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On the theory of s-Riesz sets

PhD dissertation

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Author's declaration:

I hereby declare that this dissertation is my own work.

December 9, 2020

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Supervisors' declaration:

The dissertation is ready to be reviewed

December 9, 2020

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To my family.

Abstract

In this dissertation we investigate connections of Harmonic Analysis and Geometric Measure Theory. The thesis contains results which concern systematic development of the theory of s-Riesz sets, i.e. a notion introduced in [69] in the context of the study of the regularity of vector-valued measures.

Chapter 1 contains an introduction to the topic of the dissertation and describes research methodology. In Chapter 2 we discuss selected classical results concerning connections of Harmonic Analysis and Geometric Measure Theory.

The original results presented in the dissertation come from the following three preprints:

- R. Ayoush, M. Wojciechowski, *On dimension and regularity of vector-valued measures under Fourier analytic constraints*, preprint, submitted.
- R. Ayoush, D. Stolyarov, M. Wojciechowski, *Hausdorff dimension of measures with arithmetically restricted spectrum*, accepted in Annales Academiæ Scientiarum Fennicæ Mathematica
- R. Ayoush, M. Wojciechowski, *Microlocal approach to the Hausdorff dimension of measures*, preprint, submitted.

These preprints are the essential part of Chapters 3-5 (respectively).

In Chapter 3 we focus on the problem proposed in [69], concerning the study of the regularity of vector measures subordinated to a bundle $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{G}(k, \mathbb{C}^n)$, i.e. measures whose Fourier-Stieltjes transform satisfy $\widehat{\mu}(\xi) \in \phi(\xi)$ for $\xi \neq 0$. The theorems presented there extend the main result of the paper [69] (see Theorem 3 therein) and are also related to results from [5] (Theorem 1.3. and Corollary 1.4. therein).

Chapter 4 contains a new method of estimating the lower Hausdorff dimension of measures based on arithmetical properties of elements of their spectra. It applies to the classical problem of estimating Hausdorff dimension of Riesz products, i.e. measures of the form

$$\mu_{a,q} = \prod_{k=0}^{\infty} \big(1 + a\cos(2\pi q^k x)\big), \tag{1}$$

where $q \geqslant 3$ is an integer and $a \in [-1, 1]$. Our results, for sufficiently big q's and $|a|$ sufficiently close to 1, improve bounds already known from [37], [65], [26], [9].

In Chapter 5 we present connections of Hausdorff dimension with Microlocal Analysis. We prove a criterion which gives an estimate of Hausdorff dimension based on the knowledge about the wave front set of a measure. This criterion is applied to Radon measures on the complex sphere and gives results which generalize classical theorems concerning regularity of pluriharmonic measures, due to Aleksandrov and Forelli, proved in [3] and [31].

Keywords: Fourier transform, Hausdorff dimension, rectifiability, Harmonic Analysis of measures, Riesz products, wave front set, pluriharmonic measures

AMS 2020 Subject Classification: 28A33, 28B05, 28A78, 31C10, 33C55, 42B10, 43A90

Streszczenie

Niniejsza praca podejmuje tematykę związków analizy harmonicznej z geometryczną teorią miary. Stanowi próbę systematycznego rozwinięcia teorii tzw. zbiorów s-Riesza, pojęcia wprowadzonego w pracy [69] w kontekście badania regularności miar wektorowych.

Rozdział 1 stanowi wprowadzenie do tematyki doktoratu i zawiera opis stosowanej metodologii. W Rozdziale 2 wprowadzono podstawowe definicje, a także omówiono wybrane klasyczne wyniki dotyczące związków analizy harmonicznej z geometryczną teorią miary.

Rezultaty zaprezentowane w pracy doktorskiej pochodzą z trzech prac:

- R. Ayoush, M. Wojciechowski, *On dimension and regularity of vector-valued measures under Fourier analytic constraints*, preprint, wysłano.
- R. Ayoush, D. Stolyarov, M. Wojciechowski, *Hausdorff dimension of measures with arithmetically restricted spectrum*, zaakceptowano w Annales Academiæ Scientiarum Fennicæ Mathematica
- R. Ayoush, M. Wojciechowski, *Microlocal approach to the Hausdorff dimension of measures*, preprint, wysłano.

Stanowią one zasadniczą część Rozdziałów 3-5 (odpowiednio).

W rozdziale 3 koncentrujemy się na problemie postawionym w [69], dotyczącym badania regularności miar wektorowych stowarzyszonych z wiązką $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{G}(k, \mathbb{C}^n)$, tj. miar spełniających równość $\widehat{\mu}(\xi) \in \phi(\xi)$ dla $\xi \neq 0$. Zamieszczone tutaj rezultaty stanowią rozszerzenie głównego wyniku pracy [69] (Twierdzenie 3 tamże), a także nawiązują do rezultatów z artykułu [5] (Twierdzenie 1.3. i Wniosek 1.4. tamże).

Rozdział 4 zawiera nową metodę szacowania wymiaru Hausdorffa miar na podstawie arytmetycznych własności elementów spektrum. Znajduje ona swoje zastosowanie w klasycznym problemie szacowania wymiaru Hausdorffa produktów Riesza, tj. miar postaci

$$\mu_{a,q} = \prod_{k=0}^{\infty} \big(1 + a\cos(2\pi q^k x)\big), \tag{2}$$

gdzie $q \geqslant 3$ jest liczbą całkowitą i $a \in [-1, 1]$. Uzyskane wyniki, dla dostatecznie dużych q i $|a|$ bliskich 1, poprawiają oszacowania wymiaru produktów Riesza znane z prac [37], [65], [26], [9].

W rozdziale 5 zaprezentowano związki wymiaru Hausdorffa z analizą mikrolokalną. Udowodniono kryterium dające oszacowanie wymiaru Hausdorffa na podstawie wiedzy o

zbiorze frontu falowego miary. Dzięki jego zastosowaniu dla miar Radona na sferze zespolonej uzyskujemy wyniki, które uogólniają klasyczne twierdzenia o regularności miar pluriharmonicznych, pochodzące of Aleksandrova i Forelliego, udowodnione w [3] i [31].

Podziękowania

Przede wszystkim chciałbym podziękować mojemu promotorowi, Michałowi Wojciechowskiemu, za wieloletnie próby wyjaśnienia mi czym jest, a czym nie jest prawdziwa matematyka, a także za mnóstwo cierpliwości, zaangażowanie w moją edukację matematyczną oraz za czas poświęcony mi przez te wszystkie lata.

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List of Symbols

$\mathcal{M}(\mathbb{R}^n)$	the space of finite, Borel-regular measures on $\mathbb{R}^n$
$\mathcal{M}^+(\mathbb{R}^n)$	the cone of nonnegative, finite, Borel-regular measures on $\mathbb{R}^n$
$\mathcal{M}(\mathbb{R}^n, E)$	the space of finite, Borel-regular vector measures on $\mathbb{R}^n$, taking values in a vector space E
$\mathcal{F}$, $\widehat{\ }$	the Fourier transform
$\mathcal{F}^{-1}$, $\widecheck{\ }$	the inverse Fourier transform
$C_c(\mathbb{R}^n)$	the space of compactly supported continuous functions
$\mathcal{D}(\mathbb{R}^n)$	the space of smooth, compactly supported functions
$\mathcal{D}'(\mathbb{R}^n)$	the space of Schwartz distributions
$\mathcal{S}(\mathbb{R}^n)$	the space of rapidly decaying (Schwartz) functions
$\mathcal{S}'(\mathbb{R}^n)$	the space of tempered distributions
$\mathcal{E}'(\mathbb{R}^n)$	the space of distributions with compact support
$\mathrm{WF}(\nu)$	the wave front set of a distribution ν
$\mathbb{T}$	the circle group
$\mathbb{D}$	the unit disc in $\mathbb{R}^2$
$B(x, r)$	ball of radius r centered at x
$\mathbb{G}(k, E)$	the Grassmannian of k-dimensional subspaces of a vector space E
Π_V, Π_v	orthogonal projection onto a linear subspace V or $\mathrm{span}\{v\}$, respectively
$\mathrm{supp}(\nu)$	the support of a distribution ν
$\mathrm{spec}(\nu)$	the support of $\widehat{\nu}$
$\mathrm{cl}(A)$	the closure of a set $A \subset \mathbb{R}^n$ with respect to the Euclidean topology
$\mathcal{H}^\alpha$	the α-dimensional Hausdorff measure

Contents

Chapter 1

Introduction

In this dissertation we investigate applications of Harmonic Analysis of measures to the study of their Hausdorff dimension. Its main results are theorems which give descriptions of singular sets of measures (i.e. sets which are charged by their singular parts), mainly the lower bounds of their dimension, based on various properties of the Fourier-Stieltjes transform. They can be classified as the so-called uncertainty principles (cf. [52]), i.e. theorems establishing impossibility of simultaneous sharp localization of a distribution and its Fourier transform.

Our results refer directly to the structural descriptions of the spaces of analytic functions. This (perhaps non-obvious) origin is explained by the following question which is the leitmotiv of our considerations: *Consider a characterization of some Hardy space. How regular objects would we get if we weakened some symmetry assumptions from this characterization?* The spectrum of answers for this question depends on the type of the Hardy space that we deal with. In the end, its solution requires finding examples or descriptions of the so-called *s-Riesz sets.* This notion was introduced in the paper [69], as an extension of the idea from the brothers' Riesz theorem to the case of singular measures.

Theorem 1.1 (F. and M. Riesz theorem, [68], cf. also [61], [16], [63], [20], [50])**.** *Let $\mu \in \mathcal{M}(\mathbb{T})$ be a be a finite complex Borel measure such that $\widehat{\mu}(n) = 0$ for $n < 0$. Then μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$.*

Measures described by the theorem above are boundary values of functions from $H^1(\mathbb{D})$ and for this reason are called analytic measures. It is natural to drop the analyticity assumption and ask for examples of sets other than negative integers which compel absolute continuity of measures. To the best of the author's knowledge, this was done first by Y. Meyer, who in the seminal paper [60] introduced the notion of Riesz sets and studied properties of this class. For the group $\mathbb{R}^n$, his definition translates as follows:

Definition 1.2. *We say that a closed set $A \subset \mathbb{R}^n$ is a Riesz set, if any finite Borel-regular measure $\mu \in \mathcal{M}(\mathbb{R}^n)$, such that $\mathrm{supp}(\widehat{\mu}) \subset A$, is absolutely continuous with respect to the n-dimensional Lebesgue measure.*

The original definition was proposed for measures on locally compact abelian groups. In the literature, authors in some contexts study stronger versions which, instead of being absolutely continuous, requires from measures belonging to a suitable H^1.

In [69], the authors extended this idea in a quantitative way, by taking into considerations the lower Hausdorff dimension of measures (cf. [27]):

Definition 1.3. *For any (scalar or vector) Radon measure μ on $\mathbb{R}^n$, we define its lower Hausdorff dimension by the following equality:*

$$\dim_{\mathcal{H}}(\mu) = \inf\{\alpha \colon \text{ there exists a Borel set } E \text{ such that } \dim_{\mathcal{H}} E \leqslant \alpha \text{ and } \mu(E) \neq 0\}.$$

Definition 1.4. *([69]) We say that a closed set $A \subset \mathbb{R}^n$ is an s-Riesz set if $\dim_{\mathcal{H}}(\mu) \geqslant s$ for any finite, Borel-regular μ such that $\operatorname{supp}(\widehat{\mu}) \subset A$.*

The aim of the research presented in this dissertation was to develop the theory of s-Riesz sets in a systematic way. We often draw inspiration from the existing theory concerning the theorem of F. and M. Riesz. This theorem has numerous generalizations, in the Euclidean case as well as going in completely different directions. Let us recall those which are the most important for us.

Multidimensional analog of F. and M. Riesz theorem. In the paper [70] a very general class of Riesz sets was constructed:

Definition 1.5 ([70])**.** *Let us fix some $\epsilon > 0$. We say that a set $F \subset \mathbb{R}^n$ is ϵ-asymmetric if*

$$\forall_{x \in F} \quad F \cap B(-x, \epsilon|x|) = \emptyset. \tag{1.1}$$

Theorem 1.6 ([70], Theorem 0.3.)**.** *If a set $F \subset \mathbb{R}^n$ is ϵ-asymmetric for some $\epsilon > 0$, then it is a Riesz set.*

Analogous theorem is also true for measures on $\mathbb{T}^n$.

Multiplier characterization of the space $H^1(\mathbb{R}^n)$. Theorem of Uchiyama (see [78]), apart from giving a constructive Fefferman-Stein decomposition of $BMO(\mathbb{R}^n)$, provides the answer to the question about the form of invariant operators which describe the space $H^1(\mathbb{R}^n)$.

Theorem 1.7. *Suppose that $\theta_1(\xi), \ldots, \theta_n(\xi) \in C^\infty(S^{n-1})$ and let us denote $K_{\theta_i} f = \mathcal{F}^{-1}(\theta_i(\frac{\xi}{|\xi|})\mathcal{F}(f))$. Then, the inequality*

$$\frac{1}{C}\|f\|_{H^1} \leqslant \sum_{i=1}^{n} \|K_{\theta_i} f\|_{L^1} \leqslant C\|f\|_{H^1}$$

holds for some constant C if and only if

$$\operatorname{rank}\begin{bmatrix} \theta_1(\xi) & \theta_2(\xi) & \ldots & \theta_n(\xi) \\ \theta_1(-\xi) & \theta_2(-\xi) & \ldots & \theta_n(-\xi) \end{bmatrix} \equiv 2 \tag{1.2}$$

for $\xi \in S^{n-1}$.

Equality (1.2), which (similarly as (1.1)) generalizes the assumption from the brothers' Riesz theorem, will be called further *strong antisymmetry.* Conditions (1.1) and (1.2) have also counterparts in other contexts, in particular in the ones mentioned below.

Characterization of a martingale H^1. In the paper [45], S. Janson proved a theorem which gives a necessary and sufficient condition for a certain family of martingale transforms to describe the martingale Hardy space with respect to the q–regular filtration (each atom splits into q subatoms with equal masses, which naturally induces the tree structure and relations of being a 'child' or a 'parent' among atoms). Those specific martingale transforms (see the third section in [45] for the details and strict definitions) are defined by matrices $q \times q$ acting on the space

$$\mathbb{C}_0^q = \Big\{ v \in \mathbb{C}^q \colon \sum_{i=0}^{q-1} v_i = 0 \Big\}.$$

Namely, if A is such a matrix and $\{f_n\}$ is a martingale, then (because of the presence of the tree structure) the local martingale differences (which we call the restrictions of martingale differences to parents of atoms) are in natural correspondence with vectors from $\mathbb{C}_0^q$, and the martingale transform $T_A\{f_n\}$ is given by the action of A on those vectors. To define $T_A f$, we identify f with the martingale generated by f via conditional expectations.

Theorem 1.8 (Janson, Theorem 4 in [45])**.** $H^1 = \{ f \in L^1 \colon T_{A_i} f \in L^1 \text{ for } \quad i = 1, \ldots, m\}$ *if and only if $A_1, \ldots, A_m$ do not have a common real eigenvector, i.e. an eigenvector from*

$$\mathbb{R}_0^q = \Big\{ v \in \mathbb{R}^q \colon \sum_{i=0}^{q-1} v_i = 0 \Big\}.$$

Moreover, after imposing additional invariant structure on the local martingale differences, this theorem is in full analogy with the theorem of Uchiyama and may be regarded as its full-fledged model. In particular, in this context, the strong antisymmetry translates to the mentioned constraint on the common eigenvectors (see the discussion in [45], p.149).

A microlocal theorem of F. and M. Riesz. In the paper [11], Brummelhuis proved, that if the wave front set of a measure does not contain a line (i.e. is antisymmetric), then this measure belongs to the local Hardy-Goldberg space (Theorem 1.4. in [11]). This trick enabled him to use the functional calculus of pseudodifferential operators to prove an analogue of brothers' Riesz theorem for the measures on the complex sphere, and so to construct Riesz sets with respect to $U(n)$-invariant subspaces of complex spherical harmonics (Theorem 2.1. in [11]). He also obtained results concerning regularity of boundary values of certain differential equations (Theorem 3.1. therein).

Next chapters of this work draw inspiration from the mentioned examples to varying degrees. Before we explain those parallels in a more precise way, let us mention, that apart from Fourier-analytic techniques we help ourselves also with the methods of Geometric Measure Theory. Their recapitulation is the essential part of the second chapter.

The considerations of the third chapter address the problem proposed in [69]. More specifically, the authors of the mentioned paper asked for dimensional estimates for vector-valued measures whose Fourier transform takes values in a bundle $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{G}(k, E)$, which is 0-homogeneous (see Definition 3.1 and Definition 3.2 for the details), assuming some structural properties of this bundle. This question is inspired by the classical theory of BV functions (see [4], Chapter 3), i.e. it asks whether dimensional bounds known for gradients of functions form BV are true in a more general Fourier setting. Classical proofs of dimension estimates and rectifiability of BV-gradients are based on de Giorgi characterization of sets with finite perimeter and the coarea formula, hence they cannot be immediately and very far generalized. We describe there a Fourier-analytic approach to some of those fine properties which works even in the absence of the fundamentals from the classical theory of functions with bounded variation. Structural assumptions that we impose on bundles heuristically correspond to the weakened assumption from Uchiyama's theorem. The most important for us are the following two such conditions:

$$\bigcap_{v \in \mathbb{G}(2,E)} \operatorname{span}\{\phi(V \setminus \{0\})\} = \{0\} \tag{1.3}$$

and

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \phi(\xi) = \{0\}. \tag{1.4}$$

The first one leads to an improvement of a dimension bound for generalized gradient measures from [69] and the second is connected with a new method of proving rectifiability of singular sets, based on the Besicovitch-Federer projection theorem. A classical example which satisfies (1.3) is ∇f, where $f \in BV(\mathbb{R}^3)$ (see Example 3.45), while condition (1.4) is satisfied by divergence-free measures (see Example 3.49).

In the fourth chapter we give a new construction of s–Riesz sets for positive measures, based on arithmetical properties of elements of their spectra. This theorem is proved by an adaptation of ideas from a result for martingale transforms, proved in [7], which is formulated in the setting of Janson's theorem. In comparison with Janson's theorem, the mentioned result solves a similar problem allowing presence of real vectors of some special type in the space of admissible martingale differences. Briefly speaking (see Section 1.3. in [7] for the details), it says that if μ is a $\mathbb{C}^\ell$-valued measure defined on the boundary of an infinite q-regular tree and generates a martingale for which the set

$$\operatorname{cl}\{v \in \mathbb{R}_0^{ql} : v \text{ is a local martingale difference}\} \tag{1.5}$$

does not contain matrices of the form

$$(q-1, -1, \ldots, -1) \otimes a, (-1, q-1, \ldots, -1) \otimes a, \ldots, (-1, -1, \ldots, q-1) \otimes a, \quad \forall a \in \mathbb{R}^\ell,$$

then $\dim_{\mathcal{H}}(\mu) \geqslant c > 0$. The asumption of this theorem is in a direct correspondence with 1.4, in particular, both of them provide 'separation' from Dirac delta measures. We apply the method from this theorem, in the case of positive scalar measures, to improve numerical lower bounds of the Hausdorff dimension of certain class of Riesz products. From the quantitative point of view, our goal is to estimate the constant c, which depends on the set (1.5), in the case of a specific backwards martingale whose martingale differences depend on the Fourier transform. Moreover, as a corollary we show that, for a fixed q, among integers which belong to the spectrum of a sufficiently singular measure, we can find a number whose one of divisors has any desired residue modulo q. We also obtain a quantitative form of this principle.

The fifth chapter is devoted to the techniques from Microlocal Analysis and contains a short recollection of basic facts from this theory. We extend there the method of Brummelhuis, whose point lies in microlocalizing spectral properties of measures, for dealing with Hausdorff dimension estimates. The main result of this development is a theorem which provides a bound for lower Hausdorff dimension in terms of the size of the wave front set (the assumption of size replaces the antisymmetry from Brummelhuis' theorem). We apply it to obtain a far-reaching generalization of the Aleksandrov-Forelli theorem about regularity of pluriharmonic measures.

1.0.1 Summary of the results and formal remarks

Below the main original results obtained in this dissertation are listed.

$\underline{Chapter\ 3:}$ Theorem 3.7, Theorem 3.8

$\underline{Chapter\ 4:}$ Theorem 4.2, Theorem 4.3,Proposition 4.5, Proposition 4.6, Theorem 4.25, Corollary 4.27

$\underline{Chapter\ 5:}$ Theorem 5.2, Theorem 5.6

Theorems from Chapter 3 concerning the rectifiable dimension are strengthened versions of theorems proved in the author's master's thesis and are obtained with similar proofs. Theorem 3.41 was also noticed in the author's master's thesis. Chapter 5 contains an appendix which is not present in the original article.

Chapter 2

Harmonic Analysis and Dimension Theory

The purpose of this chapter is to fix the method of measuring the level of singularity of measures. We give the definition of the Hausdorff dimension of sets and measures and compare it with other notions of the dimension such as the Fourier dimension and the energy dimension. We also present some classical applications of Fourier analysis to the dimension theory which motivate some of the methods used in further parts of this dissertation. In our presentation we follow the classical textbooks by Falconer and Mattila ([22], [57], [58]).

2.1 Various dimensions of measures and sets

The Hausdorff measure is a central notion in Geometric Measure Theory. It is a generalization of the surface measure on manifolds which allows to develop analytic theory even on rough sets such as fractals. We construct it in the following way:

Definition 2.1. *Let $A \subset \mathbb{R}^n, 0 \leqslant s \leqslant n$ and $0 \leqslant \delta \leqslant +\infty$. Let*

$$\mathcal{H}^s_\delta(A) := \inf \Big\{ \sum_{j=1}^{\infty} (\operatorname{diam} E_j)^s \colon A \subset \bigcup_{j=0}^{\infty} E_j,\ \operatorname{diam} E_j \leqslant \delta \Big\}. \tag{2.1}$$

The limit

$$\lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A) \tag{2.2}$$

is called the s-dimensional Hausdorff measure of A.

The above limit exists and is a well defined function for all subsets of $\mathbb{R}^n$. It is an outer measure. What is more, it is a metric measure, which implies that its restriction to the family of Borel sets is a countably additive measure. The family $\mathcal{H}^s$ allows to define the dimension of sets in the following way:

Definition 2.2. *For any subset $A \subset \mathbb{R}^n$ we define its Hausdorff dimension by*

$$\dim_{\mathcal{H}} A := \sup\{s : \mathcal{H}^s(A) > 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\} = \inf\{s : \mathcal{H}^s(A) < \infty\} = \inf\{s : \mathcal{H}^s(A) = 0\}. \quad (2.3)$$

In parallel to the above, it is possible to introduce the lower and upper Hausdorff dimension of a measure (cf. [27] and Chapter 10 in [22]).

Definition 2.3. *By the lower Hausdorff dimension of a non-zero (scalar or vector) measure μ we understand*

$$\underline{\dim_{\mathcal{H}}}(\mu) = \inf\{\alpha : \exists F \text{ - Borel set, } \mu(F) \neq 0,\ \dim_{\mathcal{H}} F \leqslant \alpha\}. \quad (2.4)$$

Upper Hausdorff dimension of a measure is defined by

$$\overline{\dim_{\mathcal{H}}}(\mu) = \inf\{\alpha : \exists F \text{ - Borel set, } |\mu|(\mathbb{R}^n \setminus F) = 0,\ \dim_{\mathcal{H}} F = \alpha\}. \quad (2.5)$$

Remark 2.4. *Because we are mainly interested in the lower Hausdorff dimension, we henceforth will be writing $\dim_{\mathcal{H}}(\mu)$ instead of $\underline{\dim_{\mathcal{H}}}(\mu)$*

It turns out that the Hausdorff dimension can be determined from the knowledge about the local growth of a measure. This possibility is expessed by the following facts

Definition 2.5. *The local lower and upper Hausdorff dimensions of a measure $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ at point $x \in \mathbb{R}^n$ are given (respectively) by*

$$\underline{D}\mu(x) := \liminf_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r} \quad (2.6)$$

and

$$\overline{D}\mu(x) := \limsup_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r}. \quad (2.7)$$

Theorem 2.6 (Proposition 10.2. and Proposition 10.3. in [22])**.** *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ then*

$$\dim_{\mathcal{H}}(\mu) = \sup\{s : \underline{D}\mu(x) \geqslant s \quad \text{for} \quad \mu\text{- almost every} \quad x\} \quad (2.8)$$

and

$$\overline{\dim_{\mathcal{H}}}(\mu) = \inf\{s : \underline{D}\mu(x) \leqslant s \quad \text{for} \quad \mu\text{- almost every} \quad x\}. \quad (2.9)$$

The idea behind this theorem has its counterpart in the potential theory, which in turn gives a direct link to the Fourier analysis.

Definition 2.7. *For a measure $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ and $0 < t < n$ we define its t-energy by the following formula*

$$I_t(\mu) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{-t} d\mu(x) d\mu(y). \quad (2.10)$$

The energy (Sobolev) dimension of μ is given by

$$\dim_e(\mu) := \sup\{t \in [0,n] : I_t(\mu) < \infty\}. \quad (2.11)$$

Theorem 2.8 (Frostman-type lemma, cf. Section 4.3 in [23]). *For any $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ we have*

$$\dim_e(\mu) \leqslant \dim_{\mathcal{H}}(\mu). \tag{2.12}$$

The above follows, for example, from the proof of Theorem 4.13. in [23],

Theorem 2.9 (Energy formula, cf. Theorem 3.10. in [58]). *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ and $0 < t < n$. Then*

$$I_t(\mu) = c(n,t) \int_{\mathbb{R}^n} |\widehat{\mu}(\xi)|^2 |\xi|^{t-n} d\xi. \tag{2.13}$$

The method of estimating energy integrals may be also used to prove the Mastrand's projection theorem (see the next section); for examples of other applications see Chapter 6 and Chapter 7 in [58].

The inequality in Theorem 2.8 may be in general strict. The existence of a suitable example is provided by the probabilistic model built on the so-called Riesz product measure (see Proposition 3.4. in [37], and [26]). We will investigate such measures in Chapter 4 .

Let us also mention that the potential-theoretic approach may be adapted for complex measures ([39]). Moreover, the energy formula can be generalized also on measures defined on manifolds ([38]). In particular, there is a version for spherical harmonics ([40]).

A very important role in Fourier analysis is played by even more restrictive type of the dimension, the so-called Fourier dimension. In contrast to the previously mentioned dimensions, this notion takes into account not only concentration properties, but also measures how sets (or supports of measures) are structurized in certain arithmetical sense. In particular, it shows how quickly those sets generate $\mathbb{R}^n$ as a group.

Definition 2.10. *For a closed set $A \subset \mathbb{R}^n$ we define its Fourier dimension by*

$$\dim_F(A) := \sup\{s \in [0,n] \colon \widehat{\mu}(\xi) \lesssim |\xi|^{-s/2},\ \mu \in \mathcal{M}^+(A)\}. \tag{2.14}$$

Definition 2.11. *For $\mu \in \mathcal{M}(\mathbb{R}^n)$ the number*

$$\dim_F(\mu) := \sup\{s \in [0,n] : |\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2} \quad \forall_{\xi \in \mathbb{R}^n \setminus \{0\}}\} \tag{2.15}$$

is called the Fourier dimension of μ.

Corollary 2.12. *For $\mu \in \mathcal{M}(\mathbb{R}^n)$ we have*

$$\dim_F(\mu) \leqslant \dim_e(\mu). \tag{2.16}$$

Sets whose Fourier and Hausdorff dimensions are equal are called Salem sets (see [51], [48], [21], [55], [53], [8], [36]). In Chapter 3 we use some of their basic properties.

For certain types of measures, instead of estimating Riesz integrals, it is more convenient to study asymptotic growth behaviour of their Poisson or Gauss-Weierstrass integrals. This method was presented in [80]. We conclude this section with an example of a theorem from that paper. Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $P(x,t) = 2|S^n|^{-1}t(\|x\|^2 + t^2)^{-\frac{n+1}{2}}$ where $(x,t) \in \mathbb{R}^n \times [0,+\infty]$ and $|S^n|$ denotes the area of n-dimensional unit sphere. Let us define

$$u(x,t) = P * \mu(x,t) = \int_{\mathbb{R}^n} P(x-y,t)\,\mathrm{d}\,\mu(y).$$

Theorem 2.13 (Theorem 5 (ii), [80])**.** *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$, $u = P * \mu$, $s \in [0,n]$, and let $A \subset \mathbb{R}^n$ be a Borel set, σ-finite with respect to $\mathcal{H}^s$ such that*

$$\lim_{t \to 0} t^{n-s} u(x,t) = 0$$

for μ-a.e. $x \in A$. Then $\mu(A) = 0$.

A variant of this method was used in [71] for investigating fine properties of pluriharmonic measures.

In a recent preprint [74], the author obtained a dimensional estimate for vector-valued measures by an application of Harnack-type inequalities to the Gauss-Weierstrass extension of a measure. This result, in particular, gives an alternative approach to the dimension bound obtained for PDE-constrainted measures in [5]. The method from [5] exploits certain compactness phenomenon which occurs outside the so-called wave cones connected with a differential equation (see Example 3.48 for the definition).

We remark that, especially in Dynamical Systems and Fractal Geometry, there is an extensive research concerning other notions of dimension of a measure, such as packing, Minkowski and Assouad dimension of a measure and relations between them (see Chapter 10 in [22], [23], [33], [24], [41], [27], [59] and references therein).

2.2 Other interactions of Fourier Analysis and Geometric Measure Theory

It only takes to verify the definition to see that the Hausdorff dimension cannot be increased by Lipschitz maps, in particular by orthogonal projections. One of the most elegant results belonging to both eponymous theories is the celebrated Mastrand's projection theorem. It says that (for a reasonable set of parameters) the Hausdorff dimension is generically preserved by projections.

Theorem 2.14 (Theorem I and II in [56])**.** *Let $A \subset \mathbb{R}^n$ be a Borel set such that $\dim_{\mathcal{H}} A = s$.*

- *If $s \leqslant 1$, then*

$$\dim_{\mathcal{H}}(\Pi_{\mathrm{span}\{e\}}(A)) = s \tag{2.17}$$

for $\mathcal{H}^{n-1}$-almost all $e \in S^{n-1}$.

- *If $s > 1$, then*
$$\mathcal{H}^1(\Pi_{\text{span}\{e\}}(A)) > 0 \tag{2.18}$$
for $\mathcal{H}^{n-1}$-almost all $e \in S^{n-1}$.

The proof exploits the fact that there is an easy formula for the Fourier transform of a projected measure. This observation asserts about great utility of theorems which involve projections of sets. In further chapters, to prove rectifiability of singular sets of various measures, we make a use of this observation and the Besicovitch-Federer projection theorem:

Definition 2.15. *A set $E \subset \mathbb{R}^n$ is called k-rectifiable, if there exist Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$, $i = 1, 2, ...$, such that*

$$\mathcal{H}^k(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k)) = 0.$$

A set $F \subset \mathbb{R}^n$ is called purely k-unrectifiable if $\mathcal{H}^k(F \cap E) = 0$ for every k-rectifiable E.

Theorem 2.16 (Theorem 18.1 in [57])**.** *Let $A \subset \mathbb{R}^n$ be a Borel set with $\mathcal{H}^k(A) < \infty$, where $k < n$ is an integer. Then:*

- *A is k-rectifiable if and only if $\mathcal{H}^k(\Pi_V(B)) > 0$ for almost all $V \in \mathbb{G}(k, \mathbb{R}^n)$ (with respect to the natural measure on the Grassmannian) for any measurable $B \subset A$ with $\mathcal{H}^k(B) > 0$.*
- *A is purely k-unrectifiable if and only if $\mathcal{H}^k(\Pi_V(A)) = 0$ for almost all $V \in \mathbb{G}(k, \mathbb{R}^n)$ (with respect to the natural measure on the Grassmannian).*

There is also one important theorem, which is neither used nor improved in this thesis, but plays for us a role of an important motivating example. It relates the local growth condition and the Fourier dimension of measures to arithmetical properties of its support.

Theorem 2.17 ([55], Theorem 1.2.)**.** *Suppose that $E \subset \mathbb{T}$ supports a measure $\mu \in \mathcal{M}^+(\mathbb{T})$ and the following conditions are satisfied*

- *$\mu([x, x + \epsilon]) \leqslant C_1\epsilon^{\alpha}$,*
- *$|\widehat{\mu}(k)| \leqslant C_2(1 - \alpha)^{-B}|k|^{-\frac{\beta}{2}}$ for $k \neq 0$,*

where $0 < \alpha < 1$ and $\frac{2}{3} < \beta \leqslant 1$. If $\alpha > 1 - \epsilon_0$ for some $\epsilon_0 = \epsilon_0(C_1, C_2, B, \beta)$ small enough, then E contains a 3-term arithmetic progression.

The first assumption is a variant of the so-called α-Frostman condtion and says in particular that $\dim_{\mathcal{H}}(\mu) \geqslant \alpha$. The theorem above gives a hint that the Hausdorff dimension may depend not only on the decay of Fourier transform, but also on some arithmetical properties. Inspired by this, in Chapter 4 we prove a theorem which gives a dimension estimate taking into account only divisibility properties of elements of the spectrum of measures.

Our goal was to collect facts and theorems which, in our judgement, are useful for the practical goal of dimension estimates, and so we have not mentioned many important applications of Harmonic analysis to the Dimension Theory. In particular, we have not discussed the Kakeya problem which is a central problem in this field. For the detailed informations on this topic the reader is refered to the survey article [49].

Chapter 3

On dimension and regularity of vector-valued measures

In this chapter we quantify the notion of antisymmetry of the Fourier transform of certain vector-valued measures. The introduced scale is related to the condition appearing in Uchiyama's theorem and is used to give a lower bound for the rectifiable dimension of those measures. Moreover, we obtain an estimate of the lower Hausdorff dimension assuming certain more restrictive version (in the structural sense) of the 2-wave cone condition for PDE-constrainted measures, extending its applications to a more general Fourier analytic setting. The chapter contains also a theorem concerning regularity: we prove that elements of considered class vanish on 1-purely unrectifiable sets of finite $\mathcal{H}^1$-measure.

3.1 Preliminaries and motivation

Geometric structure and dimensional properties of distributional gradients of functions from $BV(\mathbb{R}^n)$ are well studied and widely applied (cf. [1], [2], [4], [64]). It is known, for example, that their lower Hausdorff dimension is at least $n-1$ and that it is an optimal bound. Moreover, those measures cannot charge $(n-1)$-purely unrectifiable sets of finite $\mathcal{H}^{n-1}$ measure (see Lemma 3.76 and Theorem 3.78 in [4]). For the class of bundle measures, introduced in [69], we can consider analogous problems.

Definition 3.1. *By $\mathbb{G}(m, E)$ let us denote the Grassmannian of m-dimensional subspaces of some fixed, d-dimensional complex vector space E. We call a bundle any continuous function $\phi : \mathbb{R}^n \backslash \{0\} \to \mathbb{G}(m, E)$. If additionally $\phi(a\xi) = \phi(\xi)$ for any positive a, then we refer to it as a homogeneous bundle.*

This setting gives a possibility to define bundle measures by imposing Fourier analytic rigidity conditions:

Definition 3.2. *For any homogeneous bundle ϕ, by $\mathcal{M}_\phi(\mathbb{R}^n, E)$ we denote the set of finite, Borel regular and non-zero vector measures, taking values in E and satisfying $\widehat{\mu}(\xi) \in \phi(\xi)$ for each $\xi \neq 0$. We say that a vector-valued measure $\mu \in \mathcal{M}(\mathbb{R}^n, E)$ is subordinated to ϕ, if $\mu \in \mathcal{M}_\phi(\mathbb{R}^n, E)$.*

The above definition generalizes the mentioned example of gradient measures. Indeed, if $f \in BV(\mathbb{R}^n)$ then $\widehat{\nabla f}(\xi) = 2\pi i \xi \widehat{f}(\xi)$, so $\nabla f \in \mathcal{M}_\phi(\mathbb{R}^n, \mathbb{C}^n)$ for a particular bundle $\phi(\xi) = \operatorname{span}_{\mathbb{C}}\{\xi\}$. Moreover, this formalism subsumes the case of measures with generalized bounded variation, i.e. the measures which are defined similarly to BV-gradients, with ∇ replaced by a homogeneous differential operator (see Section 3.4 Example 3.46). Moreover, the above notation is complementary to the language of $\mathcal{A}$-free measures ([15]) - see Section 3.4 for the explanation. On the other hand, as crucial roles in proofs of the mentioned properties of BV gradients are played by the de Giorgi's characterization of sets with finite perimeter and by the coarea formula, in the general case we cannot make a use of the ideas from those classical proofs due to the absence of sufficiently general coarea formula.

In this chapter, for the sake of simplicity, unless explicitly stated otherwise, we treat by default the case of line bundles ($m = 1$). In other cases all reasonings can be adapted with straightforward modifications and we sketch appropriate changes in suitable places.

We propose a conjecture that links antisymmetry of a bundle with the dimension of vector measures.

Definition 3.3. *We say that a nonconstant line bundle ϕ is antisymmetric on l-dimensional subspheres or l-antisymmetric ($l = 0, 1, ..., n-1$), if for each $(l+1)$-dimensional subspace $V \subset \mathbb{R}^n$ there exist $\xi_1, \xi_2 \in V \cap S^{n-1}$ such that $\phi(\xi_1) \neq \phi(\xi_2)$. Denote*

$$a(\phi) = \min\{l : \phi \text{ is } l\text{-antisymmetric}\}.$$

Conjecture 3.4 ([6]). *If μ is a bundle measure subordinated to a smooth, nonconstant bundle ϕ, then*

$$\dim_{\mathcal{H}}(\mu) \geqslant n - a(\phi).$$

Our first result confirms correctness of Conjecture 3.4 under some additional geometric assumptions. Though not being trivial, it should be considered rather as a motivating example. It also gives some insight how reasonable bundle measures may look like. To prove this, we use the classical measure-theoretic method of blowing-up measures, modified for dealing with Fourier transforms. Similarly as in the classical version of this method, we relate the rectifiable dimension of a measure (which we define below) to the algebraic dimension of tangent spaces to the measure at points belonging to a given set. However, the fact that we deal with measures under Fourier analytic constraints forces us to modify the notion of tangent measure in a way that convergence in the vague topology of blow-ups of measures is replaced by the weak-$*$ convergence in $\mathcal{S}'(\mathbb{R}^n)$. This requires replacing $C_c(\mathbb{R}^n)$ with $\mathcal{S}(\mathbb{R}^n)$ as the set of test functions, and is absolutely necessary not only for the theoretical reasons, but also because of existence of measures for which Fourier transform is not well-defined and which are tangent measures in the classical sense (see Example 3.17). Though those modifications are relatively simple, we decided to include a detailed presentation in Section 3.2, because to our best knowledge this type of tangent space is absent in the literature.

Definition 3.5. *By the rectifiable dimension of a scalar or vector measure* μ *we understand*

$$\dim_{rect}(\mu) := \min\{k : \exists k\text{-rectifiable measure } \nu \text{ s.t. } \mu_{\llcorner F} = \nu \neq 0 \text{ for some Borel set } F\}$$

if the set on the right-hand side is non-empty, or $\dim_{rect}(\mu) := +\infty$ *otherwise.*

In the above definition we use a stronger than the usual definition of a rectifiable measure (see Definition 3.13).

Theorem 3.6. *a) Suppose that* μ *is a bundle measure subordinated to a smooth, nonconstant bundle* ϕ*. Then*

$$\dim_{rect}(\mu) \geqslant n - a(\phi).$$

b) If μ *is rectifiable then either*

$$\dim_{\mathcal{H}}(\mu) \geqslant \frac{n}{2}$$

or μ *can be identified with a scalar measure (its values belong to some one-dimensional space).*

Unfortunately, this result is non-trivial only for measures whose part is described by some analytic formula. In particular, it gives no information when μ is singular with respect to all $\mathcal{H}^k$, for $k = 0, \ldots, n$. This issue is partially bypassed in Section 3.3, where we prove two theorems concerning the Hausdorff dimension. They can be treated as extensions of the main result from [69] (Theorem 3.10).

Perhaps, the most signifficant theorem of this chapter is the following:

Theorem 3.7. *Suppose that* μ *is subordinated to a Lipschitz bundle* ϕ*. If there exist* $2-$*dimensional spaces* $V_1, \ldots, V_J$ *such that* $\bigcap_i \operatorname{span}\{\phi(V_i \setminus \{0\})\} = \{0\}$*, then*

$$\dim_{\mathcal{H}}(\mu) \geqslant 2.$$

The above condition is related to the k-wave cone scale introduced in [5] (see also Example 3.48 for the discussion). We prove also a rectifiability result which, together with Theorem 3.10 may be treated as an analogue of the Federer-Volpert theorem ([4], Theorem 3.78., Proposition 3.92.).

Theorem 3.8. *Let* ϕ *be a homogeneous bundle which is Hölder regular with exponent* $> \frac{1}{2}$ *and suppose that* ϕ *is* $(n-1)$*-antisymmetric, i.e.*

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \phi(\xi) = \{0\}.$$

Then, for any $\mu \in \mathcal{M}_\phi(\mathbb{R}^n, E)$ *and any* 1*-purely unrectifiable set* F *satisfying* $\mathcal{H}^1(F) < \infty$ *we have* $\mu_{\llcorner F} \equiv 0$.

Proofs of both results are based on the theory of s-Riesz sets from [69]. Informally, a subset $A \subset \mathbb{R}^n$ is called an s-Riesz set, if $\dim_{\mathcal{H}}(\mu) \geqslant s$ for any measure μ whose Fourier transform is small on A (cf. Definition 1.4). Typically, 'small' means being equal to zero or $\widehat{\mu}_{|A} \in L^2(\mathbb{R}^n)$. In Section 3.3, by suitable use of properties of Salem sets, we show that 'small' may be interpreted as $\widehat{\mu}_{|A} \in \cap_{\epsilon>0} W^{-\epsilon,2}(\mathbb{R}^n)$. Here $W^{s,2}(\mathbb{R}^n)$ means an L^2-based Sobolev space with order of smoothness $s \in \mathbb{R}$. With this knowledge, to prove Theorem 3.7, for each vector measure μ satisfying its assumptions we find a scalar measure ν with a comparable dimension and a 2-Riesz set A such that $\widehat{\nu}_{|A} \in \cap_{\epsilon>0} W^{-\epsilon,2}(\mathbb{R}^n)$. The existence of such ν and A is provided by the structural condition from Theorem 3.7.

The rectifiability theorem is obtained by an application of the Besicovitch-Federer projection theorem, which seems to be a new approach for this type of problems.

Section 3.4 contains examples and comparison with some known results about measures satisfying differential equations.

3.1.1 Motivation and brief history of the problem

Conjecture 3.4 is inspired by Uchiyama's theorem on multiplier characterization of Hardy spaces (Theorem 3.12) which gives a proof when $a(\phi) = 0$. It appeared while an attempt to answer a question from [69]:

Conjecture 3.9. *([69], Conjecture 1) If the Fourier transform of a bundle measure μ contains n linearly independent vectors and $\mu \in \mathcal{M}_\phi(\mathbb{R}^n, E)$ for some line bundle ϕ, then $\dim_{\mathcal{H}}(\mu) \geqslant n-1$.*

Theorem 3.10. *([69]) Let ϕ be a nonconstant line bundle, Hölder with exponent $> \frac{1}{2}$. Then $\dim_{\mathcal{H}}(\mu) \geqslant 1$ for each $\mu \in \mathcal{M}_\phi(\mathbb{R}^n, E)$.*

Theorem 3.10, which is a particular case of Theorem 3 from [69], covers the case '$a(\phi) = n-1$' which is on the endpoint opposite to Uchiyama's theorem ([78]). In this chapter we focus on the intermediate points of the scale. Conjecture 3.9 was inspired by the example of measures derived from BV, that is satisfying equation $\nabla f = \mu$ for some $f \in L^1(\mathbb{R}^n)$ in the sense of distributions. This result shows, in particular, that if in such problem we replace ∇ by any so called canceling operator (see [79] and Example 3.46), then the resulting measure has lower Hausdorff dimension at least 1. Let us also mention that a particular case of the main result from [75] is a proof of the above conjecture for measures given by $(D_1^s f, ..., D_n^s f) = \mu$ for some natural s and $f \in L^1(\mathbb{R}^n)$ ($\phi(\xi) = \operatorname{span}_{\mathbb{C}}\{(\xi_1^s, \xi_2^s, \ldots, \xi_n^s)\}$). In this situation we have $a(\phi) = 1$.

The technique used in [75] revealed strong connections of dimension estimates with embedding theorems. Briefly: the better range of an embedding connected with a differential operator, the higher lower bound of dimension it gives. It is worth mentioning that canceling and elliptic operators (see [79] or Example 3.46 for definitions) are precisely those for which critical Sobolev embedding holds true:

Theorem 3.11. *([79], Theorem 1.3.) Suppose that $\mathcal{A}(D)$ is a homogeneous differential operator of rank s on $\mathbb{R}^n$ from V to W. Then the estimate*

$$\|D^{s-1}f\|_{L^{\frac{n}{n-1}}} \leqslant C\|\mathcal{A}(D)f\|_{L^1}$$

holds for $f \in C_c^\infty(\mathbb{R}^n;V)$ if and only if $\mathcal{A}(D)$ is elliptic and canceling.

Let us also underline that the theorem of Uchiyama gives the answer when the Hardy space $H^1(\mathbb{R}^n)$ norm is equivalent to a norm given by a family of multipliers.

Theorem 3.12 ([78])**.** *Let $\theta_1(\xi),\dots,\theta_n(\xi) \in C^\infty(S^{n-1})$ and let $K_{\theta_i}f = \mathcal{F}^{-1}(\theta_i(\frac{\xi}{|\xi|})\mathcal{F}(f))$. Then the inequality*

$$\frac{1}{C}\|f\|_{H^1} \leqslant \sum_{i=1}^{n}\|K_{\theta_i}f\|_{L^1} \leqslant C\|f\|_{H^1}$$

is true for some constant C if and only if

$$\operatorname{rank}\begin{bmatrix}\theta_1(\xi) & \theta_2(\xi) & \dots & \theta_n(\xi)\\ \theta_1(-\xi) & \theta_2(-\xi) & \dots & \theta_n(-\xi)\end{bmatrix} \equiv 2$$

for $\xi \in S^{n-1}$.

The above remarks suggest that the mechanism of creating singularities and validity of some norm inequalities are governed by the same phenomenon. We hope that the study of bundle measures may give also some heuristics to the study of embedding theorems.

Conjecture 3.4 was considered also by other authors in different contexts: B. Raita (independently) in [67] posed a question analogous to Conjecture 3.4 for measures solving differential equations. For this setting there was a substantial progress: article [5] yielded dimension estimates and rectifiability results in terms of other type of antisymmetry/cancelation. Condition appearing in Theorem 3.7 (result later in time than [5]) is close to one point of an antisymmetry scale from [5] (cf. Example 3.48).

3.1.2 Conventions

Throughout this chapter, we use the following notation:
n – dimension of the ambient space $\mathbb{R}^n$,
d – dimension of E, i.e. space containing values of bundle measures,
l – degree of antisymmetry/dimension of the wave cone.

While assuming Lipschitz or Hölder continuity of a bundle we mean a suitable property of its restriction to the unit sphere.

For $f \in L^1(\mathbb{R}^n)$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$ we choose the following normalization of the Fourier transform:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi, x\rangle} f(x)dx,$$
$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi, x\rangle} d\mu(x).$$

If $f \in L^1_{loc}(\mu)$, then by $f\mu$ we understand the measure ν given by $d\nu = f d\mu$.

In this chapter we use the below definition of rectifiability

Definition 3.13. *A set $E \subset \mathbb{R}^n$ is called k-rectifiable, if there exist Lipschitz functions f_i : $\mathbb{R}^k \to \mathbb{R}^n, i = 1, 2, \ldots$, such that*

$$\mathcal{H}^k(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k)) = 0.$$

A set $F \subset \mathbb{R}^n$ is called purely k-unrectifiable if $\mathcal{H}^k(F \cap E) = 0$ for every k-rectifiable E. We call a (scalar or vector) measure μ k-rectifiable if there exist a k-rectifiable set E and a Borel function (scalar or vector) f such that $\mu = f\mathcal{H}^k_{\llcorner E}$ and f is locally integrable with respect to $\mathcal{H}^k_{\llcorner E}$.

For a vector space V and a vector u we denote Π_V, Π_u orthogonal projections on V and on $\operatorname{span}\{u\}$ respectively. A symbol $\mathcal{D}(\mathbb{R}^n)$ means for us the space of smooth functions with compact support. By the spectrum of a tempered Radon measure we understand the support of its distributional Fourier transform. We denote it by $\operatorname{spec}(\cdot)$.

3.2 Estimates for the rectifiable part

3.2.1 Tangent measures and rectifiability

The notion of tangent measure (see [66]) is extremely useful in Geometric Measure Theory. However, one has to be careful while using it in Fourier analysis. For example, it is not hard to construct a measure whose one of tangent measures, in the classical sense (see the definition below), is not a tempered distribution (see Example 3.17 and also [62] for even more pathological example). In this and the next subsection we present how to preserve Fourier analytic constraints in the limit, by modifying the definition of tangency.

Definition 3.14. *We say that a sequence of Radon measures $(\mu_j)_{j=0}^{\infty}$ converges to a Radon measure ν in the vague topology if*

$$\lim_{j\to\infty} \int \phi \, d\mu_j = \int \phi \, d\nu$$

for any $\phi \in C_c(\mathbb{R}^n)$. Here $C_c(\mathbb{R}^n)$ stands for the space of compactly supported, continuous functions on $\mathbb{R}^n$.

Definition 3.15. *([66], p. 539) For a given $r > 0$ and a Radon measure μ we define its blow-up at point x by the formula $\mu_{r,x}(A) = \mu(x + rA)$. Any measure ν which is a limit in the vague topology of a sequence of the type*

$$c_i \mu_{x,r_i} \tag{3.1}$$

for some $r_i \downarrow 0$ and $c_i > 0$ we call a tangent measure to μ at point x. We denote the set of those measures by $Tan(\mu, x)$.

The above definition can be easily extended to vector measures (in this case, convergence is understood as the coordinate-wise convergence in the vague topology). For rectifiable measures it suffices to consider normalizaitons of blow-ups given by suitable power functions.

Definition 3.16. *For a fixed $\alpha > 0$, by $Tan_\alpha(\mu, x)$ we denote the subset of $Tan(\mu, x)$ obtained by taking $c_i = r_i^{-\alpha}$. By $Tan^*(\mu, x)$ and $Tan^*_\alpha(\mu, x)$ we denote subsets of $Tan(\mu, x)$ and $Tan_\alpha(\mu, x)$, respectively, consisting of tempered Radon measures which are limits of blow-ups in in the sense of weak-* topology on $\mathcal{S}'(\mathbb{R}^n)$.*

In the next few steps we show that Tan^*_α and Tan_α coincide for some regular measures (e.g. rectifiable measures or measures of strictly positive dimension) at generic points. However, the example below shows that Tan and Tan^* may be different.

Example 3.17. *For $j = 1, 2, \dots$ let us denote $[a_j, b_j] := [\frac{1}{2^{j^2}}, \frac{1}{2^{j^2-j}}]$ and let us take*

$$\mu = \sum_{j=1}^{\infty} \lambda_j 2^{\frac{x}{a_j}} \chi_{[a_j,b_j]}(x)dx, \tag{3.2}$$

where λ_j is defined by the formula $\lambda_j 2^{\frac{b_j}{a_j}} = \exp(-\frac{1}{b_j})$. Then μ is a finite, absolutely continuous measure. Moreover,

$$\frac{1}{\lambda_j a_j} \mu_{0,a_j} \to 2^x \chi_{[1,+\infty)}(x)dx \tag{3.3}$$

in the vague topology, so $Tan(\mu, 0)$ contains a measure which is not a tempered distribution.

A straightforward generalization of Theorem 4.8 from [14] or Theorem 2.83 from [4] is the following fact:

Theorem 3.18. *Let $\mu = f\mathcal{H}^k_{\llcorner E}$ be a k-rectifiable vector measure. Then, for $\mathcal{H}^k$-a.e. $x \in E$, there exists a $k-$dimensional vector space V_x such that*

$$r^{-k}\mu_{x,r} \to f(x)\mathcal{H}^k_{\llcorner V_x}, \tag{3.4}$$

in the vague topology as $r \downarrow 0$.

From the definition of the vague convergence we can easily get the following local growth estimate:

Lemma 3.19. *Suppose that $\mu \in \mathcal{M}(\mathbb{R}^n)$ and for $x \in E$, $|\mu|(E) > 0$, there exist tempered Radon measures ν_x such that we have*

$$r^{-\alpha}\mu_{x,r} \to \nu_x,$$

as $r \downarrow 0$ in the sense of vague topology. Then $|\mu|(B(x,r)) \leqslant C_x r^\alpha$ for $|\mu| - a.e.\ x \in E$.

Proof. Because μ is finite, it suffices to prove the above for small r. Take for the test function a smooth approximation of $\chi_{B(0,1)}$ which is constant and equal to one on $B(0,1)$, and vanishes outside $B(0,2)$. If μ is positive, then convergence gives $\mu(B(x,r)) \leqslant (\nu_x(B(0,2)) + \delta)r^\alpha$ for some positive δ. In the general case we use Hahn decomposition and locality of tangent measures (Proposition 3.12. in [14])). ☐

Convergence from Theorem 3.18 is tested on functions from $C_c(\mathbb{R}^n)$. However, for our applications we need convergence in $\mathcal{S}'(\mathbb{R}^n)$. This requires extending the class of test functions to $\mathcal{S}(\mathbb{R}^n)$ and can be achieved with the following lemma:

Lemma 3.20. *Suppose that $\mu \in \mathcal{M}(\mathbb{R}^n)$ is as in the previous lemma. Then: a) $Tan^*_\alpha(\mu,x) = Tan_\alpha(\mu,x)$ for $|\mu|$-a.e. $x \in E$. b) If $g \in L^1(\mu)$ then $Tan^*_\alpha(g\mu,x) = g(x)Tan^*_\alpha(\mu,x)$ for $|\mu|$-a.e. $x \in E$.*

Proof. Let us notice that b) is implied by a) and an analogous property of $Tan_\alpha(\mu,x)$ (Proposition 3.12. in [14]).

To prove a) it suffices to use Lemma 3.19. Choose any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We can write $\varphi = \sum_{i=1}^\infty \varphi_i$, where $\varphi_i \in C^\infty$, $\operatorname{supp}(\varphi_i) \subset B(0,i)\backslash B(0,i-1)$ for $i > 1$ and $\operatorname{supp}(\varphi_1) \subset B(0,1)$. Moreover, we can assume that $\|\varphi_i\|_\infty \leqslant \|\varphi_{|B(0,i)\backslash B(0,i-1)}\|_\infty$. Then

$$\left|\frac{1}{r^\alpha}\int \varphi d\mu_{x,r} - \int \varphi d\nu_x\right| \leqslant \left|\frac{1}{r^\alpha}\int \sum_{i=1}^{j} \varphi_i d\mu_{x,r} - \int \sum_{i=1}^{j} \varphi_i d\nu_x\right| +$$

$$+ \left|\frac{1}{r^\alpha}\int \sum_{i>j} \varphi_i d\mu_{x,r}\right| + \int \sum_{i>j} |\varphi_i| d|\nu_x|.$$

Second term can be majorized by

$$\sum_{i>j} \frac{\|\varphi_i\|_\infty |\mu|(B(x,ir))}{r^\alpha} \leqslant C \sum_{i>j} i^\alpha \|\varphi_i\|_\infty,$$

(we used Lemma 3.19) and the third one is a tail of a convergent series. After taking sufficiently big j and then choosing suitable r_0, we see that for $r < r_0$ the starting expression is smaller than any a priori given positive number. ☐

3.2.2 Distributional definition of bundle measures

As tangent measures are in general unbounded, we need to extend the definition of bundle measures to the case of general tempered Radon measures. To achieve this, we exploit the observation that measures subordinated to ϕ 'annihilate' vector-valued functions taking values in $\phi^{\perp}$.

We say that a bundle $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{G}(m, E)$ is C^∞ if (locally) $\phi(x) = \mathrm{span}_{\mathbb{C}}\{e_1(x), ..., e_m(x)\}$, where $(e_1(x), ..., e_m(x))$ is an orthonormal system and $e_i(x)$ are C^∞ functions. For a bundle ϕ we can define pointwise its orthogonal complement by $\phi^{\perp}(x) := \phi(x)^{\perp}$. Of course, if ϕ is C^∞, then so is $\phi^{\perp}$ (one can see it while applying Gram-Schmidt orthogonalization). In this section all bundles are C^∞. For the sake of presentation we assume that $(E, \langle\cdot,\cdot\rangle_E)$ is isometric to $\mathbb{C}^d$ equipped with the standard Hermitian dot product.

Definition 3.21. *For a C^∞-bundle ϕ, by $\mathcal{S}_\phi(\mathbb{R}^n, E)$ we denote the set of vector-valued Schwartz functions f such that $f(x) \in \phi(x)$ for $x \in \mathbb{R}^n \setminus \{0\}$.*

Definition 3.22. *By $\mathcal{S}'_\phi(\mathbb{R}^n, E)$ we understand the class of vectors of tempered distributions $(\Lambda_1, ..., \Lambda_d)$ (d = $\dim E$) satisfying*

$$\sum_{i=1}^{d} \langle \widehat{\Lambda}_i, f_i \rangle = 0$$

for an arbitrary $(f_1, ..., f_d) \in \mathcal{S}_{\phi^\perp}(\mathbb{R}^n, E)$. This is equivalent to

$$\sum_{i=1}^{d} \langle \Lambda_i, \widehat{f_i} \rangle = 0.$$

Further we prove that this class contains bundle measures and that it is preserved by taking limits of blow-up processes. We use translation and dilation invariance of $\mathcal{M}_\phi$, and Parseval's identity (see [50], p. 145):

Theorem 3.23. *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\langle f, \mu \rangle = \int f(x)\overline{d\mu(x)} = \int \widehat{f}(\xi)\overline{\widehat{\mu}(\xi)}d\xi.$$

Lemma 3.24. *Let $\mu \in \mathcal{M}_\phi(\mathbb{R}^n, E)$. If at some point x there exists a tangent (vector) measure $\nu \in Tan^*(\mu, x)$, then it belongs to $\mathcal{S}'_\phi(\mathbb{R}^n, E)$.*

Proof. **Step 1.** We have $c_k\mu_{x,r_k} \in \mathcal{M}_\phi(\mathbb{R}^n, E)$:
Indeed, for a fixed coordinate $\mu^{(j)}$ we have

$$c_k\widehat{\mu}^{(j)}_{x,r_k}(\xi) = c_k \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi, \frac{y-x}{r_k}\rangle} d\mu^{(j)}(y) =$$

$$c_k e^{2\pi i\langle \xi, \frac{x}{r_k}\rangle} \int_{\mathbb{R}^n} e^{-2\pi i\langle \frac{\xi}{r_k}, y\rangle} d\mu^{(j)}(y) = c_k e^{2\pi i\langle \xi, \frac{x}{r_k}\rangle} \widehat{\mu}^{(j)}\Big(\frac{\xi}{r_k}\Big),$$

hence, by homogenity of the bundle, $c_k\widehat{\mu}_{x,r_k}(\xi) \parallel \widehat{\mu}(\xi)$.

Step 2. If $\theta = (\theta_1, ..., \theta_d) \in \mathcal{M}_\phi(\mathbb{R}^n, E)$, then $\theta \in \mathcal{S}'_\phi(\mathbb{R}^n, E)$:
Let $(f_1, ..., f_d) \in \mathcal{S}_{\phi^\perp}(\mathbb{R}^n, E)$. By Parseval's identity we get

$$\sum_{i=1}^{d} \langle \theta_i, \widehat{f_i}\rangle = \int \sum_{i=1}^{d} f_i(\xi)\overline{\widehat{\theta}_i(\xi)} d\xi = 0,$$

because $(f_1, ..., f_d)(\xi)$ and $(\widehat{\theta}_1, ..., \widehat{\theta}_d)(\xi)$ are orthogonal at each $\xi \neq 0$.

Step 3. Let $(f_1, ..., f_d) \in \mathcal{S}_{\phi^\perp}(\mathbb{R}^n, E)$. By previous steps we obtain

$$0 = \lim_{r_j \downarrow 0} r_j^{-\alpha} \sum_{i=1}^{d} \langle \mu^{(i)}_{x,r_j}, \widehat{f_i}\rangle = \sum_{i=1}^{d} \langle \nu^{(i)}, \widehat{f_i}\rangle.$$

□

3.2.3 Proof of Theorem 3.6

We begin with invoking a well-known fact, whose proof can be found in [43] (Theorem 7.1.25).

Lemma 3.25. *If $V \subset \mathbb{R}^n$ is a k-dimensional linear subspace, then $\widehat{\mathcal{H}^k_{\llcorner V}} = \mathcal{H}^{n-k}_{\llcorner V^\perp}$.*

Now, by using Lemma 3.20, we can reduce our considerations to the case of flat measures. In the next two lemmas V is a fixed, k-dimensional linear subspace of $\mathbb{R}^n$.

Lemma 3.26. *Suppose that a measure $\mu \in \mathcal{M}_\phi(\mathbb{R}^n, E)$ has a tangent measure in Tan^* of the form $v\mathcal{H}^k_{\llcorner V}$, where V is a linear subspace and v is some fixed non-zero vector. Then $\phi \equiv \mathrm{span}_{\mathbb{C}}\{v\}$ on $V^\perp \setminus \{0\}$.*

Proof. Denote $k = \dim V$. Let us take any vector-valued function $F \in \mathcal{S}_{\phi^\perp}(\mathbb{R}^n, E)$. Then, by the preceeding lemma and the definition of $\mathcal{S}_{\phi^\perp}(\mathbb{R}^n, E)$ we obtain

$$\int_{\mathbb{R}^n} \langle F(x), v\rangle d\mathcal{H}^{n-k}_{\llcorner V^\perp}(x) = 0$$

(here brackets under integral sign denote the standard Hermitian dot product in $\mathbb{C}^d$). Let us assume that at some $x_0 \in V^\perp \setminus \{0\}$ we have $\phi(x_0) \neq \mathrm{span}_{\mathbb{C}}\{v\}$. This implies the existence of $w \in \phi^{\perp}(x_0)$ such that $\langle w, v\rangle \neq 0$, say $\langle w, v\rangle > 0$. Take any function $g \in \mathcal{S}_{\phi^{\perp}}(\mathbb{R}^n, E)$ such that $g(x_0) = w$. Obviously, $\langle g(x), v\rangle > 0$ in some neighbourhood U_{x_0} of x_0. After multiplying g coordinatewise by a suitable mollifier supported at U_{x_0} and substituting it in place of F we get a contradiction. □

Our efforts may be summarized as follows:

Lemma 3.27. *Suppose that μ is subordinated to ϕ and it has at x a non-zero tangent measure (in Tan^*) of the form $v\mathcal{H}^k_{\llcorner V}$. Then we have*

$$\dim V \geqslant n - a(\phi).$$

Proof. By using Lemma 3.26 we get $\dim V^{\perp} \leqslant a(\phi)$. □

Now we prove the main result of this chapter.

Proof. of Theorem 3.6 Let us recall, that for rectifiable measures, the unique tangent measure at a generic point x is of the form $f(x)\mathcal{H}^k_{\llcorner V_x}$, where $f(x)$ is the density with respect to the Hausdorff measure and V_x is the tangent plane to μ at x (see Theorem 3.18). Moreover, by Lemma 3.20, those tangent measures belong to Tan^*.

a) Let ν and F be such that $\mu_{\llcorner F} = \nu$ and ν is k-rectifiable. By Lemma 3.20 b) we can assume that $\mu = \nu$, just by applying it with $g = \chi_F$. The result follows from Lemma 3.27.

b) Let $\mu = f\mathcal{H}^k_{\llcorner E}$ be such a measure and assume $k < \frac{n}{2}$. Let us observe that, by Lemma 3.26 for $\mathcal{H}^k_{\llcorner E}$-a.e. x from the set $\{y : f(y) \neq 0\}$ we have $\phi \equiv \operatorname{span}_{\mathbb{C}}\{f(x)\}$ on $V_x^{\perp}\setminus\{0\}$. But $\dim V_x^{\perp} > \frac{n}{2}$, which means $V_x^{\perp} \cap V_y^{\perp} \neq \{0\}$ and consequently $\operatorname{span}_{\mathbb{C}}\{f(x)\} = \operatorname{span}_{\mathbb{C}}\{f(y)\}$ for any two such points. Hence, the density $f(x)$ is $\mathcal{H}^k_{\llcorner E}$ - a.e. parallel to some fixed vector, which shows that μ can be identified with a scalar measure. □

3.3 Two extensions of Theorem 3.10

3.3.1 Remarks on a theorem concerning s-Riesz sets

Next theorems give examples of Riesz sets.

Theorem 3.28. (F. and M. Riesz) *If a measure $\mu \in \mathcal{M}(\mathbb{R})$ has its spectrum inside some half-line, then it is absolutely continuous with respect to the Lebesgue measure.*

Theorem 3.29. *Suppose that a measure $\mu \in \mathcal{M}(\mathbb{R}^2)$ has its spectrum inside some angle of measure strictly smaller than π. Then it is absolutely continuous with respect to the full Lebesgue measure.*

Both theorems have its higher dimensional analogues; Theorem 0.3. from [70] (see also the Introduction, p. 2) generalizes all cases mentioned above. To construct examples of s-Riesz sets, in [69] the authors used the following slicing property.

Theorem 3.30. *([69], Theorem 1) Let $A \subset \mathbb{R}^n$. If there exists a k-dimensional subspace $V \subset \mathbb{R}^n$ such that $\forall a \in \mathbb{R}^n$ $(V + a) \cap A$ is a Riesz set on $V + a$, then A is a k-Riesz set.*

Because the argument in this theorem is based on the fact that orthogonal projections do not increase Hausdorff dimension, its easy modification gives a control on projections of sets:

Theorem 3.31. *Let $A \subset \mathbb{R}^n$ and suppose that $\mu \in \mathcal{M}(\mathbb{R}^n)$ has its spectrum inside A. If there exists a k-dimensional subspace $V \subset \mathbb{R}^n$ such that $\forall a \in \mathbb{R}^n$ $(V + a) \cap A$ is a Riesz set on $V + a$, then $\mu(F) = 0$ for each F such that $\lambda_V(\Pi_V(F)) = 0$, where λ_V is the Lebesgue measure on V.*

We postpone the proof of the above theorem to the Chapter 5, where the complete argument is presented first for a particular example of set A; see Theorem 5.10.

Example 3.32. *Bounded sets are Riesz sets. Indeed, let A be a bounded set and let $f \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{f} \equiv 1$ on some ball containing A. Then we have an identity $\mu = \mu * f \in L^1(\mathbb{R}^n)$ for any μ with spectrum inside A.*

Example 3.33. *Our model set is the following: let $V \subset \mathbb{R}^n$ be a k-dimensional subspace and let $f : \mathbb{R} \to \mathbb{R}$ be any strictly increasing function such that $\lim_{x\to+\infty} f(x) = +\infty$. In coordinates $\xi = (\xi_1, \xi_2) \in V \times V^{\perp}$ let us denote*

$$B_f = \{(\xi_1, \xi_2) : |\xi_1| \geqslant 1, |\xi_2| \leqslant f(|\xi_1|)\}.$$

Then $\mathbb{R}^n \setminus B_f$ is a k-Riesz set. This is a consequence of Theorem 3.31, previous example and the fact that slices of $\mathbb{R}^n \setminus B_f$ with affine subspaces parallel to V are bounded.

Next we present a stronger version of Theorem 3.31 for tempered measures. Namely, we allow $\widehat{\mu}$ to be an L^2 function outside s-Riesz sets. In exchange, we require certain stability with respect to taking ϵ-neighbourhoods.

Corollary 3.34. *Let $A \subset \mathbb{R}^n$ and μ be a tempered Radon measure. Suppose that:*

1. *restriction (in the sense of distributions) of $\widehat{\mu}$ to $\mathbb{R}^n \setminus A$ is an L^2 function,*
2. *there exists a k-dimensional subspace $V \subset \mathbb{R}^n$ such that for some small $\epsilon > 0$, $\forall a \in \mathbb{R}^n$ $(V + a) \cap (A + B(0, \epsilon))$ is a Riesz set on $V + a$,*

then $\mu(F) = 0$ *for each* F *such that* $\lambda_V(\Pi_V(F)) = 0$. *In particular, we have* $\dim_{\mathcal{H}}(\mu) \geqslant k$.

Proof. Suppose that there exists a bounded set F contradicting the thesis. Assume first that $\widehat{\mu} = 0$ outside A. For any $\delta > 0$ we can find a function $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{f} \in \mathcal{D}(\mathbb{R}^n)$ and $|f(x) - 1| < \delta$ for $x \in F$.

Construction: Take $g \in \mathcal{D}(\mathbb{R}^n)$ such that $\int g = 1$ and denote $f = \check{g}$. Then $f(0) = 1$ and there exists U, a neighbourhood of 0, such that $|f(x) - 1| < \delta$ for $x \in U$. Of course $\forall_{r>0}\ f(\frac{x}{r})\widehat{\ } \in \mathcal{D}(\mathbb{R}^n)$. Taking big r such that $F \subset rU$ we get a suitable function. Also, for sufficiently large r, $\operatorname{spec}(f)$ is contained in arbitrarily small ball.

Denote $\nu = f d\mu$. For sufficiently small δ, $\nu(F) \neq 0$, ν is a finite measure and $\operatorname{spec}(\nu) \subset \operatorname{spec}(\mu) + \operatorname{spec}(f)$ (ν is a product of a tempered distribution μ and a Schwartz function f). Hence, the spectrum of ν is as in the Theorem 3.31, which gives a contradiction.

Now, if $\widehat{\mu} = h \neq 0$ outside A for some $h \in L^2$, then it suffices to apply previous reasoning for $\mu - \check{h}$ (changing μ by absolutely continuous measures has no impact on singular sets). □

Remark 3.35. *Sets* B_f *clearly satisfy assumption* (2) *of Corollary 3.34.*

Let us go further in weakening assumptions and ask what can be said if the restriction of $\widehat{\mu}$ to $\mathbb{R}^n \setminus A$ is close to an L^2 function in some sense? For example if it is a Fourier transform of a distribution from the fractional Sobolev space $W^{-s,2}$? If the negaitve order of smoothness $-s$ may be taken arbitrarily close to zero, then the lower bound of the Hausdorff dimension remains the same (though this trade-off formally costs us expected results about projections). This answer is obtained by the following lemma which employs a technique used in [54] and involves using properties of Salem sets.

Lemma 3.36. *Let* $\mu \in \mathcal{M}(\mathbb{R}^n)$ *and* F *be a Borel set such that* $\dim_{\mathcal{H}}(F) = \alpha$ *and* $\mu(F) \neq 0$. *Then, for any* $0 < \eta \leqslant 1$ *there exists a probability measure on* $\mathbb{R}^n$ *satisfying the following properties:*

a) $|\widehat{\nu}(\xi)| \lesssim |\xi|^{-\frac{\eta}{4}}$,

b) ν *is supported on a compact set* G *s.t.* $\dim_M(G) \leqslant 2\eta$,

c) there exists $\tilde{F}$ *such that* $\mu * \nu(\tilde{F}) \neq 0$ *and* $\dim_{\mathcal{H}}(\tilde{F}) \leqslant \alpha + 2\eta$.

Proof. To get first two properties it suffices to consider an image of a uniform measure on η-dimensional Cantor subset on $\mathbb{R}$ by the n-dimensional Brownian motion. Theorem 12.1. from [58] or Theorem 1 from Chapter 17 in [48] gives a), while b) is implied by a well known fact that trajectories of the Brownian motion are almost surely β-Hölder continuous with $0 < \beta < \frac{1}{2}$.

Now let us prove c). For simplicity, suppose that μ is real-valued and $\mu(F) > 0$. By regularity and Jordan decomposition theorem for measures, we may assume that F is compact and, for some $\delta > 0$, its δ-neighbourhood F_δ satisfies $\mu_-(F_\delta) < \frac{1}{100}\mu(F)$. It suffices to rescale previously obtained ν so that $G \subset B(0, \frac{\delta}{2})$ and take $\tilde{F} = F + G$. Indeed

$$\mu * \nu(F+G) = \int_G \mu(F+G-x)d\nu(x)$$

and $F \subset F+G-x \subset F_\delta$ for any $x \in G$, so the integral is positive. Moreover, $\dim_{\mathcal{H}}(F+G) \leqslant \dim_{\mathcal{H}}(F) + \dim_M(G) \leqslant \alpha + 2\eta$, ([54], Lemma 1.3.) which proves the lemma. □

The above immediately leads to the announced corollary:

Corollary 3.37. *Let $A \subset \mathbb{R}^n$ and μ be a tempered Radon measure. Suppose that:*

1. *for an arbitrary $s > 0$ restriction (in the sense of distributions) of $\widehat{\mu}$ to $\mathbb{R}^n \setminus A$ is a Fourier transform of an element of $W^{-s,2}$,*
2. *there exists a k-dimensional subspace $V \subset \mathbb{R}^n$ such that for some small $\epsilon > 0$ $\forall a \in \mathbb{R}^n$ $(V+a) \cap (A + B(0,\epsilon))$ is a Riesz set on $V+a$,*

then $\dim_{\mathcal{H}}(\mu) \geqslant k$.

Proof. Suppose that $\mu(F) \neq 0$, and $\eta > 0$ is such that $\dim_{\mathcal{H}}(F) + 2\eta < k$. For this η, take ν from Lemma 3.36 and convolve it with μ. Then, the restriction of $\widehat{\mu * \nu}$ to $\mathbb{R}^n \setminus A$ is in L^2, but (c) from Lemma 3.36 and Corollary 3.34 give a contradiction. □

3.3.2 Proof of Theorem 3.7

Before giving the proof in full generality we show how it works in the simplest case of a line bundle connected with gradients on $\mathbb{R}^3$, i.e. when $\mu = \nabla f$ for $f \in BV(\mathbb{R}^3)$.

Proof. (of Theorem 3.7: the case of gradients) Suppose that there exists a set F such that $\dim_{\mathcal{H}}(F) < 2$ and $\mu(F) = e \neq 0$. Without loss of generality we can assume that $e = (0,0,1)$. Let ν be a scalar measure given by the equation:

$$\nu(E) = \langle \Pi_{\mathrm{span}\{e\}}(E), (0,0,1)\rangle \quad \text{for} \quad E \subset \mathbb{R}^3.$$

Then $\nu(F) = 1$ and for $\xi \neq 0$ we have

$$|\widehat{\nu}(\xi)| = |\widehat{\mu}(\xi)| \cdot |\sin\angle(\xi, \mathrm{span}\{(1,0,0),(0,1,0)\})| = |\widehat{\mu}(\xi)| \cdot \frac{|\xi_3|}{|\xi|} \leqslant \|\mu\| \cdot \frac{|\xi_3|}{|\xi|}. \quad (3.5)$$

Let us denote

$$B = \{(\xi_1, \xi_2, \xi_3)\colon \xi_1^2 + \xi_2^2 \geqslant 1, |\xi_3| \leqslant \log(1 + \sqrt{\xi_1^2 + \xi_2^2})\}$$

and

$$R_j = B \cap \{(\xi_1, \xi_2, \xi_3)\colon 2^j \leqslant \sqrt{\xi_1^2 + \xi_2^2} < 2^{j+1}\} \quad \text{for} \quad j = 0, 1, \ldots$$

Thus, for any $s > 0$:

$$\int_B |\widehat{\nu}(\xi)|^2 |\xi|^{-2s} d\xi = \sum_{j=0}^{\infty} \int_{R_j} |\widehat{\nu}(\xi)|^2 |\xi|^{-2s} d\xi \overset{(3.5)}{\lesssim} \sum_{j=0}^{\infty} \int_{R_j} \frac{|\xi_3|^2}{|\xi|^{2+2s}} d\xi$$

$$\leqslant \sum_{j=0}^{\infty} |R_j| \log(1 + 2^{\frac{j+1}{2}})^2 \cdot 2^{-j(2+2s)} \lesssim \sum_{j=0}^{\infty} \log(1 + 2^{\frac{j+1}{2}})^3 \cdot 2^{-j(2+2s)} \cdot 2^{2j}$$

$$= \sum_{j=0}^{\infty} \log(1 + 2^{\frac{j+1}{2}})^3 \cdot 2^{-2sj} < +\infty.$$

By the inequality above and Corollary 3.37 applied to the set $A = \mathbb{R}^n \setminus B$, we obtain that $\dim_{\mathcal{H}}(\nu) \geqslant 2$, which gives a contradiction. □

Now we will show the proof for general Lipschitz bundles $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{G}(m, E)$. Let us recall that in fact we assume Lipschitz continuity of the restriction of ϕ to the unit sphere. We use the standard metric on $\mathbb{G}(m, E)$, that is

$$d_{\mathbb{G}(m,E)}(V, W) = \sup_{z \in V \cap S^{d-1}} d_E(z, W).$$

Proof. (of Theorem 3.7) For $i = 1, \ldots, J$ let us denote $W_i = \operatorname{span}\{\phi(V_i \setminus \{0\})\}$. Let us assume that for some $S \subset \mathbb{R}^n$ such that $\dim_{\mathcal{H}}(S) < 2$, we have $\mu(S) = e \neq 0$, and take j satisfying $e \notin W_j$. There exists a functional $\theta \in E^*$ satisfying $W_j \subset ker\, \theta$ and $\theta(e) \neq 0$. Its value on v may be computed as follows: project v on $\operatorname{span}_{\mathbb{C}}\{e\}$ along a subspace containing W_j (but not e) and take scalar product with e. Let $\nu \in \mathcal{M}(\mathbb{R}^n)$ be defined by the formula $\nu = \theta(\mu)$. Then we have $\widehat{\nu} = \theta(\widehat{\mu}(\xi))$, $\boldsymbol{\nu(S) \neq 0}$ and for some constant $C = C(ker\, \theta, e)$ the following estimate holds

$$|\widehat{\nu}(\xi)| \leqslant C|\widehat{\mu}(\xi)| \cdot |\sin \angle(\phi(\xi), W_j)| = C|\widehat{\mu}(\xi)| \cdot \sup_{z \in \phi(\xi) \cap S^{d-1}} d_E(z, W_j) \leqslant \tag{3.6}$$

$$\leqslant C\|\mu\| \cdot \sup_{z \in \phi(\xi) \cap S^{d-1}} d_E(z, W_j)$$

This is obvious if e is orthogonal to W_j (we can take $C = 1$). If not, we use the fact that two functionals with the same kernel are proportional. Moreover, using the inclusion

$$\phi(\xi_0) \subset \operatorname{span}\{\phi(V_j \setminus \{0\})\} = W_j \quad \forall_{\xi_0 \in V_j \cap S^{n-1}},$$

homogenity and Lipschitz continuity of a bundle, respectively, we obtain

$$\sup_{z\in\phi(\xi)\cap S^{d-1}} d_E(z, W_j) \leqslant \sup_{z\in\phi(\xi)\cap S^{d-1}} d_E(z,\phi(\xi_0)) = d_{\mathbb{G}(m,E)}(\phi(\xi),\phi(\xi_0)) =$$

$$d_{\mathbb{G}(m,E)}\left(\phi\left(\frac{\xi}{|\xi|}\right),\phi(\xi_0)\right) \lesssim d_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|},\xi_0\right)$$

By taking $\xi_0 \in V_j \cap S^{n-1}$ such that $d_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|},\xi_0\right) = d_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, V_j\cap S^{n-1}\right)$, and using above inequalities we finally get

$$|\nu(\xi)| \lesssim d_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|},\xi_0\right) \simeq d_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|},V_j\right). \tag{3.7}$$

In coordinates $\xi = (\xi_1,\xi_2) \in V_j \times V_j^{\perp}$ let B_f be given by

$$B_f = \{(\xi_1,\xi_2) : |\xi_1| \geqslant 1, |\xi_2| \leqslant f(|\xi_1|)\},$$

where $f(t) = \log(1+t)$. Then $\mathbb{R}^n \setminus B_f$ is a 2-Riesz set (see Example 3.33) and we can apply Corollary 3.37. Indeed, we will show that the inverse Fourier transform of a distribution $(\widehat{\nu}1_{B_f})$ is in $W^{-s,2}$ for an arbitrary $s > 0$.

By the inequality (3.7) we obtain

$$\int_{B_f} |\widehat{\nu}(\xi)|^2|\xi|^{-2s}d\xi \lesssim$$

$$\overset{(3.7)}{\lesssim} \int_{\{1\leqslant|\xi_1|<\infty\}}\int_{\{|\xi_2|\leqslant f(|\xi_1|)\}} d_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|},V_j\right)^2|\xi|^{-2s}d\xi_2 d\xi_1$$

$$\lesssim \int_{\{1\leqslant|\xi_1|<\infty\}} f(|\xi_1|)^{(n-2)}\left(\frac{f(|\xi_1|)}{|\xi_1|}\right)^2|\xi_1|^{-2s}d\xi_1$$

For an arbitrary $\gamma > 0$, $f(u) \leqslant C_\gamma u^\gamma$ when $u \geqslant 1$, so the last integral may be majorized, up to a constant, by

$$\int_{1\leqslant|\xi_1|<\infty} |\xi_1|^{\gamma n-2-2s}d\xi_1 = 2\pi\int_1^\infty t^{\gamma n-1-2s}dt,$$

which is finite for $\gamma < \frac{2s}{n}$. Choosing such γ we obtain $\|(\widehat{\nu}1_{B_f})^{\vee}\|_{W^{-s,2}} < \infty$, so by Corollary 3.37 we get $\boldsymbol{\nu(S) = 0}$, which gives a contradiction. □

Remark 3.38. *The proof works if we assume Lipschitz continuity of ϕ at points from $\cup_i V_i \cap S^{n-1}$ only.*

Remark 3.39. *The structural assumption in Theorem 3.7 is, by a simple compactness argument, equivalent to the condition* (1.3) *discussed in the Introduction.*

3.3.3 Proof of Theorem 3.8

As we have seen in the proof of Theorem 3.6, the homogenity condition gives us a possibility to relate geometry of singular sets with values of bundles measures. Proof of Theorem 3.8, which employs similar principles, is a consequence of the following qualitative reformulation of Theorem 3 in [69]:

Definition 3.40. *For $A \subset \mathbb{R}^n$, by $N(A)$ we denote the set*

$$\{v \in \mathbb{R}^n : \|v\| = 1,\ \lambda_v(\Pi_v(A)) = 0\}.$$

Here λ_v stands for the 1-dimensional Lebesgue measure on $\operatorname{span}\{v\}$.

Theorem 3.41. *If $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{G}(m, E)$ is a homogeneous bundle, Hölder with an exponent $> \frac{1}{2}$, then for each $\mu \in M_\phi(\mathbb{R}^n, E)$ and an arbitrary Borel set $A \subset \mathbb{R}^n$ we have*

$$N(A) \subset \phi^{-1}(\mu(A)) := \{u \in \mathbb{R}^n : \|u\| = 1,\ \mu(A) \in \phi(u)\}.$$

Note that it proves Conjecture 3.4 if we replace $\dim_{\mathcal{H}}$ by the lowest dimension of an affine subspace on which a measure does not vanish. Next we sketch the proof for general bundles.

Proof. (Sketch) Let $A \subset \mathbb{R}^n$, $\mu(A) = e$, $\lambda(\Pi_v(A)) = 0$ and assume that the thesis does not hold, i.e. $v \notin \phi^{-1}(e)$ for some v. We can choose a functional $\theta \in E^*$ satisfying $\phi(v) \subset ker\, \theta$ and $\theta(e) \neq 0$. The rest of the proof goes similarly as in Theorem 3.7. We replace V_j by $\operatorname{span}\{v\}$ and, by making a use of Hölder continuity, we prove that $\widehat{\mu}$ is square summable in B_f. Instead of Corollary 3.37 we invoke Corollary 3.34. □

To prove Theorem 3.8, we will need a part of Besicovitch-Federer projection theorem (see Theorem 18.1 in [57]):

Theorem 3.42. *Let $A \subset \mathbb{R}^n$ be a Borel set with $\mathcal{H}^m(A) < \infty$, where $m < n$ is an integer. Then A is purely m-unrectifiable if and only if $\mathcal{H}^m(\Pi_V(A)) = 0$ for almost all $V \in \mathbb{G}(m, \mathbb{R}^n)$ (with respect to the natural measure on the Grassmannian).*

Proof. (of Theorem 3.8) Suppose that $\mu(F') \neq 0$ for some $F' \subset F$. Then, by Theorem 3.41, $N(F') \subset \phi^{-1}(\mu(F'))$. But, by Besicovitch-Federer theorem, $N(F')$ is a dense subset of S^{n-1} so, by the continuity of the bundle,

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \phi(\xi) \neq \{0\}.$$

which gives a contradiction. □

Remark 3.43. *Because in the proof of Corollary 3.37 we modified a measure by convolving it with a Salem measure, analogous method does not give rectifiability in Theorem 3.7.*

Remark 3.44. *Theorems 3.7 and 3.8 can be proved for more general class than homogeneous bundles. For example, since addition of square-summable functions do not have any influence on singular sets of measures, we may admit certain error in the sense of L^2 norm (cf. original formulation of Theorem 3.10 in [69]).*

3.4 Connections with PDE-constrainted measures

In this section we show applications of our results to measures naturally arising in some classical differential problems.

Example 3.45. *If $(D_1^s f, D_2^s f, D_3^s f) = \mu \in \mathcal{M}(\mathbb{R}^3, \mathbb{R}^3)$ for some natural number s and $f \in L^1(\mathbb{R}^3)$, then we have $\widehat{\mu}(\xi) = (2\pi i)^s(\xi_1^s, \xi_2^s, \xi_3^s)\widehat{f}(\xi)$, so $\phi(\xi) = \operatorname{span}_{\mathbb{C}}\{(\xi_1^s, \xi_2^s, \xi_3^s)\}$ and $E = \mathbb{C}^3$. Let us take*

$$V_1 = \operatorname{span}\{e_2, e_3\}, V_2 = \operatorname{span}\{e_1, e_3\}, V_3 = \operatorname{span}\{e_1, e_2\}.$$

Then $\phi(V_j \setminus \{0\}) = V_j \otimes \mathbb{C}$ and $\cap_{i=1}^3 V_i = \{0\}$, so assumptions of Theorem 3.7 are fulfilled. In particular, we obtained a purely Fourier analytic proof of dimension estimate for gradients from $BV(\mathbb{R}^3)$.

Example 3.46. *(cf. [79]) Let V, W be some finitely dimensional vector spaces, $n \geqslant 1$ and $s \in \mathbb{N}$. Suppose that $\mathcal{A}(D)$ is a homogeneous differential operator of order s on $\mathbb{R}^n$ from V to W, that is*

$$\mathcal{A}(D)u = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = s} A_\alpha(\partial^\alpha u)$$

for $u \in C^\infty(\mathbb{R}^n, V)$, where $A_\alpha \in \mathcal{L}(V, W)$. We say that $\mathcal{A}(D)$ is canceling if

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \mathbb{A}(\xi)[V] = \{0\},$$

where $\mathbb{A}(\xi)$ stands for the symbol of $\mathcal{A}$. Assume that in the above $\dim\ V = 1$ and for some $f \in L^1(\mathbb{R}^n)$ we have

$$\mathcal{A}(D)f = \mu \tag{3.8}$$

in the sense of distributions. If $\mathcal{A}$ is elliptic ($\mathcal{A}(\xi) \neq 0$ for $\xi \neq 0$), then the measure μ is subordinated to a nonconstant bundle $\phi(\xi) = \operatorname{span}_{\mathbb{C}}\{\mathbb{A}(\xi)\}$. In this setting, canceling condition is equivalent to non-constancy of ϕ.

In the case of general bundles with values in $\mathbb{G}(m, E)$, the l-antisymmetry condition can be formulated as follows: *for each $(l+1)$-dimensional subspace $V \subset \mathbb{R}^n$ there exist $\xi_1, ..., \xi_j \in V \cap S^{n-1}$ such that $\phi(\xi_1) \cap ... \cap \phi(\xi_j) = \{0\}$.*

Example 3.47. *(Continuation of Example 3.46) Let us assume that $\mathcal{A}$ is as in Example 3.46 and $\dim V = m$, i.e.*

$$\mathbb{A}(\xi) = \begin{bmatrix} | & & | \\ \mathbb{A}_1(\xi) & \dots & \mathbb{A}_m(\xi) \\ | & & | \end{bmatrix}$$

and $\mathbb{A}_1(\xi), ..., \mathbb{A}_m(\xi)$ are linearly independent. Then the operator $\mathcal{A}$ satisfies the canceling condition if and only if the bundle

$$\phi(\xi) = \mathrm{span}_{\mathbb{C}}\{\mathbb{A}_1(\xi), ..., \mathbb{A}_m(\xi)\}$$

is $(n-1)$-antisymmetric.

Example 3.48. *([5], Theorem 1.3., Corollary 1.4.) Suppose that for $\mathcal{A}(D)$ as before we have*

$$\mathcal{A}(D)\mu = 0$$

in the weak sense. If $\mathbb{A}(\xi)$ has constant rank, then any such measure belongs to the class given by the bundle $\phi(\xi) = ker\{\mathbb{A}(\xi)\}$ and the empty l-wave cone condition from [5] reads as

$$\bigcap_{U \in \mathbb{G}(l,V)} \phi(U \setminus \{0\}) = \{0\}. \tag{3.9}$$

In the mentioned paper it is proved, among other things, that under this assumption, any such measure is at least l-dimensional. Hence, in this setting the constraint in Theorem 3.7 is a particular case of the 2-wave cone condition. However, we do not require any connections with differential operators or even smoothness of the bundle. In fact, our proof requires only Lipschitz continuity of ϕ at points from $\cup_i V_i \cap S^{n-1}$.

The next example contains even more concrete application of the above.

Example 3.49. *(cf. [73]) Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^n)$ be a divergence-free measure, i.e. a measure satisfying the equation*

$$div\ \mu = 0 \tag{3.10}$$

in the sense of distributions. Below we compute a bundle ϕ to which μ is subordinated. By taking the Fourier transform of (3.10) *we obtain*

$$\xi_1\widehat{\mu_1} + \xi_2\widehat{\mu_2} + \cdots + \xi_n\widehat{\mu_n} = 0, \tag{3.11}$$

hence $\langle \xi, \widehat{\mu} \rangle_{\mathbb{C}^n} = 0$. This shows that $\phi(\xi) = (\mathrm{span}_{\mathbb{C}}\{\xi\})^{\perp}$. Obviously, we have $a(\phi) = 1$, so Theorem 3 from [69] and Theorem 3.8 can be applied. In particular, $\dim_{\mathcal{H}}(\mu) \geqslant 1$.

Methods of this chapter can also be easily adapted to the study of the matrix-valued measures.

Example 3.50. *Let $f = (f_1, \ldots, f_d) \in BV(\mathbb{R}^n)^d$ be a vector-valued function with bounded variation. We have*

$$\widehat{\nabla f}(\xi) = \begin{bmatrix} \xi_1 \widehat{f_1} & \ldots & \xi_n \widehat{f_1} \\ \vdots & \ddots & \\ \xi_1 \widehat{f_d} & & \xi_n \widehat{f_d} \end{bmatrix} = \begin{bmatrix} \widehat{f_1} \\ \widehat{f_2} \\ \vdots \\ \widehat{f_d} \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}^t. \tag{3.12}$$

Thus, ∇f is subordinated to the bundle given by the formula

$$\phi(\xi) = \mathrm{span}_{\mathbb{C}}\{v \cdot \xi^t \colon v \in \mathbb{C}^d\}.$$

Example 3.51 (cf. [5], Subsection 3.2.). *Suppose that* $f = (f_1, \ldots, f_n) \in BD(\mathbb{R}^n)$ *is a function with bounded deformation, i.e. a function whose symmetrized gradient is a finite, matrix-valued measure:*

$$\frac{\nabla f + (\nabla f)^t}{2} \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^{n\times n}). \tag{3.13}$$

By (3.12)*, we have that this measure is subordinated to the bundle*

$$\phi(\xi) = \operatorname{span}_{\mathbb{C}}\{\frac{v\cdot \xi^t + \xi \cdot v^t}{2} \colon v \in \mathbb{C}^n\} = \operatorname{span}_{\mathbb{C}}\{\frac{v\otimes \xi + \xi \otimes v}{2} \colon v \in \mathbb{C}^n\}. \tag{3.14}$$

Below, for $n = 3$*, we will show that assumptions of Theorem 3.7 hold. From the formula* (3.12)*, the Fourier transform of the symmetrized gradient of* f *is the following* 3×3 *matrix:*

$$\begin{bmatrix} \xi_1\widehat{f_1} & \frac{\xi_1\widehat{f_2}+\xi_2\widehat{f_1}}{2} & \frac{\xi_1\widehat{f_3}+\xi_3\widehat{f_1}}{2} \\ \frac{\xi_1\widehat{f_2}+\xi_2\widehat{f_1}}{2} & \xi_2\widehat{f_2} & \frac{\xi_2\widehat{f_3}+\xi_3\widehat{f_2}}{2} \\ \frac{\xi_1\widehat{f_3}+\xi_3\widehat{f_1}}{2} & \frac{\xi_2\widehat{f_3}+\xi_3\widehat{f_2}}{2} & \xi_3\widehat{f_3} \end{bmatrix}. \tag{3.15}$$

Let us take for V_i $i = 1, 2, 3$ *the same spaces as in Example 3.45 and define* $W_{1,2}, W_{2,3}, W_{3,1}$ *by*

$$W_{1,2} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \colon \xi_1 = \xi_2\}, \tag{3.16}$$

$$W_{2,3} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \colon \xi_2 = \xi_3\}, \tag{3.17}$$

$$W_{3,1} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \colon \xi_3 = \xi_1\}. \tag{3.18}$$

We see that $V = \cap_{i=1}^3 \phi(V_i \setminus \{0\})$ *contains matrices with zeros on the diagonal. Moreover, entries indexed with* (i, j) *and* (j, i) *of any matrix from* $V \cap \phi(W_{i,j} \setminus \{0\})$ *must be equal to zero. Thus* $V_1, V_2, V_3, W_{1,2}, W_{2,3}, W_{3,1}$ *are the desired* 2*-dimensional subspaces.*

Chapter 4

Hausdorff dimension of measures with arithmetically restricted spectrum

In this chapter we provide an estimate from below for the lower Hausdorff dimension of measures on the unit circle based on the arithmetic properties of their spectra. We obtain those bounds via adaptation of results from [7] for vector-valued martingales on q-regular trees to a specific backwards martingale. To show the sharpness of our method, we improve the best numerical lower bound known for the Hausdorff dimension of certain Riesz products.

4.1 Preliminaries and motivation

The most common way to estimate the lower Hausdorff dimension of a measure using Harmonic Analysis tools is the so-called energy method. It involves examination of the summability properties of the Fourier coefficients of a measure. In general, however, the energy and Hausdorff dimensions may be different (see e.g. Proposition 3.4 in [37] or Chapter 13 in [57]). In this chapter, we investigate not only the size of the spectrum, but also its arithmetic properties.

By $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ we denote the circle group.

Definition 4.1. *Let $A \subset \mathbb{Z}$. We denote by $\mathcal{M}_A(\mathbb{T})$ the set of finite Borel measures satisfying $\widehat{\mu}(n) = 0$ for any $n \in \mathbb{Z} \setminus A$.*

Throughout this chapter q is a fixed integer greater than 2. The symbol $\|$ means the relation of exact division of integers. That is $a^n \parallel b$ if and only if $a^n | b$ but $a^{n+1} \nmid b$. For any $B \subset \{1, 2, \ldots, q-1\}$, let us define

$$\mathrm{C}_B = \{kq^n \colon k \in \mathbb{Z},\ k \mod q \in B,\ n \geqslant 0\} \cup \{0\}.$$

We denote the group of residues modulo q by $\mathbb{Z}_q$ and identify the set $\{0, 1, \dots, q-1\}$ with it in the natural way. Our first result may be thought of as an uncertainty principle (see [52]).

Theorem 4.2. *Let $B \subset \mathbb{Z}_q \setminus \{0\}$ and let $\mu \in \mathcal{M}_{\mathrm{C}_B}(\mathbb{T})$ be a finite non-negative measure. If $B \subset H \setminus \{0\}$ for some subgroup $H \subset \mathbb{Z}_q$, then*

$$\dim_{\mathcal{H}}(\mu) \geqslant 1 - \frac{\log |H|}{\log q}.$$

Moreover, if the inclusion $B \subset H \setminus \{0\}$ is proper, then the above inequality is strict in the following sense: there exists $\delta > 0$ independent of μ such that

$$\dim_{\mathcal{H}}(\mu) \geqslant 1 - \frac{\log |H|}{\log q} + \delta.$$

In particular, if $B \neq \mathbb{Z}_q \setminus \{0\}$, then $\dim_{\mathcal{H}}(\mu) > \delta$ for any non-negative $\mu \in \mathcal{M}_{C_B}(\mathbb{T})$.

This theorem is a corollary of more general Theorem 4.17 below. The latter theorem provides better bounds based on the arithmetic structure of the set B. In particular, it delivers simple numeric bounds for δ in Theorem 4.2. However, Theorem 4.17 requires more notation, so we leave its formulation for a while.

We confront our methods with the question about determining the dimension of Riesz products. For convenience, let us focus on the class given by

$$\mu_{a,q} = \prod_{k=0}^{\infty} \big(1 + a\cos(2\pi q^k x)\big), \tag{4.1}$$

where $a \in [-1, 1]$. One of the most important advances in the mentioned problem is contained in the seminal work [65] of Peyrière. In this paper, among other things, he proved the identity

$$\dim_{\mathcal{H}}(\mu_{a,q}) = 1 - \frac{\int_0^1 \log(1 + a\cos(2\pi x))d\mu_{a,q}}{\log q}. \tag{4.2}$$

We note that Peyrière considered Riesz products of more general type. Results of his work go beyond Hausdorff dimension estimates and shed light on random nature of those measures. Connections between random and deterministic measures were studied in a systematic way by Fan (cf. [25],[29],[26],[28]). In particular, in [28] he gave an approximation result using probabilistic methods

$$\left| \dim_{\mathcal{H}}(\mu_{a,q}) - \Big(1 - \frac{1}{\log q}\int\limits_0^1 \log\big(1 + a\cos(2\pi x)\big)\big(1 + a\cos(2\pi x)\big)\,dx\Big)\right| \leqslant \frac{8\pi^2 a}{(q+3)^2 \log q}, \tag{4.3}$$

when $|a| \leqslant \cos\big(\frac{\pi}{\lfloor\frac{q+1}{2}\rfloor + 1}\big)$.

In contrast to the above, we are mainly interested in the case of (heuristically) the most singular Riesz products, i.e when $|a|$ is close or equal to 1. For $|a|$ sufficiently close to 1 and sufficiently big q's, we improve the best numerical lower bounds for $\dim_{\mathcal{H}}(\mu_{a,q})$ derived directly from formula (4.2) (by straithforward estimates of the integral from (4.2)) and those obtained by potential-theoretic methods (see [37], Corollary 3.2. and [57], Corollary 13.4). The following theorem is a corollary of the already mentioned Theorem 4.17 below.

Theorem 4.3. *For any integer* $q \geqslant 4$ *and* $a \in [-1,1]$, *we have*

$$\dim_{\mathcal{H}}(\mu_{a,q}) \geqslant 1 - \frac{1}{q\log q}\sum_{j=1}^{q-2}\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big)\log\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big)$$

Theorem 4.3 delivers bounds which may be thought of as extensions of (4.3).

Lemma 4.4. *For any even* $q \geqslant 4$, *the following identity holds*:

$$\sum_{j=1}^{q-2}\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big)\log\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big) =$$

$$(1-\log 2)q + 2\log 2 + \frac{2}{q\cos\frac{\pi}{q}}\int_{\frac{\pi}{2}}^{\frac{q\pi}{4}}\log(\cos^2 z)\sin\frac{2z}{q}\,dz - q\log\cos\frac{\pi}{q}. \quad (4.4)$$

Proposition 4.5. *For any even integer* $q \geqslant 4$ *and* $a \in [-1,1]$, *we have*

$$\dim_{\mathcal{H}}(\mu_{a,q}) \geqslant$$

$$1-\frac{1-\log 2}{\log q}-\frac{1}{q\log q}\left(2\log 2+\frac{2}{q\cos\frac{\pi}{q}}\int_{\frac{\pi}{2}}^{\frac{q\pi}{4}}\log(\cos^2 z)\sin\frac{2z}{q}\,dz\right)+\frac{\log\cos\frac{\pi}{q}}{\log q}.$$

Proposition 4.6. *For any integer* $q \geqslant 4$ *and* $a \in [-1,1]$, *we have*

$$\dim_{\mathcal{H}}(\mu_{a,q}) \geqslant 1-\frac{1-\log 2}{\log q}-\frac{4\pi+\frac{\pi^2}{2e}}{q\log q}-\frac{1}{\log q}\Big(\frac{1}{\cos\frac{\pi}{q}}-1\Big).$$

By virtue of the identity $\int_0^1(1+\cos 2\pi x)\log(1+\cos 2\pi x)\,dx = 1-\log 2$, when $a=\pm 1$, the above expressions agree with the bound that one would expect from (4.3) up to asymptotically the most significant terms. In Proposition 4.5, the expression in the parentheses is of order $O(\frac{1}{q})$, so in the case of even q we have the same asymptotics as in (4.3) also up to lower order terms (see Remark 4.22).

We remark that the papers [9], [26], [28], [37], [47], [57], [65] treat the case of more general Riesz products

$$\prod_{k=0}^{\infty}(1+a_k\cos(2\pi q_k x)), \quad \{q_k\}_k \subset \mathbb{N}, a_k \in [-1,1],$$

assuming various size or divisibility constraints on $\{q_k\}_k$. In the most general case, the result obtained by Hare and Roginskaya in [37] assumes that $\{q_k\}_k$ is a dissociate and increasing sequence of integers. It seems impossible to get any result without the assumption $q_k | q_{k+1}$ by adapting methods from this chapter in a straightforward way. In [37] and [57] the authors already relaxed this constraint. Moreover, in the case $q_k = q^k$ and $a_k \equiv a$ our bounds are worse than most of those already known in the literature when the number a is close to zero.

Our methods are quite different from that of [26], [28], [47], and [65]; the proofs presented here are self-contained. In particular, we do not use any sort of an ergodic theorem. We adjust the methods for estimating the lower Hausdorff dimension of the so-called Sobolev martingales from [7]. Those martingales are vector-valued. The reasoning simplifies significantly in the present case of non-negative scalar measures. More specifically, we will relate a backwards martingale of periodic functions to a measure $\mu \in \mathcal{M}_{C_B}(\mathbb{T})$ and extract the estimate for $\dim_{\mathcal{H}}(\mu)$ from the growth bounds for the corresponding martingale.

4.2 Transference of results from martingale spaces

We will be representing the points of $\mathbb{T}$ in the q-ary system. We denote by $x(j)$ the j-th digit of $x \in \mathbb{T}$, that is,

$$x = \sum_{j=1}^{\infty} \frac{x(j)}{q^j}, \quad x(j) \in \{0, 1, 2, \dots, q-1\},$$

with the convention that if there are two such representations, then we choose the finite one.

4.2.1 Approximating trees and the backwards martingale

Before we give precise formulas for the martingale of periodizations, let us briefly discuss our strategy.

Our purpose is to define, for any natural N, a tree $\mathcal{T}_N$ that will be used to sample measures up to the scale $\sim q^{-N}$. Namely, the root of the tree will encode $\mathbb{T}$, the set of leaves will represent the arcs of length $\sim q^{-N}$, and the intermediate vertices will correspond to some periodic sets. This discretization procedure will allow us to obtain a bound for martingale approximations of a given measure (Lemma 4.19 below), depending on certain space of admissible martingale differences (which is computable in terms of Fourier coefficients, cf. Lemma 4.9 below). The obtained inequality will allow us to use a Frostman-type Lemma 2.4 from [75]. Unfortunately, we cannot simply refer to that lemma, so we adjust its proof to our case; in fact, the proof of Theorem 4.17 presented at the end of this section follows the lines of the proof of the said lemma.

Definition 4.7. *Let us introduce the set*

$$\alpha_{N;\emptyset} = \{x \in \mathbb{T} \colon x(j) = 0 \text{ for } j > N\}.$$

For any sequence $(i_1, \ldots, i_k)$ *with* $k \leqslant N$ *and* $i_j \in \{0, 1, \ldots, q-1\}$ *for* $j = 1, 2, \ldots, k$*, we also introduce the set*

$$\alpha_{N;i_1,i_2,\ldots,i_k} = \{x \in \alpha_{N;\emptyset} \colon x(N-j+1) = i_j \text{ for all } j = 1, 2, \ldots, k\}.$$

The above sets will be the vertices of the tree $\mathcal{T}_N$ described in the forthcoming definition. This tree will be regular (each parent has q children) and moreover, the sons of a parent will be enumerated by numbers from 0 to $q-1$.

Definition 4.8. *We define the tree* $\mathcal{T}_N$ *according to the following rules:*

1. *the root of* $\mathcal{T}_N$ *is the set* $\{\alpha_{N;\emptyset}\}$,
2. *the* j*-th child of the root is* $\alpha_{N;j}$*, here* $j = 0, \ldots, q-1$,
3. *the* j*-th child of the vertex corresponding to* $\alpha_{N;i_1,\ldots,i_{k-1}}$ *is* $\alpha_{N;i_1,\ldots,i_{k-1},j}$*, here* $j = 0, \ldots, q-1$.

For a vertex α*, we denote its* j*-th child by* $\alpha[j]$*. Let us call the set of vertices whose distance from the root is exactly* k *by* $\mathcal{T}_{k,N}$*, where* $0 \leqslant k \leqslant N$.

Note that $\mathcal{T}_N$ is a q-regular tree of heigth N such that the elements of $\mathcal{T}_{k,N}$ are q^{k-N}-periodic subsets of $\mathbb{T}$.

We recollect some basic facts about backwards martingales of periodic functions (see [13] and [35]). Consider the discrete probability space $(\alpha_{N;\emptyset}, 2^{\alpha_{N;\emptyset}}, \nu_N)$, where ν_N is the uniform probability measure on $\alpha_{N,\emptyset}$:

$$\nu_N = \frac{1}{q^N} \sum_{j=0}^{q^N-1} \delta_{\frac{j}{q^N}}. \tag{4.5}$$

Pick a function $f \in C^1(\mathbb{T})$ and define

$$f_k(x) = \frac{1}{q^{N-k}} \sum_{j=0}^{q^{N-k}-1} f\Big(x + \frac{j}{q^{N-k}}\Big), \quad k = 0, 1 \ldots, N,\ x \in \alpha_{N,\emptyset}. \tag{4.6}$$

We restrict our attention to $x \in \alpha_{N,\emptyset}$ only, even though the previous formula makes sense for arbitrary $x \in \mathbb{T}$. The function f_k is q^{k-N}- periodic, so, it is constant on each of the sets corresponding to the vertices in $\mathcal{T}_{k,N}$. That means we can identify f_k with a function on $\mathcal{T}_{k,N}$. One may verify that the sequence $f_0, f_1, \ldots, f_N$ is a martingale with respect to the filtration $\{\sigma(\mathcal{T}_{k,N})\}_{k=0}^{N}$, where $\sigma(T_{k,N})$ is the algebra of all q^{k-N}-periodic subsets of $\alpha_{N,\emptyset}$. Note that the elements of $\mathcal{T}_{k,N}$ are the atoms of $\sigma(\mathcal{T}_{k,N})$.

We may express the f_k in Fourier terms:

$$f_k(x) = \frac{1}{q^{N-k}} \sum_{j=0}^{q^{N-k}-1} \sum_{l\in\mathbb{Z}} \widehat{f}(l) e^{2\pi i l (x + \frac{j}{q^{N-k}})}$$
$$= \sum_{l\in\mathbb{Z}} \Big(\widehat{f}(l) e^{2\pi i l x} \cdot \frac{1}{q^{N-k}} \sum_{j=0}^{q^{N-k}-1} e^{2\pi i \frac{lj}{q^{N-k}}} \Big)$$
$$= \sum_{q^{N-k}|l} \widehat{f}(l) e^{2\pi i l x}, \quad (4.7)$$

for any $x \in \alpha_{N;\emptyset}$ (this relation also holds true for any $x \in \mathbb{T}$). Hence, the k-th martingale difference may be expressed as

$$df_k(x) = f_k(x) - f_{k-1}(x) = \sum_{q^{N-k}\| l} \widehat{f}(l) e^{2\pi i l x}, \quad x \in \alpha_{N;\emptyset}. \quad (4.8)$$

We use the notation

$$\mathbb{R}_0^q = \Big\{ (x_0, \dots, x_{q-1}) \in \mathbb{R}^q \colon \sum_{j=0}^{q-1} x_j = 0 \Big\}$$

and identify vectors $x \in \mathbb{R}^q$ with functions on $\mathbb{Z}_q$ in the natural way.

Lemma 4.9. *For any $\alpha \in \mathcal{T}_{k-1,N}$ we have*

$$\Big(df_k(\alpha[0]), df_k(\alpha[1]), \dots, df_k(\alpha[q-1]) \Big) =$$
$$\sum_{m=1}^{q-1} \Big(\sum_{n\in\mathbb{Z}} \widehat{f}\big((m+nq)q^{N-k}\big) e^{2\pi i (m+nq) q^{N-k} x_0} \Big) \omega_m,$$

where $x_0 \in \alpha$ and

$$\omega_m := (\omega^{mj})_{j=0}^{q-1} := \Big(e^{\frac{2\pi i m j}{q}} \Big)_{j=0}^{q-1}, \quad j = 0, 1, \dots, q-1.$$

Definition 4.10. *By the Discrete Fourier transform on $\mathbb{Z}_q$ (henceforth called $\mathbb{Z}_q$-Fourier transform for convenience) we understand the linear operator on $\mathbb{C}^q$ given by the matrix $\big(\frac{1}{q} e^{-\frac{2\pi i}{q} mn}\big)_{m,n=0}^{q-1}$.*

Remark 4.11. *Vectors ω_m are the rows of the inverse $q \times q$ Fourier matrix (DFT matrix).*

Remark 4.12. *In other words, the vector* $(df_k(\alpha[0]), df_k(\alpha[1]), \dots, df_k(\alpha[q-1]))$ *is the inverse* $\mathbb{Z}_q$*-Fourier transform of the vector* $(e_0, e_1, \dots, e_{q-1})$ *with* $e_0 = 0$ *and*

$$e_m = \sum_{n\in\mathbb{Z}} \widehat{f}((m+nq)q^{N-k}) e^{2\pi i(m+nq)q^{N-k}x_0}, \quad m = 1, 2, \dots, q-1.$$

The above lemma is standard, see, e.g. [13]. We provide its proof for completeness.

Proof of Lemma 4.9. Let us prove our formula for each coordinate individually. For any j, $j = 0, 1, \dots, q-1$, we would like to show

$$df_k(\alpha[j]) = \sum_{m=1}^{q-1} \sum_{n\in\mathbb{Z}} \widehat{f}\big((m+nq)q^{N-k}\big) e^{2\pi i(m+nq)q^{N-k}x_0} e^{\frac{2\pi imj}{q}}.$$

Note that this expression does not depend on $x_0 \in \alpha$ since $q^{N-k}(x_0 - x_0') \in \mathbb{Z}$ for any other $x_0' \in \alpha$. On the other hand, we may use (4.8) by representing $x \in \alpha[j]$ as $x = x_0 + \frac{j}{q^{N-k+1}}$, where $x_0 \in \alpha$:

$$df_k(x) = \sum_{q^{N-k}\| l} \widehat{f}(l) e^{2\pi ilx} = \sum_{m=1}^{q-1} \sum_{n\in\mathbb{Z}} \widehat{f}\big((m+nq)q^{N-k}\big) e^{2\pi i(m+nq)q^{N-k}x} =$$

$$\sum_{m=1}^{q-1} \sum_{n\in\mathbb{Z}} \widehat{f}\big((m+nq)q^{N-k}\big) e^{2\pi i(m+nq)(x_0 + \frac{j}{q^{N-k+1}})q^{N-k}} =$$

$$\sum_{m=1}^{q-1} \sum_{n\in\mathbb{Z}} \widehat{f}\big((m+nq)q^{N-k}\big) e^{2\pi i(m+nq)q^{N-k}x_0} e^{\frac{2\pi imj}{q}}.$$

□

Definition 4.13. *Let* W_B *be the linear subspace of* $\mathbb{R}_0^q$ *consisting of vectors* d *whose* $\mathbb{Z}_q$*-Fourier transform vanishes outside* B*:*

$$W_B = \Big\{ d \in \mathbb{R}_0^q \colon \forall m \in \mathbb{Z}_q \setminus B \quad \sum_{j=0}^{q-1} e^{-\frac{2\pi imj}{q}} d_j = 0 \Big\}.$$

Lemma 4.14. *Let* $f \in C^1(\mathbb{T})$ *be such that* $f\,dx \in \mathcal{M}_{C_B}(\mathbb{T})$*. For any* $\alpha \in \mathcal{T}_N$*, we have the inclusion*

$$\Big(df_k(\alpha[0]), df_k(\alpha[1]), \dots, df_k(\alpha[q-1]) \Big) \in W_B.$$

Proof. In view of Remark 4.12, $e_m = 0$ for any $m \in B$ in the terminology of that remark, provided $f\,dx \in \mathcal{M}_{C_B}(\mathbb{T})$. □

4.2.2 A general dimension estimate

Consider an auxillary function $\kappa\colon \mathbb{R}_+ \to \mathbb{R}$ defined by the rule

$$\kappa(\theta) = \sup\Big\{\theta \log\Big(\frac{1}{q}\sum_{j=1}^{q} |1+v_j|^{\frac{1}{\theta}}\Big)\colon v \in W_B \text{ and } \forall j \quad v_j \geqslant -1\Big\}. \tag{4.9}$$

Note that κ is continuous and convex, and therefore, has the left derivative at 1. Indeed, by the Hölder's inequality, for a fixed $v \in W_B$, the function

$$\theta \mapsto \theta \log\Big(\frac{1}{q}\sum_{j=1}^{q} |1+v_j|^{\frac{1}{\theta}}\Big) \tag{4.10}$$

is convex, and so is κ as a pointwise supremum of convex functions. Using this, we may compute its left derivative:

Lemma 4.15. *We have*

$$\kappa'(1) = \inf\Big\{ -\frac{1}{q}\sum_{j=1}^{q}(1+v_j)\log(1+v_j)\colon v \in W_B \textit{ and } \forall j \quad v_j \geqslant -1\Big\}, \tag{4.11}$$

where the derivative means the left derivative.

Proof. Let us call $S = \{v \in W_B \colon v_j \geqslant -1\}$ and for a fixed $v \in W_B$ denote κ_v the function given by (4.10). It is easy to verify that the right-hand side of (4.11) is equal to $\inf_{v\in S}\kappa'_v(1)$ and that $\kappa(1) = \kappa_v(1) = 0$. By the convexity of κ_v, we have

$$\frac{\kappa_v(\theta) - \kappa_v(1)}{\theta - 1} \leqslant \kappa'_v(1),$$

where $\theta \in (0,1)$. Thus,

$$\frac{\kappa(\theta) - \kappa(1)}{\theta - 1} \leqslant \inf_{v\in S}\kappa'_v(1)$$

and

$$\kappa'(1) \leqslant \inf_{v\in S}\kappa'_v(1).$$

Now we will prove the reverse inequality. In view of the fact that $\kappa'(1)$ exists, for any $\epsilon > 0$ we can find $v_\epsilon \in S$, such that for some θ we have

$$\kappa'(1) \geqslant \frac{\kappa(\theta) - \kappa(1)}{\theta - 1} \geqslant \kappa'_{v_\epsilon}(1) - \epsilon.$$

Indeed, the first inequality follows from convexity. To get the second, we replace κ with κ_{v_ϵ}(for a suitable choice of v_ϵ we make arbitrarily small error), use the mean value theorem and the fact that $\kappa'_v(\theta)$ tends to $\kappa'_v(1)$ as $\theta \to 1$ uniformly with respect to parameters $v \in S$. This obviously gives

$$\kappa'(1) \geqslant \inf_{v\in S}\kappa'_v(1).$$

□

The next lemma is simply a reformulation of the definition of κ.

Lemma 4.16. *For any $a \geqslant 0$ and any vector $b = (b_i)_i \in W_B$ such that $b_j \geqslant -a$ for any $j = 0, 1, \ldots, q-1$, we have*

$$\left(\frac{1}{q}\sum_{j=1}^{q} |a + b_j|^p\right)^{\frac{1}{p}} \leqslant a e^{\kappa(p^{-1})}.$$

Our main tool is the following principle established in [7] and adjusted to our case.

Theorem 4.17. *For any finite non-negative measure $\mu \in \mathcal{M}_{\mathrm{C}_B}(\mathbb{T})$, we have*

$$\dim_{\mathcal{H}}(\mu) \geqslant 1 + \frac{\kappa'(1)}{\log q}.$$

Let $\{\Phi_N\}_{N\geqslant 1}$ be a sequence of non-negative and smooth functions with the following properties:

$$\Phi_N(x) = \begin{cases} q^N & \text{on } [-\frac{1}{2q^N}, \frac{1}{2q^N}]; \\ \leqslant q^N & \text{on } [-\frac{1}{2q^{N-1}}, \frac{1}{2q^{N-1}}] \setminus [-\frac{1}{2q^N}, \frac{1}{2q^N}]; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\mu\Big(\Big[x - \frac{1}{2q^N}, x + \frac{1}{2q^N}\Big]\Big) \leqslant \frac{1}{q^N}\Phi_N * \mu(x) \leqslant \mu\Big(\Big[x - \frac{1}{2q^{N-1}}, x + \frac{1}{2q^{N-1}}\Big]\Big) \qquad (4.12)$$

for any $x \in \mathbb{T}$, in particular, for $x \in \alpha_{N;\emptyset}$. The inequalities (4.12) establish a relationship between metric measure structures on $\mathcal{T}_N$ and $\mathbb{T}$. Henceforth, we will be using results concerning the backwards martingale generated by the continuous function $f = \Phi_N * \mu$. Note that $f\,dx \in \mathcal{M}_{\mathrm{C}_B}(\mathbb{T})$ provided $\mu \in \mathcal{M}_{\mathrm{C}_B}(\mathbb{T})$.

Lemma 4.18. *Consider the martingale $\{f_k\}_{k=0}^N$ generated by $f = \Phi_N * \mu$ via formula* (4.6). *If $\mu \in \mathcal{M}_{\mathrm{C}_B}(\mathbb{T})$, then*

$$\|f\|_{L_p(\nu_N)} \leqslant e^{\kappa(p^{-1})N}\|f_0\|_{L_p(\nu_N)} \leqslant (q+1)e^{\kappa(p^{-1})N}\|\mu\|. \qquad (4.13)$$

We recall that ν_N is the counting measure defined in (4.5).

Proof. Let us prove the first inequality in (4.13). This inequality will follow provided we justify the single step bound

$$\|f_k\|_{L_p(\nu_N)} \leqslant e^{\kappa(p^{-1})}\|f_{k-1}\|_{L_p(\nu_N)}$$

for any $k = 1, 2 \ldots, N$. This inequality, in its turn, follows from even more localized ones: for any $\alpha \in \mathcal{T}_{k-1,N}$, we have

$$\Big(\sum_{x\in\alpha} |f_k(x)|^p\Big)^{\frac{1}{p}} \leqslant e^{\kappa(p^{-1})}\Big(\sum_{x\in\alpha} |f_{k-1}(x)|^p\Big)^{\frac{1}{p}}.$$

To prove this inequality, we note that since $\mu \geqslant 0$, the sequence $\{f_k\}_k$ consists of non-negative functions. What is more, $f_k = f_{k-1} + df_k$ and the vector

$$df_k|_\alpha = \big(df_k(\alpha[0]), df_k(\alpha[1]), \ldots, df_k(\alpha[q-1])\big)$$

lies in W_B by Lemma 4.14. So, the desired inequality is proved by application of Lemma 4.16 with $a = f_{k-1}(\alpha)$ and $b = df_k|_\alpha$.

To prove the second inequality in (4.13), we use that $f_0 \equiv \frac{1}{q^N}\sum_{x\in\mathcal{T}_{N,N}} \Phi_N * \mu(x)$ on $\alpha_{N;\emptyset}$:

$$\|f_0\|_{L_p(\nu_N)} = \frac{1}{q^N}\sum_{x\in\mathcal{T}_{N,N}} \Phi_N * \mu(x) \overset{(4.12)}{\leqslant} \sum_{x\in\mathcal{T}_{N,N}} \mu\Big(\Big[x - \frac{1}{2q^{N-1}}, x + \frac{1}{2q^{N-1}}\Big]\Big) \leqslant (q+1)\|\mu\|.$$

□

Lemma 4.19. *For any any $\beta < 1 + \frac{\kappa'(1)}{\log q}$, there exists γ such that*

$$\frac{1}{q^N}\sum_{x\in C} f(x) \lesssim \big(\#C\, q^{-\beta N}\big)^{\gamma}\|\mu\| \tag{4.14}$$

for any $C \subset \alpha_{N;\emptyset}$, with the constant independent of N.

Proof. Let $p \in (1,\infty)$ be a real to be chosen later. By Hölder's inequality and Lemma 4.18, we obtain

$$\frac{1}{q^N}\sum_{x\in C} f(x) \leqslant \|f\|_{L_p(\nu_N)}\|\chi_C\|_{L_{p'}(\nu_N)} = \|f\|_{L_p(\nu_N)}(q^{-N}\#C)^{\frac{p-1}{p}} \lesssim e^{\kappa(p^{-1})N}q^{-\frac{p-1}{p}N}(\#C)^{\frac{p-1}{p}}\|\mu\| = e^{\kappa(p^{-1})N}q^{\frac{p-1}{p}(\beta-1)N}(q^{-\beta N}\#C)^{\frac{p-1}{p}}\|\mu\|. \tag{4.15}$$

Hence (4.14) is true with $\gamma = \frac{p-1}{p}$ when $e^{\kappa(p^{-1})}q^{\frac{p-1}{p}(\beta-1)} < 1$, that is if

$$\kappa(p^{-1}) + (\beta-1)\frac{p-1}{p}\log q < 0.$$

This holds true when $(\beta - 1)\log q < \kappa'(1)$ and p is sufficiently close to 1. □

As we have already said, the reasoning presented below is very much similar to the proof of Lemma 2.4 in [75].

Proof of Theorem 4.17. Assume the contrary: there exists a Borel set F such that

$$\dim_{\mathcal{H}}(F) < \beta_1 < 1 + \frac{\kappa'(1)}{\log q} \quad \text{and } \mu(F) = c_1 > 0.$$

For each sufficiently small $\delta > 0$, there exists a covering C of F by the arcs $B(x_i, r_i)$ with centers x_i and radii r_i such that $r_i < \delta$ and $\sum_i r_i^{\beta_1} = c_2 < \infty$. For $j = 1, 2\dots$ let

$$C_j = \big\{B(x_i, r_i) \in C\colon\, q^{-j} \leqslant r_i < q^{-j-1}\big\}.$$

We have

$$\sum r_i^{\beta_1} \simeq \sum_j q^{-j\beta_1} \# C_j,$$

so, in particular, $\# C_j \lesssim c_2 q^{j\beta_1}$ for all j. By the pigeonhole principle, there exists $N \gtrsim \log \frac{1}{\delta}$ such that

$$\mu\Big(F \bigcap \Big(\bigcup_{B(x_i,r_i)\in C_N} B(x_i, r_i)\Big)\Big) \geqslant \frac{6}{\pi^2} \frac{c_1}{N^2}.$$

Since any $B(x_i, r_i) \in C_N$ can be covered by at most $q + 1$ arcs from the collection $\{x + [-\frac{1}{2q^N}, \frac{1}{2q^N}]\colon\, x \in T_{N,N}\}$, there exists a covering

$$\tilde{C}_N \subset \Big\{x + \Big[-\frac{1}{2q^N}, \frac{1}{2q^N}\Big]\colon\, x \in \mathcal{T}_N\Big\}$$

such that $\#\tilde{C}_N \leqslant \# C_N$ and

$$\mu\Big(\cup_{L\in\tilde{C}_N} L\Big) \geqslant \frac{1}{q+1}\mu\Big(F \bigcap \Big(\bigcup_{B(x_i,r_i)\in C_N} B(x_i, r_i)\Big)\Big).$$

Let us call $\mathrm{Mid}(\tilde{C}_N)$ the set of midpoints of arcs from $\tilde{C}_N$. For the previously obtained N, we apply (4.12) and Lemma 4.19 with $\beta > \beta_1$ and obtain

$$\frac{6}{\pi^2}\frac{c_1}{N^2(q+1)} \leqslant \mu(\cup_{L\in\tilde{C}_N} L) \leqslant \frac{1}{q^N} \sum_{x\in\mathrm{Mid}(\tilde{C}_N)} f(x) \lesssim \big(\# C_N\, q^{-\beta N}\big)^{\gamma} \|\mu\| \lesssim c_2^{\gamma} q^{\gamma(\beta_1-\beta)N}.$$

Hence we have $N^2 q^{-c_3 N} \geqslant c_4 > 0$ for some positive constants c_3, c_4, independent of δ and N. On the other hand, we have $N \to \infty$ when $\delta \to 0$, which leads to a contradiction. ☐

4.3 Proof of Theorem 4.2

Proof of Theorem 4.2. In view of Theorem 4.17, it suffices to show the inequality

$$\kappa'(1) \geqslant -\log|H|$$

provided $B \subset H \backslash \{0\}$ and $\kappa'(1) > -\log|H|$ in the case where the latter inclusion is proper. We will show that

$$\kappa\Big(\frac{1}{p}\Big) \leqslant \frac{p-1}{p}\log|H| \tag{4.16}$$

for any $p \in (1,\infty)$ and this inequality is strict if $B \neq H$. Until the end of the proof the Fourier transform means the Fourier transform on $\mathbb{Z}_q$. The normalization is the same as in the Definition 4.10.

Let $v \in W_B$. Then, v is the $\mathbb{Z}_q$-Fourier transform of a vector supported on H, so $v = v * \check{\chi}_H = |H| \cdot v * \chi_{H^\perp}$. Here, χ_A stands for the characteristic function of a set A and by $H^\perp$ we understand the annihilator of H, i.e.

$$H^\perp = \{m \in \mathbb{Z}_q : e^{\frac{2\pi i}{q}mx} = 1 \quad \forall x \in H\}.$$

It is easy to check that $H^\perp$ is a subgroup of $\mathbb{Z}_q$, that $\mathbb{Z}_q/H^\perp \simeq H$ and that $|H| \cdot |H^\perp| = q$. Hence, in the coordinates $(h,h') \in \mathbb{Z}_q/H^\perp \times H^\perp \simeq \mathbb{Z}_q$ (here the isomorphism sign means the natural bijection corresponding to the partition of $\mathbb{Z}_q$ by cosets of $H^\perp$) we have $v(h,h') = v(h,0)$ for all (h,h') in $\mathbb{Z}_q$, i.e. v depends on the first coordinate only. We see that each extremal point x_0 of the set

$$\Big\{x \in \mathbb{R}^q_0 \colon \forall (h,h') \in \mathbb{Z}_q \quad x(h,h') = x(h,0);\ x(h,h') \geqslant -1\Big\} \tag{4.17}$$

is characterized by the property that the function $\mathbb{Z}_q/H^\perp \ni h \mapsto x_0(h,0)$ attains the value $|H|-1$ at some h and -1 at the remaining $|H|-1$ elements. From this, the convexity of the p-norm, and formula (4.9), we get

$$\kappa\Big(\frac{1}{p}\Big) \leqslant \frac{1}{p}\log\Big(\frac{|H^\perp|}{q}|H|^p\Big) = \frac{p-1}{p}\log|H|.$$

This and the strict convexity of the L_p-norm proves that (4.16) is strict provided the inclusion $B \subset H \setminus \{0\}$ is proper. In this case, $\kappa'(1) > -\log|H|$ since the function κ is convex. □

Remark 4.20. *Theorem 4.2 is not true if we consider all complex measures; the counterexample is* $B = \{l\}$ *and* $\mu = \frac{1}{q}\sum_{k=0}^{q-1}\omega^{kl}\delta_{\{\omega^k\}}$.

4.4 Proof of Theorem 4.3

We will rely upon the simple observation that $\mu_{a,q} \in \mathcal{M}_{\mathrm{C}_{\{1,q-1\}}}(\mathbb{T})$. So, our aim is to compute the value $\kappa'(1)$ for the case $B = \{1, q-1\}$. In this case, any $v \in W_B$ is of the form

$$v = a\omega_1 + \bar{a}\omega_{q-1}, \quad \text{for some} \quad a \in \mathbb{C}.$$

The above gives

$$W_B = \left\{ c\Big(\cos\Big(\frac{2\pi j}{q} + \varphi\Big)\Big)_{j=0}^{q-1} : c \in \mathbb{R}, \varphi \in [-\pi, \pi] \right\}.$$

According to (4.11), we want to maximize a convex function

$$\mathbb{R}_0^q \ni x \mapsto \sum_{j=0}^{q-1} (1 + x_i) \log(1 + x_i)$$

over a convex region

$$\mathcal{C} = W_B \cap \{x \in \mathbb{R}_0^q : x_j \geqslant -1, \quad j = 0, \ldots, q-1\}.$$

The function above is convex because $t \to t \log t$ is convex for positive reals. Thus, our purpose is to maximize the quantity

$$\sum_{j=0}^{q-1} \Big(1 - \gamma \cos\Big(\frac{2\pi j}{q} + \varphi\Big)\Big) \log\Big(1 - \gamma \cos\Big(\frac{2\pi j}{q} + \varphi\Big)\Big), \tag{4.18}$$

where γ is chosen in such a way that all the summands are well-defined (the quantity we compute the logarithm of is non-negative) and $\varphi \in [-\frac{\pi}{q}, \frac{\pi}{q}]$ (by periodicity). The change of sign inside summands is legal since we can replace φ with $\varphi + \pi$. Without loss of generality, we may assume that at least one of the summands vanishes (as this holds for extremal points of $-\mathcal{C}$). Since $\varphi \in [-\frac{\pi}{q}, \frac{\pi}{q}]$ this leads to $\gamma = (\cos\varphi)^{-1}$.

Therefore, the supremum of (4.18) equals

$$\sup_{\varphi \in [-\frac{\pi}{q}, \frac{\pi}{q}]} \sum_{j=0}^{q-1} \Big(1 - \frac{\cos(\frac{2\pi j}{q} + \varphi)}{\cos\varphi}\Big) \log\Big(1 - \frac{\cos(\frac{2\pi j}{q} + \varphi)}{\cos\varphi}\Big) =$$

$$\sup_{\varphi \in [-\frac{\pi}{q}, \frac{\pi}{q}]} \sum_{j=0}^{q-1} \Big(1 - \cos\frac{2\pi j}{q} + \sin\frac{2\pi j}{q} \tan\varphi\Big) \log\Big(1 - \cos\frac{2\pi j}{q} + \sin\frac{2\pi j}{q} \tan\varphi\Big). \tag{4.19}$$

Consider the function g:

$$g(x) = \sum_{j=0}^{q-1} (a_j + b_j x) \log(a_j + b_j x), \quad x \in \Big[- \tan\frac{\pi}{q}, \tan\frac{\pi}{q}\Big]$$

where $a_j = 1 - \cos\frac{2\pi j}{q}$ and $b_j = \sin\frac{2\pi j}{q}$.

Lemma 4.21. *For any* $q \geqslant 3$,

$$\sup_{x\in[-\tan\frac{\pi}{q},\tan\frac{\pi}{q}]} g(x) = g\Big(\tan\frac{\pi}{q}\Big)$$

In particular, the supremum in (4.19) is attained at the endpoints since $\tan$ is a monotone function on $[-\frac{\pi}{q},\frac{\pi}{q}]$.

Proof of Lemma 4.21. Note that g is convex since the expressions $a_j + b_j x$ are linear and non-negative when $x \in [-\tan\frac{\pi}{q},\tan\frac{\pi}{q}]$, and the function $t \mapsto t\log t$ is convex on the positive semi-axis. It remains to add that g is symmetric. □

Proof of Theorem 4.3. The result follows from Theorem 4.17 and the already proved formula

$$\kappa'(1) = -\frac{1}{q}\sum_{j=1}^{q-2}\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big)\log\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big) \tag{4.20}$$

for the case $B = \{1, q-1\}$. □

4.5 Proof of Proposition 4.5

Proof of Lemma 4.4. Consider the function $f\colon \mathbb{R}\to\mathbb{R}$ defined as follows:

$$f(a) = \sum_{j=0}^{q-1}\Big(a-\cos\frac{(2j+1)\pi}{q}\Big)\log\Big|a-\cos\frac{(2j+1)\pi}{q}\Big|.$$

The sum on the left hand-side of (4.4) is then equal to

$$\frac{f(\cos\frac{\pi}{q})}{\cos\frac{\pi}{q}} - q\log\cos\frac{\pi}{q}.$$

The function f is absolutely continuous and

$$f'(a) = \log\prod_{j=0}^{q-1}\Big|a-\cos\frac{(2j+1)\pi}{q}\Big| + q = \log\Big(2^{-q+2}T_p^2(a)\Big) + q,$$

where $q = 2p$, by our assumptions, $p\in\mathbb{N}$, and T_p is the Chebyshev polynomial of order p, that is

$$T_p(x) = \cos(p\arccos x) = 2^{p-1}\prod_{j=0}^{p-1}\Big(x-\cos\Big(\frac{(j+\frac{1}{2})\pi}{p}\Big)\Big), \qquad x\in[-1,1].$$

Note that by symmetry (here we heavily use that q is even), $f(0) = 0$. Thus, since f is an absolutely continuous function,

$$f\Big(\cos\frac{\pi}{q}\Big) = \int\limits_0^{\cos\frac{\pi}{q}} \Big(\log\Big(2^{-q+2}T_p^2(a)\Big) + q\Big)\, da =$$

$$(1-\log 2)q\cos\frac{\pi}{q} + (2\log 2)\cos\frac{\pi}{q} + \int\limits_0^{\cos\frac{\pi}{q}} \log\cos^2(p\arccos a)\, da =$$

$$(1-\log 2)q\cos\frac{\pi}{q} + (2\log 2)\cos\frac{\pi}{q} + \int\limits_{\frac{\pi}{q}}^{\frac{\pi}{2}} \log\cos^2(px)\sin x\, dx =$$

$$(1-\log 2)q\cos\frac{\pi}{q} + (2\log 2)\cos\frac{\pi}{q} + \frac{2}{q}\int\limits_{\frac{\pi}{2}}^{\frac{q\pi}{4}} \log\cos^2 z\sin\frac{2z}{q}\, dz.$$

So, the sum on the left hand-side of (4.4) equals

$$(1-\log 2)q + 2\log 2 + \frac{2}{q\cos\frac{\pi}{q}}\int\limits_{\frac{\pi}{2}}^{\frac{q\pi}{4}} \log\cos^2 z\sin\frac{2z}{q}\, dz - q\log\cos\frac{\pi}{q}.$$

□

Proof of Proposition 4.5. Since $\mu_{a,q} \in \mathcal{M}_{\mathrm{C}_{\{1,q-1\}}}(\mathbb{T})$, Theorem 4.17 says that

$$\dim_{\mathcal{H}}(\mu_{a,q}) \geqslant 1 + \frac{\kappa'(1)}{\log q}.$$

Thus, it remains to combine this estimate with formula (4.20) and Lemma 4.4. □

Remark 4.22. *Proposition 4.5 shows that in Theorem 4.3, in the case of even q's, our method gives the same asymptotics as we would expect from* (4.3). *Indeed, the integral*

$$\frac{2}{q}\int\limits_{\frac{\pi}{2}}^{\frac{q\pi}{4}} \log(\cos^2 z)\sin\frac{2z}{q}\, dz$$

is equal, up to an error of size $O(\frac{1}{q})$, to the integral

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log(\cos^2 z)dz = -2\log 2,$$

and thus it cancels with $2\log 2$. *To prove this, it suffices to observe that*

$$\Big|\sum_{j=1}^{\frac{q}{2}-1}\sin\Big(\frac{j\pi}{q}\Big)\cdot\int_{\frac{j\pi}{2}}^{\frac{(j+1)\pi}{2}}\log(\cos^2 z)\,dz-\int_{\frac{\pi}{2}}^{\frac{q\pi}{4}}\log(\cos^2 z)\sin\frac{2z}{q}\,dz\Big|\leqslant$$

$$\sum_{j=1}^{\frac{q}{2}-1}\int_{\frac{j\pi}{2}}^{\frac{(j+1)\pi}{2}}\Big|\log(\cos^2 z)\Big|\Big|\sin\Big(\frac{2}{q}\cdot j\cdot\frac{\pi}{2}\Big)-\sin\frac{2z}{q}\Big|\,dz\leqslant$$

$$(\frac{q}{2}-1)\cdot\pi\log 2\cdot\frac{2}{q}\cdot\frac{\pi}{2}\leqslant\frac{\pi^2}{2}\log 2,$$

and that the expression

$$\frac{\pi}{q}\sum_{j=1}^{\frac{q}{2}-1}\sin\Big(\frac{j\pi}{q}\Big)$$

is a Riemann sum of

$$\int_0^{\frac{\pi}{2}}\sin x\,dx=1.$$

4.6 Proof of Proposition 4.6

Proof of Proposition 4.6. In view of the identity

$$\frac{1}{2\pi}\int_0^{2\pi}(1-\cos x)\log(1-\cos x)dx=1-\log 2,$$

we need to bound the expression below:

$$\Big|\frac{1}{q}\sum_{j=1}^{q-2}\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big)\log\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big)-\frac{1}{2\pi}\int_0^{2\pi}(1-\cos x)\log(1-\cos x)dx\Big|\leqslant$$

$$\Big|\frac{1}{q}\sum_{j=1}^{q-2}\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big)\log\Big(1-\frac{\cos\frac{(2j+1)\pi}{q}}{\cos\frac{\pi}{q}}\Big)-$$

$$\frac{1}{q}\sum_{j=1}^{q-2}\Big(1-\cos\frac{(2j+1)\pi}{q}\Big)\log\Big(1-\cos\frac{(2j+1)\pi}{q}\Big)\Big|+$$

$$\Big|\frac{1}{q}\sum_{j=1}^{q-2}\Big(1-\cos\frac{(2j+1)\pi}{q}\Big)\log\Big(1-\cos\frac{(2j+1)\pi}{q}\Big)-\frac{1}{2\pi}\int_0^{2\pi}(1-\cos x)\log(1-\cos x)dx\Big|$$

$$=: I+II$$

Let us denote $h(t) = (1-t)\log(1-t)$, $\theta_q = \frac{1}{\cos\frac{\pi}{q}}$, and let us define the numbers $m_{q,j}$ and $M_{q,j}$ by

$$m_{q,j} = \min\{\cos\frac{(2j+1)\pi}{q}, \theta_q \cos\frac{(2j+1)\pi}{q}\}$$

and

$$M_{q,j} = \max\{\cos\frac{(2j+1)\pi}{q}, \theta_q \cos\frac{(2j+1)\pi}{q}\}.$$

By the mean value theorem, for some $\Theta_{q,j} \in \big[m_{q,j}, M_{q,j}\big]$, $j = 1, \dots, q-2$, we have:

$$\begin{aligned} I = \frac{1}{q}\Big|\sum_{j=1}^{q-2} h\Big(\cos\Big(\frac{(2j+1)\pi}{q}\Big)\Big) - h\Big(\theta_q \cos\Big(\frac{(2j+1)\pi}{q}\Big)\Big)\Big| \\ \leqslant \frac{1}{q}\sum_{j=1}^{q-2} |1-\theta_q| \cdot \Big|\cos\Big(\frac{(2j+1)\pi}{q}\Big)\Big| \cdot |h'(\Theta_{q,j})| \\ \leqslant \frac{q-2}{q}|1-\theta_q| + \frac{|1-\theta_q|}{q}\sum_{j=1}^{q-2} |\log(1-\Theta_{q,j})| \leqslant \\ \frac{q-2}{q}\Big(|1-\theta_q| + |1-\theta_q| \cdot \Big|\log\Big(1-\theta_q\cos\Big(\frac{3\pi}{q}\Big)\Big)\Big|\Big). \end{aligned}$$

In the remaining part of calculations we will use the following three elementary inequalities:

$$1-\cos x \leqslant \frac{x^2}{2}, \quad x \in \mathbb{R}, \tag{4.21}$$

$$\sin x \geqslant \frac{2}{\pi}x, \quad x \in [0, \frac{\pi}{2}], \tag{4.22}$$

$$|x\log x| \leqslant \frac{1}{e}, \quad x \in [0,1]. \tag{4.23}$$

The first one implies the following bound

$$|1-\theta_q| = \frac{1-\cos\frac{\pi}{q}}{\cos\frac{\pi}{q}} \leqslant \frac{\frac{1}{2}\Big(\frac{\pi}{q}\Big)^2}{1-\frac{1}{2}\Big(\frac{\pi}{q}\Big)^2} \leqslant \Big(\frac{\pi}{q}\Big)^2.$$

On the other hand, by (4.22) we get

$$1-\theta_q\cos\Big(\frac{3\pi}{q}\Big) = \frac{2\sin\frac{\pi}{q}\sin\frac{2\pi}{q}}{\cos\frac{\pi}{q}} \geqslant \frac{16}{q^2}.$$

By combining the above estimates we obtain

$$I \leqslant \theta_q - 1 + \Big(\frac{\pi}{q}\Big)^2\Big|\log\frac{16}{q^2}\Big| \overset{(4.23)}{\leqslant} \theta_q - 1 + \frac{\pi^2}{2e}\cdot\frac{1}{q}.$$

Thus, it remains to prove that

$$II \leqslant \frac{4\pi}{q}.$$

This is a consequence of the following bound

$$\Big|\frac{d}{dx}(1-\cos x)\log(1-\cos x)\Big| = |\sin x(1+\log(1-\cos x))| \leqslant 2.$$

To prove the last inequality, we estimate $\sin x$ by one and $1-\cos x$ by e. □

4.7 Further examples and comments

A more general form of the backwards martingale that we used appears also as an element of the proof of the dimension estimate in [65]. In that paper, it is used to prove a version of the pointwise ergodic theorem with respect to Riesz products.

The assumption of being a non-negative measure from $\mathcal{M}_B(\mathbb{T})$ implies the symmetry of B. Theorems corresponding to the case when B is (strongly) antisymmetric were considered in [13].

Remark 4.23. *For a fixed q we can define δ_q as the best constant such that the inequality*

$$\dim_{\mathcal{H}}(\mu) \geqslant \delta_q > 0$$

is true for any finite non-negative measure from $\mathcal{M}_{C_B}(\mathbb{T})$ and $B \neq \mathbb{Z}_q \setminus \{0\}$. If q is small, then the constant δ_q may be estimated by a direct computation of the extremal points of

$$span\{\omega_m\}_{m\in B} \cap \{x \in \mathbb{R}_0^q : \forall j \quad x_j \geqslant -1\},$$

for all possible choices of symmetric sets $B \neq \mathbb{Z}_q \setminus \{0\}$. Namely, for any choice of such B, the function $\kappa'(1)$ can be bounded from below by the smallest value of

$$x \mapsto -\frac{1}{q}\sum_{j=1}^{q}(1+x_i)\log(1+x_i)$$

on the set of all such extremal points.

For example, if $q = 4$ then we may take $B = \{2\}$ or $B = \{1,3\}$. In the first case, the extremal points are $\pm(1,-1,1,-1)$, while for the second choice they are $\pm(1,1,-1,-1), \pm(1,-1,-1,1)$. This gives $\delta_4 \geqslant \frac{1}{2}$.

An obvious converse of Theorem 4.2 says that singular measures have rich spectrum in the arithmetical sense.

Corollary 4.24. *Let $\mu \in \mathcal{M}(\mathbb{T})$ be a non-negative finite measure such that*

$$\dim_{\mathcal{H}}(\mu) < \delta_q,$$

where δ_q is as in the above remark. Then for each $m \in \{1,\dots,q-1\}$ there exists $n \in \operatorname{spec}(\mu)$ such that n has a divisor with residue m modulo q.

4.8 Appendix: applications of an entropic uncertainty principle

The main result of this section is the following improvement of Theorem 4.2

Theorem 4.25. *Let $B \subset \mathbb{Z}_q \setminus \{0\}$ and let $\mu \in \mathcal{M}_{\mathrm{C}_B}(\mathbb{T})$ be a finite non-negative measure. Then*

$$\dim_{\mathcal{H}}(\mu) \geqslant 1 - \frac{\log(\#B+1)}{\log q}.$$

The proof is an immediate consequence of Theorem 4.17, Lemma 4.15 and the lemma below. The symbols dh and dm stand for the probability Haar measure and the counting measure on $\mathbb{Z}_q$, respectively. For convenience, we will be writing $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{Z}_q, dh)}$.

Lemma 4.26. *Suppose that $f \in L^1(\mathbb{Z}_q, dh)$ satisfies $\|f\|_1 = 1$ and $f \geqslant 0$. Then*

$$-\frac{1}{q} \sum_{m \in \mathbb{Z}_q} f(m) \log f(m) \geqslant -\log(\# \operatorname{spec}(f)). \tag{4.24}$$

We proceed as in [46], but instead of differentiating the Hausdorff-Young inequality, we differentiate, in a sense, the Young convolution inequality. Let us recall that the Discrete Fourier transform on $\mathbb{C}^q$ is the linear operator $F : \mathbb{C}^q \to \mathbb{C}^q$ given by the matrix $\left(\frac{1}{q} e^{-\frac{2\pi i}{q} mn}\right)_{m,n=0}^{q-1}$ and we denote by $\widehat{\ }$ the operator arising from F via identification of $\mathbb{C}^q$ with the space of complex-valued functions on the cyclic group $\mathbb{Z}_q$. With this normalization, $\widehat{\ }\colon L^2(\mathbb{Z}_q, dh) \to L^2(\mathbb{Z}_q, dm)$ is an isometry.

Proof. Denote $A = \operatorname{spec}(f)$. We have $\widehat{f} = \widehat{f} \cdot 1_A$, so $f = f * \widecheck{1_A}$. By the Young convolution inequality we have

$$\|f\|_2 = \|f * \widecheck{1_A}\|_2 \leqslant \|f\|_1 \cdot \|\widecheck{1_A}\|_2 = \|\widecheck{1_A}\|_2.$$

Moreover, by the Plancharel theorem,

$$\|\widecheck{1_A}\|_2 = \sqrt{\#A}.$$

Thus, by the Hölder inequality, for $1 \leqslant p \leqslant 2$ we have

$$\|f\|_p \leqslant \|f\|_1^{1-\theta} \|f\|_2^{\theta},$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} \Rightarrow \theta = 2 \cdot \frac{p-1}{p}.$$

This implies that

$$\|f\|_p \leqslant (\#A)^{\frac{p-1}{p}}. \tag{4.25}$$

Let us define $G(p) = \log\|f\|_p$. By (4.25) we have

$$\frac{G(p) - G(1)}{p - 1} = \frac{G(p)}{p - 1} \leqslant \frac{1}{p} \log \#A,$$

hence

$$G'_+(1) \leqslant \log \#A,$$

where G'_+ stands for the derivative from the right. On the other hand $-G'_+(1)$ is equal to the left-hand side of (4.24), so the theorem follows. □

An immediate consequence of Theorem 4.25 is the estimate of δ_q, defined in the Remark 4.23

Corollary 4.27. *For any $q \geqslant 4$ we have*

$$\delta_q \geqslant 1 - \frac{\log(q-2)}{\log q}.$$

In particular, if $\mu \in \mathcal{M}^+(\mathbb{T})$ and

$$\dim_{\mathcal{H}}(\mu) < 1 - \frac{\log(q-2)}{\log q},$$

then for each $m \in \{1, \ldots, q\}$ there exists $n \in \operatorname{spec}(\mu)$, such that n has a divisor with residue m modulo q.

Chapter 5

Microlocal approach to the Hausdorff dimension of measures

In this chapter we study the dependence of geometric properties of Radon measures, such as the Hausdorff dimension and rectifiability of singular sets, on the wave front set. We prove our results by adapting the method of Brummelhuis to the non-analytic case. As an application we obtain a general form of uncertainty principle for measures on the complex sphere which subsumes certain classical results about pluriharmonic measures.

5.1 Preliminaries and motivation

The purpose of this chapter is to extend the programme of [11] to the case of singular measures in a quantitative way. Namely, in the mentioned paper it is presented how to derive analyticity of measures (in the sense of belonging to the local Hardy-Goldberg space) from the knowledge about their wave front sets ([11], Theorem 1.4.). This was obtained by translating properties of Riesz sets to the microlocal setting. The key point in our modification is the replacement of this notion by the notion of s-Riesz sets. This operation yields the following uncertainty principle which expresses a duality between the size of the wave front set and the Hausdorff dimension:

Definition 5.1. *We say that a set F has a $k-$dimensional gap if there exists a $k-$dimensional linear space $V \subset \mathbb{R}^n$ with a conic neighbourhood N_V such that*

$$F \cap N_V \setminus B(0,r) = \emptyset$$

for some ball $B(0,r)$.

Theorem 5.2. *Let μ be a Radon measure on $\mathbb{R}^n$ such that $\mathrm{WF}_x(\mu)$ has a $k-$dimensional gap at $\mu-$almost every $x \in \mathbb{R}^n$. Then*

$$\dim_{\mathcal{H}}(\mu) \geqslant k. \tag{5.1}$$

Moreover, if a $k-$dimensional Borel set $E \subset \mathbb{R}^n$ satisfies $\mathcal{H}^k(E) < +\infty$ and $\mu(E) \neq 0$, then there exists a $k-$rectifiable set $E_r \subset E$ such that $|\mu|(E \setminus E_r) = 0$.

For the basic properties of rectifiable sets we refer the reader to Chapter 18 in [57]. The second part of the theorem asserts about additional regularity of sets minimizing (5.1). The above may be thought as a substitute of k-Riesz sets in non-Euclidean settings. In particular, it has several consequences in the study of measures on the complex sphere. Fine properties of such measures were studied, among others, in [3], [17], [18], [19], [31], [32], [40], [44] and [71] .

To state our results, we need to recall some basic notions from the Harmonic Analysis on $S^{2n-1} = \{z \in \mathbb{C}^n \colon |z| = 1\}$ (see Chapter 12 in [72] for detailed informations about this topic). In the considerations below we treat S^{2n-1} as a $(2n-1)-$dimensional submanifold of $\mathbb{R}^{2n}$ and the Hausdorff dimension is computed with respect to the Euclidean metric on $\mathbb{R}^{2n}$. As previously, we denote by $\mathcal{M}(S^{2n-1})$ the set of finite, Borel regular measures on S^{2n-1}. By $\mathbb{Z}_+$ we understand the set of non-negative integers, and for $(p, q) \in \mathbb{Z}_+^2$ the symbol $H(p, q)$ stands for the space of restrictions to S^{2n-1} of all harmonic homogeneous polynomials in $\mathbb{C}^n$ which are of degree p in $z_1, \ldots, z_n$ and of degree q in $\bar{z}_1, \ldots, \bar{z}_n$. Those spaces form an orthogonal decomposition $L^2(S^{2n-1}, \sigma) = \bigoplus_{p,q\geqslant 0} H(p, q)$, where σ is the $(2n-1)$-dimensional Hausdorff measure on S^{2n-1}. We call $\pi_{p,q} : L^2(S^{2n-1}, \sigma) \to H(p, q)$ the orthogonal projection onto $H(p, q)$. This transformation is given by the reproducing kernel $K_{p,q}$ (see [3], p. 118 for the explicit formula) and can be continued to the space of finite measures. This leads to the below definition of spectrum (which will be used by us until the end of the chapter):

Definition 5.3. *For any $\mu \in \mathcal{M}(S^{2n-1})$ we define*

$$\mathrm{spec}(\mu) = \Big\{(p, q) \in \mathbb{Z}_+^2 \colon \pi_{p,q}\mu(z) = \int_{S^{2n-1}} K_{p,q}(z, w)\, \mathrm{d}\,\mu(w) \not\equiv 0\Big\}.$$

Theorem 5.2 applies to this setting as follows:

Definition 5.4. *For $E \subset S^{2n-1}$ we write $\mathbb{T} \cdot E := \{e^{it}z \colon z \in E \text{ and } t \in [0, 2\pi]\}$.*

Definition 5.5. *For any $0 < \epsilon < 1$ we denote*

$$\kappa(\epsilon) := \Big\{(x, y) \in (0, +\infty)^2 \colon 1 - \epsilon < \frac{y}{x} < \frac{1}{1 - \epsilon}\Big\}.$$

Theorem 5.6. *Let $\mu \in \mathcal{M}(S^{2n-1})$ be a measure such that $\mathrm{spec}(\mu) \cap \kappa(\epsilon)$ is finite or empty for some $0 < \epsilon < 1$. Then μ satisfies the following regularity property:*

$$|\mu|(\mathbb{T} \cdot E) = 0 \quad \text{if } \mathcal{H}^{2n-2}(E) = 0. \tag{5.2}$$

Moreover, if $\mu(E) \neq 0$ for some $(2n-2)-$dimensional Borel set $E \subset S^{2n-1}$ such that $\mathcal{H}^{2n-2}(E) < +\infty$, then there exists a $(2n-2)-$rectifiable set $E_r \subset E$ such that $|\mu|(E \setminus E_r) = 0$.

Corollary 5.7. *For any $\mu \in \mathcal{M}(S^{2n-1})$ satisfying the assumptions of Theorem 5.6 we have*

$$\dim_{\mathcal{H}}(\mu) \geqslant 2n - 2.$$

In comparison with Theorem 2.1. in [11], the above says that, even after dropping the assumption about strong antisymmetry of spectrum, we can still maintain high regularity under relatively weak Fourier constraints. Natural examples of measures that satisfy them are the so-called pluriharmonic measures . They correspond to the case when the spectrum lies in the sum of horizontal and vertical ray, i.e.

$$\operatorname{spec}(\mu) \subset \{(p,0)\colon p \in \mathbb{Z}_+\} \cup \{(0,q)\colon q \in \mathbb{Z}_+\}$$

(cf. also Example 5.20 in the last section of this chapter). For this case, property (5.2) was proved in full generailty by Aleksandrov ([3], Theorem 3.1.2.), and by Forelli under additional assumption about positivity ([31], Corollary 1.11.). Let us point out that, by Proposition 3.3.1. in [3], positive pluriharmonic measures must vanish on $(2n-2)$–dimensional C^1 submanifolds of S^{2n-1}. This and Theorem 5.6 imply

Corollary 5.8. *If μ is a positive pluriharmonic measure, then $\mu(E) = 0$ for any $(2n-2)$–dimensional set E such that $\mathcal{H}^{2n-2}(E) < +\infty$.*

5.2 More on properties of s-Riesz sets

In this section we list several theorems that we microlocalize in further steps.

Theorem 5.9. *Let $V \subset \mathbb{R}^n$ be a k–dimensional subspace, $\alpha, \beta \in (0, +\infty)$ and let $S_{V,\alpha,\beta}$ be the complement of the set*

$$\{(\xi_1, \xi_2) \in V \times V^\perp\colon |\xi_1| \geqslant \alpha,\ |\xi_2| \leqslant \beta|\xi_1|\}.$$

Then $S_{V,\alpha,\beta}$ is a k–Riesz set.

This is a direct consequence of Theorem 1 in [69] and is sufficient for proving (5.1) and Corollary 5.7. However, for our other purposes we need also a slightly stronger form which also follows from the methods applied in [69]. We enclose its proof for completeness.

Theorem 5.10. *Let V, α, β be as in Theorem 5.9. If $\mu \in \mathcal{M}(\mathbb{R}^n)$ satisfies $\operatorname{supp}(\widehat{\mu}) \subset S_{V,\alpha,\beta}$, then*

$$\mathcal{H}^k_{\llcorner V}(\Pi_V(E)) = 0 \quad \Rightarrow \quad |\mu|(E) = 0. \tag{5.3}$$

Proof. Denote by $\pi(\mu) = (\Pi_V)_*\mu$ the pushforward of μ by Π_V, that is the measure satisfying $\pi(\mu)(A) = \mu(A \times V^\perp)$ for $A \subset V$. For $a \in \mathbb{R}^n$ let us define $\tau_a\mu$ by the formula $\mathrm{d}\,\tau_a\mu = e^{-2\pi\langle a,\cdot\rangle}\,\mathrm{d}\,\mu$. For $\xi = (\xi', 0),\ t = (t', 0) \in V \times V^\perp$ we have

$$\pi(\tau_a\mu)\widehat{\ }(\xi') = \int_V e^{-2\pi i\langle \xi', t'\rangle}\,\mathrm{d}\,\pi(\tau_a\mu)(t') = \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi, s\rangle}\,\mathrm{d}\,\tau_a\mu(s) = \widehat{\mu}(\xi + a).$$

Since $S_{V,\alpha,\beta} \cap (V + a)$ is a bounded set, it is also a Riesz set (folklore, see Example 3.32 in Chapter 3), which implies absolute continuity of $\pi(\tau_a \mu)$ with respect to the Lebesgue measure on V. In particular, for $E' = \Pi_V(E)$ such that $\mathcal{H}^k_{\llcorner V}(E') = 0$ we get

$$\tau_a \mu(E' \times V^\perp) = \pi(\tau_a \mu)(E') = 0.$$

Thus, for any $a \in \mathbb{R}^n$

$$\mu_{\llcorner E' \times V^\perp}\widehat{\ }(a) = \int_{E' \times V^\perp} e^{-2\pi i \langle a,s \rangle} d\mu(s) = \tau_a \mu(E' \times V^\perp) = 0,$$

and finally $\mu_{\llcorner E' \times V^\perp} \equiv 0$ by the uniqueness theorem. □

Proof. (of Theorem 3.31, cf. also [34]) The same as above: in the previous reasoning we replace boundednes of a slice $S_{V,\alpha,\beta} \cap (V + a)$ with the property that $A \cap (V + a)$ is a Riesz set. □

With a little help of the Besicovitch-Federer projection theorem ([57], Theorem 18.1) we can adjust the above for dealing with rectifiability of singular sets.

Theorem 5.11. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be as in the previous theorem. Then, for any $k-$dimensional Borel set E such that $\mathcal{H}^k(E) < +\infty$ and $\mu(E) \neq 0$ there exists a $k-$rectifiable set $E_r \subset E$ such that $|\mu|(E \setminus E_r) = 0$.*

Proof. Let us begin with an observation that, for $k-$dimensional vector spaces $W \in \mathbb{G}(k, \mathbb{R}^n)$, satisfying the formula

$$\forall a \in \mathbb{R}^n \quad S_{V,\alpha,\beta} \cap (W + a) \text{ is a bounded set} \tag{5.4}$$

is an open condition in the natural topology on the Grassmannian $\mathbb{G}(k, \mathbb{R}^n)$. Thus, the same proof as in Theorem 5.10 gives even stronger statement: There exists $\mathcal{O}_V \subset \mathbb{G}(k, \mathbb{R}^n)$, a neighbourhood of V of positive Haar measure, such that

$$\mathcal{H}^k_{\llcorner W}(\Pi_W(F)) = 0 \quad \Rightarrow \quad |\mu|(F) = 0 \tag{5.5}$$

is true for $W \in \mathcal{O}_V$ and any μ satisfying $\operatorname{supp}(\mu) \subset S_{V,\alpha,\beta}$.

By Theorem 18.1. in [57] we can decompose $E = E_r \cup E_u$ into disjoint sum of $k-$rectifiable and $k-$unrectifiable set. It suffies to apply the Besicovitch-Federer projection theorem and (5.5) to obtain $|\mu|(E_u) = 0$. □

5.3 Basic notions and facts from Microlocal Analysis

For the convenience of the reader we recall some basic facts from Microlocal Analysis. Let us start from the definition of the wave front set (see Chapter 8 in [43] or [77]).

If $\nu \in \mathcal{E}'(\mathbb{R}^n)$, then we define $\Sigma(\nu)$ as the set of those $\xi \in \mathbb{R}^n \setminus \{0\}$, for which there is no conic neighbourhood C such that

$$\forall_{N\in\mathbb{N}}\exists_{C_N>0} \ |\widehat{\nu}(\xi')| \leqslant C_N(1+|\xi'|)^{-N} \quad \text{for } \xi' \in C. \tag{5.6}$$

For an arbitrary $\nu \in \mathcal{D}'(\mathbb{R}^n)$ we define

$$\mathrm{WF}_x(\nu) = \bigcap_{\phi} \big\{\Sigma(\phi\nu)\colon \ \phi \in C_c^\infty(\mathbb{R}^n),\ \phi(x) \neq 0\big\}. \tag{5.7}$$

A very important property of WF_x is the following: for any set $V \subset \mathbb{R}^n$ which is a conic neighbourhood of $\mathrm{WF}_x(\nu)$, there exists a neighbourhood of x, say U_x such that

$$\mathrm{WF}_x(\nu) \subset \Sigma(\phi\nu) \subset V \quad \text{for any } \phi \in C_c^\infty(U_x),\ \phi(x) \neq 0. \tag{5.8}$$

This object can be also introduced for distributions on manifolds, namely by the following definition of a pullback: If $\Phi\colon\ M \to N$ is a C^∞–diffeomorphism between manifolds M and N, then W_x of the pullback $\Phi^*\nu$ is described by

$$\mathrm{WF}_x(\Phi^*\nu) = \{D\Phi^t(x)\eta\colon\ \eta \in \mathrm{WF}_{\Phi(x)}(\nu)\}. \tag{5.9}$$

Definition 5.12. *For any distribution $\nu \in \mathcal{D}'(M)$ on a manifold M, we define the wave front set of ν by*

$$\mathrm{WF}(\nu) = \{(x,\xi) \in T^*M \setminus \{0\} : \xi \in \mathrm{WF}_x(\nu)\}.$$

where $\mathrm{WF}_x(\nu)$ is defined locally, by the above formula for a pullback, with local maps taken for Φ.

Now let us briefly discuss the notions of pseudodifferential operator and its principal symbol.

Definition 5.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\bar{\mathrm{S}} \subset C^\infty(\Omega \times \mathbb{R}^n)$ be some set of functions. We say that the operator $p(x,D) : \mathcal{D}(\Omega) \to C^\infty(\mathbb{R}^n)$ is a pseudodifferential operator belonging to the class $OP\bar{\mathrm{S}}$ if and only if*

$$p(x,D)f(x) = \int p(x,\xi)\widehat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}d\xi$$

for some $p(x,\xi) \in \bar{\mathrm{S}}$.

The set $\bar{\mathrm{S}}$ from the definition above is called a symbol class. The most frequently used are the following:

- Let $\Omega \subset \mathbb{R}^n$ be an open set, $m \in \mathbb{R}$ and $0 \leqslant \rho, \delta \leqslant 1$. We call $S^m_{\delta,\rho}(\Omega)$ the set of those $p(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ for which the following property holds: for each compact set $K \subset \Omega$ and all multi-indices α, β there exists a constant $C = C(K,\alpha,\beta)$ such that

$$\forall_{x\in\Omega,\xi\in\mathbb{R}^n} \quad |D_x^\beta D_\xi^\alpha p(x,\xi)| \leqslant C(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

- The set $S^m_{1,0}(\Omega)$ is often denoted by $S^m(\Omega)$.
- We denote $S^m_{cl}(\Omega)$ the class of functions $p(x,\xi) \in S^m(\Omega)$ for which there exist
$$p_{m-j}(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n) \quad j = 0, 1, \dots \tag{5.10}$$
such that
$$p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x,\xi) \quad \text{ for } \ |\xi|, r \geqslant 1$$
and
$$\forall N \geqslant 0 \quad p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S^{m-N-1}(\Omega).$$
The last relation is usually denoted
$$p(x,\xi) \sim \sum_{j \geqslant 0} p_{m-j}(x,\xi)$$
and elements of $S^m_{cl}(\Omega)$ are called classical (or polyhomogeneous) symbols.

If a pseudodifferential operator T belongs to any of sets given by the above symbol classes, then we refer to the index m as the order of T.

Definition 5.14. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $p(x,D) \in OPS^m_{\rho,\delta}(\Omega)$. We call a principal symbol of $p(x,D)$ any member of equivalence class of $p(x,\xi)$ in $S^m_{\rho,\delta}(\Omega)/S^{m-(2\rho-1)}_{\rho,\delta}(\Omega)$. We denote any fixed representative of this class by $\sigma(p)$.*

For example, if $p(x,D)$ is a differential operator on $\mathbb{R}^n$ of order m
$$p(x,D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}},$$
then for its principal symbol we can take
$$\sigma(p(x,D))(x,\xi) = (2\pi i)^{|m|} \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha.$$

Pseudodifferential operators can be defined on manifolds, by the use of local maps (see [77] Chapter II or [42] Chapter XVIII), which leads to the definition of a symbol as a function on the cotangent bundle. In particular, if (M,g) is a Riemannian manifold then the principal symbol of the Laplace-Beltrami operator is equal to $\sigma(\Delta_M)(x,\xi) = -c\|\xi\|_g^2$ for some $c > 0$ depending on the normalization of the Fourier transform (we identify tangent and cotangent bundle by g).

Definition 5.15. *Let T be a pseudodifferential operator of order m on a manifold M, with a principal symbol $\sigma(T)(x,\xi)$ homogeneous in ξ in the sense that*
$$\sigma(T)(x, r\xi) = r^m \sigma(T)(x,\xi) \quad \textit{for} \quad |\xi|, r \geqslant 1.$$
Then we define the characteristic set of T by
$$\mathrm{Char}(T) = \{(x,\xi) \in T^*M \colon \sigma(T)(x,\xi) = 0\}.$$

5.4 Proofs

Proof. (of Theorem 5.2) Let us begin with proving the dimension bound. We may assume that our measure has a k–dimensional gap at every x. By the property (5.8) and the assumptions we have that for each $x \in \mathbb{R}^n$ there exists a neighbourhood U_x such that $\Sigma(\phi\mu) \subset S_{V(x),\alpha(x),\beta(x)}$ for some k–dimensional space $V(x)$, some $\alpha(x), \beta(x) \in (0, +\infty)$, and any $\phi \in C_c^\infty(U_x)$ such that $\phi(x) \neq 0$. Let us fix ϕ. After slight change of $\alpha(x)$ and $\beta(x)$, if needed, we can construct a function $\eta \in C^\infty(\mathbb{R}^n)$ such that

$$\eta(\xi) = \begin{cases} 0 & \text{on } \Sigma(\phi\mu), \\ 1 & \text{on } \mathbb{R}^n \setminus S_{V(x),\alpha(x),\beta(x)}. \end{cases}$$

From this and (5.6) we obtain that $f(x) = (\eta \cdot \widehat{\phi\mu})^{\vee} \in \mathcal{S}(\mathbb{R}^n)$ and the measure $\phi\mu - f dx$ satisfies the assumptions of Theorem 5.9. Since modifications of measures by absolutely continuous ones do not have any influence on singular sets, we get that

$$\forall_x \exists \text{ a neighbourhood } U_x \text{ s.t. } \dim_{\mathcal{H}}(\phi\mu) \geqslant k \quad \text{for } \phi \in C_c^\infty(U_x),\ \phi(x) \neq 0. \tag{5.11}$$

Suppose by contradiction that there exists F such that $\dim_{\mathcal{H}} F < k$ and $\mu(F) \neq 0$. By the regularity of μ, we may assume that F is compact, which provides the existence of a finite cover $F \subset \cup_j^N U_{x_j}$ with sets U_{x_j} satisfying (5.11). Let $\{\phi_j\}_{j=1}^N$ be a smooth partition of unity inscribed in $\{U_{x_j}\}_{j=1}^N$. We have

$$\mu(F) = \sum_{j=1}^N \phi_j \mu(F) = 0,$$

which gives the first part of the theorem. To get the rectifiability part we simply replace the use of Theorem 5.9 by Theorem 5.11 in the reasoning above. □

Before proving Theorem 5.2 let us remark that, since diffeomorphisms are locally bi-Lipschitz, we can obtain full information about dimension and rectifiability of μ from the knowledge about pushforward measures $\Phi_*\mu$, provided that we have sufficiently many good maps $\Phi\colon S^{2n-1} \to \mathbb{R}^{2n-1}$. Having in mind Theorem 5.10 and the fact that the action of $\mathbb{T}$ defines a foliation of S^{2n-1}, we can construct maps tailored to the proof of Theorem 5.6.

Recall ([3], Subsection 1.4.) that the (real) tangent space $T_z S^{2n-1}$ can be decomposed into an orthogonal sum $T_z^{\mathbb{C}} S^{2n-1} \oplus \mathbb{R}iz$, where

$$T_z^{\mathbb{C}} S^{2n-1} = \{\xi \in \mathbb{C}^n \colon \langle \xi, z \rangle_{\mathbb{C}^n} = 0\}.$$

We introduce coordinates $(\xi_1, \xi_2) \in T_z^{\mathbb{C}} S^{2n-1} \times \mathbb{R} \cong T_z S^{2n-1}$ accordingly to this splitting: $\xi = \xi_1 + \xi_2 iz$.

Lemma 5.16. *Suppose that* $\Phi\colon S^{2n-1} \to \mathbb{R}^n$ *is a smooth diffeomorphism and* $\mu \in \mathcal{M}(S^{2n-1})$. *Let* $\nu_1 = \Phi_*\mu$ *(understood as a pushforward measure) and* $\nu_2 = (\Phi^{-1})^*\mu$ *(understood as a pullback of a distribution). Then* ν_1 *and* ν_2 *are mutually absolutely continuous and* $\mathrm{WF}(\nu_1) = \mathrm{WF}(\nu_2)$.

Proof. It follows from the formula $\mathrm{d}\,\nu_2 = |\det D\Phi|\,\mathrm{d}\,\nu_1$. □

Lemma 5.17. *Suppose that* $E \subset S^{2n-1}$ *satisfies* $\mathcal{H}^{2n-2}(E) = 0$ *. Then, for any* $\mu \in \mathcal{M}(S^{2n-1})$ *and* $z_0 \in S^{2n-1}$ *there exists an open neighbourhood* $U_{z_0} \subset S^{2n-1}$ *and a smooth diffeomorphism* $\Phi\colon U_{z_0} \to \tilde{U} \subset T_{z_0}S^{2n-1}$, *such that*

1. $\Phi(z_0) = 0$ *and* $\mathrm{WF}_0(\Phi_*\mu) = \mathrm{WF}_{z_0}(\mu)$,
2. $\mathcal{H}^{2n-2}(\Pi_V\Phi(\mathbb{T}\cdot E \cap U_{z_0})) = 0$ *for* $V = T^{\mathbb{C}}_{z_0}S^{2n-1}$.

Proof. Let $\psi : S^{2n-1} \cap B(z_0, \epsilon) \to T_{z_0}S^{2n-1}$ be the orthogonal projection onto $T_{z_0}S^{2n-1}$. Here we choose small ϵ so that ψ was a bi-Lipschitz diffeomorphism. As we have already mentioned, since $\mathbb{T}$ acts on S^{2n-1} (freely) by multiplication, S^{2n-1} is foliated by leaves of the form $\{e^{it}\xi\}_{t\in[0,2\pi]}$ (see Theorem 11.3.9. in [12]). Thus, $\operatorname{im}\psi$ is foliated by leaves $\{\psi(e^{it}\xi)\}$.

Let us define a function γ on $\operatorname{im}\psi$ so that $p = (\xi_1, \xi_2) \in \operatorname{im}\psi \subset T_{z_0}S^{2n-1}$ is mapped to a point $(\tilde{\xi_1}, \xi_2)$, where $(\tilde{\xi_1}, 0)$ is the intersection point of the leaf $\{\psi(e^{it}\xi)\}$ containing p with $T^{\mathbb{C}}_{z_0}S^{2n-1}$. If ϵ is small enough, then γ is well defined as the leaves of foliation are transversal to $T^{\mathbb{C}}_{z_0}S^{2n-1}$ near 0. Moreover, γ is a diffeomorphism and $\Phi = \gamma \circ \psi$ is the desired map. Indeed, point (2) is satisfied because Φ and Π_V are Lipschitz. To prove (1), let us observe that $D\psi(z_0) = Id$ (since $D\psi^{-1}(0) = Id$) and $D\gamma(0) = Id$, as we have $\frac{d}{d\xi_2}\gamma(0) = (0, 1)$ and $\gamma_{|T^{\mathbb{C}}_{z_0}S^{2n-1}} = Id_{|T^{\mathbb{C}}_{z_0}S^{2n-1}}$. It remains to use the previous lemma and (5.9). □

Proof. (of Theorem 5.6) We essentially follow the steps of the proof of Theorem 2.1. in [11]. Our aim is to show that at each point z, $\mathrm{WF}_z(\mu)$ has a $(2n-2)$–dimensional gap given by $T^{\mathbb{C}}_zS^{2n-1}$. We consider S^{2n-1} with Euclidean metric inherited from $\mathbb{R}^{2n}$ and we identify TS^{2n-1} with T^*S^{2n-1} using this metric. Take two commuting, first order pseudodifferential operators

$$T_1 f(z) = \frac{1}{i}\frac{d}{dt}f(e^{it}z)\Big|_{t=0}$$

and

$$T_2 = \sqrt{-\Delta_{S^{2n-1}} + (n-1)^2 Id} - (n-1)Id.$$

The symbol $\Delta_{S^{2n-1}}$ stands for the spherical Laplacian. The eigenspaces of $\Delta_{S^{2n-1}}$ are $\mathcal{H}(j) = \bigoplus_{p+q=j} H(p,q)$ and the corresponding eigenvalues $\lambda_j = -j(j+2n-2)$. Principal symbols of T_1 and T_2 are

$$\sigma(T_1)(z, \xi_1, \xi_2) = \xi_2$$

and

$$\sigma(T_2)(z,\xi_1,\xi_2) = c\sqrt{|\xi_1|^2 + |\xi_2|^2}$$

for some constant c. The space $H(p,q)$ can be described as the common eigenspace of T_1 and T_2 with eigenvalues $p-q$ and $p+q$, respectively. Let $\chi \in C^\infty(\mathbb{R}^2)$ be any function having the following properties:

1. $\chi(x,y)$ is 0–homogeneous on $\kappa(\epsilon)$,
2. $\chi(x,y) \neq 0$ on $\kappa(\epsilon)$,
3. $\chi(x,y) = 0$ outside $\kappa(\epsilon) \cup B(0,\delta)$, for some small δ.

By the functional calculus from [76] (cf. also [77], Chapter 12), the operator

$$T_3 = \chi\Big(\frac{T_1+T_2}{2}, \frac{T_2-T_1}{2}\Big),$$

defined by the spectral theorem is equal to a 0–th order pseudo-differential operator with principal symbol

$$\sigma(T_3)(z,\xi_1,\xi_2) = \chi\Big(\frac{c\sqrt{|\xi_1|^2+|\xi_2|^2}+\xi_2}{2}, \frac{c\sqrt{|\xi_1|^2+|\xi_2|^2}-\xi_2}{2}\Big). \tag{5.12}$$

Moreover, by the assumptions we have

$$T_3\mu = \sum_{p,q\geqslant 0} \chi(p,q)\pi_{p,q}\mu \in C^\infty(S^{2n-1}). \tag{5.13}$$

Theorem 18.1.28 from [42] says that

$$\mathrm{WF}(\mu) \subset \mathrm{WF}(T_3\mu) \cup \mathrm{Char}(T_3). \tag{5.14}$$

From this and (5.13) we obtain that $\mathrm{WF}(\mu)$ is contained in $\mathrm{Char}(T_3)$, i.e. the set of $(z,\xi) \in T^*S^{2n-1} \setminus \{0\}$ such that

$$\sigma(T_3)(z,\xi_1,\xi_2) = 0. \tag{5.15}$$

It suffices then to show that

$$\mathrm{Char}(T_3)(z,\cdot) \subset S_{T_z^{\mathbb{C}}(S^{2n-1}),\alpha,\beta}$$

for parameters α,β depending on ϵ only. Suppose by contradiction that there exists a sequence $\xi^{(n)} = \xi_1^{(n)} + iz\xi_2^{(n)} \in \mathrm{Char}(T_3)(z,\cdot)$ such that $|\xi_2^{(n)}| \leqslant \beta_n|\xi_1^{(n)}|$ with $\beta_n \downarrow 0$. This means that

$$\frac{y_n}{x_n} := \frac{c\sqrt{|\xi_1^{(n)}|^2+|\xi_2^{(n)}|^2}-\xi_2^{(n)}}{c\sqrt{|\xi_1^{(n)}|^2+|\xi_2^{(n)}|^2}+\xi_2^{(n)}} \to 1$$

as $n \to +\infty$ and $\chi(\frac{x_n}{2},\frac{y_n}{2}) = 0$. This cannot hold by the definition of χ.

To finish the proof, it remains to use Lemma 5.17 in combination with Theorem 5.10, Theorem 5.11 and an argument analogous to the covering argument from the proof of Theorem 5.2. □

Proof. (of Corollary 5.8) By Theorem 5.6, we may assume that E is rectifiable. Whitney extension theorem ([30], Theorem 3.1.16.) says that E is contained in a countable union of $(2n-2)$–dimensional C^1–submanifolds, thus in view of Proposition 3.3.1. from [3] we have $\mu(E) = 0$. □

5.5 Examples

Example 5.18. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be the uniform Hausdorff measure on a smooth k–dimensional manifold M. Then, at each $x \in M$, $\mathrm{WF}_x(\mu)$ has a k-dimensional gap given by the tangent space.*

Example 5.19. *(cf. [10], p. 140) For any $\xi \in S^{2n-1}$, the one-dimensional Hausdorff measure on the set $\mathbb{T} \cdot \{\xi\}$ has its spectrum inside $\{(p,p) \colon p \in \mathbb{Z}_+\}$. Thus, in Theorem 5.6, the semi-axis $\{(x,x) : x \in (0,+\infty)\}$ cannot be replaced by any other one.*

Example 5.20. *Our results can be applied to the study of the so-called d–pluriharmonic measures, introduced in [17]. They are those measures from $\mathcal{M}(S^{2n-1})$, whose spectrum lies inside*

$$\{(p,q) \in \mathbb{Z}_+^2 \colon (p-d)(q-d) = 0 \text{ and } p,q \geqslant d\}.$$

In particular, the above class contains the classical pluriharmonic measures as 0–pluriharmonic measures.

We would like to finish the chapter by leaving the following open problem:

Question 5.21. *Is the dimension bound from Corollary 5.7 sharp?*

Index

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