$ we have a contradiction, because we can always choose a basis for $\tilde{W'}_1 \oplus W''$ and partition Prime, Bis, ..., Mix in a way that $v \in$ Prime. Thus, we may assume the right hand side is smaller.

We repeat the process with $v_1 \neq v_2 \in \mathcal{B}'$ and $\tilde{W'}_1$ in place of W', obtaining subspace $\tilde{W'}_2$. We do it inductively. We denote by n the smallest number i such that $\mathcal{B}' \subseteq \tilde{W'}_i$. As discussed before, we must have $R(\tilde{W'}_n \oplus W'') < R(\tilde{W'}_n) + R(W'')$ and we may assume that in the minimal decomposition of $\tilde{W'}_i \oplus W''$ all matrices from $\mathcal{B}'$ belong to Prime. After the process of digestion of all Primes of $\tilde{W'}_n \oplus W''$ we obtain $\emptyset \oplus W''$ for which rank additivity trivially holds. Thus from Lemma 3.1.3.3 (ii) we know, that $R(\tilde{W'}_n \oplus W'') = R(\tilde{W'}_n) + R(W'')$, a contradiction. ∎

As a summary, in our search for examples of failure of the additivity of the rank, in the previous section we replaced a linear space $W = W' \oplus W''$ by its repletion with respect to $v \in$ Prime (or Bis) ${}^{\mathfrak{R}(v)}W = {}^{\mathfrak{R}(v)}W' \oplus {}^{\mathfrak{R}(v)}W''$, that is possibly larger. Here in turn, we replace ${}^{\mathfrak{R}(v)}W$ by a smaller linear space $S := S' \oplus S''$, where $S' := {}^{\mathfrak{D}(v)}({}^{\mathfrak{R}(v)}W')$, $S'' := {}^{\mathfrak{D}(v)}({}^{\mathfrak{R}(v)}W'')$. In fact, $\dim S' \leq \dim W'$ and $\dim S'' \leq \dim W''$, and also $R(S) \leq R(W)$. That is, changing W into S neither makes the corresponding tensors larger nor decreases the defect.

Corollary 3.1.3.5. *Let* (W', W'') *be a pair of linear spaces,* $v \in$ Prime *and* $(S', S'') := ({}^{\mathfrak{D}(v)}({}^{\mathfrak{R}(v)}W'), {}^{\mathfrak{D}(v)}({}^{\mathfrak{R}(v)}W''))$. *Then the following inequalities holds:*

(i) $0 \leq \dim {}^{\mathfrak{R}(v)}W' - \dim W' \leq 1$,

(ii) $\dim S' = \dim {}^{\mathfrak{R}(v)}W' - 1$,

(iii) $\dim S'' = \dim W''$,

(iv) $R(S) = R(W) - 1$ *and the space* $\langle (\mathrm{Prime} \setminus v), \mathrm{Bis}, \mathrm{HL}, \mathrm{HR}, \mathrm{VL}, \mathrm{VR}, \mathrm{Mix} \rangle$ *determines a minimal decomposition of* S,

(v) $R(W') - 1 \leq R(S') \leq R(W') + (\dim {}^{\mathfrak{R}(v)}W' - \dim W')$,

(vi) $S'' = W''$, *in particular* $R(S'') = R(W'')$.

Moreover, the defect does not decrease after the process of repletion and digestion by v. *In particular if the rank additivity does not hold for* (W', W''), *then for* (S', S'') *does not hold as well.*

Proof. The proof follows directly from Proposition 3.1.2.4, Lemma 3.1.3.2 and Lemma 3.1.3.3. ∎

The following observation states, that after repletion and digestion with respect to all elements of Prime there is no a' in A'^* such that the slice $p(a') \in B' \otimes C'$ is of rank one.

Lemma 3.1.3.6. *Suppose $p(A^*) = W' \oplus W'' \subseteq (B' \oplus B'') \otimes (C' \oplus C'')$, where W' is equal to its digested and repleted version with respect to all elements of* Prime*. Then, there is no $a' \in A'^*$ such that $v := p(a') \in B' \otimes C'$ is a rank 1 matrix.*

Proof. Let us assume the opposite, there exists $a' \in A'^*$ such that $v \in B' \otimes C'$ is a rank 1 matrix. It follows from the assumption about W' that $\mathrm{Prime} = \emptyset$, thus $v \in \langle \mathrm{HL}, \mathrm{VL}, \mathrm{HR}, \mathrm{VR}, \mathrm{Bis}, \mathrm{Mix} \rangle \setminus \langle \mathrm{Bis} \rangle$. It is a contradiction with the way we partition the basis $\mathcal{B}$ to $\mathrm{Prime}, \mathrm{Bis}, ..., \mathrm{Mix}$ see Notation 3.1.1.3. ■

We may replenish and digest also in the other directions. It turns out, that we can precisely say what happens with the hook-structure (Definition 2.2.2.1) when we choose the repletion vector wisely.

Corollary 3.1.3.7. *Assume that $p(A'^*) = W' \subset B' \otimes C'$ is $(\mathbf{k}, \mathbf{l})$-hook shaped, i.e. there exists $G' \subset B', H' \subset C'$ such that $W' \subset G' \otimes C' + B' \otimes H'$, where $dim(G') = \mathbf{k}, \dim(H') = \mathbf{l}$. Assume further, that there is $\gamma \in (C'/H')^*$ such that $v := p(\gamma)$ is a rank 1 matrix. Then after the process of repletion and digestion of $p(C^*)$ with respect to v we obtain tensor $\tilde{p}$ such that:*

- *(i) $\tilde{p} = \tilde{p}' \oplus p''$,*
- *(ii) $\tilde{p}' \in \tilde{A}' \otimes B' \otimes \tilde{C}'$, where $\tilde{A}' \subseteq A'$ is such that $\tilde{p}'$ is $\tilde{A}'$-concise, $\tilde{C}' := \gamma^{\perp} \subset C'$ is the linear hyperplane $(\gamma = 0)$, and $R(\tilde{p}) = R(W) - 1$,*
- *(iii) $\tilde{p}(A'^*)$ is still $(\mathbf{k}, \mathbf{l})$-hook shaped,*
- *(iv) If the additivity of the rank does not hold for p, then it also does not hold for $\tilde{p}$,*
- *(v) If p' is A'-concise, then $\dim A' - 1 \leq \dim \tilde{A}'$,*
- *(vi) If p' is A'-concise and the set* Prime *in the decomposition of $p(A^*)$ is empty, then $A' = \tilde{A}'$.*

Proof. For $p(C^*) \subseteq A \otimes B$ we choose a minimal decomposition $V_C = \langle V_{C,\mathrm{Seg}} \rangle \subset A \otimes B$ and $\mathrm{Prime}_C, \mathrm{Bis}_C, , \ldots, \mathrm{Mix}_C$ are as in Notation 3.1.1.3 (with added the subscript "$_C$" to stress that $B \otimes C$ is changed to $A \otimes B$). Since v is a rank one matrix and is contained in $A' \otimes B'$, we can choose a minimal decomposition such that $v \in \mathrm{Prime}_C$. The process of repletion with respect to v brings no change, because v is already contained in $p(C^*)$.

The matrix v is contained in $A' \otimes G'$, so it is $(0, \mathbf{k})$-hook shaped. In the process of digestion with respect to v we obtain the new tensor $\tilde{p}$ such that $\tilde{p}(C''^*) = \tilde{p}''(C''^*) = p''(C''^*)$ and $\tilde{p}(C'^*) = \tilde{p}'(C'^*)$ which differs from $p'(C'^*)$ only in the places corresponding to $A' \otimes G'$. We conclude (i) and (iii). Items (ii) and (iv) follow from Corollary 3.1.3.5.

Let us assume that p' is A'-concise. Observe, that $p' = \tilde{p}' + v \otimes (C'/\tilde{C}') \in A' \otimes B' \otimes C'$. Since v is a rank 1 matrix, then either $v \in \tilde{A}' \otimes B'$ or there exists

$$B'\left\{\overbrace{\begin{bmatrix} & & & v_{1,4} & v_{1,5}\\ & & & v_{2,4} & v_{2,5}\\ v_{3,1} & 0 & v_{3,3} & v_{3,4} & v_{3,5}\\ v_{4,1} & 0 & v_{4,3} & v_{4,4} & v_{4,5}\\ v_{5,1} & \underbrace{v_{5,2}}_{p'(\gamma)} & v_{5,3} & v_{5,4} & v_{5,5}\end{bmatrix}}^{C'}\right. \qquad A'\left\{\overbrace{\begin{bmatrix} 0 & w_{1,2} & 0 & 0 & 0\\ w_{2,1} & w_{2,2} & w_{2,3} & w_{2,4} & w_{2,5}\\ w_{3,1} & w_{3,2} & w_{3,3} & w_{3,4} & w_{3,5}\\ w_{4,1} & w_{4,2} & w_{4,3} & w_{4,4} & w_{4,5}\\ w_{5,1} & \underbrace{w_{5,2}}_{p'(\gamma)} & w_{5,3} & w_{5,4} & w_{5,5}\end{bmatrix}}^{C'}\right. \begin{matrix}\} a \\ \\ \left.\begin{matrix}\\ \\ \\ \\ \end{matrix}\right\} \tilde{A}'\end{matrix}$$

Figure 3.2: Let a tensor $p = p' + p'' \in (A' \oplus A'') \otimes (B' \oplus B'') \otimes (C' \oplus C'')$, where $\dim(A', B', C') = (5, 5, 5)$ and $p(A'^*)$ is $(3, 2)$-hook shaped. At the figure, there are shown sample spaces of slices $p'(A'^*)$ and $p'(B^*)$. Zero elements are denoted either by a blank space or explicitly by 0. Notice, that p' is $A' -$ *concise*. We replete and digest $p(C^*)$ with respect to $p(\gamma)$, where $\gamma \in (C'/H')^*$. In result we obtain tensor $\tilde{p'} \oplus p''$ such that $p' = \tilde{p'} + p'(\gamma) \otimes \gamma \in A' \otimes B' \otimes C'$. Since $p(\gamma)$ is a rank 1 matrix, then either $v \in \tilde{A}' \otimes B'$ or there exists $a \in A' \setminus \tilde{A}'$ such that $v \in \langle \tilde{A}', a\rangle \otimes B'$. In the second case, let $\alpha \in A'^*$ be dual to a. Then $p'(\alpha) \in B' \otimes C'$ is a rank one matrix contained in $B' \otimes (\gamma = 1) \subseteq B' \otimes C'$.

$a \in A' \setminus \tilde{A}'$ such that $v \in \langle \tilde{A}', a\rangle \otimes B'$ (see Figure 3.2). In the first case $A' = \tilde{A}'$. In the second case $\langle \tilde{A}', a\rangle = A'$. We obtained (v). Let $0 \neq \alpha \in A'^*$ be such that $\alpha^{\perp} = \tilde{A}'$. Then, $p'(\alpha) \in B' \otimes C'$ is a rank one matrix contained in $B' \otimes (\gamma = 1)$. If $\mathrm{Prime}_A = \emptyset$, then we have a contradiction with Lemma 3.1.3.6. We proved (vi). ■

It follows from Corollary 3.1.3.7, that if in the pair of linear subspaces (W', W'') without the rank additivity property, one of them, say W' is $(1, \mathbf{k})$-hook shaped, then we can construct $\hat{W}'$, which is $(0, \mathbf{k})$ hook shaped and the pair $(\hat{W}', W'')$ does not poses the rank additivity property either.

Proposition 3.1.3.8. *Suppose $W' \subset B' \otimes C'$, $\mathbf{k} < \mathbf{c}'$, W' is $(1, \mathbf{k})$-hook shaped and $W'' \subset B'' \otimes C''$ is an arbitrary subspace. If the additivity of the rank fails for $W' \oplus W''$, then it also fails for smaller subspaces $\hat{W}', W''$, where $\hat{W}' \subset B' \otimes \Bbbk^{\mathbf{k}}$.*

Proof. We can assume that W' is concise. It is straightforward to verify, that for every $\gamma'^* \in (C'/F')^* \subset (C')^*$ we have $p(\gamma'^*) = p'(\gamma'^*) \in A' \otimes B'$ has rank 1. Then from Corollary 3.1.3.7, the process of repletion and digestion with respect to $p(\gamma'^*)$ leads to another pair $\widetilde{W'}, W''$, where $\widetilde{W'} \subset B' \otimes \Bbbk^{\mathbf{c}'-1}$. Subspace $\widetilde{W'}$ is again $(1, \mathbf{k})$-hook shaped. The new pair is also a counterexample to the additivity of the rank, if the starting pair was.

By a consecutive repeating the process for another $\mathbf{c}' - \mathbf{k} - 1$ times, we shrink W' to $(0, \mathbf{k})$-hook shaped $\hat{W}'$. Together with W'' it creates the desired pair $(\hat{W}', W'')$ from the statement of the proposition. ■

Proposition 3.1.3.8 allows us to generalize both [BPR20, Proposition 3.17] and Theorem 1.2.5.6. The latter one can be thought of as a theorem about $(0,2)$-hook shaped spaces. Observe, that we do not need the base field to be algebraically closed.

Corollary 3.1.3.9. *Let $W = W' \oplus W''$ where W' is a $(1,2)$-hook shaped, then the additivity of rank $R(W) = R(W') + R(W'')$ holds.*

Proof. We use Proposition 3.1.3.8 to reduce the problem to the case when $W' \subseteq B' \otimes C'$ is $(0,2)$-hook shaped. Then translating Theorem 1.2.5.6 to a language of subspaces (see Lemma 2.1.4.1), we obtain that the pair (W', W'') has rank additivity property. ∎

For a future reference, we state what we know about the case when the subspace Mix is not concise in $(E' \oplus E'') \otimes (F' \oplus F'')$. Say E'' can be replaced by a smaller one, $\widetilde{E}''$ such that $\text{Mix} \subseteq (E' \oplus \widetilde{E}'') \otimes (F' \oplus F'')$. To make the proof clearer, this time we assume that W'' is hook shaped.

Lemma 3.1.3.10. *Assume* $\text{Bis} = \emptyset$ *and* $\mathbf{k}$ *is the smallest natural number such that W'' is $(\mathbf{k}, \dim(F''))$-hook shaped. Let $\widetilde{E}'' \subseteq E''$ and $\widetilde{F}' \subseteq F'$ be the smallest subspaces such that* $\text{Mix} \subset (E' \oplus \widetilde{E}'') \otimes (\widetilde{F}' \oplus F'')$. *The additivity of ranks $R(W) = R(W') + R(W'')$ holds if all of the following conditions are fulfilled:*

(i) $\dim(\widetilde{E}'') \leq \mathbf{k} - 1$,

(ii) $\pi_{E' \oplus B''}(\text{VL})$ *is linearly independent and concise in* $(B'/E') \otimes F'$,

(iii) $\pi_{B' \oplus \widetilde{E}''} \pi_{C''}(\text{HR})$ *is linearly independent.*

Proof. Let us assume by contradiction, that additivity of ranks does not hold. Firstly we show that we can assume that there is no Prime or Bis in the decomposition of $W = W' \oplus W'' := (p' + p'')(A^*)$. If there is, we replete and digest obtaining $^{\mathfrak{D}\mathfrak{R}}W', {}^{\mathfrak{D}\mathfrak{R}}W''$ and show a contradiction for this new pair. Everything we need to know to make this assumption is given by Corollary 3.1.3.5.

To make it explicit, we have the following facts. By Corollary 3.1.3.5 (iv), if the space $\langle \text{Prime}, \text{Bis}, \text{HR}, \text{VR}, \text{VL}, \text{VR}, \text{Mix} \rangle$ determines the minimal decomposition for $W' \oplus W''$, then the subspace $\langle \text{HR}, \text{VR}, \text{VL}, \text{VR}, \text{Mix} \rangle$ determines the minimal decomposition for $^{\mathfrak{D}\mathfrak{R}}W' \oplus {}^{\mathfrak{D}\mathfrak{R}}W''$. Thus, the new pair $\{^{\mathfrak{D}\mathfrak{R}}W', {}^{\mathfrak{D}\mathfrak{R}}W''\}$ still fulfills conditions from the statement. By Corollary 3.1.3.5, if we prove the rank additivity for $\{^{\mathfrak{D}\mathfrak{R}}W', {}^{\mathfrak{D}\mathfrak{R}}W''\}$, we show it for starting tensors as well, contradicting our assumption.

From condition (i), there exists an element $w \in W''$ such that $\pi_{B' \oplus \widetilde{E}''} \pi_{C' \oplus F''}(w)$ is nonzero (cf. Figure 3.3). To present w as a linear combination of vectors from $\text{HL}, \text{HR}, \text{VL}, \text{VR}, \text{Mix}$ we need an element $h \in \text{HR}$ such that $\pi_{B' \oplus \widetilde{E}''} \pi_{C' \oplus F''}(h)$ is nonzero. Notice, that $\pi_{C''}(h)$ is nonzero. To get rid of $\pi_{C''}(h)$ in the presentation of w we have to use an element $v \in \langle \text{VL} \rangle$. Indeed, from conditions (i) and (iii) follows that we cannot restrict ourselves to elements from Mix or HR for it. Now, from

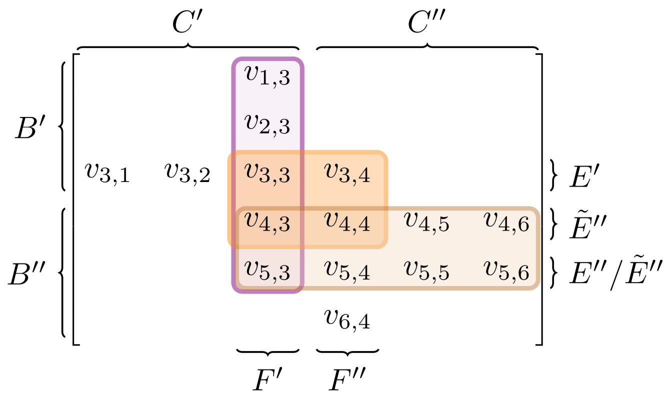

Figure 3.3: We use the notation from Lemma 3.1.3.10. At the figure, there are shown sample spaces of slices ${}^{\mathfrak{D}\mathfrak{R}}W' \oplus {}^{\mathfrak{D}\mathfrak{R}}W'' \subseteq (\Bbbk^3 \oplus \Bbbk^3) \otimes (\Bbbk^3 \oplus \Bbbk^3)$ such that $\mathbf{k} = 2$, $\dim(\tilde{E}'') = 1$. Zero elements are denoted by a blank space. We highlighted subspaces VL (purple with entries $v_{1,3}, v_{2,3}, v_{3,3}, v_{4,3}, v_{5,3}$), HR (brown with entries $v_{4,3}, v_{4,4}, v_{4,5}, v_{4,6}, v_{5,3}, v_{5,4}, v_{5,5}, v_{5,6}$), and Mix (middle orange square with entries $v_{3,3}, v_{3,4}, v_{4,3}, v_{4,4}$).

condition (ii) $\pi_{E' \oplus B''}(v) \neq 0$. Thus $\pi_{E' \oplus B''}(w) \neq 0$, which is a contradiction with the assumption that $w \in W''$.

■

3.1.4 Additivity of the tensor rank for small tensors

We conclude our discussion of the additivity of the tensor rank with the following summarizing results.

Theorem 3.1.4.1 ([BPR20, Thm 4.14.])**.** *Over an arbitrary base field $\Bbbk$, assume $p' \in A' \otimes B' \otimes C'$ is any tensor, while $p'' \in A'' \otimes B'' \otimes C''$ is concise and $R(p'') \leq \mathbf{a}''+2$. Then the additivity of the rank holds:*

$$R(p' \oplus p'') = R(p') + R(p'').$$

The analogous statements with the roles of A replaced by B or C, or the roles of $'$ and $''$ swapped, hold as well.

Proof. Since p'' is concise, the corresponding vector subspace $W'' = p''(A''^*)$ has dimension equal to $\mathbf{a}''$. By Corollary 3.1.3.9 we can restrict ourselves to the case when $\mathbf{e}'' \geq 2$ or $\mathbf{f}'' \geq 2$. Say, $\mathbf{e}'' \geq 2 \geq R(p'') - \dim W''$, then by Corollary 2.2.1.4 or Corollary 3.1.2.6 the additivity must hold. ■

Theorem 3.1.4.2 ([BPR20, Thm 4.15.])**.** *Assume the base field is $\Bbbk = \mathbb{C}$ or $\Bbbk = \mathbb{R}$ (complex or real numbers) and let $p' \in A' \otimes B' \otimes C'$ be any tensor, while $p'' \in A'' \otimes \Bbbk^3 \otimes \Bbbk^3$ for an arbitrary vector space A''. Then the additivity of the rank holds: $R(p' \oplus p'') = R(p') + R(p'')$.*

Proof. By Corollary 3.1.3.9, we can assume p'' is concise in $A'' \otimes \Bbbk^3 \otimes \Bbbk^3$. But then, by [SMS10, Thm 5 and Thm 6], the rank of p'' is at most $\mathbf{a}'' + 2$ and the result follows from Theorem 3.1.4.1. ■

Note, that in the proof above we exploit the results about maximal rank in $\Bbbk^{\mathbf{a}''} \otimes \Bbbk^3 \otimes \Bbbk^3$. In [SMS10] authors assume that the base field is $\mathbb{C}$ or $\mathbb{R}$. We are not aware of any similar results over other fields, with the unique exception of $\mathbf{a}'' = 3$, see the following proof for a discussion.

Theorem 3.1.4.3 ([BPR20, Thm 4.16.])**.** *Assume the base field $\Bbbk$ is such that the maximal rank of a tensor in $\Bbbk^3 \otimes \Bbbk^3 \otimes \Bbbk^3$ is at most 5. (For example $\Bbbk$ is algebraically closed of characteristic $\neq 2$ or $\Bbbk = \mathbb{R}$). Furthermore, let $R(p'') \leq 6$. Then independently of p', the additivity of the rank holds: $R(p' \oplus p'') = R(p') + R(p'')$.*

Proof. Without loss of generality, We can restrict ourselves to the case when p'' is concise in $A'' \otimes B'' \otimes C''$. As in the previous proof, if any of the dimensions $\dim A''$, $\dim B''$, $\dim C''$ is at most 2, then the claim follows from Theorem 1.2.5.6. On the other hand, if any of the dimensions $\mathbf{a}''$, $\mathbf{b}''$, $\mathbf{c}''$ is at least 4, then the result follows from Theorem 3.1.4.1. The remaining case $\mathbf{a}'' = \mathbf{b}'' = \mathbf{c}'' = 3$ also follows from Theorem 3.1.4.1 by our assumption on the field $\Bbbk$.

The assumption is satisfied for $\Bbbk = \mathbb{R}, \mathbb{C}$ see [BH13, Thm 5.1] or [SMS10, Thm 5]. In [BH13, top of p. 402] the authors say, that their proof is also valid for any algebraically closed field of characteristic not equal to 2. They also provide the interesting history of this question and furthermore, they show that the assumption about maximal rank in $\Bbbk^3 \times \Bbbk^3 \times \Bbbk^3$ fails for $\Bbbk = \mathbb{Z}_2$. ■

Assuming the base field is $\Bbbk = \mathbb{C}$, one of the smallest cases not covered by the above theorems would be the case of $p', p'' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$. The generic rank (that is, the rank of a general tensor) in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ is 6, moreover [AS79, p. 6] claims the maximal rank is 7 (see also [SMS10, Prop. 2]). To prove Corollary 3.1.4.13, i.e rank additivity property for the mentioned cases, we need to establish the following lemma.

Lemma 3.1.4.4. *Let us use Notation 3.1.1.3. Assume $\Bbbk$ is an algebraically closed base field, $W' \subseteq \Bbbk^{\mathbf{b}'} \otimes \Bbbk^{\mathbf{c}'}$ is of dimension $\mathbf{a}'$, $W'' \subseteq \Bbbk^{\mathbf{b}''} \otimes \Bbbk^{\mathbf{c}''}$ is of dimension $\mathbf{a}''$, the corresponding tensor is $p = p' \oplus p'' \in A' \otimes B' \otimes C' \oplus A'' \otimes B'' \otimes C''$ and:*

- *(i)* W' *is concise,*
- *(ii)* $\mathrm{Prime} = \emptyset$, $\mathrm{Bis} = \emptyset$,
- *(iii)* $\mathbf{a}' + 3 = R(W')$,
- *(iv)* $R(W') \leq \mathbf{b}' + \mathbf{c}'$,

Then either additivity of ranks holds or all the following conditions are satisfied

- $R(W') + R(W'') - 1 = R(W)$
- $R(W') = \mathbf{hl} + \mathbf{vl} + 2$
- $R(W'') = \mathbf{hr} + \mathbf{vr} + 1$
- $\mathbf{mix} = 2$

Proof. Let us assume the additivity of ranks does not hold. Corollary 2.2.1.4 and assumption (iii) imply that both $\mathbf{e}', \mathbf{f}'$ are at most 2. From Corollary 3.1.3.9 and assumption (ii) follows that $\mathbf{e}' = \mathbf{f}' = 2$.

Now we show that the defect equals one, i.e. $d := R(W') + R(W'') - R(W) = 1$. We have from Lemma 2.2.1.2 that $R(W'') + \mathbf{e}' + \mathbf{a}' \leq R(W)$, thus $R(W') + R(W'') - 1 \leq R(W)$ by the condition (iii). The opposite inequality follows from the failure of the rank additivity.

Since Prime $= \emptyset$, we must have $W' \subset \langle \pi_{C''}(\mathrm{HL}), \pi_{B''}(\mathrm{VL}), \pi_{B''}\pi_{C''}(\mathrm{Mix})\rangle \subset E' \otimes C' + B' \otimes F'$. That is, W' is $(2,2)$-hook shaped.

It follows from Proposition 3.1.3.8 that there are no integers $n < \mathbf{b}'$, $m < \mathbf{c}'$ such that W' is $(n,1)$-hook shaped or $(1,m)$-hook shaped. Moreover, we can assume that

(a) there is no $\beta \in (B'/E')^*$ such that $p(\beta)$ is a rank one matrix, and
(b) there is no $\gamma \in (C'/F')^*$ such that $p(\gamma)$ is a rank one matrix.

In particular, $\langle \pi_{F' \oplus C''}(\mathrm{HL})\rangle$ is concise in $E' \otimes (C'/F')$.

Next, we show that

$$\mathbf{b}' - 1 \leq \mathbf{vl}. \tag{3.1.4.5}$$

For this purpose we consider the projection $\pi_{E' \oplus B''} \colon B \to B'/E'$. The related map $B \otimes C \to (B'/E') \otimes C$ (which by the standard abuse we also denote $\pi_{E' \oplus B''}$), kills all the rank one tensors of types HL, HR, VR and Mix, possibly leaving a few of type VL alive. The image $\pi_{E' \oplus B''}(W) \subset (B'/E') \otimes F'$ has rank at most $\mathbf{vl}$ and is concise (otherwise, either there is $\beta \in (B'/E')^*$ such that $p(\beta)$ is a rank one matrix or p' is not concise, a contradiction in both cases). Note, that $(B'/E') \otimes F' \simeq \mathbb{C}^{\mathbf{b}'-2} \otimes \mathbb{C}^2$. Concise linear subspaces G of $\mathbb{C}^{\mathbf{b}'-2} \otimes \mathbb{C}^2$ need to have a rank at least $\mathbf{b}' - 2$.

In the case, when the rank of G is exactly $\mathbf{b}' - 2$. The tensor g corresponding to G, is contained in the space $\mathbb{C}^{\mathbf{b}'-2} \otimes (B'/E') \otimes F' \simeq \mathbb{C}^{\mathbf{b}'-2} \otimes \mathbb{C}^{\mathbf{b}'-2} \otimes \mathbb{C}^2$. It follows from Lemma 2.2.2.5 that there exists $\beta' \in (B'/E')^* \subset B'^*$ such that $g(\beta')$ has rank 1. Furthermore, both tensors $p'(\beta') \in A' \otimes C'$ and $p(\beta') \in A \otimes C$ have rank 1 as well, contradicting (a). Thus, $R(\pi_{E' \oplus B''}(W))$ must be at least $\mathbf{b}' - 1$ and consequently, $\mathbf{b}' - 1 \leq \mathbf{vl}$. Analogously, we can prove

$$\mathbf{c}' - 1 \leq \mathbf{hl}. \tag{3.1.4.6}$$

By Proposition 3.1.1.4(ii) we obtain $R(W'') \leq \mathbf{hr} + \mathbf{vr} + \mathbf{mix} = R(W) - (\mathbf{hl} + \mathbf{vl})$. Thus

$$\mathbf{hl} + \mathbf{vl} \leq R(W') - 1. \tag{3.1.4.7}$$

and similarly

$$\mathbf{hr} + \mathbf{vr} \leq R(W'') - 1. \tag{3.1.4.8}$$

At least one of $\pi_{E' \oplus B''}(\mathrm{VL})$, $\pi_{F' \oplus C''}(\mathrm{HL})$ is linearly independent. Otherwise, arguing as before, we see that $\pi_{E' \oplus B''}(W)$ has rank at most $\mathbf{vl} - 1$. Thus, $\mathbf{b}' - 1 \leq \mathbf{vl} - 1$ and similarly $\mathbf{c}' - 1 \leq \mathbf{hl} - 1$. Together with an inequality 3.1.4.7 it gives

$\mathbf{b}' + \mathbf{c}' \leq \mathbf{hl} + \mathbf{vl} \leq R(W') - 1 \leq \mathbf{b}' + \mathbf{c}' - 1$, a contradiction. The last inequality is implied by the condition (iv).

Now we will prove, that $R(W') \leq \mathbf{hl} + \mathbf{vl} + 2$. Let us assume $\pi_{E' \oplus B''}(\mathrm{VL})$ is linearly independent. (In the case when $\pi_{F' \oplus C''}(\mathrm{HL})$ is linearly independent we proceed similarly by exchanging B' with C' and HL with VL). It follows

$$W'' = \pi_{B'}(W'') \subseteq \pi_{B'}(\langle \mathrm{HR}, \mathrm{VR}, \mathrm{Mix} \rangle) \tag{3.1.4.9}$$

and $W'' \not\subseteq \pi_{B'}(\langle \mathrm{HR}, \mathrm{VR} \rangle)$ because of (3.1.4.8). We obtain from Lemma 2.1.3.2 and (3.1.4.9) that $R(W'')+1 \leq \mathbf{hr}+\mathbf{vr}+\mathbf{mix}$. Together with $\mathbf{hl}+\mathbf{hr}+\mathbf{vl}+\mathbf{vr}+\mathbf{mix} = R(W') + R(W'') - 1$ (because defect $d = 1$) it gives $\mathbf{hl} + \mathbf{vl} \leq R(W') - 2$.

Assumptions (iii), (iv) and Equations (3.1.4.5), (3.1.4.6) imply that

$$R(W') - 2 = \mathbf{a}' + 1 \leq \mathbf{b}' + \mathbf{c}' - 2 \leq \mathbf{hl} + \mathbf{vl}.$$

Hence, the inequalities can be changed into equalities. In particular $R(W') = \mathbf{hl} + \mathbf{vl} + 2$, thus

$$R(W'') = \mathbf{hr} + \mathbf{vr} + \mathbf{mix} - 1. \tag{3.1.4.10}$$

In result $\mathbf{mix} \geq 2$. Indeed, from Corollary 3.1.1.6 follows that $\mathbf{mix} \geq 1$. If we assume $\mathbf{mix} = 1$, then we look at the $\pi_{B''}\pi_{C''}(W) = W''$, which is contained in $\langle \pi_{B''}\pi_{C''}(\mathrm{HL}), \pi_{B''}\pi_{C''}(\mathrm{VL}), \pi_{B''}\pi_{C''}(\mathrm{Mix}) \rangle$. Taking the rank into account we obtain $R(W'') \leq R(W) - \mathbf{vr} - \mathbf{hr} = R(W) - R(W')$. It is against our assumption saying, that the additivity of ranks does not hold.

We get from (3.1.4.8) and (3.1.4.10) that $R(W'') \leq R(W'') + \mathbf{mix} - 2$, thus $\mathbf{mix} = 2$ which ends our proof. ∎

Remark 3.1.4.11. In Lemma 3.1.4.4 the assumption $\mathrm{Bis} = \emptyset$ can be relaxed in the following way. Let us assume that the pair (W', W'') is such that the additivity of ranks does not hold, $\mathrm{Prime} = \emptyset, \mathrm{Bis} \neq \emptyset$ and assumptions (i),(iii),(iv) of the Lemma 3.1.4.4 are fulfilled. We digest and replenish obtaining $({}^{\mathfrak{DR}}W', {}^{\mathfrak{DR}}W'') = (W', {}^{\mathfrak{DR}}W'')$. From Corollary 3.1.3.5 follows that $(W', {}^{\mathfrak{DR}}W'')$ is another pair without the rank additivity property and which fulfills all assumptions of Lemma 3.1.4.4. Thus we obtain:

- $R(W') + R({}^{\mathfrak{DR}}W'') - 1 = R(W)$
- $R({}^{\mathfrak{DR}}W'') = \mathbf{hl} + \mathbf{vl} + 2$
- $R({}^{\mathfrak{DR}}W'') = \mathbf{hr} + \mathbf{vr} + 1$
- $\mathbf{mix} = 2$

If both tensors p', p'' fulfill the assumptions of the Lemma 3.1.4.4, then the pair (p', p'') posses the rank additivity property.

Corollary 3.1.4.12. *Over an algebraically closed base field $\Bbbk$, assume $W' \subseteq \Bbbk^{\mathbf{b}'} \otimes \Bbbk^{\mathbf{c}'}$ is of dimension $\mathbf{a}'$, $W'' \subseteq \Bbbk^{\mathbf{b}''} \otimes \Bbbk^{\mathbf{c}''}$ is of dimension $\mathbf{a}''$, the corresponding tensor is $p = p' \oplus p'' \in A' \otimes B' \otimes C' \oplus A'' \otimes B'' \otimes C''$ and:*

(i) W', W'' are concise,

(ii) Prime $=$ Bis $= \emptyset$,
(iii) $\mathbf{a}' + 3 = R(W')$,
(iv) $\mathbf{a}'' + 3 = R(W'')$,
(v) $R(W') \leq \mathbf{b}' + \mathbf{c}'$,
(vi) $R(W'') \leq \mathbf{b}'' + \mathbf{c}''$.
Then additivity of ranks holds.

Proof. Assume the additivity of ranks does not hold. We obtain from Lemma 3.1.4.4 that $R(W') = \mathbf{hl} + \mathbf{vl} + 2$, $R(W'') = \mathbf{hr} + \mathbf{vr} + 1$. Now we can exchange every $'$ with $''$ and $(\mathrm{Prime}, \mathrm{HL}, \mathrm{VL})$ with $(\mathrm{Bis}, \mathrm{HR}, \mathrm{VR})$ and apply the same lemma again. This time we have (in the notation before the exchange) $R(W') = \mathbf{hl} + \mathbf{vl} + 1$, $R(W'') = \mathbf{hr} + \mathbf{vr} + 2$. A contradiction. ∎

We end the subsection with a positive answer for the question about rank additivity property for 2×2 matrix multiplication tensors (over a base field $\mathbb{C}$), i.e. $\mu_{2,2,2} \oplus \mu_{2,2,2} \in \mathbb{C}^{4+4} \otimes \mathbb{C}^{4+4} \otimes \mathbb{C}^{4+4}$. On a way to prove it we need to show the following fact. If both tensors from the pair (p', p'') have ranks less or equal 7, or if at least one of linear spaces from every triple $\{A', B', C'\}$, $\{A'', B'', C''\}$ is 3 dimensional and all other spaces are at most 4 dimensional, then the additivity of rank holds.

Corollary 3.1.4.13. *Over the base field $\mathbb{C}$, if $(\mathbf{a}', \mathbf{b}', \mathbf{c}') = (4, 4, 3)$, and either $(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'') = (4, 4, 3)$ or $(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'') = (4, 3, 4)$, then rank additivity holds.*

Proof. Assume that the rank additivity does not hold. We can reduce the problem to $(W', W'') = ({}^{\mathfrak{D}\mathfrak{R}}W', {}^{\mathfrak{D}\mathfrak{R}}W'')$ by Corollary 3.1.3.5. Further, we assume that both tensors are concise. Indeed, if at least one of the tensors is not concise then it follows from either Theorem 3.1.4.2 or Theorem 1.2.5.6 that the additivity of rank holds.

[AS79, p. 6] claims that the maximal rank of tensors from $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ is 7 (see also [SMS10, Prop. 2]). As a corollary from Theorem 3.1.4.1, we may restrict ourselves to the case $R(W') = R(W'') = 7$. Applying Corollary 3.1.4.12 we obtain a contradiction. ∎

We prove the following corollary in a similar way.

Corollary 3.1.4.14. *Over the base field $\mathbb{C}$, if both tensors have ranks less or equal 7, then rank additivity holds.*

In particular, over the base field $\mathbb{C}$, a pair of 2×2 matrix multiplication tensor has rank additivity property, i.e. $R(\mu_{2,2,2} \oplus \mu_{2,2,2}) = R(\mu_{2,2,2}) + R(\mu_{2,2,2})$.

Proof. Assume the rank additivity does not hold. We can restrict ourselves to the case $(W', W'') = ({}^{\mathfrak{D}\mathfrak{R}}W', {}^{\mathfrak{D}\mathfrak{R}}W'')$ by Corollary 3.1.3.5.

We can assume that both tensors are concise by Lemma 2.1.3.2. As a corollary from Theorem 3.1.4.1, we obtain that each of the numbers $\mathbf{a}', \mathbf{a}'', \mathbf{b}', \mathbf{b}'', \mathbf{c}', \mathbf{c}''$ is less or equal 4. It follows from Corollary 3.1.4.13 and Theorems 1.2.5.6, 3.1.4.2 that $(\mathbf{a}', \mathbf{b}', \mathbf{c}') = (4,4,4)$ and either $(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'') = (4,4,4)$ or $(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'') = (4,4,3)$ or $(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'') = (4,3,4)$. Applying Theorem 3.1.4.1 again, we obtain $R(W') = R(W'') = 7$. It contradicts Corollary 3.1.4.12.

For the last part of the statement, notice that $\mu_{2,2,2} \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ and $R(\mu_{2,2,2}) = 7$ (see Theorem 1.2.4.1 and Example 1.2.4.2). ■

The following remark follows from Corollary 3.1.4.14 and Theorem 3.1.4.1.

Remark 3.1.4.15. Over the base field $\mathbb{C}$, the minimal case in which the counterexample for the rank additivity can occur is $p' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, $p'' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ such that $R(p') = 8$, $R(p'') = 7$.

3.2 Additivity of the tensor border rank

Throughout this section we will follow Notations 2.2.0.1 and 2.2.0.2. Moreover, we restrict to the base field $\Bbbk = \mathbb{C}$. This section is based on [BPR20, Sect. 5.].

We turn our attention to the additivity of the border rank. That is, we ask for which tensors $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ the following equality holds:

$$\underline{R}(p' \oplus p'') = \underline{R}(p') + \underline{R}(p'').$$

Since the known counterexamples to the additivity are much smaller than in the case of the additivity of the tensor rank, our methods are more restricted to very small cases.

We commence with the following elementary observation.

Lemma 3.2.0.1 ([BPR20, Lem. 5.1.]). *Consider concise tensors $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ with $\underline{R}(p') \leq \mathbf{a}'$ and $\underline{R}(p'') \leq \mathbf{a}''$ (thus in fact $\underline{R}(p') = \mathbf{a}'$ and $\underline{R}(p'') = \mathbf{a}''$). Let $p = p' \oplus p''$. Then the additivity of the border rank holds $\underline{R}(p) = \underline{R}(p') + \underline{R}(p'')$.*

Proof. Since p' and p'' are concise, the linear maps $p' \colon (A')^* \to B' \otimes C'$ and $p'' \colon (A'')^* \to B'' \otimes C''$ are injective. Then also the map $p \colon A^* \to B \otimes C$ is injective and

$$\underline{R}(p) \geq \dim p(A^*) = \dim p'((A')^*) + \dim p''((A'')^*) = \underline{R}(p') + \underline{R}(p'').$$

The opposite inequality always holds. ■

Corollary 3.2.0.2 ([BPR20, Cor. 5.2.]). *Suppose both triples of integers $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ and $(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'')$ fall into one of the following cases: $(a, b, 1)$, $(a, 1, c)$, $(a, b, 2)$ with $a \geq b \geq 2$, $(a, 2, c)$ with $a \geq c \geq 2$, (a, b, c) with $a \geq bc$. Then for any concise tensors $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ the additivity of the border rank holds.*

Note, that the list of triples in the corollary is a bit exaggerated, as some of these triples have no concise tensors. However, this phrasing is convenient for further applications and search for unsolved pairs of triples.

Proof. After removing the triples that do not admit any concise tensor the list reduces to: $(a, a, 1)$, $(a, 1, a)$, $(a, b, 2)$ (for $2 \le b \le a \le 2b$), $(a, 2, c)$ (for $2 \le c \le a \le 2c$), (bc, b, c). We claim, that in all these cases $\underline{R}(p') = \mathbf{a}'$ and $\underline{R}(p'') = \mathbf{a}''$. In fact:

- The claim is clear for $(a, 1, a)$, $(a, a, 1)$, and (bc, b, c).
- For $(a, a, 2)$ and $(a, 2, a)$ the claim follows from the classification of such tensors, see the argument in the first paragraph of [BHMT18, Sect. 5.3].
- For $(a, b, 2)$ (with $2 \le b < a \le 2b$), and $(a, 2, c)$ (with $2 \le c < a \le 2c$), the claim follows from the previous case: any such concise tensor T has border rank at least a. But T is at the same time a (non-concise) tensor in a larger tensor space $\mathbb{C}^a \otimes \mathbb{C}^a \otimes \mathbb{C}^2$ or $\mathbb{C}^a \otimes \mathbb{C}^2 \otimes \mathbb{C}^a$. Thus, by Lemma 2.1.3.1 the border rank of T is at most the generic (border) rank in this larger space, which is equal to a by the previous item.

Therefore we conclude using Lemma 3.2.0.1. ■

Theorem 1.2.6.1 claims, that the additivity of the border rank holds for $\mathbf{a}, \mathbf{b}, \mathbf{c} \le 4$. Most of the cases follow from Corollary 3.2.0.2, with the exception of $(3+1, 2+2, 2+2)$ and $(3+1, 3+1, 3+1)$, which are covered in Subsections 3.2.1 and 3.2.2.

3.2.1 Case $(3 + 1, 2 + \mathbf{b}'', 2 + \mathbf{c}'')$

Assume $\mathbf{a}' = 3$, $\mathbf{b}' = \mathbf{c}' = 2$ and $\mathbf{a}'' = 1$.

Proposition 3.2.1.1 ([BPR20, Prop. 5.10.]). *For any $p' \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and $p'' \in \mathbb{C}^1 \otimes \mathbb{C}^{\mathbf{b}''} \otimes \mathbb{C}^{\mathbf{c}''}$ the additivity of the border rank holds.*

Proof. We may assume p'' is concise, so that $\underline{R}(p'') = \mathbf{b}'' = \mathbf{c}''$. Also if p' is not concise, then Corollary 3.2.0.2 shows the claim. So suppose p' is concise and thus $\underline{R}(p') = 3$.

We can write $p' = a_1 \otimes w_1' + a_2 \otimes w_2' + a_3 \otimes w_3'$ and $p'' = a_4 \otimes w_4''$, where $w_1' \dots, w_3'$ are 2×2 matrices and w_4'' is an invertible $\mathbf{b}'' \times \mathbf{b}''$ matrix.

As for p', by Example 2.3.0.3 and Lemma 2.3.0.4 we can choose the more degenerate tensor, which has the following normal form:

$$w_1' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, w_2' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, w_3' = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Write $p = \sum_{i=1}^{4} a_i \otimes w_i$, where w_i are the following $(2 + \mathbf{b}'', 2 + \mathbf{b}'')$ partitioned matrices

$$w_i = \begin{bmatrix} w_i' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}, i = 1, 2, 3, \;\; w_4 = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & w_4'' \end{bmatrix}.$$

We use the same notation as in Section 2.3.1. We claim, that the matrix representing the contraction operator $p_A^\wedge$, denoted by $M_4(w_1, w_2, w_3, w_4)$ as in (2.3.1.6), has rank $7 + 3\mathbf{b}''$. We conclude that $\underline{R}(p) \geq 3 + \mathbf{b}'' = \underline{R}(p') + \underline{R}(p'')$ by Proposition 2.3.1.3 showing the additivity.

In order to prove the claim, we observe that

$$M_4(w_1, w_2, w_3, w_4) = \begin{bmatrix} \underline{0} & w_3 & -w_2 & w_4 & \underline{0} & \underline{0} \\ -w_3 & \underline{0} & w_1 & \underline{0} & -w_4 & \underline{0} \\ w_2 & -w_1 & \underline{0} & \underline{0} & \underline{0} & w_4 \\ \underline{0} & \underline{0} & \underline{0} & -w_1 & w_2 & -w_3 \end{bmatrix} \tag{3.2.1.2}$$

can be transformed via permutations of rows and columns into the following $(6 + 3\mathbf{b}'' + 2 + \mathbf{b}'', 6 + 3\mathbf{b}'' + 2 + 2 + 2 + 3\mathbf{b}'')$-partitioned matrix

$$\begin{bmatrix} M_3(w_1', w_2', w_3') & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & N & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & -w_1' & w_2' & -w_3' & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{bmatrix},$$

where N is the following $3\mathbf{b}'' \times 3\mathbf{b}''$ matrix

$$N = \begin{bmatrix} w_4'' & \underline{0} & \underline{0} \\ \underline{0} & -w_4'' & \underline{0} \\ \underline{0} & \underline{0} & w_4'' \end{bmatrix}.$$

One can compute, that the rank of

$$M_3(w_1', w_2', w_3') = \begin{bmatrix} \underline{0} & w_3' & -w_2' \\ -w_3' & \underline{0} & w_1' \\ w_2' & -w_1' & \underline{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

equals 5. Moreover, since $\mathrm{rk}(N) = 3\mathbf{b}''$ and $\mathrm{rk}((-w_1', w_2', -w_3')) = 2$, we conclude the proof of the claim. ∎

3.2.2 Case $(3 + 1, 3 + \mathbf{b}'', 3 + \mathbf{c}'')$

Recall our usual setting: $p' \in A' \otimes B' \otimes C'$, $p'' \in A'' \otimes B'' \otimes C''$, $\mathbf{a}' := \dim A'$, etc. (Notation 2.2.0.2). In this subsection we are going to prove the following case of additivity of the border rank.

Proposition 3.2.2.1 ([BPR20, Prop. 5.11.]). *The additivity of the border rank holds for $p' \oplus p''$ if $\mathbf{a}' = \mathbf{b}' = \mathbf{c}' = 3$, and p' is concise and $\mathbf{a}'' = 1$.*

Proof. By replacing B'' and C'' with smaller spaces, we may assume p'' is also concise and in particular $\mathbf{b}'' = \mathbf{c}''$. If $\underline{R}(p') = 3$, then Lemma 3.2.0.1 implies the claim. On the other hand, by Terracini's Lemma, $\underline{R}(p') \leq 5$. Thus, it is sufficient to treat the cases $\underline{R}(p') = 4$ and $\underline{R}(p') = 5$.

Let $\{a_1, a_2, a_3\}$ be a basis of A' and let $\{a_4\}$ be a basis of $A'' \simeq \mathbb{C}$. Write

$$p' = a_1 \otimes w_1' + a_2 \otimes w_2' + a_3 \otimes w_3', \tag{3.2.2.2}$$

where $w_1', w_2', w_3' \in W' := p'((A')^*) \subset B' \otimes C'$ are 3×3 matrices. Similarly, let

$$p = a_1 \otimes w_1 + a_2 \otimes w_2 + a_3 \otimes w_3 + a_4 \otimes w_4,$$

where $w_1, w_2, w_3, w_4 \in W := p(A^*) \subset B \otimes C$ are $(3 + \mathbf{b}'', 3 + \mathbf{b}'')$ partitioned matrices:

$$w_i = \begin{bmatrix} w_i' & \underline{0} \\ \underline{0} & 0 \end{bmatrix}, \ i = 1, 2, 3, \text{ and } w_4 = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & w_4'' \end{bmatrix}. \tag{3.2.2.3}$$

We now analyse the two cases $\underline{R}(p') = 4$ and $\underline{R}(p') = 5$ separately.

The additivity holds if the border rank of p' is equal to four. Assume by contradiction, that $\underline{R}(p) \leq \mathbf{b}''+3 = \underline{R}(p')+\underline{R}(p'')-1$. By Proposition 2.3.1.2(ii), we obtain the following equations: $f_{\mathbf{b}''+3}(x', y'+y'', z') = x' \operatorname{adj}(y'+y'')z' - z' \operatorname{adj}(y'+y'')x' = \underline{0}$, for every $x', y', z' \in W' = p\left((A')^*\right)$ and $0 \neq y'' \in W'' = p\left((A'')^*\right)$. We can see, that $\operatorname{adj}(y'+y'')$ is the following $(3+\mathbf{b}'', 3+\mathbf{b}'')$ partitioned matrix

$$\operatorname{adj}(y'+y'') = \begin{bmatrix} \det(y'') \operatorname{adj}(y') & \underline{0} \\ \underline{0} & \det(y') \operatorname{adj}(y'') \end{bmatrix}.$$

Therefore we have

$$x' \operatorname{adj}(y'+y'')z' = \begin{bmatrix} \det(y'')x' \operatorname{adj}(y')z' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}.$$

Since p'' is concise, $\det(y'') \neq 0$, and thus from the vanishing of $f_{\mathbf{b}''+3}(x', y'+y'', z')$ we also obtain, that $f_3(x', y', z') = x' \operatorname{adj}(y')z' - z' \operatorname{adj}(y')x' = 0$. Therefore $\underline{R}(p') \leq 3$ by Proposition 2.3.1.2(i), a contradiction.

The additivity holds if the border rank of p' is equal to five. Consider the projection $\pi : A \otimes B \otimes C \to A' \otimes B \otimes C$ given by

$$\begin{aligned} a_i &\mapsto a_i, \ \ i = 1, 2, 3 \\ a_4 &\mapsto a_1 + a_2 + a_3. \end{aligned}$$

Consider $\bar{p} := \pi(p) \in A' \otimes B \otimes C$ and write $\bar{p} = a_1 \otimes \bar{w}_1 + a_2 \otimes \bar{w}_2 + a_3 \otimes \bar{w}_3$, where, for $i = 1, 2, 3$, $\bar{w}_i$ is the $(3 + \mathbf{b}'', 3 + \mathbf{b}'')$ partitioned matrix

$$\bar{w}_i = \begin{bmatrix} w_i' & 0 \\ 0 & w_4'' \end{bmatrix}.$$

We claim, that $\mathrm{rk}(\bar{p}^{\wedge}_{A'}) = 9 + 2\mathbf{b}''$. Indeed, by swapping both rows and columns of $M_3(\bar{w}_1, \bar{w}_2, \bar{w}_3)$ (see Equation 2.3.1.4) we obtain the following $(9 + 3\mathbf{b}'', 9 + 3\mathbf{b}'')$ partitioned matrix

$$\begin{bmatrix} p'^{\wedge}_{A'} & \underline{0} \\ \underline{0} & M_3(w''_4, w''_4, w''_4) \end{bmatrix}.$$

Since $\underline{R}(p') = 5$, the matrix $p'^{\wedge}_{A'}$ has rank 9, by Proposition 2.3.1.5. Moreover, $M_3(w''_4, w''_4, w''_4)$ has rank $2\mathbf{b}''$. Therefore, Proposition 2.3.1.3 implies $\underline{R}(\bar{p}) \geq 5 + \mathbf{b}''$. We conclude by observing, that $\underline{R}(p) \geq \underline{R}(\bar{p})$. ■

This concludes the proof of Theorem 1.2.6.1, as all possible splittings $\mathbf{a} = \mathbf{a}' + \mathbf{a}''$, $\mathbf{b} = \mathbf{b}' + \mathbf{b}''$, $\mathbf{c} = \mathbf{c}' + \mathbf{c}''$ with $\mathbf{a}, \mathbf{b}, \mathbf{c} \leq 4$ are covered either by Corollary 3.2.0.2 or one of Propositions 3.2.1.1 or 3.2.2.1.

3.2.3 Analysis of the border rank additivity of tensors living in bigger spaces

One could analyse the additivity for $\mathbf{a}, \mathbf{b}, \mathbf{c} \leq 5$ (so for the bound one more than in Theorem 1.2.6.1) by checking all 10 possible cases listed in Table 3.1. In the following Example 3.2.3.1 we solve Case 3 from the table.

Example 3.2.3.1. *If $p' \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and $p'' \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ are both concise, then the additivity of the border rank holds for $p' \oplus p''$. Indeed, by Example 2.3.0.2 there exists $q'' \in \mathbb{C}^1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ more degenerate than p'', but of the same border rank. By Lemma 2.3.0.4 it is enough to prove the additivity for $p' \oplus q''$. This is provided by Proposition 3.2.2.1.*

We conclude the chapter by presenting how Multigraded Border Apolarity Lemma 2.4.1.7 can be applied to prove border rank additivity in a subcase of Case 1. Tensor $p \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ of border rank 3, has one of the following two normal forms ([BL13, Prop. 6.2.]):

$$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes b_1 \otimes c_2 \tag{3.2.3.2}$$

and

$$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3). \tag{3.2.3.3}$$

When, up to a permutation of the factors and change of bases, $p' \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and $p'' \in \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$ have normal forms (3.2.3.2) and (3.2.3.3) correspondingly, then $p' \oplus p''$ posses the additivity of border rank property.

Lemma 3.2.3.4. *Let $p' = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_1 \otimes c_2 + a_3 \otimes b_2 \otimes c_2$ and $p'' = a_4 \otimes (b_3 \otimes c_3 + b_4 \otimes c_4) + a_5 \otimes (b_3 \otimes c_4 + b_4 \otimes c_5)$. Then for $p := p' \oplus p'' \in \mathbb{C}^{3+2} \otimes \mathbb{C}^{2+3} \otimes \mathbb{C}^{2+2}$, $\underline{R}(p) = 6$.*

Proof. We have $\underline{R}(p) \leq \underline{R}(p') + \underline{R}(p'') = 6$. Let us assume that $\underline{R}(p' \oplus p'') \leq 5$. From Multigraded Border Apolarity Lemma 2.4.1.7 there exists an ideal $I \subseteq \tilde{T}^* := \mathrm{Sym}(\mathbb{C}^5 \oplus \mathbb{C}^5 \oplus \mathbb{C}^4)^* = \mathbb{C}[\alpha_1, \alpha_2, \ldots, \alpha_5, \beta_1, \beta_2, \ldots, \beta_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4]$ such that $I \subseteq \mathrm{Ann}(p)$ and

$$\forall_{i,j,k}\ \mathrm{codim}(I_{i,j,k} \subseteq \tilde{T}^*_{i,j,k}) = \min(\binom{i+4}{4}\binom{j+4}{4}\binom{k+3}{3}, 5). \tag{3.2.3.5}$$

If $(i,j,k) = (0,1,1)$, then $\mathrm{codim}(I_{0,1,1} \subseteq \tilde{T}^*_{0,1,1}) = 5 = \mathrm{codim}(\mathrm{Ann}(p)_{0,1,1} \subseteq \tilde{T}^*_{0,1,1})$. Let us explain the last equality more precisely. Firstly, notice that $\mathrm{codim}(\mathrm{Ann}(p)_{0,1,1} \subseteq \tilde{T}^*_{0,1,1}) = \dim(\tilde{T}^* \lrcorner p)_{1,0,0}$. Secondly, observe that $\dim(\tilde{T}^* \lrcorner p)_{1,0,0} = \dim \tilde{T}^*_{1,0,0} = 5$, because of conciseness of p.

Thus, $I_{0,1,1} = \mathrm{Ann}(p)_{0,1,1} = \{\beta_1\gamma_3, \beta_1\gamma_4, \beta_1\gamma_5, \beta_2\gamma_1, \beta_2\gamma_3, \beta_2\gamma_4, \beta_2\gamma_5, \beta_3\gamma_1, \beta_3\gamma_2, \beta_3\gamma_5, \beta_3\gamma_3 - \beta_4\gamma_4, \beta_3\gamma_4 - \beta_4\gamma_5, \beta_4\gamma_1, \beta_4\gamma_2, \beta_4\gamma_3)\}$.

Using Macaulay2 we calculate, that the Hilbert function of $\tilde{T}^*/(I_{0,1,1})$ in degree $(0,1,2)$ equals 4, contradicting (3.2.3.5). ■

Multigraded Border Apolarity Lemma 2.4.1.7 can be applied to other pairs of tensors $\{p', p''\}$ from Case 1. More generally, this tool can be used to investigate cases from the Table 3.1 in which $\underline{R}(p') + \underline{R}(p'') - 1 = \mathbf{a}' + \mathbf{a}''$, i.e. Cases $1, 2, 4, 6, 7$ and partially Cases $3, 9, 10$. However, a more sophisticated method is needed. Already in Case 1, if both tensors p', p'' have a normal form (3.2.3.2), we could not prove the additivity of a border rank, arguing similarly to the proof of Lemma 3.2.3.4. It is a subject of our follow-up research with Maciej Gałązka and Tomasz Mańdziuk. There is a chance, that the minimal counterexample occur among cases from the Table 3.1.

Other, similar problems remain open. We describe one instance suggested by Landsberg.

Problem 3.2.3.6. *Let $p' \in A' \otimes B' \otimes C'$ be any tensor and $p'' \in \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$ be a nonzero tensor. Is $\underline{R}(p' \oplus p'') = R(p') + 1$?*

Another worth mentioning questions waiting for the answer are the following. When the cactus (border) rank is additive? One can ask about it also in the setting of symmetric tensors and the symmetric tensor rank, or equivalently, for homogeneous polynomials and their Waring rank. No counterexamples to this version of the problem are yet known, while some partial positive results are described in [CCC15], [CCC+18], [CCO17], [CMM18], and [Tei15].

#	$(\mathbf{a}', \mathbf{b}', \mathbf{c}')$	$(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'')$	$\underline{R}(p')$	$\underline{R}(p'')$	Remarks
1.	3, 2, 2	2, 3, 2	3	3	Partially solved in Lemma 3.2.3.4; Possibly the application of the Multigraded B. A. Lem. is a first step to prove this case
2.	3, 3, 2	2, 2, 3	3	3	Possibly the application of the Multigraded B. A. Lem. is a first step to prove this case
3.	3, 3, 3	2, 2, 2	4, 5	2	Solved in Example 3.2.3.1
4.	4, 2, 2	1, 2, 2	4	2	Possibly the application of the Multigraded B. A. Lem. is a first step to prove this case
5.	4, 2, 2	1, 3, 3	4	3	
6.	4, 3, 2	1, 2, 2	4	2	Possibly the application of the Multigraded B. A. Lem. is a first step to prove this case
7.	4, 3, 3	1, 1, 1	5	1	Possibly the application of the Multigraded B. A. Lem. is a first step to prove this case
8.	4, 3, 3	1, 2, 2	5	2	
9.	4, 4, 3	1, 1, 1	5, 6	1	
10.	4, 4, 4	1, 1, 1	5, 6, 7	1	An important case containing $\mu_{2,2,2} \oplus \mu_{1,1,1}$

Table 3.1: The list of pairs of concise tensors and their border ranks that should be checked to determine the additivity of the border rank for $\mathbf{a}, \mathbf{b}, \mathbf{c} \leq 5$. This list contains all pairs of concise tensors not covered by Corollary 3.2.0.2, or Proposition 3.2.1.1, or Proposition 3.2.2.1, together with their possible border ranks, excluding the cases covered by Lemma 3.2.0.1. The maximal possible values of border ranks above have been obtained from [AOP09, Sect. 4].

Chapter 4

Identification of points of the (Grassmann) secant variety inside the (Grassmann) cactus variety

In this chapter, we solve the problem of identification of points of the (Grassmann) secant variety inside the (Grassmann) cactus variety in minimal cases where these varieties differ. For the notions of Grassmann secant and cactus varieties, see Definitions 2.4.1.1 and 2.4.2.1.

In Section 4.1, we analyze polynomials and subspaces divisible by a large power of a linear form and prove, that they are in a specific cactus variety (Theorems 4.1.0.2 and 4.1.0.3). Remarkably, in Section 4.2 it turns out, that in the case of $\sigma_{14}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for $d \geq 5$ and $n \geq 6$ limits of a certain subset of such forms fill up the closure of the set-theoretic difference between the cactus variety and the secant variety (Theorem 4.0.0.2). For $d \geq 6$, this allows us to design an algorithm for deciding whether a point in the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ is in the secant variety $\sigma_{14}(\nu_d\mathbb{P}(\mathbb{C}^{n+1}))$ (Theorem 4.0.0.4). In Section 4.3 we prove results analogous to those of Section 4.2 for the case of the Grassmann secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}(\mathbb{C}^{n+1})))$ for $n \geq 4$. This chapter is based on the work with Tomasz Mańdziuk and Maciej Gałązka [GMR20].

We use the notation coming from Definitions 2.4.0.1, 2.4.0.2 and the following one.

Definition 4.0.0.1. Let T be defined as in Definition 2.4.0.1, i.e. $T = \Bbbk_{dp}[x_0, x_1, \ldots, x_n]$. Given positive integers $d \geq m$ and $f \in T_{\leq m}$, we define

$$f^{\blacktriangledown d} := (d-m)!F_m + (d-m+1)!F_{m-1} + \ldots + d!F_0,$$

where $f = F_m + F_{m-1} + \cdots + F_0$ and $F_j \in T_j$.

Similarly, for a linear subspace $W \subseteq T_{\leq m}$

$$W^{\blacktriangledown d} := \{f^{\blacktriangledown d} \mid f \in W\}.$$

Now we can state the generalized versions of Theorems 1.3.1.1 and 1.3.1.3.

Theorem 4.0.0.2 ([GMR20, Thm 1.4.]). *Let $n \geq 6$ and $d \geq 5$ be integers and consider the polynomial ring $T = \mathbb{C}[x_0, \ldots, x_n]$. Then $\kappa_{14}(\nu_d(\mathbb{P}T_1))$ has two irreducible components, one of which is $\sigma_{14}(\nu_d(\mathbb{P}T_1))$, and we denote the other one by $\eta_{14}(\nu_d(\mathbb{P}T_1))$. Let $\psi\colon \mathbb{P}T_1 \times \mathbb{P}T_3 \to \mathbb{P}T_d$ be given by $([y_0], [P]) \mapsto [y_0^{d-3}P]$ and let $q\colon (T_1 \setminus \{0\}) \times (T_3 \setminus \{0\}) \to \mathbb{P}T_1 \times \mathbb{P}T_3$ be the natural map. Let*

$$\mathcal{C} := \{(y_0, P) \in T_1 \times T_3 \mid \textit{there exists a completion of } y_0 \textit{ to a basis } (y_0, y_1, \ldots, y_n) \textit{ of } T_1 \textit{ such that } \operatorname{Apolar}((P|_{y_0=1})^{\blacktriangledown d}) \textit{ has Hilbert function } (1,6,6,1)\}.$$

Then $\eta_{14}(\nu_d(\mathbb{P}T_1))$ is the closure of $\psi(q(\mathcal{C}))$.

Let $\operatorname{Gr}(k, V)$ be the Grassmannian of k-dimensional subspaces of a linear space V.

Theorem 4.0.0.3 ([GMR20, Thm 1.5.]). *Let $n \geq 4$ and $d \geq 5$ be integers and consider the polynomial ring $T = \mathbb{C}[x_0, \ldots, x_n]$. Then $\kappa_{8,3}(\nu_d(\mathbb{P}T_1))$ has two irreducible components, one of which is $\sigma_{8,3}(\nu_d(\mathbb{P}T_1))$, and we denote the other one by $\eta_{8,3}(\nu_d(\mathbb{P}T_1))$. Let $\psi\colon \mathbb{P}T_1 \times \mathrm{Gr}(3, T_2) \to \mathrm{Gr}(3, T_d)$ be given by $([y_0], [U]) \mapsto [y_0^{d-2}U]$ and let $q\colon (T_1 \setminus \{0\}) \to \mathbb{P}T_1$ be the natural map. Let*

$$\mathcal{C} := \{(y_0, [U]) \in T_1 \times \mathrm{Gr}(3, T_2) \mid \text{there exists a completion of } y_0 \text{ to a basis } (y_0, y_1, \ldots, y_n) \text{ of } T_1 \text{ such that } \mathrm{Apolar}((U|_{y_0=1})^{\blacktriangledown d}) \text{ has Hilbert function } (1, 4, 3)\}.$$

Then $\eta_{8,3}(\nu_d(\mathbb{P}T_1))$ is the closure of $\psi((q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)})(\mathcal{C}))$.

We prove Theorems 4.0.0.2 and 4.0.0.3 in Sections 4.2 and 4.3, respectively. Theorem 1.3.1.1 follows from Theorem 4.0.0.2, since for $n = 6$ the closure of $q(\mathcal{C})$ is $\mathbb{P}T_1 \times \mathbb{P}T_3$. For more details, see the last paragraph of the Proof of Theorems 1.3.1.1 and 4.0.0.2 in Section 4.2.1. For $6 < n$ the closure $\overline{\mathcal{C}}$ is a proper subset of $\mathbb{P}T_1 \times \mathbb{P}T_3$. Similarly, Theorem 1.3.1.3 follows from Theorem 4.0.0.3, since for $n = 6$ the closure of $q \times Id_{\mathrm{Gr}(3,T_2)}(\mathcal{C})$ is $\mathbb{P}T_1 \times \mathrm{Gr}(3, T_2)$. For more details, see the last paragraph of the Proof of Theorems 1.3.1.3 and 4.0.0.3 in Section 4.3.1. For $6 < n$ the closure $\overline{q \times Id_{\mathrm{Gr}(3,T_2)}(\mathcal{C})}$ is a proper subset of $\mathbb{P}T_1 \times \mathrm{Gr}(3, T_2)$.

The following theorem contains an algorithm to compute whether a point in the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}^n))$ is in the secant variety $\sigma_{14}(\nu_d(\mathbb{P}^n))$ for $d \geq 6, n \geq 6$. The case of $n = 6$ is implemented in Macaulay2, see Subsection 4.4.

Theorem 4.0.0.4 ([GMR20, Thm 1.6.]). *Let $T := \mathbb{C}[x_0, \ldots, x_n]$ be a polynomial ring with $n \geq 6$. Given an integer $d \geq 6$ and $[G] \in \kappa_{14}(\nu_d(\mathbb{P}T_1)) \subseteq \mathbb{P}T_d$ the following algorithm checks if $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1))$.*

Step 1 *Compute the ideal $\mathfrak{a} := \sqrt{((\mathrm{Ann}\, G)_{\leq d-3})} \subseteq T^* := \mathbb{C}[\alpha_0, \ldots, \alpha_n]$.*

Step 2 *If $\mathfrak{a}_1$ is not n-dimensional, then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1))$ and the algorithm terminates. Otherwise compute $\{K \in T_1 \mid \mathfrak{a}_1 \lrcorner K = 0\}$. Let y_0 be a generator of this one dimensional $\mathbb{C}$-vector space.*

Step 3 *Let e be the maximal integer such that y_0^e divides G. If $e \neq d-3$, then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1))$ and the algorithm terminates. Otherwise let $G := y_0^{d-3}P$, pick a basis $(y_0, y_1, \ldots, y_n)$ of T_1 and compute $f := P|_{y_0=1} \in R := \mathbb{C}[y_1, \ldots, y_n]$.*

Step 4 *Let $I := \mathrm{Ann}(f^{\blacktriangledown d}) \subseteq R^*$. If the Hilbert function of R^*/I is not equal to $(1, 6, 6, 1)$, then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1))$, and the algorithm terminates.*

Step 5 *Compute $r := \dim_{\mathbb{C}} \mathrm{Hom}_{R^*}(I, R^*/I)$. Then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1))$ if and only if $r > 14n - 8$.*

Let us look at an example of application of Theorem 4.0.0.4. As it was mentioned in Chapter 1, Theorem 1.3.1.1 implies that $G = x_0^3(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3) \in \kappa_{14}(\nu_6(\mathbb{C}[x_0, x_1, ..., x_6]))$. Now we are able to check, if the polynomial G belongs also to $\sigma_{14}(\nu_6(\mathbb{C}[x_0, x_1, ..., x_6]))$.

Corollary 4.0.0.5. *Let $G := x_0^3(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3) \in \mathbb{C}[x_0, x_1, ..., x_6]$, then $G \in \sigma_{14}(\nu_6(\mathbb{C}[x_0, x_1, ..., x_6]))$.*

Proof. Using notation from Theorem 4.0.0.4, in our case $d := 6$. One can check, that $(\operatorname{Ann} G)_{\leq 3} = (\{\alpha_i\alpha_j\}_{1\leq i<j\leq 6}, \{\alpha_i^3 - \alpha_6^3\}_{1\leq i\leq 5})$. Now $\mathfrak{a} = \sqrt{((\operatorname{Ann} G)_{\leq 3})} = (\alpha_1, \alpha_2, \dots, \alpha_6)$, thus $\dim(\mathfrak{a}_1) = 6$.

In Step 2, x_0 is the distinguished generator, i.e. $\langle x_0\rangle = \{K \in T_1 | \mathfrak{a}_1 \lrcorner K = 0\}$ and the maximal power e such that x_0^e divides G equals 3.

Therefore, $f = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 \in R := \mathbb{C}[x_1, x_2, ..., x_6]$. Notice, that $I := \operatorname{Ann}(F_3) = \operatorname{Ann}(f) = (\operatorname{Ann} G)_{\leq 3}$, thus $I_1 = 0$.

Finally, we check using Macaulay2 [GS], that $\dim_{\mathbb{C}} \operatorname{Hom}_{R^*}(I, R^*/I) = 112 > 76$, so the algorithm ends and we obtain $G \in \sigma_{14}(\nu_6(\mathbb{C}[x_0, x_1, ..., x_6]))$. ∎

We prove Theorem 4.0.0.4 in Section 4.2. In Theorem 4.3.1.5 we present an analogous algorithm to compute whether a point in the cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}^n))$ is in the secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}^n))$ for $d \geq 5$, $n \geq 4$.

4.1 Results regarding the cactus rank and the border cactus rank of W^{hom,d_2}

In this section we apply the results of Section 2.5 to prove Theorems 4.1.0.2 and 4.1.0.3. These imply Theorems 1.3.1.4 and 1.3.1.5.

The following notation will be used. Fix a positive integer n and let $S^* := \Bbbk[\alpha_1, \dots, \alpha_n] \subseteq T^* := \Bbbk[\alpha_0, \dots, \alpha_n]$ be polynomial rings with graded dual rings $S := \Bbbk_{dp}[x_1, \dots, x_n] \subseteq T := \Bbbk_{dp}[x_0, \dots, x_n]$. Recall definitions of an $(s, n+1)$-standard Hilbert function and W^{hom,d_2} (see Definitions 2.4.0.3 and 2.5.0.1 correspondingly).

We frequently use the following lemma.

Lemma 4.1.0.1 ([GMR20, Lem. 4.1.]). *Let $J, K \subseteq T^*$ be homogeneous ideals such that $J_s = K_s$ for a positive integer s. If K is generated in degrees at most s, then $J_t \supseteq K_t$ for $t \geq s$.*

Proof. We have $J_t \supseteq (J_s)_t = (K_s)_t = K_t$. ∎

Theorems 4.1.0.2 and 4.1.0.3 provide upper bounds for the cactus rank of polynomials and subspace of polynomials, which are divisible by a power of a linear form. Item (iii) says, that under certain circumstances, the ideal I from Weak Border Cactus Apolarity Lemma 2.4.2.3 is unique up to a saturation.

Theorem 4.1.0.2 (Subspace case, [GMR20, Thm 4.2.]). *Let $W \subseteq S_{\leq d_1}$ be a linear subspace and $r := \dim_{\Bbbk} S^*/\operatorname{Ann}(W)$. We have the following:*

(i) The cactus rank $\operatorname{cr}(W^{hom,d_2})$ of W^{hom,d_2} is at most r.

(ii) If $d_2 \geq d_1$, then there is no homogeneous ideal $J \subseteq \operatorname{Ann}(W^{hom,d_2})$ such that T^*/J has an $(r-1, n+1)$-standard Hilbert function. In particular, the border cactus rank $\underline{\operatorname{cr}}(W^{hom,d_2})$ of W^{hom,d_2} equals r.

(iii) If $d_2 \geq d_1 + 1$, and $J \subseteq \operatorname{Ann}(W^{hom,d_2})$ is a homogeneous ideal such that T^*/J has an $(r, n+1)$-standard Hilbert function, then $J^{sat} = \operatorname{Ann}(W)^{hom}$.

Proof. (i) We have $\operatorname{Ann}(W)^{hom} \subseteq \operatorname{Ann}(W^{hom,d_2})$ by Lemma 2.5.0.13(i). Since the Hilbert polynomial of $T^*/\operatorname{Ann}(W)^{hom}$ is r by Corollary 2.5.0.5 and the ideal $\operatorname{Ann}(W)^{hom}$ is saturated by Lemma 2.5.0.3, the claim follows from the Cactus Apolarity Lemma 2.4.2.2.

(ii) It follows from Corollary 2.5.0.5 that $H(T^*/\operatorname{Ann}(W)^{hom}, d_1) = r$. Therefore, by Lemma 2.5.0.13(ii)

$$H(T^*/\operatorname{Ann}(W^{hom,d_2}), d_1) = r.$$

Thus, there exists no ideal $J \subseteq \operatorname{Ann}(W^{hom,d_2})$ such that T^*/J has an $(r-1, n+1)$-standard Hilbert function. By the Weak Border Cactus Apolarity Lemma 2.4.2.3 we get $\underline{\operatorname{cr}}(W^{hom,d_2}) \geq r$, which together with Part (i) implies that $\underline{\operatorname{cr}}(W^{hom,d_2}) = r$.

(iii) Assume that $J \subseteq \operatorname{Ann}(W^{hom,d_2})$ is such that T^*/J has an $(r, n+1)$-standard Hilbert function. By Lemma 2.5.0.13(ii) and Corollary 2.5.0.5

$$H(T^*/\operatorname{Ann}(W^{hom,d_2}), d_2) = H(T^*/\operatorname{Ann}(W)^{hom}, d_2) = r.$$

In particular, $J_{d_2} = (\operatorname{Ann}(W)^{hom})_{d_2}$. Since $\operatorname{Ann}(W)^{hom}$ is generated in degrees less than or equal to $d_1 + 1 \leq d_2$, it follows from Lemma 4.1.0.1, that $J_d \supseteq (\operatorname{Ann}(W)^{hom})_d$ for every $d \geq d_2$.
Ideals J and $\operatorname{Ann}(W)^{hom}$ have the same Hilbert polynomial. Hence we obtain $J^{sat} = (\operatorname{Ann}(W)^{hom})^{sat} = \operatorname{Ann}(W)^{hom}$. The last equality is true by Lemma 2.5.0.3.

■

Now we can state Theorem 1.3.1.4 in the following form. The differences between this theorem and trivial implication of Theorem 4.1.0.2 for one dimensional $W = \langle f \rangle$ are in points (ii) and (iii) when we additionally treat cases $d_2 = d_1 - 1$ and $d_2 \geq d_1 - 1$ correspondingly.

Theorem 4.1.0.3 (Polynomial case, [GMR20, Thm 4.3.]). *Let $f = F_{d_1} + F_{d_1-1} + \ldots + F_0 \in S = \Bbbk_{dp}[x_1, ..., x_n]$ be a degree $d_1 \geq 1$ polynomial, $r := \dim_{\Bbbk} S^*/\operatorname{Ann}(f)$. For a non-negative integer d_2, we have the following:*

(i) The cactus rank $\operatorname{cr}(f^{hom,d_2})$ of f^{hom,d_2} is at most r.

(ii) If $d_2 \geq d_1$, then there is no homogeneous ideal $J \subseteq \operatorname{Ann}(f^{hom,d_2})$ such that T^*/J has an $(r-1, n+1)$-standard Hilbert function. Moreover, the same is true for $d_2 = d_1 - 1$ if we assume further that F_{d_1} is not a power of a linear form.
In particular, in both cases the border cactus rank $\underline{\operatorname{cr}}(f^{hom,d_2})$ of f^{hom,d_2} equals r.

(iii) Assume that F_{d_1} is not a power of a linear form. If $d_2 \geq d_1$ and $J \subseteq \operatorname{Ann}(f^{hom,d_2})$ is a homogeneous ideal such that T^*/J has an $(r, n+1)$-standard Hilbert function, then $J^{sat} = \operatorname{Ann}(f)^{hom}$. Moreover, the same is true for $d_2 = d_1 - 1$ if we assume further that $r > 2d_1$.

Proof. (i) It follows directly from Theorem 4.1.0.2(i).

(ii) If $d_2 \geq d_1$, then the claim follows from Theorem 4.1.0.2(ii).
Assume that $d_2 = d_1 - 1$ and F_{d_1} is not a power of a linear form. If $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r$, then there is no ideal $J \subseteq \operatorname{Ann}(f^{hom,d_2})$ such that T^*/J has an $(r-1, n+1)$-standard Hilbert function. Suppose that

$$H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) \neq r. \tag{4.1.0.4}$$

It follows from Lemma 2.5.0.7 that $\operatorname{Ann}(f)^{hom}$ is generated in degrees at most d_1. Then (4.1.0.4) and Lemma 2.5.0.9(iii) together imply that $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r - 1$ and $\operatorname{Ann}(f^{hom,d_2})$ has no minimal generator of degree greater than d_1. Let $J \subseteq \operatorname{Ann}(f^{hom,d_2})$ be a homogeneous ideal such that T^*/J has an $(r-1, n+1)$-standard Hilbert function. Then we have $J_{d_1} = \operatorname{Ann}(f^{hom,d_2})_{d_1}$ since $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r - 1 = H(T^*/J, d_1)$. Therefore, $J_d \supseteq \operatorname{Ann}(f^{hom,d_2})_d$, for every $d \geq d_1$, by Lemma 4.1.0.1. This gives a contradiction since $H(T^*/\operatorname{Ann}(f^{hom,d_2}), t) = 0$ for $t \gg 0$.
It follows from the Weak Border Cactus Apolarity Lemma 2.4.2.3 that $\underline{\operatorname{cr}}(f^{hom,d_2}) \geq r$ and from (i) we have an equality.

(iii) Suppose that $J \subseteq \operatorname{Ann}(f^{hom,d_2})$ is such that T^*/J has an $(r, n+1)$-standard Hilbert function. We will consider the following five cases:
- (I) $d_2 \geq d_1$;
- (II) $d_2 = d_1 - 1$ and $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r$;
- (III) $d_2 = d_1 - 1$, $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r - 1$ and $H(T^*/J, d_1) = r - 1$;
- (IV) $d_2 = d_1 - 1$, $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r - 1$, $H(T^*/J, d_1) = r$ and $J_{d_1} = (\operatorname{Ann}(f)^{hom})_{d_1}$;
- (V) $d_2 = d_1 - 1$, $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r - 1$, $H(T^*/J, d_1) = r$ and $J_{d_1} \neq (\operatorname{Ann}(f)^{hom})_{d_1}$.

We explain that these are the only possible cases. Suppose that $d_2 = d_1 - 1$ and

$$H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) \neq r.$$

Then

$$H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r - 1$$

by Lemma 2.5.0.9(iii). It suffices to show that if $H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_1) = r - 1$, then $H(T^*/J, d_1) \in \{r-1, r\}$. This holds since T^*/J has an $(r, n+1)$-standard Hilbert function and $J \subseteq \operatorname{Ann}(f^{hom,d_2})$.
We prove, that $J^{sat} = \operatorname{Ann}(f)^{hom}$ in each case.
- (I) By Lemma 2.5.0.13(ii) and Corollary 2.5.0.5

$$H(T^*/\operatorname{Ann}(f^{hom,d_2}), d_2) = H(T^*/\operatorname{Ann}(f)^{hom}, d_2) = r.$$

In particular $J_{d_2} = (\operatorname{Ann}(f)^{hom})_{d_2}$. It follows from Lemma 2.5.0.7 that $\operatorname{Ann}(f)^{hom}$ is generated in degrees smaller or equal $d_1 \leq d_2$. Together with Lemma 4.1.0.1 it implies that $J_d \supseteq (\operatorname{Ann}(f)^{hom})_d$ for $d \geq d_2$.

Ideals J and $\operatorname{Ann}(f)^{hom}$ have the same Hilbert polynomial. Hence, we obtain $J^{sat} = (\operatorname{Ann}(f)^{hom})^{sat} = \operatorname{Ann}(f)^{hom}$. The last equality is true by Lemma 2.5.0.3.

(II) We have $J_{d_1} = \operatorname{Ann}(f^{hom,d_2})_{d_1} = (\operatorname{Ann}(f)^{hom})_{d_1}$. The ideal $\operatorname{Ann}(f)^{hom}$ is generated in degrees at most d_1 by Lemma 2.5.0.7. Thus, from Lemma 4.1.0.1, there is a containment $J_d \supseteq (\operatorname{Ann}(f)^{hom})_d$ for $d \geq d_1$. Hilbert polynomial of ideals J and $\operatorname{Ann}(f)^{hom}$ are the same, therefore $J^{sat} = (\operatorname{Ann}(f)^{hom})^{sat} = \operatorname{Ann}(f)^{hom}$. The last equality is true by Lemma 2.5.0.3.

(III) We have $J_{d_1} = \operatorname{Ann}(f^{hom,d_2})_{d_1}$ and the ideal $\operatorname{Ann}(f^{hom,d_2})$ is generated in degrees at most d_1 by Lemmas 2.5.0.9(iii) and 2.5.0.7. Thus, for $d \geq d_1$ there is a containment $J_d \supseteq \operatorname{Ann}(f^{hom,d_2})_d$, by Lemma 4.1.0.1. This is a contradiction, as Hilbert polynomial of T^*/J is not zero.

(IV) Proof is as in case (II).

(V) The ideal J has a generator of the form $\alpha_0^{d_1} + \rho$ where $\rho \in T^*_{d_1}$ has degree smaller than d_1 with respect to α_0 (again by Lemma 2.5.0.9(iii)). Since $\operatorname{codim}_{\operatorname{Ann}(f^{hom,d_2})_{d_1}} J_{d_1} = 1$ we have

$$\operatorname{codim}_{(\operatorname{Ann}(f^{hom,d_2})^c)_{d_1}} (J^c)_{d_1} \leq 1.$$

Here, K^c denotes $K \cap S^*$ for any ideal $K \subseteq T^*$. We shall consider $I = \operatorname{Ann}(F_{d_1}) \subseteq S^*$. There is a containment $I_{d_1} \subseteq (\operatorname{Ann}(f^{hom,d_2})^c)_{d_1}$ and $H(S^*/I, d_1) = 1$. Therefore,

$$H(S^*/J^c, d_1) \leq H(S^*/\operatorname{Ann}(f^{hom,d_2})^c, d_1) + 1 \leq H(S^*/I, d_1) + 1 = 2.$$

Since $d_1 \geq 2$, from the Macaulay's bound ([BH98], Theorem 4.2.10) follows that for $d \geq d_1$ there is inequality $H(S^*/J^c, d) \leq 2$. Hence

$$H(T^*/J, d) \leq H(S^*/J^c, d) + \ldots + H(S^*/J^c, d - (d_1 - 1)) \leq 2d_1 < r$$

for $d \geq 2d_1 - 1$. We used here Lemma 2.5.0.6. This gives a contradiction since the Hilbert polynomial of T^*/J is equal to r.

■

As an example of application of Theorem 4.1.0.3, we can calculate the cactus rank and the border cactus rank of the polynomial $x_1^2 x_2 x_0^2 + x_2 x_0^4$.

Corollary 4.1.0.5. *Let* $G := x_1^2 x_2 x_0^2 + x_2 x_0^4 \in \mathbb{C}[x_0, x_1, x_2]$, *then*

$$\operatorname{cr}(G) = \underline{\operatorname{cr}}(G) = 6.$$

Proof. We have $G = 2x_1^2 x_2 \frac{x_0^2}{2!} + 4! x_2 \frac{x_0^4}{4!}$, so $G = f^{\mathrm{hom},d_2}$ for $f := 2x_1^2 x_2 + 24x_2$, $d_2 = 2$. Notice, that $d_2 = d_1 - 1$ and the leading form F_{d_1} is not a power of a linear form. Thus, from Theorem 4.1.0.3 (i) and (ii), the cactus rank and border cactus rank equal $\dim_{\mathbb{C}} S^*/\operatorname{Ann}(f)$. One can check, that $\operatorname{Ann}(f) = (\alpha_2^2, \alpha_1^3)$, therefore we obtain $\dim_{\mathbb{C}} \mathbb{C}[\alpha_1, \alpha_2]/\operatorname{Ann}(f) = 6$. ■

The following examples show that the assumptions of Theorem 4.1.0.3 are in general as sharp as possible.

Example 4.1.0.6. *Let* $S := \Bbbk_{dp}[x_1, x_2]$, $f := x_1^{[2]} + x_1x_2$ *and assume* $d_2 = d_1 - 2 = 0$. *Then* $r = 4$ *and* $\mathrm{Ann}(f^{hom,0}) = (\alpha_0, \alpha_1^2 - \alpha_1\alpha_2, \alpha_2^2)$. *Consider the ideal* $J := (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2 - \alpha_1\alpha_2)$. *Then* $\Bbbk[\alpha_0, \alpha_1, \alpha_2]/J$ *has Hilbert function* $h_{3,3}$ *and* $J \subset \mathrm{Ann}(f^{hom,0})$. *Therefore, the assumption* $d_2 \geq d_1 - 1$ *in Theorem 4.1.0.3 (ii) cannot be weakened in general.*

Example 4.1.0.7. *As in Example 4.1.0.6, let* $S := \Bbbk_{dp}[x_1, x_2]$ *and* $f := x_1^{[2]} + x_1x_2$. *Then* $r = 4 = 2d_1$. *If* $d_2 = 1$, *then* $\mathrm{Ann}(f^{hom,1}) = (\alpha_0^2, \alpha_1^2 - \alpha_1\alpha_2, \alpha_2^2)$. *The ideal* $J := (\alpha_0^2, \alpha_1^2 - \alpha_1\alpha_2)$ *is saturated and* $\Bbbk[\alpha_0, \alpha_1, \alpha_2]/J$ *has Hilbert function* $h_{4,3}$. *However,* J *does not contain* $\alpha_2^2 \in \mathrm{Ann}(f)^{hom}$. *Therefore, the assumption* $r > 2d_1$ *in Theorem 4.1.0.3 (iii) cannot be skipped.*

Example 4.1.0.8. *Let* $S := \Bbbk_{dp}[x_1, x_2, x_3]$ *and* $f := x_1x_2x_3$. *Then* $r = 8 > 6 = 2d_1$. *If* $d_2 = d_1 - 2 = 1$, *then* $\mathrm{Ann}(f^{hom,1}) = (\alpha_0^2, \alpha_1^2, \alpha_2^2, \alpha_3^2)$. *Consider the ideal* $J := (\alpha_0^3, \alpha_0^2\alpha_1, \alpha_1^2, \alpha_0^2\alpha_2, \alpha_2^2, \alpha_0^2\alpha_3)$. *Then* $\Bbbk[\alpha_0, \alpha_1, \alpha_2, \alpha_3]/J$ *has Hilbert function* $h_{8,4}$ *and* $J^{sat} = (\alpha_0^2, \alpha_1^2, \alpha_2^2) \neq \mathrm{Ann}(f)^{hom}$. *Therefore, the assumption* $d_2 \geq d_1 - 1$ *in Theorem 4.1.0.3 (iii) cannot be weakened in general.*

4.2 14-th cactus variety of d-th Veronese embedding of $\mathbb{P}^n$

In this section we assume that $d \geq 5$ and $n \geq 6$ are integers. We show that the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}^n))$ has 2 components one of which is the secant variety $\sigma_{14}(\nu_d(\mathbb{P}^n))$ and we describe the other one (see Theorem 4.0.0.2). Furthermore, for $n \geq 6$ and $d \geq 6$ we present an algorithm (Theorem 4.0.0.4) for deciding whether $[G] \in \kappa_{14}(\nu_d(\mathbb{P}^n))$ is in $\sigma_{14}(\nu_d(\mathbb{P}^n))$.

Since the methods of this section rely on the results of [Jel18] in which the author works over the field of complex numbers, in this section we will assume that the base field $\Bbbk$ is $\mathbb{C}$. In that case, the graded dual ring $\mathbb{C}_{dp}[x_0, x_1, \ldots, x_n]$ of a polynomial ring $T^* = \mathbb{C}[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is isomorphic to a polynomial ring $\mathbb{C}[x_0, x_1, \ldots, x_n]$. Thus, from now on we use notation $T := \mathbb{C}[x_0, x_1, \ldots, x_n]$.

Let $\mathcal{H}ilb_r^{Gor}(X)$, where $X := \mathbb{A}^n$ or $\mathbb{P}^n$, denote the open subset of the Hilbert scheme of r points on X consisting of Gorenstein subschemes, and let $\mathcal{H}ilb_r^{Gor,sm}(X)$ denote its smoothable component. For $r \leq 13$, we have $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n) = \mathcal{H}ilb_r^{Gor,sm}(\mathbb{A}^n)$ by [CJN15, Thm A]. In particular, $\mathcal{H}ilb_r^{Gor}(\mathbb{P}^n)$ is irreducible for such r. Therefore, in that case, $\kappa_r(\nu_d(\mathbb{P}^n)) = \sigma_r(\nu_d(\mathbb{P}^n))$. Indeed, we have

$$\kappa_r(\nu_d(\mathbb{P}^n)) = \overline{\bigcup\{\langle\nu_d(R)\rangle \mid [R] \in \mathcal{H}ilb_r^{Gor}(\mathbb{P}^n)\}} \qquad (4.2.0.1)$$

by [BB14, Prop. 2.2]. Therefore, irreducibility of $\mathcal{H}ilb_r^{Gor}(\mathbb{P}^n)$ implies $\kappa_r(\nu_d(\mathbb{P}^n)) = \sigma_r(\nu_d(\mathbb{P}^n))$. Note, that a description of the cactus variety, similar to the one given by Equation (4.2.0.1), works over an arbitrary field (see [BJ17, Cor. 6.20]).

4.2.1 Characterization of the irreducible components

We will consider the polynomial ring $T^* := \mathbb{C}[\alpha_0, \alpha_1, \dots, \alpha_n]$, and its graded dual $T := \mathbb{C}[x_0, x_1, \dots, x_n]$, where $n \geq 6$. Given $f \in T$, by F_j is denoted its homogeneous part of degree j. Recall Definitions 2.4.0.1 and 2.4.0.2.

Our goal is to characterize the closure of the set-theoretic difference between the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}T_1))$ and the secant variety $\sigma_{14}(\nu_d(\mathbb{P}T_1))$ for $d \geq 5$ and $n \geq 6$. For $d \geq 5$ and $n = 6$ this closure consists of points $[G] \in \mathbb{P}T_d$ with G divisible by $(d-3)$-th power of a linear form. However, for $n > 6$ the situation is more complicated.

For $d \geq 3$ we will define a subset $\eta_{14}(\nu_d(\mathbb{P}^n))$ of the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}^n))$. Later, in Theorem 4.0.0.2 it will be shown, that for $d \geq 5$

$$\kappa_{14}(\nu_d(\mathbb{P}^n)) = \sigma_{14}(\nu_d(\mathbb{P}^n)) \cup \eta_{14}(\nu_d(\mathbb{P}^n))$$

is the decomposition into irreducible components.

Consider the following rational map φ, which assigns to a scheme R its projective linear span $\langle v_d(R)\rangle$

$$\varphi : \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n) \dashrightarrow \mathrm{Gr}(14, T_d).$$

Let $U \subseteq \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ be a dense open subset on which φ is regular.

Consider the projectivized universal bundle $\mathbb{P}\mathcal{S}$ over $\mathrm{Gr}(14, T_d)$, given as a set by

$$\mathbb{P}\mathcal{S} = \{([P], [p]) \in \mathrm{Gr}(14, T_d) \times \mathbb{P}(T_d) \mid p \in P\},$$

together with the inclusion $i : \mathbb{P}\mathcal{S} \hookrightarrow \mathrm{Gr}(14, T_d) \times \mathbb{P}(T_d)$. We pull the commutative diagram

$$\begin{array}{ccc} \mathbb{P}\mathcal{S} & \xhookrightarrow{\quad i \quad} & \mathrm{Gr}(14, T_d) \times \mathbb{P}(T_d) \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & \mathrm{Gr}(14, T_d) & \end{array}$$

back along φ to U, getting the commutative diagram

$$\begin{array}{ccc} \varphi^*(\mathbb{P}\mathcal{S}) & \xhookrightarrow{\quad \varphi^* i \quad} & U \times \mathbb{P}(T_d) \\ & {\scriptstyle \varphi^*\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U. & \end{array}$$

Let Y be the closure of $\varphi^*(\mathbb{P}\mathcal{S})$ inside $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n) \times \mathbb{P}(T_d)$. The scheme Y has two irreducible components, Y_1 and Y_2, corresponding to two irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$, the schemes $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{P}^n)$ and $\mathcal{H}_{1661}$, respectively. For the description of irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ see [CJN15].

Then

$$\begin{aligned}\sigma_{14}(\nu_d(\mathbb{P}^n)) &= \mathrm{pr}_2(Y_1), \text{ and we define}\\ \eta_{14}(\nu_d(\mathbb{P}^n)) &= \mathrm{pr}_2(Y_2).\end{aligned}$$

In Proposition 4.2.1.2 we bound from above the dimension of the irreducible subset $\eta_{14}(\nu_d(\mathbb{P}^n))$ by $14n+5$. Later, we prove Theorem 4.0.0.2, which identifies a $(14n+5)$-dimensional subset of $\kappa_{14}(\nu_d(\mathbb{P}^n)) \setminus \sigma_{14}(\nu_d(\mathbb{P}^n))$. It will allow us to conclude that the closure of this subset is $\eta_{14}(\nu_d(\mathbb{P}^n))$.

Lemma 4.2.1.1 ([GMR20, Lem. 5.4]). *For $n \geq 6$ the component $\mathcal{H}_{1661}$ has dimension $14n-8$. Let $[R] \in \mathcal{H}_{1661} \subseteq \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$. If $[R]$ is a non-smoothable subscheme, then the dimension of the tangent space $\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ equals $14n-8$. If $[R]$ is a smoothable subscheme, then $\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ is larger than $14n-8$.*

Proof. For $m \geq 6$ we use the notation $\mathcal{H}_{1661}^m$ for the non-smoothable component of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^m)$. By [BJJM19, Prop. A.4] we have $\dim \mathcal{H}_{1661}^n = 14n + \dim \mathcal{H}_{1661}^6 - 84$. Thus $\dim \mathcal{H}_{1661}^n = 14n-8$, since $\dim \mathcal{H}_{1661}^6 = 76$, see [Jel18, Thm 1.1].

Let $R' \subseteq \mathbb{P}^6$ be a subscheme abstractly isomorphic with R. It follows from [CN09, Lem. 2.3] that

$$\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n) = 14n + T_{[R']}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) - 84.$$

From [BJ17, Thm 1.1] R' is non-smoothable, hence $\dim T_{[R']}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) = 76$ by [Jel18, Claim 3].

By [CJN15], the scheme $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^n)$ has two irreducible components $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$ and $\mathcal{H}_{1661}$. In the case when $[R] \in \mathcal{H}_{1661} \subseteq \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ is a smoothable subscheme, $[R]$ is a singular point, since it lies on the intersection of two components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$. Thus the dimension of the tangent space $\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ has to be larger than $\dim \mathcal{H}_{1661} = 14n-8$. ∎

Proposition 4.2.1.2 ([GMR20, Prop. 5.5]). *Dimension of $\eta_{14}(\nu_d(\mathbb{P}^n))$ is less or equal $14n+5$.*

Proof. We have the following commutative diagram

$$\begin{array}{ccccc}
\mathbb{P}(T_d) \supseteq \sigma \cup \eta & \longleftarrow & Y_1 \cup Y_2 & \dashrightarrow & \mathbb{P}\mathcal{S} \\
 & & \downarrow \chi & & \downarrow \\
\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n) & = & \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{P}^n) \cup \mathcal{H}_{1661} & \dashrightarrow & \mathrm{Gr}(14, T_d)
\end{array}$$

where σ and η denote $\sigma_{14}(\nu_d(\mathbb{P}^n))$ and $\eta_{14}(\nu_d(\mathbb{P}^n))$ respectively, and $\chi : Y_1 \cup Y_2 \to \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ is the projection. Then $\dim \eta_{14}(\nu_d(\mathbb{P}^n)) \leq \dim(Y_2) = m + 13$, where $m := \dim \mathcal{H}_{1661}$ and 13 is the dimension of the general fiber of the map $\chi|_{Y_2} : Y_2 \to \mathcal{H}_{1661}$. It follows from Lemma 4.2.1.1, that $m = 14n - 8$ and therefore $\dim \eta_{14}(\nu_d(\mathbb{P}^n)) \leq 14n + 5$. ■

In the rest of the section we use the notation $f^{\blacktriangledown d}$ introduced in Definition 4.0.0.1.

Proposition 4.2.1.3 ([GMR20, Prop. 5.6])**.** *Let T be defined as at the beginning of this subsection and let $(y_0, y_1, \dots, y_n)$ be a $\mathbb{C}$-basis of T_1. Assume that $G := y_0^{d-3}P$ for some natural number $d \geq 5$ and $P \in T_3$. Define $f := P|_{y_0=1} = F_3+F_2+F_1+F_0 \in R := \mathbb{C}[y_1, \dots, y_n]$. If f satisfies the following conditions:*
(a) $\operatorname{Apolar}(f^{\blacktriangledown d})$ *has Hilbert function* $(1, 6, 6, 1)$*,*
(b) $[\operatorname{Spec} \operatorname{Apolar}(f^{\blacktriangledown d})] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$*,*
then $[G] \in \eta_{14}(\nu_d(\mathbb{P}^n)) \setminus \sigma_{14}(\nu_d(\mathbb{P}^n))$*.*

Proof. By Condition (a) we have $\dim_{\mathbb{C}}(R^*/\operatorname{Ann}(f^{\blacktriangledown d})) = 14$. Therefore, from Theorem 4.1.0.3 (i)

$$\operatorname{cr}(G) = \operatorname{cr}\Big(\sum_{i=0}^{3} y_0^{[d-i]} F_i^{\blacktriangledown d}\Big) \leq 14.$$

From the Border Apolarity Lemma 2.4.1.5, if $[G] \in \sigma_{14}(\nu_d(\mathbb{P}^n))$, then there exists $J \subseteq \operatorname{Ann}(G)$ with $[J] \in \operatorname{Slip}_{14,\mathbb{P}T_1} \subseteq \operatorname{Hilb}_{T^*}^{h_{14,n+1}}$. Thus $[\operatorname{Proj}(T^*/J^{sat})] \in \mathcal{H}ilb_{14}^{sm}(\mathbb{P}^n)$. The Hilbert function of $R^*/\operatorname{Ann}(F_3^{\blacktriangledown d})$ is $(1, 6, 6, 1)$ by [CJN15, Thm 2.3 and the following remarks]. In particular, $F_3^{\blacktriangledown d}$ is not a power of a linear form. It follows from Theorem 4.1.0.3 (iii) that $J^{sat} = \operatorname{Ann}(f^{\blacktriangledown d})^{hom}$, so

$$[\operatorname{Spec}(R^*/\operatorname{Ann}(f^{\blacktriangledown d}))] \in \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n).$$

This contradicts Condition (b). ■

Finally we present the proof of the characterization of points of the second irreducible component of the cactus variety.

Proof of Theorems 1.3.1.1 and 4.0.0.2. First we prove Theorem 4.0.0.2. We start with showing, that the closure of the set of points from the statement is contained in $\eta_{14}(\nu_d(\mathbb{P}^n))$. We define

$$\begin{aligned}\mathcal{D} := \{(y_0, P) \in T_1 \times T_3 \mid\ & \text{there exists a completion of } y_0 \text{ to a basis } (y_0, y_1, \dots, y_n) \\ & \text{of } T_1 \text{ such that } \operatorname{Apolar}((P|_{y_0=1})^{\blacktriangledown d}) \text{ has Hilbert function} \\ & (1,6,6,1) \text{ and } [\operatorname{Spec} \operatorname{Apolar}((P|_{y_0=1})^{\blacktriangledown d})] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)\}.\end{aligned}$$

We claim, that the set $\mathcal{C}$ is irreducible and $\mathcal{D}$ is dense in $\mathcal{C}$, and that $\dim \mathcal{D} = \dim \mathcal{C} = 14n + 7$. In order to prove the claim, let us consider the morphism

$\varphi : GL(T_1) \times T_3 \to T_3$, which is given by a change of basis. Then we have a product morphism

$$\tau : GL(T_1) \times T_3 \to T_1 \times T_3, \text{ defined by } (a, P) \mapsto (a(x_0), \varphi(a, P)).$$

Recall the sets $\mathcal{A}$, and $\mathcal{B}$ from Lemma 2.6.0.5. Let $\mathcal{A}' := \{f^{\blacktriangledown d} \in S_{\leq 3} \setminus \{0\} \mid [f] \in \mathcal{A}\}$ and $\mathcal{B}' := \{f^{\blacktriangledown d} \in S_{\leq 3} \setminus \{0\} \mid [f] \in \mathcal{B}\}$. We identify $S_{\leq 3}$ with T_3. There are equalities $\tau(GL(T_1) \times \mathcal{A}') = \mathcal{C}$ and $\tau(GL(T_1) \times \mathcal{B}') = \mathcal{D}$. It follows from Lemma 2.6.0.5, that $\mathcal{C}$ is irreducible, $\mathcal{D}$ is dense in $\mathcal{C}$, and $\dim \mathcal{D} = \dim \mathcal{C} = 14n + 7$. The equality $\dim \mathcal{C} = (n+1) + (13n+6)$ comes from the fact that $\dim(GL(T_1) \times \mathcal{A}') = (n+1)^2 + 13n + 6$ and the fiber over a general point of $\tau|_{GL(T_1)\times \mathcal{A}'}$ is isomorphic to the $n \times (n+1)$ matrix of full rank and has a dimension $n(n+1)$. Therefore, it is enough to show that if $(y_0, P) \in \mathcal{D}$ and $G = y_0^{d-3}P$, then $[G] \in \eta_{14}(\nu_d(\mathbb{P}^n))$. This follows from Proposition 4.2.1.3.

Now we prove, that in fact $\overline{\psi(q(\mathcal{C}))} = \eta_{14}(\nu_d(\mathbb{P}^n))$. It follows from Proposition 4.2.1.2, that for every $d \geq 5$ we have

$$\dim(\eta_{14}(\nu_d(\mathbb{P}^n))) \leq 14n + 5 \leq \dim(q(\mathcal{C})) = \dim(\overline{q(\mathcal{C})}) = \dim \overline{\psi(q(\mathcal{C}))}.$$

The last equality follows from [Vak17, Thm 11.4.1], since the fibers of ψ are finite. Hence $\overline{\psi(q(\mathcal{C}))} = \eta_{14}(\nu_d(\mathbb{P}^n))$.

Now we prove Theorem 1.3.1.1. Assume that $n = 6$. Then the closure of $q(\mathcal{C})$ in $\mathbb{P}T_1 \times \mathbb{P}T_3$ has the maximal dimension $14 \cdot 6 + 5 = 89$. Thus $\overline{q(\mathcal{C})} = \mathbb{P}T_1 \times \mathbb{P}T_3$. It follows that $\eta_{14}(\nu_d(\mathbb{P}^6)) = \overline{\psi(q(\mathcal{C}))} = \psi(\mathbb{P}T_1 \times \mathbb{P}T_3)$. ■

Proposition 4.2.1.4 ([GMR20, Prop. 5.7.])**.** *Let $d \geq 4$ be an integer, $y_0 \in T_1$, $Q \in T_2$. Define $G := y_0^{d-2}Q \in T_d$. If $[G] \in \kappa_r(\nu_d(\mathbb{P}^n))$ for a positive integer r, then $[G] \in \sigma_r(\nu_d(\mathbb{P}^n))$.*

Proof. Complete y_0 to a basis $(y_0, y_1, \ldots, y_n)$ of T_1. Let $S := \mathbb{C}[y_1, ..., y_n]$ and $q := Q_2 + Q_1 + Q_0 \in S$ be such that $G = Q_2 y_0^{[d-2]} + Q_1 y_0^{[d-1]} + Q_0 y_0^{[d]}$. By Theorem 4.1.0.3(ii) we have $\dim_{\mathbb{C}} S^*/\operatorname{Ann}(q) = s$ for some $s \leq r$. Therefore,

$$[\operatorname{Proj} T^*/\operatorname{Ann}(q)^{hom}] \in \mathcal{H}ilb_s(\mathbb{P}^n).$$

By [CEVV09, Prop. 4.9] this subscheme is smoothable. Hence, it follows from Lemma 2.5.0.15, that $[G] \in \sigma_r(\nu_d(\mathbb{P}^n))$. ■

The following lemma provides a description of the set-theoretic difference of the cactus variety and the secant variety. We need it to give a clear proof of Theorem 4.0.0.4.

Lemma 4.2.1.5 ([GMR20, Lem. 5.8.])**.** *Let $d \geq 6, n \geq 6$. The point $[G] \in \kappa_{14}(\nu_d(\mathbb{P}^n))$ does not belong to $\sigma_{14}(\nu_d(\mathbb{P}^n))$ if and only if there exists a linear form $y_0 \in T_1$, and $P \in T_3$ such that $G = y_0^{d-3}P$ and for any completion of y_0 to a basis $(y_0, \ldots, y_n)$ of T_1 we have:*

(a) $\operatorname{Apolar}((P|_{y_0=1})^{\blacktriangledown d})$ *has Hilbert function* $(1, 6, 6, 1)$,
(b) $[\operatorname{Spec}\operatorname{Apolar}((P|_{y_0=1})^{\blacktriangledown d})] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$.

Proof. If $y_0 \in T_1$ and $P \in T_3$ are such that $G = y_0^{d-3}P$, and there exists a completion of y_0 to a basis $(y_0, \ldots, y_n)$ of T_1, for which Conditions (a),(b) hold, we get

$$[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^n))$$

by Proposition 4.2.1.3.

Assume that $[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^n))$. Then by Theorem 4.0.0.2 there exists a linear form $y_0 \in T_1$ such that $y_0^{d-3}|G$. Using Proposition 4.2.1.4 we conclude that G is not divisible by y_0^{d-2}. Hence $G = y_0^{d-3}P$ for some $P \in T_3$. Extend y_0 to a basis $(y_0, y_1, \ldots, y_n)$. Let $f := P|_{y_0=1}$. Suppose $f = F_3 + F_2 + F_1 + F_0$.

Now we prove that Conditions (a), (b) hold. Recall, that $f^{\blacktriangledown d} = F_3^{\blacktriangledown d} + F_2^{\blacktriangledown d} + F_1^{\blacktriangledown d} + F_0^{\blacktriangledown d} \in \mathbb{C}[y_1, \ldots, y_n]$ where $F_i^{\blacktriangledown d} := (d-i)!F_i$. We have

$$G = \sum_{i=0}^{3} y_0^{[d-i]} F_i^{\blacktriangledown d}.$$

By Lemma 2.5.0.9 (i)

$$\operatorname{Ann}(f^{\blacktriangledown d})^{hom} \subseteq \operatorname{Ann}(G).$$

If $\dim_{\mathbb{C}}(\operatorname{Apolar}(f^{\blacktriangledown d})) \leq 13$, then $\operatorname{cr}(G) \leq 13$ by the Cactus Apolarity Lemma 2.4.2.2, since $\operatorname{Ann}(f^{\blacktriangledown d})^{hom}$ is saturated by Lemma 2.5.0.3. Therefore, $[G] \in \kappa_{13}(\nu_d(\mathbb{P}^n)) = \sigma_{13}(\nu_d(\mathbb{P}^n)) \subseteq \sigma_{14}(\nu_d(\mathbb{P}^n))$, a contradiction.

One obtains from Theorem 4.1.0.3(ii) that $\dim_{\mathbb{C}}(\operatorname{Apolar}(f^{\blacktriangledown d})) \leq 14$. We proved, that $\dim_{\mathbb{C}}(\operatorname{Apolar}(f^{\blacktriangledown d})) = 14$. Since we assume that $[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^n))$, it follows from Lemma 2.5.0.15, that $\operatorname{Spec}\operatorname{Apolar}(f^{\blacktriangledown d})$ is not smoothable. This implies Condition (b) holds. By [CJN15, Thm 2.3] and [CJN15, Prop. 6.11] the algebra $\operatorname{Apolar}(f^{\blacktriangledown d})$ has Hilbert function $(1, 6, 6, 1)$. Thus, we proved Condition (a) holds. ■

Steps 2–5 of the algorithm from Theorem 4.0.0.4 verify if G satisfies necessary conditions to be of the form given by Lemma 4.2.1.5.

Proof of Theorem 4.0.0.4. Assume that $[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^n))$. Then there exist a basis $(y_0, \ldots, y_n)$ of T_1 and $P \in T_3$ as in Lemma 4.2.1.5. Let $f := P|_{y_0=1}$ and recall, that $f^{\blacktriangledown d} = F_3^{\blacktriangledown d} + F_2^{\blacktriangledown d} + F_1^{\blacktriangledown d} + F_0^{\blacktriangledown d} \in \mathbb{C}[y_1, \ldots, y_n]$ where $F_i^{\blacktriangledown d} := (d-i)!F_i$. Therefore, $G = y_0^{[d-3]}F_3^{\blacktriangledown d} + y_0^{[d-2]}F_2^{\blacktriangledown d} + y_0^{[d-1]}F_1^{\blacktriangledown d} + y_0^{[d]}F_0^{\blacktriangledown d}$. It follows from Lemma 2.5.0.9(ii) that $\operatorname{Ann}(G)_{\leq d-3} = (\operatorname{Ann}(f^{\blacktriangledown d})^{hom})_{\leq d-3}$. Moreover, by Lemma 2.5.0.7,

$$((\operatorname{Ann}(f^{\blacktriangledown d})^{hom})_{\leq d-3}) = \operatorname{Ann}(f^{\blacktriangledown d})^{hom} \tag{4.2.1.6}$$

(The assumptions of the lemma are satisfied since $\operatorname{Apolar}(F_3^{\blacktriangledown d})$ and $\operatorname{Apolar}(f^{\blacktriangledown d})$ have the same Hilbert function by [CJN15, Thm 2.3 and the following remarks].)

Therefore, $\mathfrak{a} = \sqrt{(\mathrm{Ann}(G)_{\leq d-3})} = \sqrt{\mathrm{Ann}(f^{\blacktriangledown d})^{hom}} = (\beta_1,\ldots,\beta_n)$, where $\beta_1,\ldots,\beta_n \in T_1^*$ are dual to $y_1,\ldots,y_n \in T_1$. This shows that if the $\mathbb{C}$-linear space $\big(\sqrt{(\mathrm{Ann}(G)_{\leq d-3})}\big)_1$ is not n-dimensional then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}^n))$. Therefore, in that case, algorithm stops correctly at Step 2.

Assume that the algorithm did not stop at Step 2. Then if G is of the form as in Lemma 4.2.1.5, then y_0 divides G exactly $(d-3)$-times. Otherwise $[G] \in \sigma_{14}(\nu_d(\mathbb{P}^n))$ and the algorithm stops correctly at Step 3.

Assume that the algorithm did not stop at Step 3. Then the algorithm does not stop at Step 4 if and only if Condition (a) of Lemma 4.2.1.5 is fulfilled. Therefore, if the Hilbert function of $R^*/\mathrm{Ann}(f^{\blacktriangledown d})$ is not equal to $(1,6,6,1)$, the algorithm stops correctly at Step 4.

Assume that the algorithm did not stop at Step 4. Then P satisfies Condition (a) from Lemma 4.2.1.5. Hence $[G]$ is in $\sigma_{14}(\nu_d(\mathbb{P}^n))$ if and only if P does not satisfy Condition (b). Using Lemma 4.2.1.1, this is equivalent to

$$\dim_{\mathbb{C}} \mathrm{Hom}_{R^*}(I, R^*/I) > 14n - 8.$$

The left term is the dimension of the tangent space to the Hilbert scheme $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^n)$ at the point $[\mathrm{Spec}\, R^*/I]$ (see [Har09, Prop. 2.3] or [MS05, Thm 18.29]). ■

Remark 4.2.1.7. The algorithm is stated for $d \geq 6$ even though it is based on Theorem 4.0.0.2 which works for $d \geq 5$. The reason for this is a necessity of the bound $d \geq 6$ to obtain Equation (4.2.1.6) and for Lemma 4.2.1.5 to work. We do not know a counterexample for the algorithm in case $d = 5$.

Equations defining the cactus variety $\kappa_{14}(\nu_6(\mathbb{P}^n))$ for $n \geq 6$ are unknown and there is no example of an explicit equation of the secant variety $\sigma_{14}(\nu_6(\mathbb{P}^n))$ which does not vanish on the cactus variety. We present some known results about 14-th secant and cactus varieties of Veronese embeddings of $\mathbb{P}^6$.

Remark 4.2.1.8. Let V be a 7-dimensional complex vector space. The catalecticant minors define a subscheme of $\mathbb{P}(\mathrm{Sym}^6 V)$ one of whose irreducible components is the secant variety $\sigma_{14}(\nu_6(\mathbb{P}V))$ (see [IK99, Thm 4.10A]). Moreover, these equations are known to vanish on the cactus variety $\kappa_{14}(\nu_6(\mathbb{P}V))$ (see [BB14, Prop. 3.6], or [Gał17]). Example 4.2.1.9 shows that the catalecticant minors do not define $\kappa_{14}(\nu_6(\mathbb{P}V))$ set-theoretically. However, if we consider the d-th Veronese for $d \geq 28$, then the catalecticant minors are enough to define $\kappa_{14}(\nu_d(\mathbb{P}V))$ set-theoretically, see [BB14, Thm 1.5]. The article [LO13] gives an extensive list of results on equations of secant varieties but in the case of $\sigma_{14}(\nu_6(\mathbb{P}V))$ it does not improve the result in [IK99].

Example 4.2.1.9. *Let* $F := x_0^6 + x_1^2x_2^2x_3^2 + x_4^3x_5^2x_6 \in T := \mathbb{C}[x_0,\ldots,x_6]$. *Then Hilbert function of* $T^*/\mathrm{Ann}(F)$ *is* $(1,7,12,14,12,7,1)$ *but there is only one minimal homogeneous generator of* $\mathrm{Ann}(F)$ *in degree 4. Therefore, there is no homogeneous* *ideal J in T^* such that T^*/J has an $(14,7)$-standard Hilbert function and J is contained in* $\operatorname{Ann}(F)$. *Thus,* $\underline{\operatorname{cr}}(F) > 14$ *by the Weak Border Cactus Apolarity Lemma 2.4.2.3, even though the Hilbert function of $T^*/\operatorname{Ann}(F)$ is bounded by* 14.

4.3 (8,3)-th Grassmann cactus variety of Veronese embeddings of $\mathbb{P}^n$

In this section we show that the Grassmann cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}^n))$ has 2 components for $d \geq 5$ and $n \geq 4$ one of which is the Grassmann secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}^n))$ and the other one is described in Theorem 4.0.0.3. Furthermore, we present an algorithm (Theorem 4.3.1.5) for deciding whether $[V] \in \kappa_{8,3}(\nu_d(\mathbb{P}^n))$ is in $\sigma_{8,3}(\nu_d(\mathbb{P}^n))$. As in the previous section, we will assume that $\Bbbk = \mathbb{C}$ because of technical reasons.

The following remark explains briefly why we focus on studying $\kappa_{8,3}(\nu_d(\mathbb{P}^n))$ for $n \geq 4$.

Remark 4.3.0.1. By [CEVV09], we know that $\sigma_{r,k}(\nu_d(\mathbb{P}^n)) = \kappa_{r,k}(\nu_d(\mathbb{P}^n))$ for $r \leq 7$, and any k, n, d, and that $\sigma_{8,k}(\nu_d(\mathbb{P}^n)) = \kappa_{8,k}(\nu_d(\mathbb{P}^n))$ for $n \leq 3$, and any k, d. In addition, we claim that $\sigma_{8,2}(\nu_d(\mathbb{P}^n)) = \kappa_{8,2}(\nu_d(\mathbb{P}^n))$ for any n. We sketch the proof. All local algebras of length at most 8 and socle dimension at most 2 are smoothable, see [CEVV09, Thm 1.1]. Hence the claim follows from the fact, that $\kappa_{r,k}(\nu_d(\mathbb{P}^n))$ is the closure of the following set $\{R \hookrightarrow \mathbb{P}^n \mid \operatorname{length} R \leq r$ and $H^0(R, \mathcal{O}_R)$ is a product of local algebras of socle dimension at most $k\}$ (a generalization of [BB14, Prop. 2.2]). The detailed proof of this fact is outside the main interests of this thesis, hence we skip it.

4.3.1 Characterization of irreducible components

We will consider the polynomial ring $T^* := \mathbb{C}[\alpha_0, \alpha_1, \ldots, \alpha_n]$, and its graded dual $T := \mathbb{C}[x_0, x_1, \ldots, x_n]$, where $n \geq 4$. Recall Definitions 2.4.0.1, 2.4.0.2, 2.5.0.1. We say that $p \in T$ divides $V \subseteq T$ or V is divisible by p and write $p|V$ if there exist $U \subseteq T$ and a positive number d such that $p^d U := \{p^d q \mid q \in U\} = V$.

Our goal is to characterize for $d \geq 5$ and $n \geq 4$ the closure of the set-theoretic difference between the cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}T_1))$ and the secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}T_1))$. For $n = 4$ and $d \geq 5$ this closure consists of points $[V] \in \operatorname{Gr}(3, T_d)$ with V divisible by $(d-2)$-th power of a linear form. However for $n > 4$ the situation is more complicated.

We start with showing, that points of $\operatorname{Gr}(3, T_d)$ corresponding to subspaces divisible by $(d-1)$-th power of a linear form are in the Grassmann secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}T_1))$.

Proposition 4.3.1.1 ([GMR20, Prop. 6.4.]). *Let $d \geq 2$ and $n \geq 4$ be integers, $y_0 \in T_1$ and $[U] \in \mathrm{Gr}(3, T_1)$. Define $V := y_0^{d-1}U \in \mathrm{Gr}(3, T_d)$. Then $\mathrm{cr}(V) \leq 4$, so $[V] \in \kappa_{4,3}(\nu_d(\mathbb{P}T_1)) = \sigma_{4,3}(\nu_d(\mathbb{P}T_1)) \subseteq \sigma_{8,3}(\nu_d(\mathbb{P}T_1))$.*

Proof. Up to a linear change of variables, V is of one of the following forms
(i) $V = \langle x_0^{d-1}x_1, x_0^{d-1}x_2, x_0^{d-1}x_3\rangle$ or
(ii) $V = \langle x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2\rangle$.
Then $V = W^{hom,d_2}$ for $d_2 = d-1$, where W is correspondingly
(i) $W = \langle x_1, x_2, x_3\rangle$ or
(ii) $W = \langle 1, x_1, x_2\rangle$.
In either case, $\dim_{\mathbb{C}} S^*/\operatorname{Ann}(W) \leq 4$, so $\mathrm{cr}(V) = \mathrm{cr}(W^{hom,d_2}) \leq 4$ by Theorem 4.1.0.2(i). ∎

For $d \geq 2$ we will define a subset $\eta_{8,3}(\nu_d(\mathbb{P}^n))$ of the Grassmann cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}^n))$. Later, in Theorem 4.0.0.3, it will be shown, that for $d \geq 5$

$$\kappa_{8,3}(\nu_d(\mathbb{P}^n)) = \sigma_{8,3}(\nu_d(\mathbb{P}^n)) \cup \eta_{8,3}(\nu_d(\mathbb{P}^n))$$

is the decomposition into irreducible components.

Consider the following rational map φ, which assigns to a scheme R its projective linear span $\langle v_d(R)\rangle$

$$\varphi : \mathcal{H}ilb_8(\mathbb{P}^n) \dashrightarrow \mathrm{Gr}(8, T_d).$$

Let $U \subseteq \mathcal{H}ilb_8(\mathbb{P}^n)$ be a dense open subset on which φ is regular. Consider the projectivized incidence bundle $\mathbb{P}\mathcal{S}$ over the Grassmannian $\mathrm{Gr}(8, T_d)$, given as a set by

$$\mathbb{P}\mathcal{S} = \{([V_1], [V_2]) \in \mathrm{Gr}(8, T_d) \times \mathrm{Gr}(3, T_d) \mid V_2 \subseteq V_1\},$$

together with the inclusion $i : \mathbb{P}\mathcal{S} \hookrightarrow \mathrm{Gr}(8, T_d) \times \mathrm{Gr}(3, T_d)$. We pull the commutative diagram

$$\begin{array}{ccc} \mathbb{P}\mathcal{S} & \xhookrightarrow{\ i\ } & \mathrm{Gr}(8, T_d) \times \mathrm{Gr}(3, T_d) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & \mathrm{Gr}(8, T_d) & \end{array}$$

back along φ to U, getting the commutative diagram

$$\begin{array}{ccc} \varphi^*(\mathbb{P}\mathcal{S}) & \xhookrightarrow{\ \varphi^* i\ } & U \times \mathrm{Gr}(3, T_d) \\ & {\scriptstyle\varphi^*\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U. & \end{array}$$

Let Y be the closure of $\varphi^*(\mathbb{P}\mathcal{S})$ inside $\mathcal{H}ilb_8(\mathbb{P}^n) \times \mathrm{Gr}(3, T_d)$. The scheme Y has two irreducible components, Y_1 and Y_2, corresponding to two irreducible components of $\mathcal{H}ilb_8(\mathbb{P}^n)$, the schemes $\mathcal{H}ilb_8^{sm}(\mathbb{P}^n)$ and $\mathcal{H}_{143}$, respectively, see [CEVV09, Thm 1.1]. Then for $d \geq 2$

$$\sigma_{8,3}(\nu_d(\mathbb{P}^n)) = \mathrm{pr}_2(Y_1), \text{ and we define}$$
$$\eta_{8,3}(\nu_d(\mathbb{P}^n)) = \mathrm{pr}_2(Y_2).$$

In the following proposition we bound from above the dimension of the irreducible subset $\eta_{8,3}(\nu_d(\mathbb{P}^n))$ by $8n+8$. Later, in Theorem 4.0.0.3, a $(8n+8)$-dimensional subset of $\kappa_{8,3}(\nu_d(\mathbb{P}^n)) \setminus \sigma_{8,3}(\nu_d(\mathbb{P}^n))$ is identified. We will conclude that the closure of this subset is $\eta_{8,3}(\nu_d(\mathbb{P}^n))$.

Proposition 4.3.1.2 ([GMR20, Prop. 6.5.]). *Dimension of $\eta_{8,3}(\nu_d(\mathbb{P}^n))$ is less or equal $8n+8$.*

Proof. We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Gr}(3,T_d) \supseteq \sigma \cup \eta & \longleftarrow & Y_1 \cup Y_2 & \dashrightarrow & \mathbb{P}\mathcal{S} \\ & & \downarrow \chi & & \downarrow \\ \mathcal{H}ilb_8(\mathbb{P}^n) & = & \mathcal{H}ilb_8^{sm}(\mathbb{P}^n) \cup \mathcal{H}_{143} & \dashrightarrow & \mathrm{Gr}(8,T_d), \end{array}$$

where σ and η denote $\sigma_{8,3}(\nu_d(\mathbb{P}^n))$ and $\eta_{8,3}(\nu_d(\mathbb{P}^n))$ respectively, and $\chi : Y_1 \cup Y_2 \to \mathcal{H}ilb_8(\mathbb{P}^n)$ is the projection. Then $\dim \eta_{8,3}(\nu_d(\mathbb{P}^n)) \leq \dim(Y_2) = m + 15$, where $m := \dim \mathcal{H}_{143}$ and 15 is the dimension of the general fiber of the map $\chi|_{Y_2} : Y_2 \to \mathcal{H}_{143}$. It follows from [CEVV09, Thm 1.1], that $m = 8n-7$ and therefore $\dim \eta_{8,3}(\nu_d(\mathbb{P}^n)) \leq 8n+8$. ■

In the rest of the section we use the notation $W^{\blacktriangledown d}$ from Definition 4.0.0.1. Note, that it depends on d, which is the degree of polynomials in V.

In the following proposition we identify many points from the Grassmann cactus variety which are outside of the Grassmann secant variety. In fact, the closure of the set of these points is the second irreducible component of the Grassmann cactus variety. This will be established in Theorem 4.0.0.3.

Proposition 4.3.1.3 ([GMR20, Prop. 6.6.]). *Let T be defined as at the beginning of this subsection and let $(y_0, y_1, \dots, y_n)$ be a $\mathbb{C}$-basis of T_1. Assume that $V = y_0^{d-2}U$ for some natural number $d \geq 5$ and $[U] \in \mathrm{Gr}(3, T_2)$. Define $[W] := [U|_{y_0=1}] \in \mathrm{Gr}(3, R_{\leq 2})$ where $R := \mathbb{C}[y_1, \dots, y_n]$. If W satisfies the following conditions:*
(a) $\mathrm{Apolar}(W^{\blacktriangledown d})$ *has Hilbert function* $(1,4,3)$,
(b) $[\mathrm{Spec}\,\mathrm{Apolar}(W^{\blacktriangledown d})] \notin \mathcal{H}ilb_8^{sm}(\mathbb{A}^n)$,
then $[V] \in \eta_{8,3}(\nu_d(\mathbb{P}^n)) \setminus \sigma_{8,3}(\nu_d(\mathbb{P}^n))$.

Proof. By Condition (a) we have $\dim_{\mathbb{C}}(R^* / \mathrm{Ann}(W^{\blacktriangledown d})) = 8$. Therefore, from Theorem 4.1.0.3 (i)

$$\mathrm{cr}(V) = \mathrm{cr}((W^{\blacktriangledown d})^{hom, d-2}) \leq 8.$$

From the Border Apolarity Lemma 2.4.1.5, if $[V] \in \sigma_{8,3}(\nu_d(\mathbb{P}^n))$, then there exists $J \subseteq \operatorname{Ann}(V)$ with $[J] \in \mathrm{Slip}_{8,\mathbb{P}T_1} \subseteq \mathrm{Hilb}^{h_{8,n+1}}_{T^*}$. Thus $[\operatorname{Proj}(T^*/J^{sat})] \in \mathcal{H}ilb^{sm}_8(\mathbb{P}^n)$. It follows from Theorem 4.1.0.2 (iii) that $J^{sat} = \operatorname{Ann}(W^{\blacktriangledown d})^{hom}$, so

$$[\operatorname{Spec}(R^*/\operatorname{Ann}(W^{\blacktriangledown d}))] \in \mathcal{H}ilb^{sm}_8(\mathbb{A}^n).$$

This contradicts Condition (b). ■

Finally we present the proof of the characterization of points of the second irreducible component of the Grassmann cactus variety.

Proof of Theorems 1.3.1.3 and 4.0.0.3. First we prove Theorem 4.0.0.3. Let us start with showing, that the closure of the set of points from the statement is contained in $\eta_{8,3}(\nu_d(\mathbb{P}^n))$. We define

$$\begin{aligned}\mathcal{D} := \{(y_0,[U]) \in T_1 \times \mathrm{Gr}(3,T_2) \mid\ & \text{there exists a completion of } y_0 \text{ to a basis} \\ & (y_0,y_1,\dots,y_n) \text{ of } T_1 \text{ such that } \operatorname{Apolar}((U|_{y_0=1})^{\blacktriangledown d}) \\ & \text{has Hilbert function } (1,4,3) \text{ and} \\ & [\operatorname{Spec}\operatorname{Apolar}((U|_{y_0=1})^{\blacktriangledown d})] \notin \mathcal{H}ilb^{sm}_8(\mathbb{A}^n)\}.\end{aligned}$$

We claim, that the set $\mathcal{C}$ is irreducible, $\mathcal{D}$ is dense in $\mathcal{C}$, and that $\dim\mathcal{D} = \dim\mathcal{C} = 8n+9$. Consider the morphism $\varphi : GL(T_1) \times \mathrm{Gr}(3,T_2) \to \mathrm{Gr}(3,T_2)$, given by a change of basis. We have a product morphism

$$\tau : GL(T_1) \times \mathrm{Gr}(3,T_2) \to T_1 \times \mathrm{Gr}(3,T_2), \text{ defined by } (a,[U]) \mapsto (a(x_0), \varphi(a,[U])).$$

Recall the sets $\mathcal{A},\mathcal{B}$ from Lemma 2.7.0.2. Let $\mathcal{A}' := \{[W^{\blacktriangledown d}] \in \mathrm{Gr}(3,S_{\le 2}) \mid [W] \in \mathcal{A}\}$ and $\mathcal{B}' := \{[W^{\blacktriangledown d}] \in \mathrm{Gr}(3,S_{\le 2}) \mid [W] \in \mathcal{B}\}$. We identify $S_{\le 2}$ with T_2. There are equalities $\tau(GL(T_1) \times \mathcal{A}') = \mathcal{C}$ and $\tau(GL(T_1) \times \mathcal{B}') = \mathcal{D}$. It follows from Lemma 2.7.0.2, that $\mathcal{C}$ is irreducible, $\mathcal{D}$ is dense in $\mathcal{C}$, and that $\dim\mathcal{D} = \dim\mathcal{C} = 8n+9$. The equality $\dim\mathcal{C} = (n+1) + (7n+8)$ comes from the fact that $\dim(GL(T_1) \times \mathcal{A}') = (n+1)^2 + 7n + 8$ and the fiber of $\tau|_{GL(T_1)\times\mathcal{A}'}$ over a general point is isomorphic to the $n \times (n+1)$ matrix of full rank and has a dimension $n(n+1)$. Therefore, it is enough to show that if $(y_0,[U]) \in \mathcal{D}$ and $V = y_0^{d-2}U$, then $[V] \in \eta_{8,3}(\nu_d(\mathbb{P}^n))$. This follows from Proposition 4.3.1.3.

Now, we prove that in fact, $\overline{\psi(q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)}(\mathcal{C}))} = \eta_{8,3}(\nu_d(\mathbb{P}^n))$. It follows from Proposition 4.3.1.2, that for every $d \ge 5$ one has

$$\begin{aligned}\dim(\eta_{8,3}(\nu_d(\mathbb{P}^n))) \le 8n+8 \le \dim(q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)}(\mathcal{C})) &= \dim(\overline{q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)}(\mathcal{C})}) \\ &= \dim\overline{\psi(q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)}(\mathcal{C}))}.\end{aligned}$$

The last equality follows from [Vak17, Thm 11.4.1], since the fibers of ψ are finite. Hence $\overline{\psi(q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)}(\mathcal{C}))} = \eta_{8,3}(\nu_d(\mathbb{P}^n))$.

Now we prove Theorem 1.3.1.3. Assume that $n = 4$. Then the closure of $q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)}(\mathcal{C})$ in $\mathbb{P}T_1 \times \mathrm{Gr}(3,T_2)$ has the maximal dimension $8 \cdot 4 + 8 = 40$. Thus $\overline{q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)}(\mathcal{C})} = \mathbb{P}T_1 \times \mathrm{Gr}(3,T_2)$. It follows that $\eta_{8,3}(\nu_d(\mathbb{P}^4)) = \overline{\psi(q \times \mathrm{Id}_{\mathrm{Gr}(3,T_2)}(\mathcal{C}))} = \psi(\mathbb{P}T_1 \times \mathrm{Gr}(3,T_2))$. ■

In order to perform the last step of the algorithm in Theorem 4.3.1.5 we need to know the dimension of the tangent space to $\mathcal{H}_{143}$ in a generic point. The following Lemma 4.3.1.4 will also be used later to prove Proposition 4.3.1.7, that is the analogue of the slice technique (Lemma 2.1.4.1) for border cactus rank does not hold.

Lemma 4.3.1.4 ([GMR20, Lem. 6.7.]). *Let $n \geq 4$ and $[R] \in \mathcal{H}_{143} \subseteq \mathcal{H}ilb_8(\mathbb{P}^n)$. If $[R]$ is a non-smoothable subscheme, then the dimension of the tangent space $\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_8(\mathbb{P}^n)$ equals $8n - 7$. If $[R]$ is a smoothable subscheme, then $\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_8(\mathbb{P}^n)$ is larger than $8n - 7$.*

Proof. Let $R' \subseteq \mathbb{P}^4$ be a subscheme abstractly isomorphic to R. We have from [CN09, Lem. 2.3] that

$$\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_8(\mathbb{P}^n) = 8n + T_{[R']}\mathcal{H}ilb_8(\mathbb{P}^4) - 32.$$

From [BJ17, Thm 1.1] R' is non-smoothable, hence $\dim T_{[R']}\mathcal{H}ilb_8(\mathbb{P}^4) = 25$ by [CEVV09, Thm 1.3 and the comment above].

By [CEVV09, Thm 1.1], the scheme$\mathcal{H}ilb_8(\mathbb{P}^n)$ has two irreducible components $\mathcal{H}ilb_8^{sm}(\mathbb{P}^n)$ and $\mathcal{H}_{143}$. In the case when $[R] \in \mathcal{H}_{143} \subseteq \mathcal{H}ilb_8(\mathbb{P}^n)$ is a smoothable subscheme, $[R]$ is a singular point, since it lies on the intersection of two components of $\mathcal{H}ilb_8(\mathbb{P}^n)$. Thus the dimension of the tangent space $\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_8(\mathbb{P}^n)$ has to be larger than $\dim \mathcal{H}_{143} = 8n - 7$. ■

Using the description of the irreducible component η from Theorem 4.0.0.3, we are able to determine algorithmically if a given point from the Grassmann cactus variety is in the Grassmann secant variety.

Theorem 4.3.1.5 ([GMR20, Thm 6.8.]). *Let n be at least 4 and $T := \mathbb{C}[x_0, \dots, x_n]$ be a polynomial ring. Given an integer $d \geq 5$ and $[V] \in \kappa_{8,3}(\nu_d(\mathbb{P}T_1)) \subseteq \mathrm{Gr}(3, T_d)$ the following algorithm checks if $[V] \in \sigma_{8,3}(\nu_d(\mathbb{P}T_1))$.*

Step 1 *Compute the ideal $\mathfrak{a} := \sqrt{((\mathrm{Ann}\, V)_{\leq d-2})} \subseteq T^* = \mathbb{C}[\alpha_0, \dots, \alpha_n]$.*

Step 2 *If $\mathfrak{a}_1$ is not n-dimensional, then $[V] \in \sigma_{8,3}(\nu_d(\mathbb{P}T_1))$ and the algorithm terminates. Otherwise compute $\{K \in T_1 \mid \mathfrak{a}_1 \lrcorner K = 0\}$. Let y_0 be a generator of this one dimensional $\mathbb{C}$-vector space.*

Step 3 *Let e be the maximal integer such that y_0^e divides V. If $e \neq d-2$, then $[V] \in \sigma_{8,3}(\nu_d(\mathbb{P}T_1))$ and the algorithm terminates. Otherwise let $V := y_0^{d-2}U$, pick a basis $(y_0, y_1, \dots, y_n)$ of T_1 and compute $W := U|_{y_0=1} \subseteq R := \mathbb{C}[y_1, \dots, y_n]$.*

Step 4 *Let* $I = \operatorname{Ann}(W^{\blacktriangledown d}) \subseteq R^*$. *If the Hilbert function of* R^*/I *is not* $(1,4,3)$, *then* $[V] \in \sigma_{8,3}(\nu_d(\mathbb{P}T_1))$, *and the algorithm terminates.*

Step 5 *Compute* $r := \dim_{\mathbb{C}} \operatorname{Hom}_{R^*}(I, R^*/I)$. *Then* $[V] \in \sigma_{8,3}(\nu_d(\mathbb{P}T_1))$ *if and only if* $r > 8n-7$.

The following lemma provides a description of the set-theoretic difference of the Grassmann cactus variety and the Grassmann secant variety. We need it to give a clear proof of Theorem 4.3.1.5.

Lemma 4.3.1.6 ([GMR20, Lem. 6.9.]). *Let* $d \geq 5, n \geq 4$. *The point* $[V] \in \kappa_{8,3}(\nu_d(\mathbb{P}^n))$ *does not belong to* $\sigma_{8,3}(\nu_d(\mathbb{P}^n))$ *if and only if there exists a linear form* $y_0 \in T_1$, *and* $U \in \operatorname{Gr}(3, T_2)$ *such that* $V = y_0^{d-2}U$ *and for any completion of* y_0 *to a basis* $(y_0, \ldots, y_n)$ *of* T_1 *(with dual basis equal to* $(\beta_0, \ldots, \beta_n)$*) we have:*

(a) $\operatorname{Apolar}((U|_{y_0=1})^{\blacktriangledown d})$ *has Hilbert function* $(1,4,3)$,

(b) $[\operatorname{Spec}\operatorname{Apolar}((U|_{y_0=1})^{\blacktriangledown d})] \notin \mathcal{H}ilb_8^{sm}(\mathbb{A}^n)$.

Proof. If $y_0 \in T_1$ and $U \in \operatorname{Gr}(3, T_2)$ are such that $V = y_0^{d-2}U$, and there exists a completion of y_0 to a basis $(y_0, \ldots, y_n)$ of T_1, for which Conditions (a),(b) hold, we get

$$[V] \notin \sigma_{8,3}(\nu_d(\mathbb{P}^n))$$

by Proposition 4.3.1.3.

Assume that $[V] \notin \sigma_{8,3}(\nu_d(\mathbb{P}^n))$. Then by Theorem 4.0.0.3 there exists a linear form $y_0 \in T_1$ such that $y_0^{d-2}|V$. Using Proposition 4.3.1.1 we conclude that V is not divisible by y_0^{d-1}. Hence it shows that $V = y_0^{d-2}U$ for some $U \in \operatorname{Gr}(3, T_2)$. Extend y_0 to a basis $(y_0, y_1, \ldots, y_n)$. Let $W := U|_{y_0=1}$.

Now we prove Conditions (a), (b) hold. There is equality

$$V = (W^{\blacktriangledown d})^{hom, d-2}.$$

By Lemma 2.5.0.13 (i)

$$\operatorname{Ann}(W^{\blacktriangledown d})^{hom} \subseteq \operatorname{Ann}(V).$$

If $\dim_{\mathbb{C}}(\operatorname{Apolar}(W^{\blacktriangledown d})) \leq 7$, then $\operatorname{cr}(V) \leq 7$ by the Cactus Apolarity Lemma 2.4.2.2, since $\operatorname{Ann}(W^{\blacktriangledown d})^{hom}$ is saturated by Lemma 2.5.0.3. Therefore, $[V] \in \kappa_{7,3}(\nu_d(\mathbb{P}^n)) = \sigma_{7,3}(\nu_d(\mathbb{P}^n)) \subseteq \sigma_{8,3}(\nu_d(\mathbb{P}^n))$, a contradiction.

We obtain from Theorem 4.1.0.2(ii) that $\dim_{\mathbb{C}}(\operatorname{Apolar}(W^{\blacktriangledown d})) \leq 8$. We proved, that $\dim_{\mathbb{C}}(\operatorname{Apolar}(W^{\blacktriangledown d})) = 8$. Because of the assumption, that $[V] \notin \sigma_{8,3}(\nu_d(\mathbb{P}^n))$, it follows from Lemma 2.5.0.15, that $\operatorname{Spec}(\operatorname{Apolar}(W^{\blacktriangledown d}))$ is not smoothable. This implies Condition (b) holds. From [CEVV09, Thm 4.20], the algebra $\operatorname{Apolar}(W^{\blacktriangledown d})$ has Hilbert function $(1,4,3)$. We proved Condition (a) holds. ∎

Steps 2–5 of the algorithm verify if V satisfies necessary conditions to be of the form given by Lemma 4.3.1.6.

Proof of Theorem 4.3.1.5. Assume that $[V] \notin \sigma_{8,3}(\nu_d(\mathbb{P}^n))$. Then there exist a basis $(y_0, \dots, y_n)$ of T_1 and $U \subseteq \mathbb{C}[y_0, \dots, y_n]$ as in Lemma 4.3.1.6. Let $W := U|_{y_0=1} \subseteq \mathbb{C}[y_1, y_2, ..., y_n]$. Recall, that

$$W^{\blacktriangledown d} := \{(d-2)!F_2 + (d-1)!F_1 + d!F_0, \text{ where } F_2 + F_1 + F_0 \in W, \text{ and } F_i \in \mathbb{C}[y_1, y_2, \dots, y_n]_i\}.$$

Then, in the notation from Definition 2.5.0.1, we get

$$V = (W^{\blacktriangledown d})^{hom, d-2}.$$

By Lemma 2.5.0.13(ii), we have $\mathrm{Ann}(V)_{\leq d-2} = (\mathrm{Ann}(W^{\blacktriangledown d})^{hom})_{\leq d-2}$. Moreover, since $W^{\blacktriangledown d} \subseteq \mathbb{C}[y_1, \dots, y_n]_{\leq 2}$, and $d \geq 5$, we obtain $((\mathrm{Ann}(W^{\blacktriangledown d})^{hom})_{\leq d-2}) = \mathrm{Ann}(W^{\blacktriangledown d})^{hom}$. Therefore, there is a sequence of equalities

$$\mathfrak{a} = \sqrt{(\mathrm{Ann}(V)_{\leq d-2})} = \sqrt{\mathrm{Ann}(W^{\blacktriangledown d})^{hom}} = (\beta_1, \dots, \beta_n),$$

where $\beta_1, \dots, \beta_n \in T_1^*$ are dual to $y_1, \dots, y_n \in T_1$. This shows that if the $\mathbb{C}$-linear space $\left(\sqrt{(\mathrm{Ann}(V)_{\leq d-2})}\right)_1$ is not n-dimensional, then $[V] \in \sigma_{8,3}(\nu_d(\mathbb{P}^n))$. Therefore, in that case, the algorithm stops correctly at Step 2.

Assume that the algorithm did not stop at Step 2. Then, if V is of the form as in Lemma 4.3.1.6, then y_0 divides V exactly $(d-2)$-times. Otherwise $[V] \in \sigma_{8,3}(\nu_d(\mathbb{P}^n))$ and the algorithm stops correctly at Step 3.

Assume that the algorithm did not stop at Step 3. Then the Hilbert function of R^*/I computed in Step 4 is $(1, 4, 3)$ if and only if Condition (a) of Lemma 4.3.1.6 is fulfilled. Therefore, if it is not $(1, 4, 3)$, the algorithm stops correctly at Step 4.

Assume that the algorithm did not stop at Step 4. Then V satisfies Condition (a) from Lemma 4.3.1.6. Hence, $[V]$ is in $\sigma_{8,3}(\nu_d(\mathbb{P}^n))$ if and only if V does not satisfy Condition (b). Using Lemma 4.3.1.4, this is equivalent to

$$\dim_{\mathbb{C}} \mathrm{Hom}_{R^*}(I, R^*/I) > 8n - 7.$$

The left term is the dimension of the tangent space to the Hilbert scheme $\mathcal{H}ilb_8(\mathbb{A}^n)$ at the point $[\mathrm{Spec}\, R^*/I]$ (see [Har09, Prop. 2.3.] or [MS05, Thm 18.29]). ■

As it was announced earlier, the analogue of the slice technique (2.1.4.1) for the border cactus rank does not hold. We give an example of the tensor p such that its border cactus rank differ from the border cactus rank of a space spanned by slices $p((\mathbb{C}^3)^*)$. Recall the Segre-Veronese variety from Remark 2.4.2.9.

Proposition 4.3.1.7. *Let* $W := \langle y_2y_3 + y_1y_4, y_3y_4, y_1y_2 \rangle \subseteq S := \mathbb{C}[y_1, y_2, y_3, y_4]$ *and* $p = x_0 \otimes (y_0^3y_2y_3 + y_0^3y_1y_4) + x_1 \otimes y_0^3y_3y_4 + x_2 \otimes y_0^3y_1y_2 \in \mathrm{Sym}^1(\mathbb{C}^3) \otimes T_5$, *where* $T = \mathbb{C}[y_0, y_1, \dots, y_4]$. *Then* $9 \leq \underline{\mathrm{cr}}_{SV_{1,5}(\mathbb{P}^2 \times \mathbb{P}^4)}(p)$, *while* $\underline{\mathrm{cr}}_{\nu_5(\mathbb{P}^4)}(p((\mathbb{C}^3)^*)) = \underline{\mathrm{cr}}_{\nu_5(\mathbb{P}^4)}(W^{hom,3}) = 8$.

Proof.
Hilbert function of Apolar(W) equals $(1,4,3)$. It follows from Theorem 4.1.0.2 that

$$\underline{\mathrm{cr}}_{\nu_5(\mathbb{P}^4)}(W^{hom,3}) = 8.$$

Before we prove that $9 \leq \underline{\mathrm{cr}}_{SV_{1,5}(\mathbb{P}^2\times\mathbb{P}^4)}(p)$, let us demonstrate first that $9 \leq \underline{R}_{\nu_5(\mathbb{P}^4)}(W^{hom,3})$. If this is not the case, then by Border Apolarity Lemma 2.4.1.5 there exists an ideal $[J] \in \mathrm{Slip}_{8,\mathbb{P}T_1}$ such that $J \subseteq \mathrm{Ann}(W^{hom,3})$. Furthermore, from Theorem 4.1.0.2 (iii) we obtain, that

$$J^{sat} = \mathrm{Ann}(W)^{hom}. \tag{4.3.1.8}$$

The following equality holds

$$\dim_{\mathbb{C}} \mathrm{Hom}_{S^*}(\mathrm{Ann}(W), S^*/\mathrm{Ann}(W)) = 25. \tag{4.3.1.9}$$

The left term is the dimension of the tangent space to the Hilbert scheme $\mathcal{H}ilb_8(\mathbb{A}^4)$ at the point $[\mathrm{Spec}\,\mathrm{Apolar}(W)]$ (see [Har09, Prop. 2.3.] or [MS05, Thm 18.29]). By Lemma 4.3.1.4 and (4.3.1.9), the point $[\mathrm{Spec}\, S^*/\mathrm{Ann}(W)]$ is contained in $\mathcal{H}_{143} \setminus \mathcal{H}ilb_8^{sm}(\mathbb{A}^4) \subseteq \mathcal{H}ilb_8(\mathbb{A}^4)$. Together with (4.3.1.8) it gives $[\mathrm{Proj}(T^*/J^{sat})] \notin \mathcal{H}ilb_8^{sm}(\mathbb{P}^4)$, which contradicts the assumption saying, that $[J] \in \mathrm{Slip}_{8,\mathbb{P}T_1}$. Thus, the border rank of $W^{hom,3}$ is greater than its border cactus rank

$$8 = \underline{\mathrm{cr}}_{\nu_5(\mathbb{P}^4)}(W^{hom,3}) < \underline{R}_{\nu_5(\mathbb{P}^4)}(W^{hom,3}). \tag{4.3.1.10}$$

The variety $\kappa_r(SV_{1,5}(\mathbb{P}^2\times\mathbb{P}^4))$ equals $\overline{\bigcup\{\langle SV_{1,5}(R)\rangle \mid R \in \mathcal{H}ilb_r^{Gor}(\mathbb{P}^2\times\mathbb{P}^4)\}}$ (see [BB14, Proposition 2.2]). The paper [CJN15, Thm A] shows that for $r < 14$ and any n, the scheme $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$ is irreducible. Thus, $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n) = \mathcal{H}ilb_r^{Gor,sm}(\mathbb{A}^n)$ for any $r < 14$ and therefore $\mathcal{H}ilb_r^{Gor}(\mathbb{P}^2\times\mathbb{P}^4) = \mathcal{H}ilb_r^{Gor,sm}(\mathbb{P}^2\times\mathbb{P}^4)$ and

$$\kappa_r(SV_{1,5}(\mathbb{P}^2\times\mathbb{P}^4)) = \sigma_r(SV_{1,5}(\mathbb{P}^2\times\mathbb{P}^4)). \tag{4.3.1.11}$$

Assume that the tensor $p \in \mathrm{Sym}^1(\mathbb{C}^3)\otimes T_5$ has border cactus rank $\underline{\mathrm{cr}}_{SV_{1,5}(\mathbb{P}^2\times\mathbb{P}^4)}(p)$ at most 8. It follows from (4.3.1.11) that

$$\underline{R}_{SV_{1,5}(\mathbb{P}^2\times\mathbb{P}^4)}(p) = \underline{\mathrm{cr}}_{SV_{1,5}(\mathbb{P}^2\times\mathbb{P}^4)}(p) \leq 8.$$

By Lemma 2.1.4.1, we know that $\underline{R}_{SV_{1,5}(\mathbb{P}^2\times\mathbb{P}^4)}(p) = \underline{R}_{\nu_5(\mathbb{P}^4)}(W^{hom,3})$. Thus, $\underline{R}_{\nu_5(\mathbb{P}^4)}(W^{hom,3}) \leq 8$ and we obtain a contradiction with (4.3.1.10). ■

The tensor from Proposition 4.3.1.7 is a minimal example of $p \in \mathbb{C}^N \otimes T_d$ such that $\underline{\mathrm{cr}}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)) \neq \underline{\mathrm{cr}}_{SV_{1,d}(\mathbb{P}^{N-1}\times\mathbb{P}^n)}(p)$. To show this fact, we introduce following lemmas.

Lemma 4.3.1.12. *If $p \in \mathbb{C}^N \otimes T_d$ and $\underline{\mathrm{cr}}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)) \leq 7$ then*

$$\underline{\mathrm{cr}}_{SV_{1,d}(\mathbb{P}^{N-1}\times\mathbb{P}^n)}(p) \leq \underline{\mathrm{cr}}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)). \tag{4.3.1.13}$$

Proof. If $\underline{\mathrm{cr}}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)) \leq 7$ then

$$\underline{\mathrm{cr}}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)) = \underline{R}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)).$$

This follows from the fact that for $r \leq 7$ and any k, the scheme $\mathcal{H}ilb_r(\mathbb{P}^k)$ is irreducible [CEVV09].

In the case of tensor border rank, the slice technique (Lemma 2.1.4.1) says that

$$\underline{R}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)) = \underline{R}_{SV_{1,d}(\mathbb{P}^{N-1}\times\mathbb{P}^n)}(p).$$

Since the border cactus rank is less or equal the tensor border rank, we obtain $\underline{\mathrm{cr}}_{SV_{1,d}(\mathbb{P}^{N-1}\times\mathbb{P}^n)}(p) \leq \underline{\mathrm{cr}}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*))$. ■

To prove the opposite inequality to (4.3.1.13) we need the following lemma about cactus rank of space of slices.

Lemma 4.3.1.14. *Let A, V be vector spaces over $\mathbb{C}$. If $X \subseteq \mathbb{P}(V)$ and $p \in A \otimes V$, then*

$$\mathrm{cr}_X(p(A^*)) \leq \mathrm{cr}_{\mathbb{P}A\times X}(p).$$

Proof. Let $\pi_2 : \mathbb{P}(A) \times X \to X$ be the projection, and $R \subseteq \mathbb{P}(A) \times X$. It is enough to show that if $[p] \in \langle R\rangle$ then $\mathbb{P}(p(A^*)) \subseteq \langle \pi_2(R)\rangle$.

Let $W \subseteq V$ be such that $\mathbb{P}(W) = \langle \pi_2(R)\rangle$. Notice, that $\langle R\rangle \subseteq \mathbb{P}(A\otimes W)$. Since $[p] \in \langle R\rangle$, it follows that $p \in A \otimes W$. Thus $\forall_{\alpha\in A^*}\ p(\alpha) \in W$. In other words, we obtained $\mathbb{P}(p(A^*)) \subseteq \langle \pi_2(R)\rangle$. ■

Lemma 4.3.1.15. *Let A, V be vector spaces over $\mathbb{C}$ and A be finite dimensional. If $X \subseteq \mathbb{P}(V)$ and $p \in A \otimes V$, then*

$$\underline{\mathrm{cr}}_X(p(A^*)) \leq \underline{\mathrm{cr}}_{\mathbb{P}A\times X}(p).$$

Proof. Let $p_t \in A \otimes V$ be a curve of tensors such that $\lim_{t\to 0} p_t = p$ and $\underline{\mathrm{cr}}_{\mathbb{P}A\times X}(p) = \mathrm{cr}_{\mathbb{P}A\times X}(p_t)$. We have

$$p(A^*) \subseteq \lim_{t\to 0} p_t(A^*) \tag{4.3.1.16}$$

Indeed, let $w \in p(A^*)$ and $\alpha \in A^*$ be such that $p(\alpha) = w$. Then $p(\alpha) = \lim_{t\to 0} p_t(\alpha) \in \lim_{t\to 0} p_t(A^*)$. We obtain $\underline{\mathrm{cr}}_X(p(A^*)) \leq \underline{\mathrm{cr}}_X(\lim_{t\to 0} p_t(A^*))$ as a consequence of (4.3.1.16).

To complete the proof it is enough to demonstrate the following two inequalities.

$$\underline{\mathrm{cr}}_X(\lim_{t\to 0} p_t(A^*)) \leq \mathrm{cr}_X(p_t(A^*)) \leq \mathrm{cr}_{\mathbb{P}(A)\times X}(p_t) = \underline{\mathrm{cr}}_{\mathbb{P}(A)\times X}(p)$$

The first one follows from the definition of the border cactus rank of the space (see Definition 2.4.2.1). The second one is a corollary of Lemma 4.3.1.14. ■

It follows from Lemmas 4.3.1.12, 4.3.1.15 that for $p \in \mathbb{C}^N \otimes T_d$ such that $\underline{\mathrm{cr}}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)) \leq 7$ we have an equality

$$\underline{\mathrm{cr}}_{\nu_d(\mathbb{P}^n)}(p((\mathbb{C}^N)^*)) = \underline{\mathrm{cr}}_{SV_{1,d}(\mathbb{P}^{N-1}\times\mathbb{P}^n)}(p).$$

4.4 Implementation of the algorithm identifying the secant variety inside the cactus variety

We end the thesis with a presentation of the code of the algorithm from Theorem 4.0.0.4 (for $n = 6$) written in Macaulay2 [GS], [GMR20, Sect. A].

```
KK=ZZ/7919
T=KK[x_0..x_6]

completeToBasis = (y) -> {
  use T;
  L := {y,x_0,x_1,x_2,x_3,x_4,x_5,x_6};
  A := {x_0,x_1,x_2,x_3,x_4,x_5,x_6};
  for i from 1 to #L-1 do{
    (M,C) := coefficients(matrix{drop(L, {i,i})},Monomials=> A);
    if rank(C) == 7 then return drop(L, {i,i});
  }
}

triangle = (d,f) -> {
  C := terms(f);
  C = apply(C, g -> (d-(degree g)#0)! * g);
  return sum(C);
}

generatorsUpToDegree = (d,I) -> {
  E := entries mingens I;
  E = E#0;
  E = select(E, (i)->((degree i)#0 <= d));
  return ideal E;
}

annihilatorUpToDegree = (d,G) -> {
  J := inverseSystem(G);
  return generatorsUpToDegree(d, J);
}

dualLinearGenerator = (I) -> {
  J := generatorsUpToDegree(1,I);
  K := inverseSystem(J);
  J = generatorsUpToDegree(1,K);
  y := entries mingens J;
  y = y#0;
```

```
  return y#0;
}

howManyTimes = (y,G) -> {
  i := 0;
  while (G % y) == 0 do{
    G=G//y;
    i=i+1;
  };
  return i;
}

dehomogenizationWrtBasis = (G, L) -> {
  y := L#0;
  R := T/ideal(y-1);
  G = substitute(G, R);
  Q := KK[L#1, L#2, L#3, L#4, L#5, L#6];
  q := map(R, Q, {L#1,L#2,L#3,L#4,L#5,L#6});
  J := preimage_q(ideal(G));
  E := entries mingens J;
  E = E#0;
  return (E#0, Q);
}

homogeneousPart = (d, G) -> {
  E := terms G;
  E = select(E, (i)->((degree i)#0 == d));
  return sum E;
}

localHilbertFunction = I -> {
  S := ring I;
  m := ideal vars S;
  R := S/I;
  m = sub(m, R);
  return apply({ R/m, m/m^2, m^2/m^3, m^3/m^4}, degree);
}


isInSecant = (G) -> {
  --Step 1:
  d := (degree(G))#0 - 3;
  I := annihilatorUpToDegree(d,G);
```

```
  J := radical(I);
  --Step 2:
  if (hilbertFunction(1, module(J)) != 6) then return true;
  y := dualLinearGenerator(J);
  --Step 3:
  if (howManyTimes(y, G) != d) then return true;
  --Step 4:
  for i from 0 to d-1 do G=G//y;
  L := completeToBasis(y);
  (f, R) := dehomogenizationWrtBasis(G, L);
  ftriangle = triangle(d+3,f);
  K := inverseSystem(ftriangle);
  if (localHilbertFunction(K) != {1,6,6,1}) then return true;
  --Step 5:
  deg := degree Hom(K, R/K);
  return (deg > 76);
}
```

Bibliography

[Ådl87] Bjørn Ådlandsvik. Joins and higher secant varieties. *Math. Scand.*, 61(2):213–222, 1987.

[AFT11] Boris Alexeev, Michael A. Forbes, and Jacob Tsimerman. Tensor rank: some lower and upper bounds. In *26th Annual IEEE Conference on Computational Complexity*, pages 283–291. IEEE Computer Soc., Los Alamitos, CA, 2011.

[AIK89] Yu. I. Manin Alexei I. Kostrikin. *Linear Algebra and Geometry.* Algebra, Logic and Applications, Volume 1. CRC Press, 1 edition, 1989.

[AOP09] Hirotachi Abo, Giorgio Ottaviani, and Chris Peterson. Induction for secant varieties of Segre varieties. *Trans. Amer. Math. Soc.*, 361(2):767–792, 2009.

[Arn06] Morten Mørup; Lars Kai Hansen; Christoph S. Herrmann; Josef Parnas; Sidse M. Arnfred. Parallel factor analysis as an exploratory tool for wavelet transformed event-related EEG. *NeuroImage*, 29, 2006.

[AS79] M. D. Atkinson and N. M. Stephens. On the maximal multiplicative complexity of a family of bilinear forms. *Linear Algebra Appl.*, 27:1–8, 1979.

[BB14] Weronika Buczyńska and Jarosław Buczyński. Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes. *J. Algebraic Geom.*, 23:63–90, 2014.

[BB15] Weronika Buczyńska and Jarosław Buczyński. On differences between the border rank and the smoothable rank of a polynomial. *Glasg. Math. J.*, 57(2):401–413, 2015.

[BB19] Weronika Buczyńska and Jarosław Buczyński. Apolarity, border rank and multigraded Hilbert scheme. arXiv:1910.01944, to appear in Duke Mathematical Journal, 2019.

[BB20] Weronika Buczyńska and Jarosław Buczyński. Apolarity for border cactus decomposition in case of Veronese embedding, 2020. Working notes provided as a temporary reference for other authors, available at https://www.mimuw.edu.pl/~jabu/CV/publications/abcd_for_Veronese.pdf.

[BBKT15] Weronika Buczyńska, Jarosław Buczyński, Johannes Kleppe, and Zach Teitler. Apolarity and direct sum decomposability of polynomials. *Michigan Math. J.*, 64(4):675–719, 2015.

[BCS97] Peter Bürgisser, Michael Clausen, and M. Amin Shokrollahi. *Algebraic complexity theory*, volume 315 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997. With the collaboration of Thomas Lickteig.

[BFZ20] Wojciech Bruzda, Shmuel Friedland, and Karol Życzkowski. Tensor rank and entanglement of pure quantum states. 2020. arXiv: 1912.06854.

[BH98] W. Bruns and H.J. Herzog. *Cohen-Macaulay Rings.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[BH13] Murray R. Bremner and Jiaxiong Hu. On Kruskal's theorem that every $3 \times 3 \times 3$ array has rank at most 5. *Linear Algebra Appl.*, 439(2):401–421, 2013.

[BHMT18] Jarosław Buczyński, Kangjin Han, Massimiliano Mella, and Zach Teitler. On the locus of points of high rank. *Eur. J. Math.*, 4(1):113–136, 2018.

[BJ17] Jarosław Buczyński and Joachim Jelisiejew. Finite schemes and secant varieties over arbitrary characteristic. *Differential Geom. Appl.*, 55:13–67, 2017.

[BJJM19] Jarosław Buczyński, Tadeusz Januszkiewicz, Joachim Jelisiejew, and Mateusz Michałek. Constructions of k-regular maps using finite local schemes. *Journal of the European Mathematical Society*, 21:1775–1808, 2019.

[BJMR17] Alessandra Bernardi, Joachim Jelisiejew, Pedro Macias Marques, and Kristian Ranestad. On polynomials with given Hilbert function and applications. *Collectanea Mathematica*, 69(1):39–64, Jan 2017.

[BL13] Jarosław Buczyński and J.M. Landsberg. Ranks of tensors and a generalization of secant varieties. *Linear Algebra Appl.*, 438(2):668–689, 2013.

[Ble15] Grigoriy Blekherman. Typical real ranks of binary forms. *Found. Comput. Math.*, 15(3):793–798, 2015.

[BPR20] Jarosław Buczyński, Elisa Postinghel, and Filip Rupniewski. On Strassen's rank additivity for small three-way tensors. *SIAM Journal on Matrix Analysis and Applications*, 41(1):106–133, 1 2020.

[BR87] A. Berele and A. Regev. Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras. *Adv. in Math.*, 64(2):118–175, 1987.

[BR13] Alessandra Bernardi and Kristian Ranestad. On the cactus rank of cubics forms. *J. Symbolic Comput.*, 50:291–297, 2013.

[Bro33] J. Bronowski. The sums of powers as simultaneous canonical expressions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 29(2):245–256, 1933.

[CC01] Luca Chiantini and Marc Coppens. Grassmannians of secant varieties. *Forum Math.*, 13(5):615–628, 2001.

[CC08] Ciro Ciliberto and Filip Cools. On Grassmann secant extremal varieties. *Adv. Geom.*, 8(3):377–386, 2008.

[CCC15] Enrico Carlini, Maria Virginia Catalisano, and Luca Chiantini. Progress on the symmetric Strassen conjecture. *J. Pure Appl. Algebra*, 219(8):3149–3157, 2015.

[CCC$^+$18] Enrico Carlini, Maria Virginia Catalisano, Luca Chiantini, Anthony V. Geramita, and Youngho Woo. Symmetric tensors: rank, Strassen's conjecture and e-computability. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(1):363–390, 2018.

[CCO17] Enrico Carlini, Maria Virginia Catalisano, and Alessandro Oneto. Waring loci and the Strassen conjecture. *Advances in Mathematics*, 314:630–662, jul 2017.

[CEVV09] Dustin A. Cartwright, Daniel Erman, Mauricio Velasco, and Bianca Viray. Hilbert schemes of 8 points. *Algebra Number Theory*, 3(7):763–795, 2009.

[CGLM08] Pierre Comon, Gene Golub, Lek-Heng Lim, and Bernard Mourrain. Symmetric tensors and symmetric tensor rank. *SIAM J. Matrix Anal. Appl.*, 30(3):1254–1279, 2008.

[CGO14] Enrico Carlini, Nathan Grieve, and Luke Oeding. Four lectures on secant varieties. In *Connections between algebra, combinatorics, and geometry*, volume 76 of *Springer Proc. Math. Stat.*, pages 101–146. Springer, New York, 2014.

[CHL19] Austin Conner, Alicia Harper, and J. M. Landsberg. New lower bounds for matrix multiplication and the 3x3 determinant. arXiv:1911.07981v1, 2019.

[CJN15] Gianfranco Casnati, Joachim Jelisiejew, and Roberto Notari. Irreducibility of the Gorenstein loci of Hilbert schemes via ray families. *Algebra Number Theory*, 9(7):1525–1570, 2015.

[CLO15] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, 4/e (Undergraduate Texts in Mathematics)*. Springer-Verlag, Berlin, Heidelberg, 2015.

[CMM18] Alex Casarotti, Alex Massarenti, and Massimiliano Mella. On Comon's and Strassen's conjectures. 2018.

[CN09] Gianfranco Casnati and Roberto Notari. On the Gorenstein locus of some punctual Hilbert schemes. *J. Pure Appl. Algebra*, 213(11):2055–2074, 2009.

[Com02] Pierre Comon. Tensor decompositions: state of the art and applications. In *Mathematics in signal processing, V (Coventry, 2000)*, volume 71 of *Inst. Math. Appl. Conf. Ser. New Ser.*, pages 1–24. Oxford Univ. Press, Oxford, 2002.

[CW87] D. Coppersmith and S. Winograd. Matrix multiplication via arithmetic progressions. *Proceedings of the Nineteenth Annual ACM Symposium on Theory of Computing*, page 1–6, 1987.

[DLDM97] Lieven De Lathauwer and Bart De Moor. From matrix to tensor: Multilinear algebra and signal processing. *Mathematics in Signal Processing IV*, pages 1–15, 01 1997.

[DLDMV00] Lieven De Lathauwer, Bart De Moor, and Joos Vandewalle. On the best rank-1 and rank-(r1, r2,..., rn) approximation of higher-order tensors. *SIAM Journal on Matrix Analysis and Applications*, 21, 01 2000.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[Far88] R.W. Farebrother. *Linear least squares computations, STATISTICS: Textbooks and Monographs,*. New York : Marcel Dekker, 1988, 1988.

[Fri13] Shmuel Friedland. On tensors of border rank l in $\mathbb{C}^{m\times n\times l}$. *Linear Algebra Appl.*, 438(2):713–737, 2013.

[Gał17] Maciej Gałązka. Vector bundles give equations of cactus varieties. *Linear Algebra Appl.*, 521:254–262, 2017.

[Gał20] Maciej Gałązka. Multigraded apolarity. to appear in Mathematische Nachrichten, arXiv:1601.06211v3, 2020.

[GMR20] Maciej Gałązka, Tomasz Mańdziuk, and Filip Rupniewski. Distinguishing secant from cactus varieties. arXiv: 2007.16203, 2020.

[GOV19] Fulvio Gesmundo, Alessandro Oneto, and Emanuele Ventura. Partially symmetric variants of Comon's problem via simultaneous rank. *SIAM Journal on Matrix Analysis and Applications*, 40(4):1453–1477, 2019.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Har09] Robin Hartshorne. *Deformation Theory*. Graduate Texts in Mathematics. Springer New York, 2009.

[Has90] Johan Hastad. Tensor rank is NP-complete. *Journal of Algorithms*, 11, 1990.

[Hea10] Thomas L. Heath. *Diophantus of Alexandria, a Study in the History of Greek Algebra*. Cambridge University Press, 1910.

[HKT05] Wolfgang Hackbusch, B Khoromskij, and E Tyrtyshnikov. Hierarchical Kronecker tensor-product approximations. *Journal of Numerical Mathematics, v.13, 119-156 (2005)*, 13, 06 2005.

[HMV20] Hang Huang, Mateusz Michałek, and Emanuele Ventura. Vanishing Hessian, wild forms and their border VSP. *Mathematische Annalen*, 370(3):1505–1532, Dec 2020.

[HS04] Mark Haiman and Bernd Sturmfels. Multigraded Hilbert schemes. *J. Algebraic Geom.*, 13(4):725–769, 2004.

[IK99] Anthony Iarrobino and Vassil Kanev. *Power sums, Gorenstein algebras, and determinantal loci*, volume 1721 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999. Appendix C by Iarrobino and Steven L. Kleiman.

[Jel18] Joachim Jelisiejew. VSPs of cubic fourfolds and the Gorenstein locus of the Hilbert scheme of 14 points on $\mathbb{A}^6$. *Linear Algebra and its Applications*, 557, 11 2018.

[JT86] Joseph Ja'Ja' and Jean Takche. On the validity of the direct sum conjecture. *SIAM J. Comput.*, 15(4):1004–1020, 1986.

[KB09] Tamara G. Kolda and Brett W. Bader. Tensor decompositions and applications. *SIAM Review*, 51, 08 2009.

[KBK05] Tamara G. Kolda, Brett W. Bader, and Joseph P. Kenny. Higher-order web link analysis using multilinear algebra. *Proceedings - IEEE International Conference on Data Mining, ICDM*, pages 242–249, 01 2005.

[Lan08] J. Landsberg. Geometry and the complexity of matrix multiplication. *Bulletin of the American Mathematical Society*, 45(2):247–284, jan 2008.

[Lan12] J. M. Landsberg. *Tensors: geometry and applications*, volume 128 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[Lan17] J. M. Landsberg. *Geometry and Complexity Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017.

[LG14] François Le Gall. Powers of tensors and fast matrix multiplication. In *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, ISSAC '14, page 296–303, New York, NY, USA, 2014. Association for Computing Machinery.

[LM08] J. M. Landsberg and L. Manivel. Generalizations of Strassen's equations for secant varieties of Segre varieties. *Comm. Algebra*, 36(2):405–422, 2008.

[LM17] J. M. Landsberg and Mateusz Michałek. Abelian tensors. *J. Math. Pures Appl. (9)*, 108(3):333–371, 2017.

[LO13] J. M. Landsberg and Giorgio Ottaviani. Equations for secant varieties of Veronese and other varieties. *Ann. Mat. Pura Appl. (4)*, 192(4):569–606, 2013.

[Mań20] Tomasz Mańdziuk. Identifying limits of ideals of points in the case of projective space. arXiv:2011.14692, 2020.

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[Ott07] Giorgio Ottaviani. Symplectic bundles on the plane, secant varieties and Lüroth quartics revisited. In *Vector bundles and low codimensional subvarieties: state of the art and recent developments*, volume 21 of *Quad. Mat.*, pages 315–352. Dept. Math., Seconda Univ. Napoli, Caserta, 2007.

[Pal06] Francesco Palatini. Sulle superficie algebriche i cui $s_h(h+1)$-seganti non riempiono lo spazio ambiente. *Atti della R. Acc. delle Scienze di Torino*, 41:634–640, 1906.

[Rup17] Filip Rupniewski. Tensors of border rank 3 and Strassen's additivity conjecture. Master's thesis, Institute of Mathematics, Warsaw University, 2017. Supervisor: Jarosław Buczyński. Available at https://www.mimuw.edu.pl/~jabu.

[Sch81] A. Schönhage. Partial and total matrix multiplication. *SIAM J. Comput.*, 10(3):434–455, 1981.

[Sha13] Igor R Shafarevich. *Basic Algebraic Geometry; 3rd ed.* Springer, Berlin, 2013.

[Shi19] Yaroslav Shitov. A counterexample to Strassen's direct sum conjecture. *Acta Mathematica*, 222(2):363–379, 2019.

[SMS10] Toshio Sumi, Mitsuhiro Miyazaki, and Toshio Sakata. About the maximal rank of 3-tensors over the real and the complex number field. *Ann. Inst. Statist. Math.*, 62(4):807–822, 2010.

[SPY06] Jimeng Sun, Spiros Papadimitriou, and Philip Yu. [IEEE Sixth International Conference on Data Mining (ICDM'06) - Hong Kong, China (2006.12.18-2006.12.22)] - Window-based Tensor Analysis on High-dimensional and Multi-aspect streams. 2006.

[Str69] Volker Strassen. Gaussian elimination is not optimal. *Numer. Math.*, 13:354–356, 1969.

[Str73] Volker Strassen. Vermeidung von Divisionen. *J. Reine Angew. Math.*, 264:184–202, 1973.

[Str88] Volker Strassen. The asymptotic spectrum of tensors. *J. Reine Angew. Math.*, 384:102–152, 1988.

[Tei14] Zach Teitler. Geometric lower bounds for generalized ranks. arXiv: 1406.5145, 2014.

[Tei15] Zach Teitler. Sufficient conditions for Strassen's additivity conjecture. *Illinois J. Math.*, 59(4):1071–1085, 2015.

[Ter15] Alessandro Terracini. Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari. *Annali di Matematica Pura ed Applicata (1923 -)*, 24:1–10, 1915.

[Vak17] Ravi Vakil. The Rising Sea: Foundations of Algebraic Geometry. Available at math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf, Nov 2017.

[VT02] M. Alex O. Vasilescu and Demetri Terzopoulos. Multilinear analysis of image ensembles: Tensorfaces. *2003 IEEE Computer Society Conference on Computer Vision and Pattern Recognition, 2003. Proceedings*, 09 2002.

[War91] Edward Waring. *Meditationes algebraicæ*. American Mathematical Society, Providence, RI, 1991. Translated from the Latin, edited and with a foreword by Dennis Weeks, With an appendix by Franz X. Mayer, translated from the German by Weeks.

[Zak93] F. L. Zak. *Tangents and secants of algebraic varieties*, volume 127 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993. Translated from the Russian manuscript by the author.