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Report on the thesis "On q-deformed von Neumann algebras"
submitted by Mateusz Wasilewski

by Éric Ricard

Mateusz Wasilewski presents a thesis about two q-deformations in the context of free products of von Neumann algebras. The main goal is to show that these q-deformations do not affect various structural properties.

Description of the main results

In free probability, Voiculescu's gaussian functor $\Gamma_0(H)$ is a central object that admits many variants. Bożejko and Speicher introduced its q-deformation $\Gamma_q(H)$ in 1991 for $-1 < q < 1$. It is a natural question to wonder if the properties that are known for the free group algebras still hold for them. The combinatorics in the q-case is much more involved and it is difficult to make the free arguments work and very often new ideas are necessary. Among interesting properties, it is known that the q-deformed algebras are, in full generality, non injective (Nou 2004) factors (myself 2005), they have the complete metric approximation property (Avsec 2011). With some restrictions on the number of generators and the index q, they are solid (Dabrowski 2014, Avsec 2011). Using free transportation, Guionnet and Shlyakhtenko (2015) showed that actually $\Gamma_q(\mathbb{R}^n)$ is isomorphic to $\Gamma_0(\mathbb{R}^n)$ for small q. Very recently, in relation with Popa's works, the subject of studying MASAs in $\Gamma_q(H)$ attracted some attention. The radial MASAs are singular (Bikram and Mukherjee 2017) and they are not conjugated, this last result is Theorem D of the thesis obtained in collaboration with Caspers and Skalski.

There is a type III deformation of $\Gamma_0(H)$ called the free Araki-Woods algebras $\Gamma_0(H,(U_t))$ introduced by Shlyakhtenko in 1997. They are tractable examples of type III von Neumann algebras and their classification has been a very active subject recently. Among properties, there are also factors (Shlyakhtenko 1997) with the complete metric approximation property (Houdayer and myself 2011) and are solid (Boutonnet, Houdayer and Vaes 2015). Shortly after Shlyakhtenko, Hiai combined the two deformations to get the q-Araki-Woods algebras $\Gamma_q(H,(U_t))$. The situation is more intricate for them and few techniques are available. For instance, it is still unknown if $\Gamma_q(\mathbb{R}^2,(U_t))$ is a factor in general. Brent (2015) showed that free transportation is still available so that $\Gamma_q(\mathbb{R}^n,(U_t)) = \Gamma_0(\mathbb{R}^n,(U_t))$ for small q. This thesis brings new important informations : they have both the Haagerup and the complete metric approximation properties. The first result was published in 2017 by the author and the second is based on a joint paper with Avsec and Brannan (2018). They open a door to tackle extra properties like absence of Cartan subalgebras or solidity.

The last topic of this manuscript is about another kind of q-deformations of the group algebra of the free power $(\mathbb{Z}/2\mathbb{Z})^{*n}$. It is possible to deform the product of the algebra using some parameter q to obtain a new type II_1 algebra, this construction works in general for any right-angled Coxeter group W. Using the regular representation, one can define the Hecke-von Neumann algebra $vN_q(W)$, this was first considered by Dymara in 2006. As before, it is interesting to establish some properties that are analogous to that of free group algebras. For example, the factoriality depends on q and W (Garncarek 2016). Beyond the fact that they are free products, the study of these algebras is really starting. For free group

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algebras, Rădulescu (1991) proved that the so called radial MASA is singular and computed its Pukánsky invariant. With Boca (1992), he adapted this to free product of groups. Following the same approach, the last contribution of the thesis, in collaboration with Caspers and Skalski, is to define the radial MASA in $vN_q(\mathbb{Z}/2\mathbb{Z}^{*n})$ to show its singularity and compute its Pukánsky invariant.

Description and comments on the thesis

The thesis consists in 6 sections and is based on three papers mentioned above.

The Introduction in Polish and English states the main results. I particularly appreciate the nice discussion on q-deformations in mathematics that goes far beyond the content of the thesis.

The Preliminaries deals with operator algebras/spaces from the very beginning up to some very recent techniques that are necessary to get the main results. Mateusz Wasilewski tries to give some insight of the theories and chooses to present some proofs that are relevant. He almost gives full arguments for Popa's intertwining techniques. This part is very pleasant to read for specialists of operator algebras and may also be very interesting for people not so familiar with the subject. This section gives evidence of maturity even if it contains more misprints than the rest of the manuscript (this is not disturbing).

Section 3 is very short and is devoted to introduce approximation properties. It is worth to note that the definition of the Haagerup property in type III von Neumann algebras has been clarified only recently by works of Caspers-Skalski (2015) and Okayasu-Tomatsu (2015).

Section 4 introduces the main object. It starts with the q-Gaussian and q-Araki-Woods algebras. Some efforts have been made so that anyone can follow. Once again proofs of fundamental facts are given or sketched. It also contains new results. Mateusz Wasilewski gives a very general version of the second quantization for q-Araki-Woods algebras, this is a key result toward approximation properties. He also gets the second quantization for the q-Toeplitz algebra. The proof is not that difficult and surprisingly this was not established before for $|q| > .45$ (up to my knowledge) and is more general. This could be further used for the study of q-Toeplitz algebras themselves. Next, he gives a very clever model for the q-Araki-Woods algebras based on a central limit procedure. This construction is elegant and powerful, it may have some implications beyond approximation properties. Next, the Hecke von Neumann algebras of $(\mathbb{Z}/2\mathbb{Z})^{*n}$ are introduced as well as their radial subalgebras.

Section 5 contains the proofs of the main results. For the Haagerup property of q-Araki-Woods algebras, this is done pretty fast and follows an usual scheme as second quantization is available. The proof for the CMAP is more involved and relies on the model from section 4. It is used to show that radial multipliers do not see the type III deformation. I find this result very satisfactory. This was already known for $q = 0$, but here the argument are of a very different nature. Thus, the CMAP follows from that of Avsec for the q-Gaussian and second quantization established above. The rest of the thesis deals with masas. First it is shown that the radial algebra in $vN_q(\mathbb{Z}/2\mathbb{Z})^{*n}$ is a masa. The arguments closely follows that of Pytlik for the free group. Then, its Pukánsky invariant is computed. The arguments are based on those of Rădulescu for the free group; even if the ideas are not really new, this is the most technical of part where lots of computations appear. One has to figure how to make them, this is not obvious and is done with a lot care. Finally, the non conjugacy of generator masas in q-Gaussian algebras is shown. This is done as a quick application of Popa's intertwining techniques, especially in the orthogonal case. The proof of the general case is a bit more involved and more sketchy. The writing of this section is very good, the argument are really well explained.

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Recommendation

The material presented in this thesis covers a very large part of what is known about q-deformed von Neumann algebras. This shows a very strong background on operator algebras/spaces. Some new significant results are obtained. They were already published in two papers in JFA and PAMS. I have no doubt that results on masas will be accepted for publication. Moreover, Mateusz Wasilewski also has a paper in Bull. Lond. Math. Soc. on operator spaces that is not part of this thesis. He already attended more than 15 conferences and has contacts with many mathematicians. I am convinced that he has all the qualities to keep on doing good researches in the futur. Therefore, **I consider that Mateusz Wasilewski deserves to obtain the Ph.D. degree in mathematics, and in my opinion all formal and traditional requirement for PhD theses are fully satisfied.**

Éric Ricard, March 2018

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List of misprints and comments to improve the presentation

- It could be nice to quote the paper of Brent on free transport without a trace somewhere as it proves Theorem A in some cases. For instance in 4.3.1.
- The paper by Boca and Radulescu could also be quoted as it seems that they prove there Theorem C for $q = 0$. It could also be explained why the radial algebra here is different from their (recalling that the underlying algebra is a free product)
- The paper by Dykema and Nica on q-Toeplitz algebra could also be quoted as it gives a proof for 4.3.11 when $|q| < 0.44$
- page 13, line -9 : $A \to B$
- page 15, line 22 : $E(\Omega)e_n$
- page 16, line 17 : is to check
- page 18, line 13 : it is always
- page 18, line 24 : (u_λ^2) is not necessarily increasing
- page 19, line 8 : If M
- page 22, line 1 : $\hat{G}$ instead of Γ
- page 22, line 14 : λ_g, λ_h instead of δ_g, δ_h.
- page 22, line -11, -7: it is better to use e rather than 0
- page 22, line -1 : $\int_G$
- page 23, line -9 : antilinear
- page 24, Theorem 2.3.5 : it seems that something is missing here, where is the uniqueness? Some σ_t have to be τ_t
- page 26, line -7 : expectation
- page 26, line -6 : N not $\mathbb{N}$
- page 27, line 10: $\mathbb{E}'(x)$.
- page 29, line -4: $\varphi(u_n^2 z_n)$
- page 30, line 9 : homomorphism
- page 31, line 4 : $B(L^2(\mathrm{M}))$
- page 31, line -8 : measurable function
- page 35, line 7 : $\mathbf{vN}(\mathbb{F}_n)$
- page 40, line -15 : ultraweak topology
- page 40, line -2 : its absence

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- page 45, (4.2.1) : ξ_1 instead of x_1. The statement that multiples of Ω are the only vectors that is annihilated by annihilation is more difficult than the rest of the proof.
- page 46, line 8 : we can write
- page 51, line 2 : $a_q a_q^* ... a_q^* a_q^*$ has to be corrected
- page 52, line 16 : The
- page 52, (4.2.1) : a $\sum$ is missing. Next line : orthonormal tensors
- page 53 (4.2.13) + 1: the formula $\Phi_n(\xi) = W(\xi)$ may not be what is meant
- page 56, line -4 : brackets could be had for associativity of tensor products
- page 56, line 4 : $\mathcal{F}_q(U_T)$
- page 60, line 14 : one has to say that the action is pointwise continuous
- page 60, line -14 : $x\Omega \in \mathbf{K}$. The case $n = 0$ should be explained a bit more to show how the assumption on the kernel is used.
- page 61, line 15 : a_q instead of a
- page 62 1st paragraph : the estimate on $\|R_{n+k,k}\|$ could be found in [Nou04]
- page 65, line -7 : $\mathbf{vN}_q(W)$ instead of $\mathbf{vN}_q(H)$
- page 67, line -14 : for some $\lambda, \mu > 1$
- page 68, line 19 : to the identity on L^2
- page 70, line -13 : $e_k(1)$ instead of $e_k(0)$
- page 70, line -11 : extra)
- page 74, line 15 : intertwines
- page 74, Proof of Theorem B : it could have been a better idea to use the same letters as in the proof of Theorem A especially for Q_n
- page 76, (5.3.3) : a justification is required to get this estimate, one has to justify that all e_{xazby} are different.
- page 77, line -12 : this is where you use $L \geqslant 3$, this has to be noted
- page 79, line 14 : $m \geqslant$
- page 82, line 15 : k is $m + n$
- page 82, line -3,-2,-1 : some more explanations could be given here
- page 84, line 7 : $(\mathbf{B} \cup \mathbf{B}_r)'$
- page 84, line 12 : $(\mathbf{B} \cup \mathbf{B}_r)'$ not $(\mathbf{B} \cup \mathbf{B}_r)''$
- page 85, line 8: one has to use 5.3.8 here

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- page 85 : I think [PSW16] only deals with small q
- page 86 : Is it possible to deduce the general case from the orthogonal one? An idea could be : If $W(e_1)''$ and $W(v)''$ are conjugated, then $W(a)''$ and $W(b)''$ are conjugated in $\Gamma_q(\ell_2)$ for any a, b with the same angle α as e_1, v by second quantization. Then it seems possible to find some n and vectors $a_1, ..., a_n$ so that the angle between a_i, a_{i+1} is α and a_1, a_n are orthogonal.

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