**REFEREE REPORT:** *GROUND STATE, BOUND STATE AND NORMALIZED SOLUTIONS TO SEMILINEAR MAXWELL AND SCHRÖDINGER EQUATIONS*, **PHD THESIS, JACOPO SCHINO**

This thesis consists of eight chapters divided in two parts and contains the results of graduate work of Jacopo Schino and which otherwise are included in one published paper and three manuscripts (one more article is at the stage of preparation). In general terms its objective is to show the existence of solutions to a class of semilinear partial differential equations and to this end a variety of methods and tools of calculus of variations, nonlinear analysis, functional analysis etc. are employed. After the preliminary section which summarizes some basic tools and notations used later, the stage for the rest is set in the Introduction to Part I. It is very well written, exhaustive and informative and personally I found it very helpful in understanding the main ideas of the program of investigation of Jacopo. Below I give my general opinion of each of the remaining sections and at the end I list some minor errors and recommendations.

Part I. Chapter 3.

The problem studied in this part is that of the existence of solutions of the equation

$$\nabla \times \nabla \times U = f(x, U), \qquad U \colon \mathbb{R}^3 \to \mathbb{R}^3,$$

where $f(x,u) = \nabla_u F(x,u)$, $F \colon \mathbb{R}^{\times}\mathbb{R}^3 \to \mathbb{R}$ is a given function. Because of the form of the differential operator it is referred to as the curl − curl type problem and arises naturally in studying Maxwell's equations (the time harmonic Maxwell equation). As for the nonlinear potential $F$ usually a series of assumptions is made but roughly speaking it should be convex and $F(U) = \min\{|U|^q, |U|^p\}$, $2 < p < 6 < q$ (double power nonlinearity) has been considered in earlier works of Benci et al. that originated the interest in the problem. In the present work, more generally, it is assumed that $F$ is non autonomous and depends $\mathbb{Z}^3$ periodically on $x$ (this of course includes autonomous nonlinearities). More importantly the hypothesis (F3) says that $F$ is controlled by an $N$-function $\Phi$ restricted by (N1)–(N3). This allows to work in a very general functional analytic setting of the Orlicz spaces $L^{\Phi}(\mathbb{R}^3, \mathbb{R}^3)$ for the solution and examples of functions which can not be controlled by the double power nonlinearity but satisfy (F3) are given.

The major difficulty in dealing with the curl − curl problems is that the operator $\nabla \times \nabla \times$ has an infinite dimensional kernel consisting of the conservative vector fields. To eliminate this difficulty first the Helmholtz decomposition is used to write $U = v + w$ where $v$ is the solenoidal and $w$ the irrotational component. Second he problem is set up variationaly for

$$J(v, w) = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla v|^2 - \int_{\mathbb{R}^3} F(x, v + w)$$

and then it is reduced to finding critical points of $J$. The path taken from here could be described as sort of variational Lyapunov-Schmidt reduction: for a given $v$ the map $v \mapsto m(v)$ from the solenoidal fields to irrotational fields is defined via

$$\int_{\mathbb{R}^3} F(x, m(v)) = \inf_w \int_{\mathbb{R}^3} F(x, v + w)$$

(in the manuscript $m(v)$ is found as the maximum of $J(v, w)$ with $v$ fixed) and then the problem is reduced to finding critical points of $J(m(v))$. To execute this plan an abstract critical point theory described in section 3.3 is employed. This part of the thesis is based on the earlier work of Bartsch and Mederski (references [18] and [19]) but I imagine that it was adapted to the non-compact setting. (It this respect I would like to recommend including a paragraph commenting on this, in particular in the context of the proof of Theorem 3.3.5, which is the key result of this section. Minor suggestion: indicate right before the Lemma 3.3.6 the beginning of the proof in a series of Lemmas. ) The results of this abstract theory are applied in section 3.4 which amounts to checking various assumptions made on the functional $J$ in the specific case. The choice of the functional analytic context (Orlicz spaces) is a very nice, original contribution of this work however it makes the proofs more involved than the usual Sobolev space theory. I think that this point (advantages/disadvantages) deserves some discussion, otherwise it looks like an ad hoc choice.

After all the preparation of previous section the last one containing the proof of the main existence theorem is rather short. In reality two theorems are proven: first shows the existence of solutions without assuming any symmetry and for a general class of nonlinearities and the second is the multiplicity result showing that for $F$ even in $u$ there exists an infinite sequence of distinct solutions. The former is shown by the min/max argument and the latter by the topological argument based on Krasnoselskii genus. These are clearly exceptionally good results in terms of their novelty, generality and difficulty.

Part I. Chapter 4

The results in this chapter concern the vectors fields $U$ with cylindrical symmetry, which in the special case $U \colon \mathbb{R}^3 \to \mathbb{R}^3$ means

$$U(x) = u(x)(-x_2/r, x_1/r, 0), \quad r^2 = x_1^2 + x_2^2,$$

where $u \colon \mathbb{R}^3 \to \mathbb{R}$. In fact, more general, still symmetric fields are allowed in some of the results, but in any event the original $\mathrm{curl} - \mathrm{curl}$ problem is reduced to the Schrödinger equation

$$-\Delta u + \frac{u}{r^2} = \tilde{f}(x, u)$$

with an explicit relation between $f$ y $\tilde{f}$ (given the symmetries assumed). This idea goes back to Banci et al. Two specific types of nonlinearities are considered: 1) subcritical (hypothesis (F1)–(F4); 2) critical in dimension 3: $f(U) = |U|^5$.

After the introduction containing the statement of the main results the equivalence between the two problems is carefully shown. Next, (1) is considered and the idea here is to work with the Schrödinger equation. Using the abstract variational approach of the previous chapter two existence theorems are shown for this equation and as a corollary the existence result for the curl − curl equation is proven. Its novelty (with respect to Benci et al.) is to allow more general nonlinearities and also sign changing solutions. As for the critical problem a very nice idea, inspired by Ding and also Clapp-Pistoia, is to recover compactness by putting the problem in $\mathbb{S}^3$. After this, rather then working with the Schrödinger equation the existence is shown directly for $U$, using the mountain pass structure and the Palais principle of symmetric criticality. Although this approach seems to be limited to $\mathbb{R}^3$ I found it very interesting and elegant.

Part II. Chapters 5,6,7

The problem considered in these three chapters is the following constrained nonlinear Schrödinger equation/system

$$-\Delta u_i + \lambda_i u_i = \partial_i F(u), \qquad \text{in} \quad \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} u_i^2 \ = \rho_i^2$$

where $i = 1, \ldots, K$, $1 \leq K < 2_{\#} = 2 + 4/N$, and $F(u) = F(u_1, \ldots, u_K)$ is a nonlinear function and $\rho_i > 0$ are given constants. As pointed out in the introduction this problem has a rather long history and reach literature. The main issue of interest is the existence of solutions/ground state solutions and the difficulty is of course to guarantee that the minimizing sequences of the associated constrained variational problem

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - \int_{\mathbb{R}^N} F(u) \longrightarrow \min_{\|u_i\|_2^2 = \rho_i^2}$$

converges to a function that itself satisfy the constraint. Additionally, depending on the nonlinearity the problem may be classified as mass subcritical, critical or supercritical (when $F(u) = |u|^p/p$, $u$ scalar functions this means, respectively, $2 < p < 2_{\#}$, $p = 2_{\#}$, $2_{\#} < p < 2^*$). In the mass critical case invariant scaling of the problem is the source of the extra loss of compactness and in the supercritical case the functional is not bounded from below. Partly due to this several additional assumptions are made on $F$ and although they exclude the pure power nonlinearity in the critical case and are not optimal but on the other hand allow to extend the existence of solutions to a new class of nonlinearities. The loss of compactness of the embedding $H^1(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is in the present paper dealt with by minimizing in a smaller space (assuming for example radial symmetry) and, in the mass supercritical case, by constrained minimization on the Nehari-Pohozhaev manifold of the problem.

The main idea behind the proofs is to relax the original problem by requiring

$$\int_{\mathbb{R}^N} u_i^2 \leq \rho_i^2, \qquad i = 1, \ldots, K,$$

(idea introduced by Bieganowski and Mederski in a recent work). This simplifies the argument (because of the weak lower semicontinuity of the $L^2$ norm) but requires an extra step to show that in fact the relaxed is equivalent to the original problem. In the language of constrained optimization this amounts to showing that all the constraints are active. As a matter of fact the argument relies on am abstract nonlinear optimization result due to Clark which gives automatically $\lambda_i \geq 0$. Intuitively the active constraints correspond to $\lambda_i \neq 0$.

This is easier to show when $K = 1$ and the existence theorems are proven in subcritical, critical and supercritical cases (as defined in the paper they are not easy to interpret since $F$ is in general not a pure power). As for the subcritical and critical cases (Chapter 6) the key assumption has to do with the behavior of the nonlinearity $F(u)$ for $u \to 0^+$ and $u \to +\infty$ (stated as hypothesis in Theorem 6.1.1). In the case $2 \leq K < 2_\#$ additional assumptions on $F$ are made but still they are quite flexible (Theorem 6.1.3). In general the existence of a radially symmetric ground state solution is shown, and the existence of a second solution is also proven when in the case $K = 1$, $F$ even. In the supercritical case (Chapter 7) the existence of the ground state solution is proven first for the relaxed problem is solved under several assumptions on $F$ and for the original problem when $N = 3, 4$. These hypothesis are quite technical.

In my opinion this part is a very valuable contribution to the very extensively studied problem since it gives it an alternative, straightforward and natural (from the nonlinear optimization point of view) setting and new results are obtained. The relaxed problem is in fact interesting in its own right and I am convinced that the approach presented will inspire further research in this direction.

Part II. Chapter 8

This chapter contains some consequences of the developments of the Chapter 6 translated to the setting of the constrained curl − curl problem with cylindrical symmetry which (Chapter 4) is equivalent to the constrained Schrödinger equation

$$-\Delta u + \frac{u}{r^2} + \lambda u = f(u), \qquad \text{in} \quad \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} u^2 = \rho^2.$$

Again $\rho > 0$ is given and $\lambda$ is the Lagrange multiplier. As for the nonlinear function $f$ several assumptions made mirror those of Chapters 6 and 7, and as before the mass subcritical, critical and supercritical cases are treated. Because the method used in Chapter 6 requires sometimes radial symmetry to recover compactness the method does not work in these cases when $N = 3$ and some results are restricted to $N \geq 4$. Still the results contained in this section are original with short proofs building up on the previous parts.

Conclusions

Overall I have a very high opinion about this work. In fact its content, difficulty, breadth and the results are well beyond what I would normally expect from the PhD thesis. Already Part I by itself would make very nice thesis. I am very impressed with the range of techniques of analysis and calculus of variations that Jacopo manages. His writing is clear and very careful and in spite of the technical difficulties of the proofs I found them quite readable. I also find remarkable that he was able to do so much since 2017. I am convinced that this thesis will be an excellent springboard for the future scientific career of Jacopo. Because I considered Jacopo's thesis exceptionally good I would like to propose to award his dissertation a special distinction.

In summary, in my view Jacopo Schino's doctoral thesis meets all legal and customary requirements of a Phd Thesis. I move to proceed with the conferment of the doctoral degree of the author.

List of errors and specific recommendations

(1) p. 19, line 2 from the top and also below in Theorem 1.1.5 $m$ should be $k$.
(2) page 31. line 12 from the bottom "such a paper", could be "their argument" ?
(3) page 34, line 3 from the bottom, "peculiar" change to "particular"
(4) page 37, condition (N2): please check if this condition is consistent with the example involving $W(t^2)$ in the second paragraph on page 39. Be specific about the values of $p, q$.
(5) page 40, line 11 from the top, "Rabinowitz".
(6) page 40 (a) of Theorem 3.1.1 $E'(U) = 0$.
(7) page 42 line 6 from the top, "needs not be closed", the correct form of this expression is "need not be" (it is also used later in the manuscript, please correct).
(8) page 42, line 8 from the bottom, the phrase beginning with "However..." is not clear, please clarify.
(9) page 55, line line 9 from the bottom $\mathcal{J}'(u_n^-)$, is $^-$ correct ?
(10) page 58, line 8 from the bottom, "We show that".
(11) page 61, are $\mathcal{M}$, $\mathcal{N}$ the same as defined before ? Also, check the font of $\mathcal{N}$ in line 16 from the top.
(12) page 69, the last line on this page, please explain better the contradiction.
(13) page 78, line 6 from the bottom, should be $\bar{u} < 0$ ?
(14) page 80, line 9 from the bottom, please clarify what is $\tilde{g}_1$.
(15) page 81, line 3 from the top "to have as follows" does not sound grammatically correct.
(16) page 86, line 5 from the top, should be $U_\tau$ perhaps.
(17) page 93, condition (I3), please check the arrows.
(18) page 95, first line in the proof of Lemma 4.3.5, check $X_{\mathcal{O}}$.
(19) page 96, line 5 from the to should be $t_n$ ?
(20) page 97, line line 7 from the bottom, add parenthesis in the third member of the expression.
(21) page 101, line 7 from the top, check $\mathcal{D}_{\mathcal{O}(2\times 2)}$.
(22) page 102, please explain a bit the displayed lines starting from "Let us compute".
(23) page 116, line 1 from the top, please comment on the reason of the restriction $K < 2_\#$ with more details.
(24) page 120, is $a$ in (6.2.2) correct ?
(25) page 120, (b) of the statement of Lemma 6.2.5, is $\min_{\mathcal{D}}$ correct ?
(26) page 122, please clarify the restriction of $N$ in the Lemma 6.2.9.
(27) page 122-123, the statement of Lemma 6.2.10, the is a repetition of "For every $\ell \in \{1, \dots, L\}$...", please rephrase it. Also, recall that $u^*$ is the Schwartz symmetrization.
(28) page 125, line 12 from the bottom, should be $\partial_j F(u)$ ?
(29) page 125, displayed formula staring from $J(u)$, I do not see what happens with the coupling terms.

(30) page 131, line 10 from the bottom, I do not understand the example $F_j(u)$, doesn't it give a decoupled problem ?
(31) page 132, formula (7.2.1) means weak convergence in $H^1$ ?
(32) , page 132, first line in the first displayed formula of the proof of Lemma 7.2.3: is the $\int$ sign missing ?
(33) page 132, I am not sure that the inequality in line 4 from the bottom give the conclusion since it holds when $|\nabla u|_2 = 0$, am I missing something ?
(34) page 135 line 1 from the top, $L^{2_N}$ correct ?