Ground state, bound state, and normalized solutions to semilinear Maxwell and Schrödinger equations

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Abstract

This Ph.D. thesis is concerned with the existence of entire solutions to semilinear elliptic equations and is divided into two parts, dealing with unconstrained and constrained problems respectively. Chapter 1 contains a series of recalls about notions and properties used throughout this work and lies before the aforementioned division into parts.

In Part I, we study existence and multiplicity results for equations of the form

$$\nabla \times \nabla \times \mathbf{U} = f(x, \mathbf{U}), \quad \mathbf{U} \colon \mathbb{R}^N \to \mathbb{R}^N,$$

where $N \geq 3$ and $f = \nabla F \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is the gradient (with respect to $\mathbf{U}$) of a given nonlinear function $F \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$. Here, when $N \geq 4$, $\nabla \times \nabla \times \mathbf{U}$ is defined using the identity, valid when $N = 3$, $\nabla \times \nabla \times \mathbf{U} = \nabla(\nabla \cdot \mathbf{U}) - \Delta \mathbf{U}$. Such problems are known as *curl-curl* problems and arise, when $N = 3$, from the nonlinear Maxwell equations in absence of electric charges, electric currents, and magnetization. The main issue is that the kernel of the differential operator $\nabla \times \nabla \times$ consists of the subspace of gradient fields and is therefore infinite-dimensional. Historically, two approaches have been used to tackle curl-curl problems by means of variational methods and both make use of divergence-free vector fields. The reason is that $\nabla \times \nabla \times \mathbf{U} = -\Delta \mathbf{U}$ for every divergence-free field $\mathbf{U}$ and the vector Laplacian is a differential operator easier to handle.

In Chapter 2, we give an accurate physical derivation of curl-curl problems and then survey important results throughout the last decades, from the first works to the current days, including those illustrated in this Ph.D. thesis.

In Chapter 3, based on [3], we focus on the (physically relevant) case $N = 3$. The nonlinearity F is controlled, from above and from below, by a suitable nice Young function: in particular, Sobolev-supercritical at zero, Sobolev-subcritical but superquadratic at infinity, and satisfying the Δ_2 and ∇_2 conditions globally. Our approach makes use of a Helmholtz-type decomposition of the function space we work with into a divergence-free subspace and a curl-free subspace (the aforementioned kernel), i.e., $u = v + w$ with $\nabla \cdot v = \nabla \times w = 0$ and (v, w) uniquely determined; then we build a homeomorphism from the former subspace

to a certain topological submanifold (of the whole space) that contains all the nontrivial solutions. This somehow allows us to work only with the divergence-free subspace, although the "curl-free" part must be taken care of; in fact, that is what causes the most difficulties in the methods we use. We prove the existence of a least-energy solution and, if f is odd, of infinitely many distinct solutions. Unlike Chapter 4, we do not use any symmetries; in particular, we provide the first multiplicity result about curl-curl problems in unbounded domains without any symmetry assumptions.

In Chapter 4, based on [1], we consider the general case $N \geq 3$. Under certain symmetry assumptions about the nonlinearity, we exploit suitable group actions to reduce the curl-curl problem to the Schrödinger equation with singular potential

$$-\Delta u + \frac{a}{|y|^2} u = \tilde{f}(x,u), \quad u \colon \mathbb{R}^N \to \mathbb{R},$$

with $x = (y,z) \in \mathbb{R}^K \times \mathbb{R}^{N-K}$, $K = 2$, and $a = 1$, studying as well the general case $2 \leq K < N$ and $a > -(K/2-1)^2$. More in detail, we require that $f(\cdot,\alpha w) = \tilde{f}(\cdot,\alpha)w$ for every $\alpha \in \mathbb{R}$ and every $w \in \mathbb{S}^{N-1}$ and that $\tilde{f}(gx,\cdot) = \tilde{f}(x,\cdot)$ for a.e. $x \in \mathbb{R}^N$ and every $g \in \mathcal{SO}(2) \times \{I_{N-2}\}$. Extending to the case of weak solutions a well-known equivalence property that puts in a 1-to-1 correspondence the classical solutions to the two problems via the formula $\mathbf{U}(x) = u(x)/|y|(-x_2,x_1,0)$, we prove new existence results (nontrivial solutions, least-energy solutions relatively to the functions with the same symmetry, infinitely many distinct solutions) about both, in the Sobolev-critical and -noncritical cases; in particular, we work with the curl-curl equation in the former case, using the same symmetry machinery to reduce $\nabla\times\nabla\times\mathbf{U}$ to $-\Delta\mathbf{U}$, and with the Schrödinger equation in the latter. The most prominent result is the existence, when $N = 3$, of a divergent sequence of solutions in the critical case, obtained with the aid of another group action, which restores compactness; this is the first multiplicity result for curl-curl problems in unbounded domains in the Sobolev-critical case. Concerning the existence, in the noncritical case, of a least-energy solution and of infinitely many distinct solutions, we exploit an abstract critical point theory built in Chapter 3.

In Part II, we look for least-energy solutions to autonomous Schrödinger systems of the form

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \partial_j F(u) \\ \int_{\mathbb{R}^N} u_j^2 \, dx = \rho_j^2 \end{cases} \quad \forall j \in \{1,\dots,K\}, \quad u \colon \mathbb{R}^N \to \mathbb{R}^K,$$

where $N, K \geq 1$, $\rho = (\rho_1,\dots,\rho_K) \in]0,\infty[^K$ is given, and $\lambda = (\lambda_1,\dots,\lambda_K) \in \mathbb{R}^K$ is part of the unknown. Solutions to such problems are called *normalized* due to the L^2-constraints, which are what causes the quantity λ to appear as a K-tuple of Lagrange multipliers. Equations of this type arise when seeking standing wave solutions to similar time-dependent problems and come from areas of Physics such as nonlinear optics and Bose–Einstein condensation. Their importance lies in the physical meaning of the masses (the L^2 norms squared) and the fact that such quantities are conserved in time in the corresponding evolution equations.

In Chapter 5, we introduce the problem, briefly comment some seminal papers and other results in the literature, and provide useful preliminary properties.

Depending on the assumptions about F and, sometimes, on the value of ρ, the associated energy functional exhibits different behaviours: it can be bounded from below for all, some, or no values of ρ and these cases are known, respectively, as mass-subcritical, -critical, and -supercritical. The first two are studied in Chapter 6, based on [4], while the last is studied in Chapter 7, based on [2]. In both cases, we consider a minimizing sequence for the energy functional and work out proper assumptions so that such a sequence converges to a solution to the system. In the mass-supercritical case, since the functional is unbounded from below, we restrict it to a natural manifold, given by a suitable linear combination of the Nehari and the Pohožaev identities to get rid of the unknown quantity λ, in order to recover such boundedness. The outcome consists of a least-energy solution, relatively to the functions with the same symmetry or in the general sense depending on the structure of the nonlinearity.

The novelty of this approach consists of considering the $L^2(\mathbb{R}^N)$ balls

$$\left\{ u \in H^1(\mathbb{R}^N) \;\middle|\; |u|_2 \le \rho_j \right\}, \quad j \in \{1, \dots, K\}$$

instead of the $L^2(\mathbb{R}^N)$ spheres

$$\left\{ u \in H^1(\mathbb{R}^N) \;\middle|\; |u|_2 = \rho_j \right\}, \quad j \in \{1, \dots, K\}$$

in order to work with a weakly closed subset and have, a priori, additional information about the sign of the components of λ, which is due to the fact that the constraints are given by inequalities and that the critical points we obtain are minimizers.

When $K \ge 2$, we need particular hypotheses about the nonlinearity in order to make use of the Schwarz symmetric rearrangements; nevertheless, we can still deal with rather generic functions, which is new about systems.

Finally, Chapter 8 contains new results and deals with normalized solutions both to curl-curl problems and to nonautonomous Schrödinger equations with singular potential as in Chapter 4, but always with autonomous nonlinearities. Such results are obtained combining the symmetry and the equivalence from Chapter 4 with the outcomes from Chapters 6 and 7. In particular, the symmetry allows us to reduce the curl-curl problem to a vector-valued autonomous Schrödinger equation, with a single L^2-constraint and which we study directly, while the equivalence provides analogous results for the scalar-valued Schrödinger equation with singular potential. Again, we obtain least-energy solutions relatively to the functions with the same symmetry.

References

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