Report on PhD dissertation of Rami Ayoush

Maria Roginskaya

1 The content survey

The dissertation under consideration is based on 3 papers of the author, all joint with other authors, one of which is at the moment of writing published, and another are submitted for publication.

All the results address so called Uncertainty Principle of Fourier Analysis, which says that a function/measure can not be restricted in both its value and its Fourier transform. In the dissertation, by "restriction of measure" one understand how concentrated it can be on sets of small Hausdorff measure, i.e. the lower estimate on its Hausdorff dimension.

The first paper connect the low Hausdorff dimension of vector valued measures with restriction on directions of its vector valued Fourier transform. The model for such a restriction are action (in the sense of distribution) of differential operators on the same function. The most significant result of the paper is finding a condition under which the lower dimension is 2 (instead of previously know estimate by 1).

The second paper address low Hausdorff dimension measures with arithmetical restriction on well known class of measures called the Riesz products. This is a class of measures with a well known singular behaviour, which was extensively studied. The estimates of their low Hausdorff dimension was started, as far to my knowledge, in 1975 by Jacques Peyriere, and then developed on by Brown, Moran, and Pearce in 80'th and A. Fan in 90'th. While there is a sharp estimate on the low Hausdorff dimension of such measures, it is expressed in terms of the measure itself and is impossible to compute from the coefficients explicitly. A non-sharp explicit estimate was obtained by the K. Hare and the referee in 2003. The defendant with collaborators suggest another explicit estimate for regular Riesz products, which on one hand can be better than the one from 2003 in some cases, and on the other hand uses completely different approach, more similar in spirit to the one by Peyriere.

The third paper gives a condition for an estimate of Hausdorff dimension on a complex sphere as manifold in terms of spherical harmonics. Namely if spectrum of a measure on n-dimensional complex sphere has no more than finite intersection with a cone around the main diagonal, then the Hausdorff dimension of the measure is at least 2n-2. This result generalises a long known result of Alexandrov about pluriharmonic measures.

2 The value of the results

The vector-valued measures satisfying a "directional" condition on their Fourier transform are not extensively studied. The results of the section 3 (first paper) take us a step further from the known results with estimate of lower Hausdorff dimension by 1 and $n-1$. While the application of this result is less obvious then the previous ones, it gives us a hint about what conditions are important in order to have the estimate for all intermediate steps. The condition found for the case of 2 is by no mean obvious generalization of the ones for 1 and $n-1$. Furthermore, one obtains additional information about rectifiability of the measure under a "directional" condition, which is a new direction of query.

The question of Hausdorff dimension of Riesz products is already well studied. Here the value of the results in section 4 lie in the new approach, which gives an explicit estimate on the dimension in terms of coefficients of the Riesz product.

The results of section 5 are probably the most interesting, as they make an attempt to extend the Uncertainty principle from Euclidian space to a manifold (in this case a sphere). The analog of Fourier coefficients in this case are spherical harmonics. This is a generalization which, to my knowledge was not successfully investigated before. Finding a right form of the conditions and a working analytical tool opens a wide range of interesting questions.

3 Possible development

A natural question for the section 3 is if and under which condition one can obtain estimates of lower Hausdorff dimensions by all numbers between 0 and n. Another question which is natural is to see how sharp the estimates are.

The results of section 4 use only very regular Riesz transforms. A natural question is: Whither the technique can be adapted to less regular Riesz products?

The results of section 5 open a wide range of questions: Is it possible to extend this form of Uniqueness Principle to Manifolds, and in this case what will be a generalization of the Fourier transform. Real spherical harmonics are eigenfunctions of Laplacian, and so are natural generalization of Fourier frequencies. But the condition of the results in section 5 demands additional subdivision to $H(p, q)$, which is connected to the complex structure. Is it possible to find a similar classification of spherical harmonics, which would be applicable also to real case? Can the condition be generalized to cones which do not include the main diagonal $q = p$? Which shapes other than cone have this $(2n - 2)$-Riesz property? Is there similar estimates on other manifolds? All those questions are very natural, and lack an answer.

4 Assessment

The first paper is a development on the result of the referee and the supervisor of the defendant. While for the most the paper use the same ideas as in the previous work, it involves both deep understanding of the original ideas and fluent use of advanced analytical techniques. I find it encouraging that the defendant can demonstrate such a good command of important for researcher tools.

The proofs of the second paper make use of martingales. This is a very uncommon tool in this settings, which to my knowledge was previously used only by Svante Janson.

The last section involves a less known techniques from different branches of analysis as well as some clever estimates. The result presents a turn to a unexpected direction of investigation. I think it is very impressive.

My conclusion is that the thesis presented by Mr. Rami Ayoush clearly fulfills expectation for a PhD thesis and furthermore is outstanding. If I compare it to the PhD's which were granted by my department in the recent 10 years, then there were a couple which were better, but in my opinion Mr. Rami Ayoush's thesis will fit in the top 20%.