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Report on the Thesis:
On the Theory of $S-$Riesz sets

submitted by
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To
IMPAN
Institute of Mathematics
Polish Academy of Sciences (Warsaw)

Rami M. Ayoush's PhD-thesis *"On the Theory of s-Riesz sets"* is the result of excellent mathematical work supervised by Prof. Dr. M. Wojciechowski at IMPAN Warsaw.

The thesis is based on four finished papers with Professors M. Wojciechowski and D. Stolyarov as co-authors. Two are published in **Revista Mat. Iberoamericana**, and **Annales Acad. Sci. Fennicae** respectively. The remaining two are available as ArXiv-preprints; each of very high quality.

Chapter 1. This chapter forms the introduction to the work by Rami Ayoush. It provides the background and the motivation for the problems addressed in the thesis.

P.M.

Chapter 2. This chapter presents a summary of tools used in the core chapters of the thesis. It covers extensive background material selected from harmonic analysis and geometric measure theory.

Chapter 3. This chapter of the dissertation deepens and extends results contained in the master's thesis of Rami Ayoush. The chapter presents his significant contributions to the field of compensated compactness. More precisely to the study of $\mathbb{R}^n$ valued measures arising in the range of homogeneous linear differential operators.

The main results formalize the notion of anti-symmetry (Definition 3.3.) of the Fourier-transform of a vector valued measure μ and derive from it a lower estimate of the Hausdorff dimension of μ (Theorem 3.6. and Theorem 3.7) as well as conditions for the absence of $1-$purely rectifiable subsets charged by μ. (Theorem 3.8.) The impressive list of applications to measures with differential constraints, obtained by suitable specialisation, is displayed in Sections 3.4. It covers results of G. Alberti, Rademacher, Arroyo-Rabasa, de Philippis, J. Hirsch, F. Rindler.

Chapter 4. (Riesz products and transference to martingale spaces.) The center role of this chapter is played by the Riesz products

$$\mu_{a,q}(A) = \int_A \prod_{k=0}^{\infty} (1 + a\cos(2\pi q^k x))dx,$$

where $a \in [-1, 1]$. In Theorem 4.3., by reduction to Theorem 4.17, Rami Ayoush obtains the following lower bounds on the Hausdorff dimension,

$$dim_{\mathcal{H}}(\mu_{a,q}) > 1 - \frac{1}{q\log q}\sum_{j=1}^{q-2}(1 - \frac{\cos((2j+1)\pi/q)}{\cos(\pi/q)})\log(1 - \frac{\cos((2j+1)\pi/q)}{\cos(\pi/q)}).$$

Exploiting this estimate through Lemma 4.4., Proposition 4.5. and Proposition 4.6., he is able to present the limiting behaviour of $dim_{\mathcal{H}}(\mu_{a,q})$, when $|a|$ is close to 1, and $q >> 1$. Thus he succeeds in improving upon famous estimates of J. Peyriere. See Eq.: (4.1). This remarkable feat results from a probabilistic approach, based on tree approximation and transference to a (backward) martingale sequence in Section 4.2. Finally, Section 4.8. sharpens the lower dimension estimates of Theorem 4.2., obtained initially by reduction to Theorem 4.17.

P. M.

Chapter 5. Classical microlocal analysis techniques, in particular the wavefront sets $WF_x(\mu)$, were used by R. Brummelhuis to determine that a given measure is absolutely continuous with respect to Lebesque measure. In this chapter of the thesis Rami Ayoush further deepens the microlocal approach by obtaining lower bounds on the Hausdorff dimension of the measure μ in terms of information on the wavefront set $WF_x(\mu)$. (Theorem 5.2.) We note an impressive degree of novelty in the chapter *microlocal approach to the Hausdorff dimension of measures* where pseudo differential operators and spherical harmonics are employed to establish connections between the *wavefront set* of a measure and its geometric properties such as $k-$rectifiability. (Theorem 5.6. Corollary 5.8)

Assessment and Recommendation

The Ph.D. thesis submitted by Rami M. Ayoush documents his excellent achievements pertaining to the fields of harmonic analysis, geometric measure theory, martingales and compensated compactness. By means of hard, impressive results, and their convincing presentation Rami M. Ayoush proved to be a first rate, young mathematician, who solved important problems conjectured by leading experts in the area.

In summary, the excellent Ph.D thesis, of Rami M. Ayoush meets the requirements for **Ph.D. with distinction** (**summa cum laude**).

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