A REPORT ON THE THESIS "**SOME PROBLEMS IN NOCOMMUTATIVE ANALYSIS**"
SUBMITTED BY **HAONAN ZHANG** TO
UNIVERSITÉ DE FRANCHE-COMTÉ - ÉCOLE DOCTORALE CARNOT-PASTEUR
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The thesis presents four different results whose common denominator is that they concern discrete or finite quantum structures and belong to the broadly understood field of noncommutative analysis. Two of these results are closely related. They concern states on finite quantum groups and have already appeared in the form of research papers published in Communication in Algebra and Mathematische Zeitschrift, respectively. The third result is about L_p-Fourier multipliers on discrete groups von Neumann algebras, that is multipliers on noncommutative L_p-spaces coming from quantum duals to discrete groups. The fourth result, which also appeared in a separate article (posted on arXive and submitted for publication), solves a problem which is fundamental for the notion of quantum relative entropy.

The thesis consists of six chapters. The first two summarise the main results, discuss their context, and describe some necessary background on idempotent states, compact quantum groups, quantum information theory, noncommutative L_p-spaces and Plancherel triples.

The third chapter is based on candidate's article [73]. Here the author successfully describes all idempotent states, together with their natural partial order, on the family of Sekine quantum groups. This answers a question of Franz and Skalski stated in 2009 [28]. Sekine quantum groups are finite (genuine) quantum groups $\mathcal{A}_k$, $k \geq 2$, arising as bicrossed products of classical cyclic groups $\mathbb{Z}_k$. They provide good test examples exhibiting various quantum phenomena. Franz and Skalski [28] identified all Haar idempotent states on $\mathcal{A}_k$, i.e states that Haar states for some quantum subgroups of $\mathcal{A}_k$, and showed that there exist a class of non-Haar idempotents (such a phenomena can not occur in classical setting). In fact, they gave a system of linear equation whose solutions give all idempotent states on $\mathcal{A}_k$, but they where not able to solve it in general. This is achieved by the candidate in his thesis. He does that by means of linear algebra and elementary number theory, and with remarkable case-by-case computations that are several pages long. In particular, he discovers one more interesting class of idempotent states on $\mathcal{A}_k$.

The forth chapter presents the result published in [74]. The author introduces a notion of *Poisson state* for any compact quantum group (Definition 4.8). Such states are defined with respect to an idempotent state, and so they might not correspond to a subgroup in the general quantum setting. First piece of evidence that this is the right definition is a characterisation of Poisson states in terms of convolutions semigroups; bi-invariant, conditionally positive functionals; and bi-invairant states (Theorem 4.9). However, the main result in this chapter is coincidence of Poisson states with infinitely divisible states for finite quantum groups (Theorem 4.21). This theorem generalizes and unifies two classical results: Bóge's theorem (1957) and Parthasarathy's theorem (1970), that characterise infinitely divisible measures and infinitely divisible positive definite functions on finite groups, respectively (in fact, Parthasarathy considers general metric compact groups). Both of the aforementioned results in this chapter are non-trivial and rely heavily on the analysis of a Plancherel triple induced from an idem-

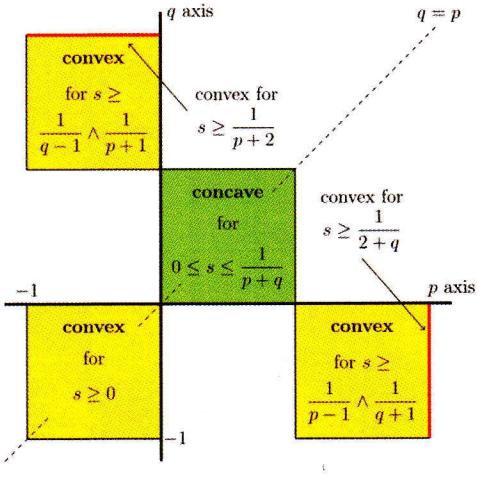
potent state. The main theorem is proved in two different ways, which might be important for the future developments and generalisations.

In the fifth chapter the author gives two sufficient conditions for L_p-boundedness of Fourier multipiers on discerete groups. One result is general and one concerns radial Fourier multipliers on the free group $\mathbb{F}_\infty$. The proofs relay on an inventive use of some facts and tools of interpolation theory. The limitations is that such arguments apply only to discrete groups.

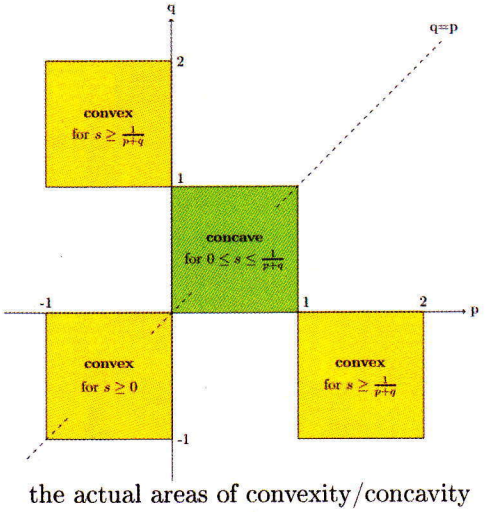
In the last chapter the author proves Carlen-Frank-Lieb conjecture which has remarkable consequences for α-z Rényi relative entropies $D_{\alpha,z}(\rho||\sigma)$ for density matrices ρ, σ. These entropies, depending on parameters $\alpha \in \mathbb{R} \setminus \{1\}$ and $z > 0$, seem to contain all reasonable quantum analogues of classical relative entropies. However, $D_{\alpha,z}$ might have a physical meaning only if it satisfies the *Data Processing Inequality*: $D_{\alpha,z}(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq D_{\alpha,z}(\rho||\sigma)$ where $\mathcal{E}$ is a completely positive trace preserving map (quantum channel). Thus it is crucial to determine the parameters α, z for which $D_{\alpha,z}$ is monotone with respect to every quantum channel. It is known that this problem can be phrased in terms of joint convexity/concavity of the trace function

$$\Psi_{p,q,s}(A,B) := \mathrm{Tr}(B^{\frac{q}{2}}K^*A^pKB^{\frac{q}{2}})^s,$$

defined on pairs of positive semi-definite matrices A and B, where $p, q, s \in \mathbb{R}$ are parameters and K is a fixed $N \times N$ matrix. The function $\Psi_{p,q,s}$ is interesting in its own right and plays an important role in mathematical physics and quantum information theory. Carlen, Frank and Lieb, based also on the work of others, managed to determine the range of parameters p, q, s for which $\Psi_{p,q,s}$ is jointly convex/concave except from the region $(p,q) \in [1,2] \times [-1,0] \setminus \{(1,-1)\}$ where they conjectured that $\Psi_{p,q,s}$ is convex for every $s \geq \frac{1}{p+q}$. The candidate proved this conjecture is true and so he completely cleared the picture of the regions of convexity/concavity of $\Psi_{p,q,s}$, cf. the figure below.



Areas of convexity/concavity of $\Psi_{p,q,s}$ known before Zhang's thesis



the actual areas of convexity/concavity of $\Psi_{p,q,s}$ as proved by Haonan Zhang

As consequence the candidate also solved the problem of determining the parameters for which Rényi relative entropies $D_{\alpha,z}$ are monotone. The main tool to establish these results is a clever use of variational method formalised in Theorem 6.12. In fact, the author uses it to give a full

and independent proof of joint convexity/concavity regions of $\Psi_{p,q,s}$ as described in the figure above (not only in the region conjectured by Carlen, Frank and Lieb).

Summary and the recommendation:

The results presented in the thesis of Haonan Zhang are more than sufficient for a doctoral degree. The candidate demonstrates his broad knowledge in the advanced field of noncommutative harmonic analysis and his ability to solve abstract mathematical problems. He has both computational skills and very good insight in literature to either apply complicated direct computations or use clever tricks and ideas to attack the problems successfully. I believe the results presented in papers [72], [73], [74] will receive considerable interest in the noncommutative analysis community, and the proof of Carlen-Frank-Lieb conjecture [72] should position the candidate amongst one of the frequently cited authors by mathematical physicists.

The presentation in the thesis is mostly careful and detailed, giving satisfactory background, motivations and references to earlier relevant work. However, I have a few complaints on the structure and editorial part. It seems to me that the process of combining the separate articles into one thesis could go better. Dividing the previous material to preliminaries and actual chapters did not make the text easier to digest (making the reader to jumb back and forth). Also it resulted in a number of repetitions (some notions such as of an idempotent state are explained over and over again) while some notation is not introduced or is introduced after it is used. Writing the remarks and definitions, which are sometimes more than half-page long, in italics is also inappropriate. To be honest, sometimes it was easier for me to read the candidate papers first and then the thesis, rather than the other way round. The thesis contains a few typographic errors, sentences that are stylistically incorrect or use an informal language, and there are some places where the author could give more details, but this does not affect the general impression that much.

Taking into account the above facts **I consider the thesis submitted by Haonan Zhang as satisfying all formal and traditional requirements for a doctoral thesis and with full conviction recommend that it be admitted to the public defense. I believe it deserves a recommendation of the Magna cum Laude.**

Białystok, 2019-08-06