REPORT ON THE DOCTORAL THESIS "SOME PROBLEMS IN NONCOMMUTATIVE ANALYSIS" SUBMITTED BY HAONAN ZHANG

1. Introduction

The doctoral thesis of Haonan Zhang concerns noncommutative analysis, quantum groups and quantum information theory. Three papers and one short note have been written from the thesis materials. For all of them, Haonan Zhang is the single author. One paper is already published in *Communications in Algebra*, one is accepted in *Mathematische Zeitschrift* and one is under review for publication and available on arXiv.

The manuscript is organized in six chapters. The first one contains the statements of the original results of the thesis and the second one contains the main notations, definitions and results used in the thesis. The last four chapters contain the original results and proofs of the thesis.

2. About Chapter 1 and 2

The first chapter is an overview of the thesis. In the first and second sections, the reader will find a brief historical background and an overview of some fundamental definitions and results with references. The presentation is clear and goes directly to the point. The citations are correct and most of the important references are given.

The last section of chapter 1 contains a more specialized state of the art review combined with the statements of the original results of the thesis. The motivations are clearly exposed which makes the statements of the original results limpid.

The second chapter contains three preliminary sections on non-commutative L_p-spaces, compact quantum groups and Plancherel triples. Again, the presentation is clear and concise.

3. About chapter 3

The third chapter of the thesis deals with the classification of idempotent states on Sekine quantum groups. These finite dimensional quantum groups have been introduced by Sekine in 1996. An idempotent state on a compact quantum group G is a state on the C*-algebra $C(G)$ which is an idempotent element with respect to the convolution product on $C(G)^*$. On a classical compact group G, every indempotent state (i.e. probability measure on G) is actually the probability coming from the Haar measure on a closed subgroup of G. In the quantum world, there exists idempotent states that do not come for the Haar measure of a closed quantum subgroup. Idempotent states coming from subgroups are called *Haar idempotent states*. Such Haar idempotent states on Sekine quantum groups have been completely classified by Franz and Skalski in 2009 and some examples of non Haar idempotent states have been given. Franz and Skalski have also written a system of linear equations characterizing when a given state on a Sekine quantum group is an idempotent state and they have asked for a complete computation of such states. In this chapter, Haonan Zhang solves this system of linear equations using elementary arithmetics and linear algebra, thus answering to the question of Franz and Skalski by computing explicitly all the idempotent states on Sekine quantum groups. **Haonan Zhang demonstrates here his ability to conduct**

long and difficult algebraic computations. Haonan Zhang also computes, for Sekine quantum groups, the order relation on idempotent states introduced by Franz and Skalski. Finally, he gives a necessary condition for the sequence of convolution powers of a state on a Sekine quantum group to be convergeant.

4. About chapter 4

The fourth chapter of the thesis deals with states of Poisson type and infinitely divisible states on finite quantum groups. It is well known that states of Poisson type on a compact quantum group are infinitely divisible i.e., admits a n-th root with respect to the convolution product, for all $n \geq 1$. It is known that the converse holds for finite groups (by the work of Bödge in 1959) and for duals of finite groups (by the work of Parthasarathy in 1970). Haonan Zhang shows that the converse holds for actually any finite quantum group and he presents two different proofs. The classical proofs (for finite groups as well as for duals of finite groups) do not generalize well in the quantum context. **Haonan Zhang overcomes this difficulty by introducing new theoretical and technical ideas. Haonan Zhang's approach to this proof exposes his capacity to find a nice point of view on a mathematical problem for the sake of developping a clear and conceptual proof.**

5. About chapter 5

The fifth chapter contains two elegant and new remarks about l_p-boundedness of Fourier multipliers on discrete groups. The first one shows that a real interpolation argument (combined with standard non commutative L_p-spaces inequalities) suffices to deduce that the Fourier multiplier of a function in $l_{q,\infty}(G)$ is $l_p(G)$-bounded, when G is a discrete group, $1 < p < \infty$ and $1/q = |1/2 - 1/p|$. The second remark shows that it suffices to apply a complex interpolation argument with, on the one hand, the norm inequality proved by Haagerup which shows rapid decay for $\mathbb{F}_\infty$ and, on the other hand, another Haagerup's type inequality for $\mathbb{F}_\infty$ proved by Ricard and Xu in 2016 to deduce a simple necessary condition for a radial Fourier multiplier on $\mathbb{F}_\infty$ to be bounded in $l_p(\mathbb{F}_\infty)$, for $2 < p < \infty$. Haonan Zhang also deduces some new examples of l_p-bounded Fourier multipliers on finitely generated groups with exponential growth or polynomial growth and on $\mathbb{F}_\infty$. **Haonan Zhang succeeds here to obtain new results in a very active domain.**

6. About Chapter 6

The last chapter of the manuscript deals with quantum information theory. Haonan Zhang studies quantum versions of the classical *α-Rényi relative entropies* used in information theory, which are themselves generalizations of the classical *Shannon entropy*. There are obviously many ways to define quantum analogues of the classical α-Rényi relative entropy and it is not clear which quantum relative entropy should be the right one. The fundamental property a meaningful quantum relative entropy should satisfy is the monotonicity under completely positive trace preserving maps: it is called the *Data Processing Inequality* (DPI).

Some of the most important quantum versions of relative entropies, which generalized many of the previously studied quantum entropies, are the *quantum α-Rényi entropy* $D_\alpha(\rho||\sigma)$, introduced by Müller-Lennert, Dupuis, Szehr, Fehr and Tomamichel in 2013, and the *sandwiched α-Rényi entropy* $\widetilde{D}_\alpha(\rho||\sigma)$, introduced by Wilde, Winter and Yang in 2014. They both depend on $\alpha \in]0,1[\cup]1,+\infty[$ and on a couple of positive trace one matrices of the same size (ρ,σ) and take extended real values. An fundamental problem is to identify which of those entropies satisfy the DPI. This problem has been solved for the entropies D_α and $\widetilde{D}_\alpha$. The point is to remark that the DPI is equivalent to the joint convexity/concavity of a certain trace function.

In 2015, Audenaert and Datta have introduced the *α-z Rényi relative entropy* which depends of an extra parameter $z > 0$ and which unifies and generalizes both D_α and $\widetilde{D}_\alpha$. Audenaert and Datta have obtained the DPI for certain values of (α, z). Haonan Zhang completely identifies in his thesis the set of (α, z) for which $D_{\alpha,z}$ satisfies the DPI by proving a conjecture of Audenaert and Datta. Indeed, it is known that the DPI is equivalent to the joint concavity/convexity of a certain trace function $\Psi_{p,q,s}(A, B)$ depending on a couple of positive trace one matrices of the same size (A, B) and with real parameters p, q and $s = \frac{1}{p+q}$. This remark has motivated Audeaert and Datta to state a conjecture (in 2015) on the jointly convexity of $\Psi_{p,q,\frac{1}{p+q}}$ for certain values of (p, q) which implies the DPI of $D_{\alpha,z}$. The joint convexity/concavity of the function $\Psi_{p,q,s}$ for different values of (p, q, s) has been studied by many authors such as Ando, Carlen, Frank, Hiai and Lieb. In 2018, Carlen, Frank and Lieb have conjectured the joint convexity of $\Psi_{p,q,s}$ for some remaining open cases. Their conjecture strengths the Audenaert-Datta conjecture.

Haonan Zhang proves the Carlen, Frank and Lieb conjecture thus, deducing the full range of (p, q, s) for which $\Psi_{p,q,s}$ is convex/concave. His result implies in particular the Audenaert-Datta conjecture and gives the full range of (α, z) for which $D_{\alpha,z}$ satisfies the DPI. To prove the conjecture, Haonan Zhang reduces the problem of joint convexity/concavity of the two variables function $\Psi_{p,q,s}(A, B)$ to convexity/concavity of a certain one variable trace function (which was already well studied) by applying the so-called *variational method.*

Haonan Zhang manages to completely solve an important conjecture in a highly competitive domain, studied by many mathematicians. To prove it, he did understand and further develop a difficult technic of analytical nature.

7. Conclusion

In his thesis, Haonan Zhang brings into play many sophisticated mathematical objects: compact quantum groups, non-commutative L_p-spaces, Fourier multipliers on discrete group von Neumann algebras, interpolation theory, matrix analysis and the variational method.

He shows a great mastery of these delicate analytical notions and I am impressed by his capacity to give clear and concise proofs and by the quantity and quality of his results.

Haonan Zhang has made a significant contribution to the theory of compact quantum groups and to quantum information theory. The candidate did show all the required qualities to become a researcher in mathematics. **I deeply recommend the thesis to be defended.**

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