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Report on the thesis of Lei Jin
"Embedding problems for real and discrete time flows on compact spaces"

The dissertation under review concerns five problems belonging to the field of topological dynamics:

1. Vanishing of entropy of predictable actions of countable amenable groups (*Hochman's problem*);
2. Conditions implying *mean Li-Yorke chaos* (a.k.a. *distributional chaos of type 2*) for $\mathbb{Z}$ actions;
3. Mean Li-Yorke chaos for actions of countable amenable groups with positive entropy;
4. The *realization of assigments* problem in zero-dimensional systems;
5. Embeddings of real flows ($\mathbb{R}$ actions) into some universal flows.

The dissertation is based on four papers: [GJ17], [GJT16], [HJ16], and [HJY16][1] (listed as published or accepted for publication) and two preprints: [DJLQ17] and [GJ16] (available on `arxiv`). All four advisors (Tomasz Downarowicz, Yonatan Gutman, Wen Huang, Xiandong Ye) together with four other mathematicians are (in various configurations) co-authors of these works. The content of the dissertation mostly coincides with the text of these papers and preprints. The journals which accepted the papers are very good (*Ergodic Theory and Dynamical Systems*, *Proceedings of the AMS*) or good (*Entropy*, *Journal of Dynamics and Differential Equations*).

The main contributions presented in the thesis are:

1. Theorem 2.1.1 saying that the predictability property implies zero topological entropy for actions of torsion-free locally nilpotent groups;
2. Theorem 3.1.1 providing sufficient conditions for the existence of a Cantor uniformly mean Li-Yorke chaotic (DC2-chaotic) set for $\mathbb{Z}$-actions;

[1] All references are the same as in the dissertation.

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3. Theorem 4.2.1 asserting that every action of a countable bi-orderable amenable group with positive entropy is mean Li-Yorke chaotic;
4. A construction (Theorem 5.1.1) of a zero-dimensional system whose simplex of invariant measures is a copy (appropriately understood) of the simplex of invariant measures $\mathcal{M}_T(X)$ of an aperiodic $\mathbb{Z}$ action (X, T) with zero-dimensional closed or sigma-compact set of ergodic measures, in other words the zero-dimensional system realizes an assigment provided by (X, T);
5. An improved version of the classical Bebutov-Kakutani theorem (Theorem 6.2.5) and an explicit construction of a universal real flow (Theorem 6.3.3).

The variety of problems tackled in this thesis shows that the author has mastered a broad spectrum of methods in ergodic theory and topological dynamics. He works in the classical setting of actions of $\mathbb{Z}$ and $\mathbb{R}$, as well as in a more demanding context of actions of countable amenable groups. None of the main results seems to be a breakthrough in dynamical systems. Most of the proofs require a skilful and knowledgeable application of existing techniques. I think that the main theorems are nontrivial and somewhat interesting. **In my opinion, the thesis is a valuable addition to the topological dynamics and deserves to be accepted after a public defense**.

That said, I will proceed with a more detailed description and further remarks.

As I said above, the author studies actions of groups on compact metric spaces by homeomorphisms with emphasis on actions of $\mathbb{Z}$, $\mathbb{R}$, and countable amenable groups. This very general setting is the only coherent common thread, which can be identified in the thesis. This broad approach allows the author to demonstrate his extensive knowledge and undeniable work ethic, but as a result, we see a dissertation which is uneven and looks like a random collection of theorems. Furthermore, copying the content from the articles the author concentrated on results and shortened introductions, discussions of the motivation and related problems, in other words, all the background information which would allow the reader to put the results in a broader context. In effect we get a work which is (at times) very tersely written and the reader can only guess (or consult the original articles) the motivation behind the problems studied in the thesis. (Here, Chapter 5 is a notable exception. Chapter 6 contains at least some motivation, but the articles on which Chapter 6 is based presents much broader point of view.)

I will now move on to more specific remarks.

The introduction is the union of short paragraphs merely describing the content of the following chapters 1–6. No motivation and only minimal amount of historical or background information can be found here.

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Chapter 1 contains preliminaries: a list of definitions and some results for further reference. An unusual description of a Choquet simplex is provided on page 22. According to it, a Choquet simplex is a convex and compact subset of a metric-linear space endowed with a convex metric satisfying the barycenter condition. A more common definition calls a set a Choquet simplex if it is a compact, convex, and metrizable subset of a locally convex topological vector space satisfying the barycenter condition. In particular, Phelps book [Phe01] cited in the dissertation as the source for a lesser known characterization of a Choquet simplex uses, in fact, the second one.

Chapter 2 presents a partial solution to the problem proposed by Mike Hochman [Hoc12]. It is based on a paper by Huang, Jin, and Ye [HJY16]. The authors use the Rhemtulla-Formanek theorem guaranteeing that any unit-free subsemigroup of a torsion-free locally nilpotent group G is a subset of some algebraic past of G. This allows the authors to use the method used by Hochman almost verbatim. Hochman showed that predictable actions of $\mathbb{Z}^d$ have zero topological entropy. The existence of a past enables a representation of the entropy of a partition as a conditional entropy of the partition with respect to the 'past'. The chapter also contains direct proofs of two particular cases of the Rhemtulla-Formanek theorem (for countable abelian torsion-free groups and the integer Heisenberg group). Both reasonings are quite straightforward and very technical. On the other hand, the proof given by Formanek in [For73] seems to me to be much shorter and clearer. Furthermore, I do not see much merit in these proofs, and I think that they could be omitted without any harm to the overall value of the thesis. Unfortunately, no example of a predictable action of any group is discussed, so applicability of Theorem 2.1.1 is unknown. It is a severe drawback of the whole theory, and it is surprising that this issue is not addressed. The result of Hochman which proves that there are plenty of predictable $\mathbb{Z}$-actions is not even mentioned.

Chapter 3 is based on a joint work of the author and Felipe García Ramos [GJ17]. The proof of Theorem 3.1.1 cleverly combines the proof of the implication $1 \implies 4$ in Proposition 5.1 of Li, Tu and Ye article [LTY15] with ideas of Huang and Ye paper [HY02]. The author seems to neglect the fact that what he calls the *mean Li-Yorke chaos* was at least since 2005 studied under the name *distributional chaos of type 2* or *DC2 chaos* for short. He should be aware of that because he cites papers which explicitly elaborate on this connection. A sentence on page 43 proves that the author knew of that different terminology, but he chosen to skip any discussion on that matter and therefore, in my opinion, he did not acknowledge the earlier work of other authors appropriately.

The last three chapters are in my opinion the best part of the thesis.

Chapter 4 is based on the joint work of the author and Wen Huang [HJ16] which appeared in the *Ergodic Theory and Dynamical Systems*. The authors prove a version of the Theorem 1.1 from [HLY14] for the bi-orderable amenable groups. The overall strategy is similar as in [HLY14], but the technical

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details differ considerably. The chapter also contains examples of applications of the main theorem.

Chapter 5 contains the results presented in [DJLQ17], which is a joint work of the author of the thesis with Tomasz Downarowicz, Wolfgang Lusky, and Yixiao Qiao. The proof of the main theorem is quite intricate and heavily relies on the theory developed by Downarowicz and his collaborators over the past 20 years. The authors cleverly anticipated possible comments from the reviewer and nicely explained their motivation in Remark 5.3.7.

The final chapter contains two main theorems. The first one is a joint work of the author of the thesis, Yonatan Gutman, and Masaki Tsukamoto and comes from [GJT16], which is accepted for publication in the *Journal of Dynamics and Differential Equations*. The result improves the classical Bebutov-Kakutani Theorem. It says that a real flow Φ can be embedded into Bebutov translation flow

$$\mathbb{R} \times L(\mathbb{R}) \ni (s, f(t)) \mapsto f(s+t) \in L(\mathbb{R})$$

where $L(\mathbb{R})$ is the space of all functions $f\colon \mathbb{R} \to [0,1]$ satisfying $|f(x) - f(y)| \leqslant |x-y|$ if and only if the set of fixed points of Φ can be embedded into the interval $[0,1]$. In the Bebutov-Kakutani Theorem the role of the *compact* space $L(\mathbb{R})$ is played by the *non-compact* space $C(\mathbb{R})$ of all continuous functions $f\colon \mathbb{R} \to [0,1]$. The method of proof is original and quite elegant. The author notes in Remark 6.2.7 that much simpler example of a Bebutov translation flow on a compact function space containing $C(\mathbb{R})$ which are universal for all real flows with the set of fixed points embeddable into $[0,1]$ can be obtained in a much easier way. This immediately raises a question: What is the importance of finding an embedding into $L(\mathbb{R})$? One can only regret that this question is not discussed. The same remarks apply to the second main theorem of Chapter 6, which comes from a joint work of the author of the dissertation and Gutman [GJ16]. The result is a construction of a universal compact flow, that is, a compact flow Ψ such that every real flow is isomorphic with a subflow of Ψ. Other proofs of the existence of such a universal object are known, and it is not explained what can be gained from such an explicit construction (apart from aesthetical satisfaction).

My overall opinion about the work of Lei Jin is positive, the dissertation under review fulfils all the requirements for a PhD thesis, and I recommend to accept it.

Sincerely yours

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