Warsaw, July 19, 2021

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Referee report on Ph.D. thesis
Ground state, bound state and normalized solutions to semilinear Maxwell and Schrödinger equations
by Jacopo Schino

The thesis is devoted to the study of a semilinear elliptic system, whose highest order linear term has a big kernel. More specifically, the author analyses the steady states of the nonlinear Maxwell system

$$\nabla \times \nabla \times \mathbf{U} = f(x, \mathbf{U}) \qquad \text{in } \mathbb{R}^3. \tag{1}$$

The operator on the left-hand-side of (1) has an infinitely dimensional kernel. The observation,

$$\nabla \times \nabla \times \mathbf{U} = \nabla(\operatorname{div} \mathbf{U}) - \Delta \mathbf{U}$$

permits to offer simultaneous treatment of (1) and a nonlinear stationary Schrödinger eq.

$$\Delta \mathbf{U} + f(x, \mathbf{U}) = 0 \qquad \text{in } \mathbb{R}^3, \tag{2}$$

provided that the vector field $\mathbf{U}$ is divergence-free.

The main purpose of this thesis is to show existence of solutions to (1) and (2) and to count them. The fact that the kernel of the curl operator is infinitely dimensional makes the study difficult and interesting. Another difficulty is the fact that (1) and (2) are considered on the whole $\mathbb{R}^3$ making any compactness in the function spaces hard to get.

I find this area underdeveloped and offering challenges to researchers venturing there. I wonder if the theory developed in this thesis could be applied to other interesting elliptic operators with large kernel in higher dimensions.

The thesis is composed of two separate parts and an introductory Chapter I, where the theoretical background is presented. It is based on four papers, among them three are written with his advisor and for one article Mr Jacopo Schino is the sole author.

I begin with my comments on Part I of the thesis, which is devoted to (1) and (2). Due to the structure of the leading linear term the number of space dimensions is restricted to 3. However, this limitation is frequently lifted while discussing (2).

Part I is opened by a chapter presenting the motivation to study (1) and the relationship between (1) and (2). This definitely helps to follow the content.

Chapter 3 of the thesis is based on paper:

J. Mederski, J. Schino, A. Szulkin, Multiple solutions to a semilinear curl-curl problem in $\mathbb{R}^3$, *Arch. Ration. Mech. Anal.*, **236** (2020), no. 1, 253–288.

One of the merits of this chapter is to expose the variational structure of eq. (1). For this purpose the author assumes that there is $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ such that $f(x, \mathbf{U}) = \nabla_{\mathbf{U}} F(x, \mathbf{U})$. In this case (1) becomes formally the Euler-Lagrange eq. of functional E defined by

$$E(\mathbf{U}) = \int_{\mathbb{R}^3} (\frac{1}{2}|\nabla \times \mathbf{U}|^2 - F(x, \mathbf{U}))\, dx.$$

It is also worth commenting on the growth assumptions on F with respect to $\mathbf{U}$. They are made in the spirit of the theory of Orlicz spaces. Thus, the growth of F is expressed in term of an N-function Φ whose behavior near zero may be different from that at infinity. However, for large arguments of F the growth is polynomial. In other words the exponential or logarithmic type of growth are excluded. However, the assumptions on F make E the difference of competing term and E is not bounded from below.

The function spaces considered in the thesis must accommodate the growth of f. At the same time it is nice to deal with reflexive and separable spaces, where smooth functions are dense. In order to achieve that the author imposes the Δ_2 and ∇_2 conditions on Φ. In the case considered in the thesis the growth of Φ and F at ∞ is slower than dictated by the critical Sobolev exponent. Such a statement provokes questions about a feasibility of extending the theory to deal with arbitrary growth of F. A related question is about possibility of relaxing the convexity assumption on F?

A natural domain of definition of E is

$$\mathcal{D}(\text{curl}, \Phi) := \{\mathbf{U} \in L^{\Phi} :\ \nabla \times \mathbf{U} \in L^2(\mathbb{R}^3; \mathbb{R}^3)\},$$

where Φ is the N-function controlling the growth of f. As we mentioned this space is too big. A way to resolve this issue is to use the Helmholtz decomposition of $\mathcal{D}(\text{curl}, \Phi)$ into a space of divergence free fields, $\mathcal{V}$, and gradient fields, $\mathcal{W}$. Subsequently a counterpart of E on $\mathcal{V} \times \mathcal{W}$ is defined by

$$J_1(v, w) = \int_{\mathbb{R}^3} (\frac{1}{2}|\nabla v|^2 - F(x, v + w))\, dx.$$

(I have to modify slightly the notation since the one used in the thesis is not consistent).

Space $\mathcal{V} \times \mathcal{W}$ is equipped with the natural norm $\|(v, w)\|^2 = \|\nabla v\|^2_{L^2(\mathbb{R}^3)} + |w|^2_{\Phi}$.

One of the main results of Chapter 3 is Theorem 3.1.1. Under the assumptions of periodicity of f with respect to x, appropriate convexity on F, monotonicity on f the author proves existence of a ground state. Since, E is not bounded from below E has no global minimizers. A ground state means a minimizers over a Nehari–Pankov manifold $\mathfrak{N}$. If in addition F is even with respect to $\mathbf{U}$, then the author proves existence of countably many geometrically distinct solutions $\mathbf{U}_n$ to (1) called bound states.

Theorem 3.1.1 is proved with the help of abstract tools of the calculus of variations. The part dealing with existence of a ground state exploits the mountain pass geometry of J. In order to prove existence of the bound states, the author uses even more difficult theory of Krasnosel'skiĭ genus. I really appreciate the technical effort to implement the abstract theory in the specific context of the nonlinear Maxwell system.

Chapter 4 is based on:

M.Gaczkowski, J. Mederski, J. Schino, Multiple solutions to cylindrically symmetric curl-curl problems and related Schrödinger equations with singular potentials, *arXiv:2006.03565*.

It is devoted to the study of

$$-\Delta u + \frac{a}{r^2}u = f(x,u) \qquad \text{in } \mathbb{R}^N \tag{3}$$

and

$$\nabla \times \nabla \times \mathbf{U} = h(x,\mathbf{U}) \qquad \text{in } \mathbb{R}^3. \tag{4}$$

Unfortunately, the notation used in (4) is different from that used in (1).

In this chapter, the nonlinearities in (3) and (4) enjoy cylindrical symmetries. This means that these eqs are invariant under the action of $\mathcal{SO} := \mathcal{SO}(K) \times \{Id_{N-K}\}$, where $2 \le K < N$. In fact, the most interesting results are shown for $K = 2$.

This invariance justifies the definition of r, namely $r^2 = \sum_{i=1}^K x_i^2$. The function space, X, which is natural from the weak solution point of view is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_X^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + \frac{u^2}{r^2})\, dx,$$

and $X_{\mathcal{SO}}$ denotes the subspace of elements of X which are invariant with respect to $\mathcal{SO}$.

The analysis of this chapter is performed under the assumption that f grows subcritically at infinity and vanishes subcritically at zero. If in addition to this the author assumes superquadratic growth at infinity and monotonicity of $u \mapsto \frac{f(x,u)}{|u|}$, $a > -(K-2)^2/4$, then the existence of a ground state in $X_{\mathcal{SO}}$ is shown. A ground state here is a solution with minimal energy J_2 defined by

$$J_2(u) = \int_{\mathbb{R}^N} (\frac{1}{2}|\nabla u|^2 + \frac{a}{2r^2}u^2 - F(x,u))\, dx,$$

where $\nabla_u F(x,u) = f(x,u)$ and $F(x,0) \equiv 0$. An additional assumption of f being odd in u gives us infinitely many geometrically distinct solutions.

Theorem 4.1.2 is another result of this sort proved here. It is stated for $a > -(K-2)^2/4$, but for different assumptions on f that is a weaker version of the Ambrosetti-Rabinowitz condition. Then, there is a nontrivial solution to (3).

The proofs of these two results are based on the abstract theory developed in Chapter 3. The main effort focuses on checking that the assumptions specified there (including Theorem 3.3.5 (b)) are satisfied.

The author is able to treat (3) and (4) simultaneously for $K = 2$ and special nonlinearity h. More precisely, the author assumes that

$$h(\cdot,0) = 0, \qquad h(\cdot,\alpha w) = f(\cdot,\alpha)w \qquad \forall \alpha \in \mathbb{R}, w \in \mathbb{S}^{N-1}. \tag{5}$$

The author makes a very interesting observation about the relationship between solutions to (3) and (4). Namely, let us suppose that:
(a) the growth of f is subcritical;
(b) f invariant with respect to $\mathcal{SO}$ and $\{Id_2\} \times \mathbb{Z}^{N-2}$;
(c) h is given by (5);
(d) functions $u \in X$ and $\mathbf{U} \in \dot{H}^1(\mathbb{R}^N, \mathbb{R}^N)$ are related by

$$\mathbf{U}(x) = \frac{u(x)}{r}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad x_1, x_2 \in \mathbb{R}. \tag{6}$$

Then, u is a solution to (3) with $a = 1$ if and only if $\mathbf{U}$ is a solution to (4) and div $\mathbf{U} = 0$.

Another topic picked up here is the nonlinearity with the critical growth, from the point of view of Sobolev embedding, when $N = 3$ and $K = 2$ for a special nonlinearity, $f = u^5$. The notion invariance of solutions is quite interesting here. Namely, a vector field $\mathbf{U} : \mathbb{R}^3 \to \mathbb{R}^3$ is called invariant with respect to $(g_1, g_2) \in \mathcal{SO}(2) \times \mathcal{SO}(2)$ if the field $\tilde{\mathbf{U}} : \mathbb{S}^3 \to \mathbb{S}^3$, obtained due to the identification of $\mathbb{R}^3$ and $\mathbb{S}^3 \setminus \{(1,0,0,0)\}$ by the stereographic projection is invariant with respect to $\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$.

Mr Schino proves that there is a sequence of solutions, $\mathbf{U}_n$, to

$$\nabla \times \nabla \times \mathbf{U} = |\mathbf{U}|^4 \mathbf{U} \qquad \text{in } \mathbb{R}^3, \tag{7}$$

which are invariant with respect to $\mathcal{SO}(2) \times \mathcal{SO}(2)$ in the sense described above and which have the form (6). Moreover, the energy of $\mathbf{U}_n$ goes to ∞.

In the proof of this result the author notices that the embedding of the space of H^1 functions with the required invariance into $L^6(\mathbb{R}^3, \mathbb{R}^3)$ is compact. Basically, the existence result is shown with the help of the compactness result mentioned above applied to Palais-Smale sequences.

In the second part of the thesis Mr Schino studies variational functionals like J_3 defined by

$$J_3(u) = \int_{\mathbb{R}^N} (\frac{1}{2}|\nabla u|^2 - F(u))\, dx$$

with an additional two-sided constraints,

$$\int_{\mathbb{R}^N} u_j^2\, dx = \rho_j^2, \qquad j = 1, \ldots, K. \tag{8}$$

These constraints result in Lagrange multipliers appearing in the Euler-Lagrange equations,

$$-\Delta u_j + \lambda_j u_j = \partial_j F(u) \qquad \text{in } \mathbb{R}^N, \quad j = 1, \ldots, K. \tag{9}$$

The goal is to study the least energy solutions to (9) combined with (8). The difficulty associated with the variational tools is that the constraints (8) do not survive after taking a weak limit in L^2. The idea of the author is to relax (8) and consider one-sided constraints,

$$\int_{\mathbb{R}^N} u_j^2\, dx \le \rho_j^2, \qquad j = 1, \ldots, K, \tag{10}$$

which are easier to satisfy. This approach was first applied to a single equation by Bieganowski-Mederski. The contribution of Mr Schino is to apply to system (9). Let me stress that such an extension of this method requires additional effort.

Chapter 5, which opens Part II, is devoted to explaining the content of Chapter 6 and Chapter 7 as well as to introducing auxiliary results.

Chapter 6 is based on a single-author paper:

> J. Schino, Normalized ground states to a cooperative system of Schrödinger equations with generic L^2–subcritical or L^2–critical non-linearity, arXiv:2101.03076.

The problem stated above is addressed in this chapter, when we have a single equation in (9) (see Theorem 6.1.1) or we have system (see Theorem 6.1.2), but the number of equations K is not big, i.e. $2 \le K < 2_\# = 2 + \frac{4}{N}$. The existence result depends on four components of the problem:

1) the amount of mass, i.e. ρ; 2) the behavior of the nonlinearity at $u = 0$; 3) the behavior of the nonlinearity when $u \to \infty$, 4) the dimension N and the universal constant in Gagliardo-Nirenberg inequality.

In addition, in case of systems a special structure of the nonlinearity is required. A model nonlinearity for system in (9) is

$$F(u) = \sum_{j=1}^{K} \left(\frac{\nu_j}{2_\#} |u_j|^{2_\#} + \frac{\bar{\nu}_j}{p_j} |u_j|^{p_j} \right) + \alpha \prod_{j=1}^{K} |u_j|^{r_j} + \beta \prod_{j=1}^{K} |u_j|^{\bar{r}_j}, \tag{11}$$

where $\bar{\nu}_j, \alpha, \beta \geq 0$, $\bar{\nu}_j, \alpha + \beta > 0$, $\bar{r}_j, r_j > 1$, $2 < p_j < 2_\#$, $\sum_{j=1}^{K} r_j = 2_\#$, $2 < \sum_{j=1}^{K} \bar{r}_j < 2_\#$.

The main claim is: if the total amount of mass is small but not too small, then there exists a solution to (9), (8) with negative energy having the prescribed symmetry, e.g. radial one or cylindical one. Actually, theorem statements are more precise.

Let me comment first on the scalar case, $K = 1$. The proof of this result is based on the direct method of the calculus of variations. One important step is to show that functional J_3 is bounded from below. Since J_3 is a difference of competing terms establishing this fact requires some work.

The proof of Lemma 6.2.5 bothers me. While the weak compactness of the minimizing sequence in $H^1(\mathbb{R}^N)$ is clear. A possibility of extracting an a.e. subsequence is not clear. I suspect that this follows from the symmetries of the problem, however, an additional comment is necessary here.

Let me also notice that particular effort is necessary to show that the Lagrange multipliers λ_j are not trivial. The Pohožaev identity plays an important role here.

When we deal with a single equation in (9), it is possible to use the Schwarz rearrangement to show that solutions are radially symmetric.

In the vectorial case the existence of minimizers part is the same however additional steps are required to show that $\lambda_j \neq 0$ and $u_j \neq 0$.

Chapter 7 deals with system (9) with two-sided (see (8)) or one-sided constraints, (see (10)), when some of the restrictions on the total mass are lifted. It is based on the following paper:

> J. Mederski, J. Schino, Least energy solutions to a cooperative system of Schrödinger equations with prescribed L^2-bounds: at least L^2-critical growth, arXiv:2101.02611.

Minimizers are sought in a manifold containing the critical points of J_3, which is defined with the help of Nehari and Pohožaev identities. The nonlinearities considered are special and are close to the model F defined in (11).

At the technical level the content is close in spirit to the previous chapter. A method of scaling is exploited, where for $0 < s \in \mathbb{R}$ the scaling $s \star u(x) := s^{N/2} u(sx)$ permits one to play with the different scaling properties of the terms in J_3. I will not offer comments about the details, because to some extent they are in the same category as in Chapter 6.

The last chapter provides a link between the equations studied in Chapter 6 and 7 with the Maxwell eq. with constraints like (8). The proofs are combination of the result presented earlier.

After describing the content let us evaluate the results presented in the thesis. Mr Schino presented a series of non-trivial results on existence of ground states or bound states of the nonlinear stationary Maxwell system and the nonlinear stationary Schröringer equation in $\mathbb{R}^N$.

The main treat of the thesis is the use of variational methods, which are difficult to use in the context of the present thesis due to the large kernel of the curl operator. The variational functional is a difference of two competing components which make the methods based on the mountain pass theorem suitable. However, important modifications are necessary. I'm very much impressed with the use of Krasnosel’skiĭ genus. I like very much studying equations with prescribed symmetries. On the one hand this add another layer of difficulty but on the other hands this sometimes helps, because one could get extra compactness, this was the case of Chapter 4.

The proofs included in the thesis are quite complex. Theorem statements often involve many variants omitted in my writing for the sake of simplicity of exposition. Let me also stress the sheer volume of this work, which belongs to the category of long theses.

There are also some deficiencies related to the presentation and gaps in the proofs which are possibly pasted from the published articles. For instance see my comments on the proof of Lemma 6.2.5. However, after taking all, known to me, aspects of the thesis I claim that the thesis of Mr Jacopo Schino fulfills all the legal and customary requirements. Moreover, taking into account the scientific value of the results and the contribution of Mr. Schino in reaching them I propose granting an honorary distinction of the thesis.

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