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Report on the PhD thesis:

Singular limits and rough behavior in evolutionary equations arising in physics and biology

by Jakub Skrzeczkowski

Contrary to what the title and introduction of this thesis suggest, its content contributes not only to two but to three distinctive areas in the analysis of partial differential equations (PDE).

In the first part, the author considers the fast reaction limit for a system of reaction-diffusion equations where the reaction function is non-monotone. This limit is much simpler when the reaction function is monotone, and it was solved a decade ago in the work of Bothe et al. [41], which was nicely described in the review chapter by Hilhorst et al. [171], as mentioned by the author.

In the paper [Perthame, Skrzeczkowski, *Comm. Pure Appl. Math.* 2023] being discussed as first in Chapter 3 of the thesis, the authors concentrate on a N-shaped reaction function F with certain non-degenracy condition (condition 3.2.6) that excludes piecewise affine functions. This assumption was later removed in Jakub's single-authored paper [Skrzeczkowski, *Comptes Rendus Math.* 2022] described in the second part of Chapter 3. Both results are restricted to the singular limit of the system when only one component diffuses, so the equations are reduced to ODE-PDE system in which the solution v_ε to the PDE converges to the limit strongly, and the solution u_ε to the ODE converges only weakly. The first result is quite surprising as there are no a-priori estimates from which this could be deduced. The author solves the problem by using the theory of Young measures. He considers the Young measures generated by sequences of solutions $v_\varepsilon, u_\varepsilon$ (labelled with the reaction rate parameter ε) aiming to identify the first one with the Dirac measure.

To this purpose, Jakub exploits the family of energies, in the spirit of Murat and Tartar's work on compensated compactness techniques in conservation laws. He further adapts the argument of Plotnikov from 1994, which was excavated from the manuscript [232], available only in Russian. It is a work dedicated to certain regularisation of the second order evolutionary equation with "variable direction of parabolicity" similar, in shape, to the function F from the thesis. Following this approach, Jakub extends the standard weak convergence result for monotone functions to piecewise monotone case, thanks to clever reformulation in terms of Young measures (see Eq. (3.4.1)). This language, along with careful exploitation of structural assumptions on F, allows him to push forward the measure associated with sequence u_ε in Eq. (3.4.1) on the branches where F is invertible, and to deduce that v_ε must converge strongly.

The fact that the other sequence u_ε converges only weakly is then a consequence of non-monotonicity of function F, which has been well illustrated by a one-dimensional example plotted in Fig. 3.1.

All in all, two features of this part of the thesis seem to be especially impressive: the clever use of compensated compactness techniques in the singular limit result for reaction-diffusion equations and a significant generalization of Plotnikov's non-degeneracy assumption, allowing Jakub to extend this technique to more physical affine functions F.

In the second part of his thesis, the author discuses various aspects related to degenerate (linear mobility) Cahn-Hiliard equation and systems. In this part of the thesis, commencing from Chapter 4, he first derives the degenerate Cahn-Hilliard equation from the kinetic Vlasov-type PDE. He rigorously justifies the limit in which the probability distribution function converges to Maxwellian multiplied by total density. This leads to the non-local degenerate Cahn-Hilliard equation, where non-locality refers to the approximation of the Vlasov's force term by double convolution with a mollifying kernel, which would otherwise give rise to the laplacian in the macroscopic equation for the chemical potential μ.

The nonlocality in this case is different from the non-local Cahn-Hilliard equation studied in Chapter 5.The author discusses the reason for this in a couple of remarks throughout the thesis and also in Section 4.5. While it would have been nice to complete the derivation of the local degenerate Cahn-Hilliard equations starting from the nonlocality that guarantees the well-posedness of the Vlasov-Cahn-Hilliard equation, the main theorem from Chapter 4 is a remarkable first rigorous result that provides the link between the micro and macro descriptions for systems with degenerate mobility. This result was published in one of the best journals in applied analysis, Comm. Math. Phys.

Similarly, the main result from the following Chapter 5, which is devoted to the passage from non-local to local Cahn-Hilliard equation, also meets high standards. The study of degenerate Cahn-Hilliard equation is substantially more difficult than the case with mobility separated from zero, which provides much stronger a priori estimates. However, Jakub's choice of the non-local kernel, in conjunction with Bourgain-Br'ezis-Mironescu and Ponce compactness results, allows him to deduce strong convergence of the sequence approximating the solution and its gradient. This is sufficient to pass to the limit in the weak formulation. It would be interesting to provide a similar result for other than periodic boundary conditions. Some ideas of how to progress in this direction have been highlighted by the author in the concluding Section 5.6.

Finally, in Chapter 6, the author considers yet another singular limit for the Cahn-Hilliard system, complementing the previous results. This time he links the first-order nonlocal equations with the second-order expansion given by the Euler-Korteweg (nonlocal) equation through the high friction limit. He uses the method of relative entropies for well-prepared initial data to provide the rigorous convergence of the dissipative masurevalued solutions to the primitive system to the global-in-time regular solutions of the nonlocal Cahn-Hilliard equation. This gives him an advantage over the previous results of Lattanzio and Tzavaras, who identified the same limit only locally in time.

In the last part of the thesis, the author focuses on existence theory for $p(t,x)$-Laplace generalisation of the heat equation. A-priori estimates for such equations suggest working in the setting of Orlicz and Musielak-Orlicz spaces, whose definitions and properties have been revised in Chapter 7. Chapter 8, based on the result [Bulicek, Gwiazda, Skrzyczkowski, *JDE*, 2021], provides an existence and uniqueness result in the situation where the restriction on the t-growth of $p(t,x)$ has been removed. Continuity in time of the exponent $p(t,x)$ was required in the previous results to commute the regularizing convolution with the operation of taking $p(t,x)$-power in the estimates. The key observation of Jakub and his coauthors is that the time regularization is not necessary, and that the regularity of solution with respect to time can be obtained directly from the form of the equation. Additional difficulty in the case of non-Newtonian fluid equations, discussed in Chapter 9, is that, to derive the basic estimates, one needs to test the momentum equation by regular solutions that are solenoidal. On top of that, all embedding inequalities and the Korn inequality (the stress tensor depends on the symmetric part of the gradient) need to be performed locally in space. Arbitrary regularizations of the solution, truncations or convolutions do not maintain the divergence free condition. Therefore, the method of harmonic pressure of Wolf [262] needed to be adapted to recover the pressure and to be able to test the momentum equation by arbitrary smooth functions. It is remarkable that despite these problems the authors provide the existence result with only minimal assumptions on the growth exponent in the stress tensor, necessary to bound the convective term.

However, the real highlight of this part, in my opinion, is discussed at the end in Chapter 10. The corresponding result has been published in ARMA by the same set of coauthors. They have managed to extend the results from Chapters 7 and 8 to complement the work of Colombo and Mingione [82] devoted to the double phase variational integrals with the so called (p,q)-growth, see Eq. (10.1.1). They proved that in the range of parameters when p is not bigger than the dimension, the Lavrentiev phenomenon does not occur. This means that the infimum of the functionals can be approximated by the smooth(-er) functions with the right boundary conditions. A key element of the proof was showing the density of smooth functions in appropriate Musielak-Orlicz space, with the middle step using the fact that it is enough to consider the intersection of Musielak-Orlicz space with the space of L^∞ functions. This allows using better estimates of the solution to bound the gradient of the mollified solution in L^∞, see Eq. (10.3.5), which was a bottleneck for the previous results. I find this an extremely nice example of how the PDE theory can be applied to solve a problem in the calculus of variations.

The length of Jakub's thesis fortunately correlates with the quality of his research. The thesis encompasses a diverse and impressive collection of publications, demonstrating Jakub's mastery and in-depth understanding of modern PDE techniques, such as advanced compensated compactness theory and measure theory, with an emphasis on Young measures. The thesis is well-structured, with a clear introduction, numerous intuitive examples, and elaboration on the relevance of the presented problems. These demonstrate the author's keen interest in applications and a genuine desire to solve meaningful problems. The author also reflected on the limitations of their techniques, discussed possible generalizations and further open problems, for example in Sections 3.6, 4.5, and 10.4. This carefully crafted and insightful thesis reveals Jakub's exceptional research abilities as a mathematician. It is clear that he possesses a rare combination of creativity, rigour, and technical expertise, which are all essential qualities of a successful researcher. I am confident that Jakub will continue to make significant contributions to the field.

Therefore, it is with great enthusiasm and without hesitation that I recommend him for a PhD degree with distinction.

Yours sincerely,

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