Report on the PhD thesis of Mariusz Tobolski, titled

The local triviality of noncommutative principal bundles

This thesis collects the candidate's research in noncommutative topology and quantum groups. These are relatively young fields of pure mathematics sitting in the intersection of functional analysis and abstract algebra, which grew out of the theory of operator algebras in late 20th century. Incorporating viewpoints from various fields of mathematics, they continue to provide new and interesting insights.

A fundamental notion in these fields is that of C*-algebras, which are certain Banach algebras admitting realization as system of bounded operators on Hilbert spaces.

One of the guiding principles in this theory is the correspondence between C^*-algebras and (locally) compact topological spaces. Concretely, the Gelfand–Naimark theorem says that a commutative C^*-algebra with multiplicative unit is always isomorphic to the algebra of continuous functions on a compact (Hausdorff) space. This motivates the viewpoint of interpreting noncommutative C^*-algebras as system of functions on 'noncommutative spaces', bringing in ideas from topology to understand the structure of operator algebras. One particularly relevant development is the theory of regularity and dimensions for C^*-algebras due to Rieffel and Winter among others, which bring in the idea of covering dimension for topological spaces with various flavors of approximation theory for operator systems.

Another fundamental framework for this thesis is that of compact quantum groups. This encapsulates the notion of compact noncommutative spaces with group laws, encoded by bialgebra structures $(C(G), \Delta)$ with certain regularity. Pursuing the analogy with compact spaces, action of such objects on C^*-algebras can be formulated in terms of coaction homomorphisms. This opens a viewpoint of noncommutative spaces with quantum symmetry. One particularly important subclass is that of free actions, which represent G-principal bundles.

This thesis is motivated by problems around noncommutative analogues of the Borsuk–Ulam theorem, recently popularized by Baum, Hajac, and Dąbrowski. The original Borsuk–Ulam theorem states that, when $n < m$, there is no continuous map from S^m to S^n which commutes with the antipodal actions of $\mathbb{Z}/2\mathbb{Z}$ on these spheres. Generalization of this result to free actions of finite groups was quite actively studied in algebraic topology in the 70s and 80s.

In the yoga of noncommutative topology, it is quite natural to ask for analogous statements for free actions of finite groups, or more generally, of compact groups, on C*-algebras. A recent breakthrough was given by Passer, who combined the classical Borsuk–Ulam theorem with ideas from regularity theory to show that such results indeed hold.

Turning back to algebraic topology, the Borsuk–Ulam theorem has reformulation in terms of joins and equivariant maps between them. This carries over in a natural way to compact quantum groups, and this thesis works on this front in connection with locally trivial principal bundles.

Let me briefly go through the content of the thesis, which is based on the following preprints:

[1] Alexandru Chirvasitu, Ludwik Dąbrowski, Mariusz Tobolski, *The weak Hilbert–Smith conjecture from a Borsuk–Ulam-type conjecture*, `arXiv:1612.09567`

[2] Eusebio Gardella, Piotr M. Hajac, Mariusz Tobolski, Jianchao Wu, *The local-triviality dimension of actions of compact quantum groups*, `arXiv:1801.00767`

[3] Alexandru Chirvasitu, Benjamin Passer, Mariusz Tobolski, *Equivariant Dimensions of Graph C^*-algebras*, `arXiv:1907.10010`

Chapter 1 is dedicated to a review of compact group actions on compact spaces, and motivating results for later sections. One particularly important concept is the Schwartz genus g_G for free actions of compact groups an compact spaces: when G is a compact group and X is a (free) compact G-space, $g_G(X)$ is the smallest integer n such that X admits an open covering by $n+1$ open G-invariant sets which are total spaces of trivial principal G-bundles.

The author gives an equivalent formulation of this invariant as the local triviality dimension $\dim_{\mathrm{LT}}^G$ introduced in the paper [2]: $\dim_{\mathrm{LT}}^G X$ is defined as the smallest integer n such that there exist continuous equivariant maps $\rho_i\colon X \to \mathcal{C}G$ for $i = 0, \dots, n$ satisfying $\sum_i \mathbb{t} \circ \rho_i = 1$. Here, $\mathcal{C}G$ is the unreduced cone $[0, 1] \times G/\{0\} \times G$, and $\mathbb{t}\colon \mathcal{C}G \to [0, 1]$ is the projection onto the first component.

This invariant can be captured by the iterated join spaces. Given topological spaces X and Y, the topological join $X * Y$ is the quotient of $[0, 1] \times X \times Y$ by collapsing $\{0\} \times \{x\} \times Y$ for $x \in X$ and $\{1\} \times X \times \{y\}$ for $y \in Y$.

This leads to $E_nG = G * \cdots * G$ with $(n+1)$-copies of G, which are behind the Milnor construction of the classifying space of G. A first important result in this chapter is the following correspondence: $g_G(X) = \dim_{\mathrm{LT}}^G X$ is the smallest integer n such that there is a G-equivariant continuous map $X \to E_nG$.

Chapter 2 is about compact group actions on unital C*-algebras. The above formulation of local triviality dimension has direct analogue in the context of noncommutative C*-algebras with free action of G, as explored in the paper [2]. Namely, if A is a unital G-C*-algebra, $\dim_{\mathrm{LT}}^G A$ is the smallest integer n such that there exist G-equivariant $*$-homomorphisms $\rho_i\colon C(\mathcal{C}G) \to A$ for $i = 0, \dots, n$ such that $\sum_i \rho_i(\mathbb{t}) = 1$. Because of noncommutativity of A, they obtain several variants of $\dim_{\mathrm{LT}}^G$.

This notion of dimension was motivated by the theory of regularity for C*-algebras, which plays an essential role in the classification program for simple nuclear C*-algebras. Recall that the $*$-homomorphisms from $C_0((0,1]) \otimes B$ to A correspond to the completely positive order zero maps from B to A, which are the fundamental ingredients of the nuclear dimension of C*-algebras due to Winter and Zacharias. The equivariant analogue of this notion, the Rokhlin dimension $\dim_{\mathrm{Rok}}^G$, for compact group actions were defined by Gardella following the work of Hirshberg, Winter, and Zacharias for finite groups (and integers). From the above correspondence, for any separable G-algebra A, one has $\dim_{\mathrm{Rok}}^G A = \dim_{\mathrm{LT}}^G(A_\infty \cap A')$. Generalizing Gardella's previous work on compact Lie groups, they give estimate of equivariant Rokhlin dimension for commutative G-algebras $C(X)$ in terms of the dimension of the quotient X/G.

This chapter ends with the discussion of free product variant $C(E_n^{\boxast}G)$ of E_nG is discussed, replacing the tensor product $C(G)^{\otimes n+1}$ which serves as an ambient algebra of $C(E_nG)$ by the (unital) free product of C*-algebras.

Finally, Chapter 3 is about compact quantum group actions on unital C*-algebras. The above C*-algebraic formulation of $\dim_{\mathrm{LT}}^G$ admit generalization to quantum group actions, by interpreting $C(\mathcal{C}G)$ as the unitization $\mathbb{C} + C_0((0,1]) \otimes C(G)$. Similarly the join construction $X * G$ admits an analogue called equivariant noncommutative join $A \circledast^\delta C(G)$ for a G-C*-algebra $(A, \delta\colon A \to A \otimes C(G))$, modeled as a subspace of $A \otimes C(G)$.

Now, let us summarize the main results of thesis.

The main result of first chapter is comparison between Borsuk–Ulam type theorems and the local triviality dimension, or Schwartz genus, for E_nG. Namely, there is no G-equivariant map from $X * G$ to X for any G-space X with $\dim_{\mathrm{LT}}^G X < \infty$ (that is, a Borsuk–Ulam type theorem holds for G), if and only if $g_G(E_nG) = n$ for all n, as proved in the paper [1]. The proof relies on structure theory of compact abelian groups. The case of $G = \mathbb{Z}/2\mathbb{Z}$ is the original Borsuk–Ulam theorem, since $E_n(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to S^n as a $\mathbb{Z}/2\mathbb{Z}$-space.

The main result of second chapter is the following C*-algebraic analogue of the above result. Given a compact group G and a G-algebra (A, δ) such that $\dim_{\mathrm{LT}}^G A$ is finite, there is no G-equivariant $*$-homomorphism $A \to A \circledast^\delta C(G)$. The proof follows the strategy of the previous chapter, but instead relies on a characterization of $\dim_{\mathrm{LT}}^G A$ in terms of equivariant homomorphisms from $C(E_n^{\boxast}G)$ and estimate for $\dim_{\mathrm{LT}}^G(C(E_n^{\boxast}G))$.

The main result of the third chapter is the following quantum group analogue of the above. Let G be a compact quantum group which has a classical subgroup $H < G$ satisfying $\dim_{\mathrm{SLT}}^H(C(G)) < \infty$. Then, for any unital G-C*-algebra (A, δ) with $\dim_{\mathrm{LT}}^G A < \infty$, there is no G-equivariant $*$-homomorphism from A to $A \circledast^\delta C(G)$.

I found the thesis to be very interesting and well written. One sees a vibrant interaction of paradigms from different fields around operator algebras, particularly from recent advances in the classification program for C*-algebras, and from classical topology. This culminated in important contributions towards actively researched problems in the field of noncommutative topology, pushing the boundary of the theory.

The exposition was also clear and well motivated. I hope this will serve as a good entry point into noncommutative topology theory for researchers of next generation.

I only have some minor suggestions for improvement:

- below (2.8): the subalgebra is equal to $\mathcal{C}A$, rather than just isomorphic.
- Proposition 2.2.7: the notation $\dim_{\mathrm{WSLT}}$ is not explained.
- p. 66, above Definition 3.1.3: $C^*_{\ell_p}(\mathbb{F}_d)$ for $2 < p < \infty$ due to Brown–Guentner[1] is such an example, as follows. The standard coproduct descends to a $*$-homomorphism from $C^*_{\ell_{p/2}}(\mathbb{F}_d)$ to $C^*_{\ell_p}(\mathbb{F}_d) \otimes C^*_{\ell_p}(\mathbb{F}_d)$ by their Remark 2.4, and the canonical quotient map from $C^*_{\ell_p}(\mathbb{F}_d)$ to $C^*_{\ell_{p/2}}(\mathbb{F}_d)$ is not injective for such p by Okayasu[2] Corollary 3.2.
- p. 69: both Vaes and Roy–Woronowicz work with reduced models and injective coaction homomorphisms. I see that injectivity is indeed assumed in `arXiv:1801.00767`.

[1] Nathanial P. Brown and Erik P. Guentner, *New C^*-completions of discrete groups and related spaces*, Bull. Lond. Math. Soc. **45** (2013), no. 6, 1181–1193.

[2] Rui Okayasu, *Free group C^*-algebras associated with ℓ_p*, Internat. J. Math. **25** (2014), no. 7, 1450065, 12.

- Example 3.3.3: this presents $U_q(2)$ as a direct product of $U(1)$ and $SU_q(2)$, which is not the case. Indeed in the classical case $q = 1$, by continuous group homomorphisms $U(2)$ only maps onto $PU(2)$, not to $SU(2)$.
- p. 74: associativity does not make sense for the equivariant noncommutative join operation, because you only have $A \circledast^{\delta} C(G)$. I suppose this remark applies instead to free noncommutative joins, as in `arXiv:1801.00767`.

Turning to the candidate's quality, one can see that he has been very productive with a wide variety of research topics. I also remark that his collaboration spans over a broader community of mathematicians and theoretical physicists, culminating in fruitful exchange of ideas from different disciplines. This shows that the candidate will continue to perform as a successful researcher, and play an important role in development of the research fields he is involved in. There is no question that he should be awarded PhD degree from any institution in the world.

In this light, I recommend this thesis to be accepted at the Institute of Mathematics of the Polish Academy of Sciences with distinction.

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