Referee report on the PhD Dissertation:
Thermodynamic formalism and multifractal analysis for matrix cocycles and solenoids
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The doctoral dissertation of Mr Reza Mohammadpour Bejargafsheh consists of four chapters. In the first one the new results are stated. The second chapter is called Preliminaries. In this Chapter, Mr Reza Mohammadpour Bejargafsheh provides a thorough account of the large number of tools used in the thesis. The third chapter is about the multifractal analysis of the matrix cocycles. Finally, the last chapter is about the dimension of thin nonlinear solenoids.

The results presented in chapter 3 are about linear cocycles. Given a topological dynamical system $(X, T)$. That is a compact metric space $X$ with metric $d$ and a continuous mapping $T : X \to X$. Moreover, we are given a measurable map $\mathcal{A} : X \to GL(k, \mathbb{R})$, the space of $k \times k$ invertible matrices. The corresponding matrix cocycle is

$$\mathcal{A}^n(x) = \mathcal{A}\left(T^{n-1}(x)\right) \dots \mathcal{A}(T(x))\mathcal{A}(x).$$

We say that the linear cocycle is a locally constant cocycle if $X = \{1, \dots, q\}^{\mathbb{Z}}$, $T$ is the left shift, moreover,there is a set of matrices $\{A_1, , \dots, A_q\}$ such that $\mathcal{A} : X \to GL(k, \mathbb{R})$ is given by $\mathcal{A}(x) := A_{x_0}$ for an $x = (\dots, x_{-n}, \dots, x_0, \dots, x_n, \dots) \in X$. Let $T : \Sigma \to \Sigma$ be a topological mixing subshift of finite type. We write $H^r(\Sigma, GL(k, \mathbb{R}))$ for the family of $r$-Hölder continuous mappings $\mathcal{A} : \Sigma \to GL(k, \mathbb{R})$. The collection of fibre bunched elements of $H^r(\Sigma, GL(k, \mathbb{R}))$ is denoted by $H^r_b(\Sigma, GL(k, \mathbb{R}))$. We write $\mathcal{W}$ for the set of those elements of $H^r_b(\Sigma)$ which are pinching and twisting (definitions pinching and twisting are given in Section 3.4.2). For a continuous function $f : \Sigma \to \mathbb{R}$ the Birkhoff spectrum is

$$L := \left\{\alpha \in \mathbb{R} : \exists x \in X \text{ and } \lim_{n\to\infty} \frac{1}{n} S_n f(x) = \alpha\right\},$$

and the $\alpha$-level set of the Birkhoff spectrum is

$$E_f(\alpha) = \left\{x \in X : \frac{1}{n} S_n f(x) \to \alpha \text{ as } n \to \infty\right\},$$

where $S_n f(x) = \sum_{k=0}^{n-1} f(T^k(x))$. In Theorem 1.21 and Theorem 1.2.2 the followings were proved: For an $\mathcal{A} \in \mathcal{W}$ the function $\alpha \mapsto h_{top}(E(\alpha))$ in concave in the interior of $L$ and $L = \overline{\{\alpha : h_{top}(E(\alpha)) > 0\}}$. Furthermore,

$$h_{top}(E(\alpha)) = \sup \{h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T), \chi(\mu, \mathcal{A}) = \alpha\}$$
$$= \inf \{P_{\Phi_{\mathcal{A}}}(q) - \alpha \cdot q : q \in \mathbb{R}\} \, \forall \alpha \in \Omega := \left\{ \int f d\mu : \mu \in \mathcal{M}(X, T) \right\}.$$

Let $\chi(x, \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\|$ . We write $\chi(\mu, \mathcal{A}) = \int \chi(., \mathcal{A}) d\mu$. If the measure $\mu$ is ergodic then $\chi(x, \mathcal{A}) = \chi(\mu, \mathcal{A})$ for $\mu$-almost all $x$.

Let $\beta(\mathcal{A})$ be the maximum of $\chi(x, \mathcal{A})$. That is

$$\beta(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x \in X} \|\mathcal{A}^n(x)\| .$$

Moreover, let $\mathcal{M}_{\max}(\mathcal{A})$ be the set of measures which maximize the Lyapunov exponent:

$$\mathcal{M}_{\max}(\mathcal{A}) := \{\mu \in \mathcal{M}(X, T) : \beta(\mathcal{A}) = \chi(\mu, \mathcal{A})\}.$$

In Theorems 1.2.3 and 1.2.4 for a $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ sub-additive potential, the family of equilibrium measures $\mu - t$ for $t\Phi$ is considered as $t \to \infty$. It is proved that for any accumulation point $\mu$ of $\mu_t$ as $t \to \infty$, $\mu$ is Lyapunov maximizing measure for $\Phi$ and

- $\chi(\mu, \Phi) = \lim_{t\to\infty} \chi(\mu_t, \Phi)$
- $h_\mu(T) = \lim_{t\to\infty} h_{\mu_t}(T) = \max \{h_\nu(T), \nu \in \mathcal{M}_{\max}(\Phi)\}$

The last result of this chapter is an extension of the results from the references [Ol], [Feng1], [FFW] in which similar assertions were proved for additive potentials by Olsen, Feng, Fan, Feng, Wu. In Theorem 1.2.6 The continuity of the topological pressure is proved. This extends earlier results of Feng and Shmerkin who proved the same for locally constant potentials. This result also extends a recent result of Park.

In Section 4 the Hausdorff dimension is computed for thin nonlinear solenoids. Solenoids are very important examples of hyperbolic attractors. In the linear case when on the stable slices the mapping is non-conformal, the problem of finding the dimension was open for more than a decade.

First result was obtained by Bothe then the referee and then Hasselblat and Schmelling made important achievement related the dimension of the attractor. It was pointed out by Michal Rams (the PhD supervisor of Mr Reza Mohammadpour Bejargafsheh ) and the referee that for thin linear solenoids, the SBR measure is equivalent to the (appropriate dimensional) packing dimension and the appropriate dimensional Hausdorff measure of the attractor is zero.

Mr Reza Mohammadpour Bejargafsheh with his co-authors, Michal Rams and Felix Przytycki, obtained the following results related to the dimension theory of solenoids: Let $M = S^1 \times \mathbb{D}$ be the solid torus, where $\mathbb{D} = \{v \in \mathbb{R}^2 : | v | < 1\}$. We write $d_1$, $d_2$ for the distance on $S^1$ and $\mathbb{D}$ respectively and put $d := d_1 \times d_2$. Given a $C^{1+\alpha}$ injective mapping $f : M \to M$ of the form:

$$(x, y, z) \mapsto (\eta(x) \bmod 2\pi, \lambda(x, y) + u(x), \nu(x, y, z) + v(x)),$$

where $\lambda(x, 0) = \nu(x, 0, 0) = 0$, and for every $x, y, z$

$$\frac{\partial}{\partial x}\eta(x, y, z) > 1, \quad \frac{\partial}{\partial y}\lambda(x, y, z) < 1, \quad \chi\left(\mu_{t_0}, \frac{\partial}{\partial z}\nu\right) < \chi\left(\mu_{t_0}, \frac{\partial}{\partial x}\lambda\right),$$

where $t_0$ is the affinity dimension of the stable slices defined on page 73, and $\mu_{t_0}$ is the geometric measure introduced in Section 4.4 Moreover, the functions $\lambda, \nu$ and $\eta(x, y, z) - d \times x$ are $2\pi$ -periodic with respect to $x$, and $d \geq 2$ is always assumed. The attractor is denoted by $\Lambda$. The chapter starts with a good explanation of the holonomy map. Figure 8 is really useful. Solenoids are very important examples of hyperbolic attractors.

The main result of the chapter is Theorem 4.4 which asserts that if

1. $\frac{\partial}{\partial x}\eta$ is a constant,
2. $\sup \frac{\partial}{\partial y}\lambda(p) < \left(\frac{\partial}{\partial x}\eta\right)^{-1}(p) = 1/d \left(d \text{ the is degree of } \frac{\partial}{\partial x}\eta\right)$ for $p \in \Lambda$,
3. The unstable lines of the $\pi_{x,y}(\Lambda)$ intersect each other transversal where $\pi_{x,y}$ is the projection to the $(x, y)$ lane.

Then $\dim_H(\Lambda) = 1 + \dim_H(\Lambda_x) = 1 + t_0$ for every $x \in S^1$.

This theorem relates to a conjecture of Hasselblatt and Schmeling which conjecture states that the Hausdorff dimension of a hyperbolic set is the sum of the dimensions of the stable and unstable slices. This conjecture is verified for solenoids in Theorem 4.4.4.

In summary, this is an excellent PhD thesis. The candidate has a deep understanding of a number of fields of the theory of dynamical systems of dimension theory. The results are really interesting and important and the proofs are correct. The thesis is well written. I suggest awarding Mr Reza Mohammadpour Bejargafsheh the PhD degree on summa cum laude (with highest distinction) level.

Suggested corrections

1. p. 28 l. 7 $v \in E_{i+1} \setminus \{0\}$,
2. p. 37 the distance $d(\cdot, \cdot)$ appears in formula (3.4.1.1). I did not find the precise definition. Probably it is presented somewhere I just did not find it. I believe it is $d(x, y) = \omega^{|x \wedge y|}$, where $\omega \in (0, 1)$ is used in Definition 3.4.2.
3. p. 43 very end of Example 3.4.13. A typo in the formula $\sigma^2 < 2^{\alpha\ 1}$.
4. p.63 l. 10 typo: injective instead of invective,

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